Invent. math. 88, 125–151 (1987)

Inventiones mathematicae

Rigidity and the lower bound Theorem 1

Gil Kalai

Institute of Mathematics, Hebrew University, Jerusalem, Israel and
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Summary. For an arbitrary triangulated $(d-1)$-manifold without boundary C with f_0 vertices and f_1 edges, define $\gamma(C)=f_1-\mathrm{d}f_0+\binom{d+1}{2}$. Barnette proved that $\gamma(C)\geqq 0$. We use the rigidity theory of frameworks and, in particular, results related to Cauchy's rigidity theorem for polytopes, to give another proof for this result. We prove that for $d\geqq 4$, if $\gamma(C)=0$ then C is a triangulated sphere and is isomorphic to the boundary complex of a stacked polytope. Other results: (a) We prove a lower bound, conjectured by Björner, for the number of k-faces of a triangulated $(d-1)$-manifold with specified numbers of interior vertices and boundary vertices. (b) If C is a simply connected triangulated d-manifold, $d\geqq 4$, and $\gamma(\mathrm{lk}(v, C))=0$ for every vertex v of C, then $\gamma(C)=0$. ($\mathrm{lk}(v,C)$ is the link of v in C.) (c) Let C be a triangulated d-manifold, $d\geqq 3$. Then $\mathrm{skel}_1(\Delta_{d+2})$ can be embedded in $\mathrm{skel}_1(C)$ iff $\gamma(C)>0$. (Δ_d is the d-dimensional simplex.) (d) If P is a 2-simplicial d-polytope then $f_1(P)\geqq \mathrm{d}f_0(P)-\binom{d+1}{2}$. Related problems concerning pseudomanifolds, manifolds with boundary and polyhedral manifolds are discussed.

1. The lower bound theorem

Barnette's *lower bound theorem* (LBT) ([9, 10]) asserts that if P is a simplicial d-polytope with n vertices, then $f_k(P)$, the number of k-dimensional faces of P, satisfies the inequality $f_k(P)\geqq \varphi_k(n,d)$, where

$$\varphi_k(n,d)=\begin{cases}\binom{d}{k}n-\binom{d+1}{k+1}k & \text{for } 1\leqq k\leqq d-2\\ (d-1)n-(d+1)(d-2) & \text{for } k=d-1.\end{cases} \tag{1.1}$$

Barnette's theorem settled an old conjecture in the theory of convex polytopes. (See [31, pp. 183–188] for the history of this conjecture.)

The main purpose of this paper is to show the connection between the lower bound theorem and the *rigidity theory of frameworks*. The basic idea is quite simple. Let P be a simplicial d-polytope, $d \geqq 3$, with n vertices. The inequality $f_1(P) \geqq \varphi_1(n,d)$ follows from the fact that P is *rigid*. This means that every small perturbation of the vertices of P, which does not change the length of the edges of P, is induced by an affine rigid motion of $\mathbb{R}^d$. The crucial result is Cauchy's rigidity theorem ([22]) which gives the rigidity of simplicial 3-polytopes. The result for higher dimensions follows by a simple inductive argument. (See [66, 60, p. 119]). We use rigidity theory to prove several extensions of the lower bound theorem and to study the cases of equality.

Barnette's inequality $f_k(P) \geqq \varphi_k(n,d)$ is sharp, and equality holds for every $1 \leqq k < d$ if P belongs to the family of *stacked* polytopes defined as follows: A d-simplex is stacked, and each simplicial d-polytope obtained from a stacked d-polytope with one fewer vertex by adding a pyramid over some facet is stacked. Alternatively, a simplicial d-polytope P is stacked if P is the union of simplices $S_1, S_2, \ldots, S_t$ such that each $(d-2)$-face of any of these simplices is a face of P.

Let P be a simplicial d-polytope. The set $\mathscr{B}(P)$ of proper faces of P forms a triangulation of the boundary of P. Thus, $\mathscr{B}(P)$ can be regarded as an abstract triangulation of S^{d-1}, the $(d-1)$-dimensional sphere. $\mathscr{B}(P)$ is called the *boundary complex* of P, [31, *Sect.* 3.2]. Define a *stacked* $(d-1)$-*sphere* to be a triangulated $(d-1)$-sphere which is isomorphic to the boundary complex of a stacked d-polytope.

A few years before Barnette proved the LBT, Walkup ([63]) settled the cases $d \leqq 5$. Walkup considered arbitrary triangulated $(d-1)$-manifolds and proved the case $d=4$ of the following theorem.

Theorem 1.1. *Let C be triangulated $(d-1)$-manifold, $d \geqq 4$, with n vertices, then:*

(i) $f_k(C) \geqq \varphi_k(n,d)$ *for* $1 \leqq k \leqq d-1$,

(ii) *If* $f_k(C) = \varphi_k(n,d)$ *for some* k, $1 \leqq k < d$, *then C is a stacked* $(d-1)$-*sphere.*

Note that the situation for $d=3$ is quite simple. A triangulated 2-manifold C with n vertices has $3n-3\chi(C)$ edges and $2n-2\chi(C)$ triangles, where $\chi(C)$ is the Euler characteristics of C. For every 2-manifold M, $\chi(M) \leqq 2$ and $\chi(M) = 2$ iff M is a 2-sphere. Thus, $f_i(C) = \varphi_i(n,3)$ for $i=1$ or $i=2$ iff C is a triangulated 2-sphere.

Our first purpose is to prove Theorem 1.1 for every $d \geqq 4$. Major portions of this result have been proved before by other methods: Part (i) and the special case $k=d-1$ of part (ii) were proved by Barnette (see [10, p. 354], [11]). Part (ii) for the special case of simplicial d-polytopes was proved by Billera and Lee in [15]. Their proof relies on the (necessity part of the) "g-theorem" – the complete characterization of f-vectors of simplicial polytopes, which was conjectured by McMullen ([47, 48]), and was proved by Stanley (necessity, [55]) and Billera and Lee (sufficiency, [15])). However, it was not known before that $f_k(C) = \varphi_k(n,d)$ occurs only if C is a triangulated sphere, (this was conjectured by Walkup [63, p. 77]). Nor was it known whether equality may holds for non-polytopal spheres.

A well-known and easy reduction due to McMullen, Perles and Walkup (see Sect. 5) reduces Theorem 1.1 to the case $k=1$.

We recall some definitions on rigidity of graph embeddings (frameworks). (See [5, 51, 28, 26, 33]). Given a graph $G=\langle V, E\rangle$, a *d-embedding* of G is a map $\psi\colon V\to\mathbb{R}^d$. A d-embedding ψ is *rigid* if any small perturbation φ of ψ which keeps the distances fixed between the images of adjacent vertices in G, keeps the distances fixed between every pair of vertices of G (and thus extends to an isometry of $\mathbb{R}^d$). A graph G is generically d-rigid if "almost all" embeddings of G into $\mathbb{R}^d$ are rigid. Such a graph having n vertices must have at least $dn-\binom{d+1}{2}$ edges. (Detailed definitions are given in Sect. 3.)

The inquality $f_1(C)\geqq\varphi_1(n,d)=dn-\binom{d+1}{2}$ for a triangulated $(d-1)$-manifolds C, $d\geqq 4$, with n vertices, follows from

Theorem 1.2. *The graph* (1-*skeleton*) *of every triangulated* $(d-1)$-*manifold*, $d\geqq 4$, *is generically* d-*rigid*.

The proof is given in Sect. 6. Using some basic results on rigidity we reduce Theorem 1.2 to the generic 3-rigidity of graphs of triangulated 2-spheres which was proved by Gluck [28] (see Sect. 4). (Compare Gromov [67, Ch. 2.4.10].)

For a triangulated $(d-1)$-manifold C define $\gamma(C)=f_1(C)-dn+\binom{d+1}{2}$. (The same definition applies to simplicial d-polytopes.) For $d\geqq 4$, $\gamma(C)$ is, by Theorem 1.2, the dimension of the space of *stresses* of a generic d-embedding of the graph of C.

In Sect. 7 we study those triangulated manifolds C for which $\gamma(C)=0$. We prove that if $\gamma(C)=0$ then $\gamma(\mathrm{lk}(v,C))=0$ for every vertex v of C. ($\mathrm{lk}(v,C)$ is the link of v in C, see Sect. 2.) Using this result, we reduce Theorem 1.1 (ii) to the known case $k=d-1$. A direct proof of Theorem 1.1 (ii) is given in Sect. 9.

In Sect. 8 we determine the class of triangulated d-manifolds C, $d\geqq 4$, which satisfy the condition: $\mathrm{lk}(v,C)$ is a stacked $(d-1)$-sphere for every vertex v of C. This condition implies a severe restriction on C, and, in particular, if C is simply-connected, then C itself must be a stacked d-sphere. We also derive a useful combinatorial characterization of stacked spheres among all triangulated manifolds.

Klee proved in [42] that the inequality $f_{d-1}(C)\geqq\varphi_{d-1}(n,d)$ holds for an arbitrary $(d-1)$-pseudomanifold C. Other cases of Theorem 1.1 are still open for this general setting. In Sect. 10 we show how the assertion of Theorem 1.1 for arbitrary $(d-1)$-pseudomanifold reduces to the old standing conjecture:

Conjecture G [28, 25]. *The graph of every triangulated* 2-*manifold is generically* 3-*rigid*.

In Sect. 11 we prove a sharp lower bound, conjectured by Björner [17], for the number of k-faces of a triangulated manifold with boundary, when the numbers of interior vertices and boundary vertices are specified.

Theorem 1.3. *Let C be a triangulated $(d-1)$-manifold $d\geqq 3$ with non-empty boundary. If C has n_i vertices in the interior and n_b vertices on the boundary then*

$f_k(C) \geqq \varphi_k^{\mathbf{b}}(n_i, n_b, d)$, *where*

$$\varphi_k^{\mathbf{b}}(n_i, n_b, d) = \begin{cases} \binom{d-1}{k} n_b + \binom{d}{k} n_i - \binom{d}{k+1} k & \text{for } 1 \leqq k \leqq d-2 \\ n_b + (d-1) n_i - (d-1) & \text{for } k = d-1. \end{cases} \tag{1.2}$$

Equality occurs only for a special type of triangulated balls. Theorem 1.3 for the special case when C is the dual of an unbounded simple polyhedra was proved (using the "g-theorem",) by Billera and Lee [16].

In Sect. 12 we discuss an extension of the LBT to arbitrary polytopes and polyhedral manifolds. For a polyhedral complex C, let $f_2^k(C)$ denotes the number of 2-faces of C which are k-gons. The following theorem, which was conjectured in [35, p. 67], extends the lower bound theorem to arbitrary d-polytopes.

Theorem 1.4. *If P is a d-polytope with n vertices then*

$$f_1(P) + \sum_{k \geqq 3} (k-3) f_2^k(P) \geqq dn - \binom{d+1}{2} \tag{1.3}$$

The analogous statement for arbitrary polyhedral $(d-1)$-manifolds (even polyhedral $(d-1)$-spheres,) is still open.

Theorem 1.4 follows from a recent theorem of Whiteley ([66], Sect. 4) on infinitesimal rigidity of certain embedded graphs associated with d-polytopes. (See Sect. 4.) Previously, it was proved for rational d-polytopes (namely, d-polytopes whose vertices have rational coordinates,) using some deep results from algebraic geometry ([58, Ch. 4, 46, 59]). In the second part of this paper ([38]) we study the class of d-polytopes which satisfy (1.3) as an equality.

Grünbaum proved ([31, p. 200],) that the graph of every d-polytope contains a refinement of the complete graph on $d+1$ vertices. Barnette extended this result ([11]) to arbitrary polyhedral $(d-1)$-manifolds. In Sect. 13 we prove

Theorem 1.5. *The graph of a triangulated $(d-1)$-manifold C, $d \geqq 4$, contains a refinement of the complete graph on $d+2$ vertices* iff *C is not a stacked $(d-1)$-sphere.*

In Sect. 14 we present a few open problems which were raised during this research. In particular, we briefly consider the LBT in the context of McMullen-Walkup "generalized lower bound conjecture" and discuss related problems on f-vectors of triangulated manifolds.

The basic reference (and source of inspiration) for convex polytope theory is Grünbaum's book [31]. We try to follow the definitions and notations of [31]. Other books on the subject are [48] and [21].

I would like to thank Richard Stanley for many valuable discussions on f-vectors of polytopes, and for introducing to me the recent exotic applications of algebraic geometry. I am thankful to Margaret Bayer, Louis Billera, Anders Björner, Robert Connelly, Henri Crapo, Micha Perles, and Walter Whiteley for helpful discussions during the various stages of this work. I would like to thank Lou Billera also for the warm hospitality during my visit at Cornell in summer 1984. Being familiar to some of Perles' unpublished work was, as usual, a great advantage. This research was supported by the Weizmann postdoctoral fellowship. My work at Cornell was supported in part by NSF grant DMS 8403225.

2. Preliminaries

We shall use the following definitions and notation on simplicial complexes: Let C be a finite abstract simplicial complex on the vertex set V. Thus, C is a collection of subsets of V (called the *faces* of C,) and if $T \in C$ and $S \subseteq T$ then $S \in C$. For $S \in C$ the *dimension* of S is $\dim S = |S| - 1$. $f_k(C)$ denotes the number of k-dimensional faces (briefly k-faces) of C. The *f-vector* of C is the vector $f(C) = (1, f_0(C), f_1(C), \ldots)$. The k-th dimensional *skeleton* of C, $\text{skel}_k(C)$ is defined by

$$\text{skel}_k(C) = \{S \in C : \dim S \leqq k\}.$$

$V(C)$ denotes the set of vertices (0-faces) of C. (Thus, $V(C) \subseteq V$.) 1-faces of C are called *edges* and $\text{skel}_1(C)$ is called the *graph* of C and is denoted by $G(C)$.

For a face $S \in C$ the *link* of S in C, $\text{lk}(S, C)$, is defined by:

$$\text{lk}(S, C) = \{T \setminus S : T \in C, T \supset S\}.$$

($\text{lk}(S, C)$ is also called the quotient complex of C by S.) Let V be a set of vertices and A be a family of subsets of V. $\bar{A}$ denotes the simplicial complex spanned by A. (I.e., $\bar{A} = \{S \subset V : S \subset T \text{ for some } T \in A\}$.) For a face $S \in C$, the *star* of S in C is defined by $st(S, C) = \overline{\{T \in C : T \supset S\}}$. The *antistar* of S in C is defined by $ast(S, C) = \{T \in C : T \cap S = \emptyset\}$.

Let C and D be simplicial complexes with $V = V(C)$, $U = V(D)$ and $V \cap U = \emptyset$. $C * D$, the *join* of C and D is defined by:

$$C * D = \{T \in V \cup U : T \cap V \in C, T \cap U \in D\}.$$

Note that $st(S, C) = \bar{S} * \text{lk}(S, C)$.

A simplicial complex C is *pure* if all its maximal faces have the same size. Maximal faces of a pure simplicial complex are called *facets*. Two facets S, T of a pure simplicial complex are *adjacent* if they intersect in a maximal proper face of each. A pure simplicial complex C is *strongly connected* if for every two facets S and T of C, there is a sequence of facets $S = S_0, S_1, \ldots, S_m = T$, such that S_i and S_{i+1} are adjacent, $0 \leqq i < m$.

A *d-pseudomanifold* is a strongly connected d-dimensional simplicial complex, such that every $(d-1)$-face is contained in exactly two facets. A *d-pseudomanifold with boundary* is a strongly connected d-dimensional simplicial complex, such that every $(d-1)$-face is contained in at most two facets. For a d-pseudomanifold with boundary C, the boundary of C, ∂C, is the $(d-1)$-dimensional pure simplicial complex whose facets are those $(d-1)$-faces of C which are included in a unique facet of C.

Let C be a pure simplicial complex and let F be a facet of C. The *stellar subdivision* of C at the facet F is defined by $C[F] = (C \setminus F) \cup \{R \cup \{u\} : R \subset F, R \neq F\}$. Here, u is a new vertex.

C is a *triangulated manifold* if $|C|$ is a manifold. ($|C|$ is the topological space associated with C. See, [53, Ch. 3]). It is usually more convenient to consider the larger class of homology manifolds. A pure d-dimensional complex C is a *homology manifold* if for every $\phi \neq S \in C$, $|S|=k$, the link of S in C has the same homology groups as a $(d-k)$-dimensional sphere. A homology d-manifold which has the same homology groups as a d-sphere is called a *homology d-sphere.*

3. Rigidity of frameworks

Let $G=\langle V,E\rangle$ be a graph with vertex set $V=V(G)$ and edge set $E=E(G)$. A *d-embedding* of G into $\mathbb{R}^d$ is a map $\varphi\colon V\to\mathbb{R}^d$. A *framework* $\mathscr{F}$ is a pair $\mathscr{F}=(G,\varphi)$ where $G=\langle V,E\rangle$ is a graph and φ is a d-embedding of G.

Two d-embeddings φ and ψ of a graph G are *isometric* if for every two vertices $a,b\in V$, $d(\varphi(a),\varphi(b))=d(\psi(a),\psi(b))$. ($d(x,y)$ denotes the Euclidian distance between x and y.) Equivalently, φ and ψ are isometric if there is an affine rigid motion T of $\mathbb{R}^d$ such that $\varphi=T(\psi)$. Two d-embeddings φ and ψ of a graph G are *G-isometric* if for every two adjacent vertices $a,b\in V$, $d(\varphi(a),\varphi(b))=d(\psi(a),\psi(b))$. (The vertices a and b are called *adjacent* if $\{a,b\}\in E(G)$.)

For two d-embeddings φ and ψ of G define their distance $d(\varphi,\psi)=\max_{a\in V} d(\varphi(a),\psi(a))$.

Definition 3.1. A *d-embedding* φ of a graph G is *rigid* if there is an $\varepsilon>0$ such that every embedding ψ of G which is G-isometric to φ and satisfies $d(\varphi,\psi)<\varepsilon$ is isometric to φ. φ is *flexible* if it is not rigid.

Definition 3.2. A graph G is *generically d-rigid* if the set of rigid d-embeddings of G is an open dense set in the set of all embeddings. (The set of all embeddings is a topological vector space of dimension $|V|\times d$.)

Remarks. (1) We will freely use these definitions for an arbitrary simplicial (or more general) complex C and they will apply to the graph of C. (2) When we consider rigidity of d-polytopes or embedded manifolds this will be (unless stated otherwise) w.r.t. the given embedding.

A systematic study of rigidity of frameworks may be found in [5, 6, 28, 51]. We shall need the following basic facts:

0. If $H=\langle V,E'\rangle$ is generically d-rigid and $G=\langle V,E\rangle$ where $E\supset E'$ then G is generically d-rigid (obvious).

1. If G is not generically d-rigid then the set of rigid d-embeddings of G has empty interior. (In this case G is *generically d-flexible.*)

2. *If G is a generically d-rigid graph with n vertices and e edges then $e\geq dn-\binom{d+1}{2}$.*

3. Let $G=\langle V(G),E(G)\rangle$ be a graph and let u be a vertex not in $V(G)$. Define $G^*\{u\}=\langle V',E'\rangle$ where $V'=V(G)\cup\{u\}$ and $E'=E(G)\cup\{\{u,v\}\colon v\in V(G)\}$. $G^*\{u\}$ is called a *cone* over G.

Cone Lemma (Whiteley, [65]). *G is generically d-rigid* iff *$G * \{u\}$ is generically $(d+1)$-rigid.*

4. **Replacement Lemma.** *Let $G = \langle V, E\rangle$ be a graph and let U be a subset of V. If the restriction of G to U is generically d-rigid and $G \cup K(U)$ is generically d-rigid then G is generically d-rigid. ($K(U)$ denotes the complete graph on U.)* (The proof is easy.)

Given a fixed set V of vertices, the set of edges of minimal (w.r.t. inclusion) generically d-rigid graphs on V, is the set of bases of a matroid $\mathscr{R}_d^n$ of rank $\binom{n}{2} - \binom{n-d}{2}$ $\left(= dn - \binom{d+1}{2} \text{ for } n \geqq d\right)$.

Definition 3.3. A graph $G = \langle V, E\rangle$ is (generically) *d-acyclic*[1] if the set of its edges is independent in $\mathscr{R}_d^n$.

For the reader who is not familiar with matroid theory terminology (a good reference is Welsh [64]), here is an equivalent definition: Let φ be a d-embedding of a graph G. An edge $\{a,b\}$, not in $E(G)$ depends on G (w.r.t. φ), if for every embedding ψ which is G-isometric and close enough to φ, $d(\psi(a), \psi(b)) = d(\varphi(a), \varphi(b))$. G is d-acyclic if for a generic d-embedding of G no edge E of G depends on $G' = < V(G), E(G) \backslash E>$.

An important variant of rigidity is the notion of *infinitesimal rigidity.* The definition given below follows Connelly [26]. For the geometric motivation behind the definition and a full treatment of the relations between the different notions of this section see [26] and [51].

Let φ be a d-embedding of a graph G. An *infinitesimal flex* of φ is a d-embedding ψ of G such that for every two adjacent vertices a and b of G, $(\varphi(a) - \varphi(b)) \cdot (\psi(a) - \psi(b)) = 0$. (Here, $\cdot$ is the usual scalar product.) An infinitesimal flex ψ of φ is *trivial* of for every two vertices a, b of G, $(\varphi(a) - \varphi(b)) \cdot (\psi(a) - \psi(b)) = 0$. A d-embedding φ of G is *infinitesimally rigid* if every infinitesimal flex of φ is trivial.

Infinitesimally rigid frameworks are rigid, and the generic behavior w.r.t. rigidity and infinitesimal rigidity coincide. If a graph G is infinitesimally rigid w.r.t. one d-embedding then it is generically d-rigid. (In particular, $|E(G)| \geqq d\,|V(G)| - \binom{d+1}{2}$.)

Given a d-embedding φ of a graph G, a *stress* of G w.r.t. φ is a function $w: E(G) \to R$ such that for every vertex $v \in V$

$$\sum \{w(\{v,u\})(\varphi(v) - \varphi(u)) \colon \{v,u\} \in E(G)\} = 0.$$

For a graph $G = \langle V, E\rangle$, $a_d(G)$ will denote the rank of G in $\mathscr{R}_d^n$. Alternatively, $a_d(G)$ is the number of edges of a maximal d-acyclic subgraph of G. (All maximal d-acyclic subgraphs of G have the same number of edges.) Define $b_d(G) = |E(G)| - a_d(G)$. $b_d(G)$ is the dimension of the space of stresses of G w.r.t. a generic d-embedding. In particular, G is d-acyclic if a generic d-embedding of G has no non-zero stress.

---

[1] This definition is slightly different from the definition in [37] which relies on a different matroid

4. Theorems of Cauchy, Steinitz, Alexandrov, Gluck and Whiteley

We make an essential use on the following theorem of Gluck [28].

Theorem G. *A triangulated 2-sphere is generically 2-rigid.*

Let us give a quick survey of Gluck's proof. Theorem G follows from the fundamental theorems of Cauchy and Steinitz. Cauchy's rigidity theorem ([22]) asserts that if P and Q are two convex 3-polytopes and φ: $V(P) \to V(Q)$ is a combinatorial isomorphism, which induces an isometry between every face of P and its image in Q, then P and Q are isometric. Steinitz's theorem (see [61, 31, p. 235, 14]) asserts that every polyhedral 2-sphere is combinatorially isomorphic to the boundary complex of a 3-polytope.

Cauchy's theorem implies that every simplicial 3-polytope P is rigid. Since the set of embeddings of P which actually realize P as a convex polytope is an open subset of the set of all embeddings, the graph of P is generically 3-rigid. By Steinitz's theorem every triangulated 2-sphere is isomorphic to the boundary complex of a simplicial 3-polytope and is therefore generically 3-rigid.

A *d-polytopal framework* is an embedded graph obtained from the graph of a d-polytope P by triangulating the 2-faces of P in an arbitrary way.

Alexandrov ([1]) extended Cauchy's arguments and proved that every 3-polytopal framework is infinitesimally rigid. (Note that Alexandrov's theorem combined with Steinitz's theorem give an even more direct proof of Theorem G. This is the variant in [28].)

Whiteley ([66]) have recently found a significant generalization of Alexandrov's theorem to higher dimensions

Theorem W. *A d-polytopal framework, $d \geqq 3$, is infinitesimally rigid.*

The basic connection between rigidity and the LBT can be seen at this point. Note that in a d-polytopal framework $\mathscr{F}(P)$, based on a d-polytope P, there are $(k-3)$ additional edges for each k-gonal 2-face. Thus, $\mathscr{F}(P)$ has exactly $f_1(C) + \sum_{k \geqq 3} (k-3) f_2^k(C)$ edges. Theorem 1.4 follows from Theorem W and the basic inequality $e \geqq dn - \binom{d+1}{2}$ for the number e of edges of an infinitesimally rigid d-embedded graph with n vertices. In particular, this gives the essential case $k = 1$ of the lower bound inequalities for simplicial polytopes.

Remark. Gluck's proof of the generic 3-rigidity of triangulated 2-spheres is unusual. Convexity is not involved in the assertion of the theorem but is very much present in the proof. Steinitz's theorem is a sort of a low dimensional miracle, and Cauchy's theorem gives a much stronger rigidity property than needed. Recently, Tay and Whiteley ([62]) found a direct proof for Gluck's theorem which does not depend on Cauchy's or Steinitz's theorems. Graver's approach ([30]) may also supply a direct proof for Gluck's theorem.

5. The MPW-reduction

The result of this section were found (independently) by McMullen, Perles and Walkup (see [10, 49]). Recall that $\varphi_k(n,d)$ is the number of k-faces in a stacked

d-polytope with n vertices and is given by formula (1.1). For a pure $(d-1)$-dimensional simplicial complex C with n vertices define $\gamma(C)=f_1(C)-\varphi_1(n,d)$. Thus, for $d\geqq 3$, $\gamma(C)=f_1(C)-dn+\binom{d+1}{2}$ and for $d=2$, $\gamma(C)=f_1(C)-n$. Define also

$$\gamma_k(C)=f_k(C)-\varphi_k(n,d),$$

and

$$\gamma^k(C)=\sum\{\gamma(\mathrm{lk}(S,C))\colon S\in C, |S|=k\}.$$

Thus, $\gamma_1(C)=\gamma^0(C)=\gamma(C)$.

Proposition 5.1. *Let C be a $(d-1)$-dimensional simplicial complex, and let k, d be integers, $1\leqq k\leqq d-1$. There are* positive *constants $w_i(k,d)$, $0\leqq i\leqq k-1$, such that*

$$\gamma_k(C)=\sum_{i=0}^{k-1} w_i(k,d)\,\gamma^i(C). \tag{5.1}$$

Proof. First note that

$$(k+1)f_k(C)=\sum_{i=1}^{n} f_{k-1}(\mathrm{lk}(v,C)). \tag{5.2}$$

Put $\varphi_k(n,d)=a_k(d)\,n+b_k(d)$. (Thus, $a_k(d)=\binom{d}{k}$ for $1\leqq k\leqq d-2$ and $a_{d-1}(d)=d-1$.) Easy calculation gives

$$2\left(dn-\binom{d+1}{2}\right)a_{k-1}(d-1)+n\,b_{k-1}(d-1)=(k+1)\,\varphi_k(n,d). \tag{5.3}$$

Let C be a pure $(d-1)$-dimensional simplicial complex, $d\geqq 3$, with n *vertices* $v_1,\dots,v_n$. Assume that the degree of v_i in $G(C)$ is n_i (i.e., $f_0(\mathrm{lk}(v_i,C))=n_i$). Note that $\sum_{i=0}^{n} n_i=2f_1(C)=2\left(dn-\binom{d+1}{2}+\gamma(C)\right)$. Therefore

$$\begin{aligned}\sum_{1=1}^{n}\varphi_{k-1}(n_i,d-1)&=a_{k-1}(d-1)\sum_{i=1}^{n} n_i+n\,b_{k-1}(d-1)\\ &=a_{k-1}(d-1)\,2\left(dn-\binom{d+1}{2}\right)+2a_{k-1}(d-1)\,\gamma(C)+nb_{k-1}(d-1).\\ &=(k+1)\,\varphi_k(n,d)+2a_{k-1}(d-1)\,\gamma(C).\end{aligned} \tag{5.4}$$

From (5.2) and (5.4) we get

$$(1+k)\,\gamma_k(C)=2a_{k-1}(d-1)\,\gamma(C)+\sum_{i=1}^{n}\gamma_{k-1}(\mathrm{lk}(v_i,C)). \tag{5.5}$$

Repeated applications of formula (5.5) give (5.1). The value of $w_i(k,d)$ is

$$w_i(k,d)=\begin{cases}2(a_{k-i-1}(d-i-1))/(k+1)\binom{k}{i} & 0\leqq i\leqq k-2,\\ 2/(k+1)\,k & i=k-1\end{cases} \tag{5.6}$$

Corollary (the MPW-reduction). *Let $d \geqq 2$ be an integer. Let C be a $(d-1)$-dimensional simplicial complex with n vertices, such that $\gamma(\mathrm{lk}(S,C)) \geqq 0$ for every $S \in C$, $|S|<k$. Then* (i) $f_k(C) \geqq \varphi_k(n,d)$. (ii) *If $f_k(C)=\varphi_k(n,d)$ then $\gamma(C)=0$.*

Remark. Note that if C is a $(d-1)$-pseudomanifold then $\gamma^{d-2}(C)=0$.

6. The lower bound inequalities for triangulated manifolds

For $d \geqq 3$ define a class $\mathscr{C}_d$ of $(d-1)$-pseudomanifolds inductively as follows: $\mathscr{C}_3$ is the class of triangulated 2-spheres. For $d \geqq 4$, a $(d-1)$-pseudomanifold C belongs to $\mathscr{C}_d$ if for every vertex v of C, $\mathrm{lk}(v,C) \in \mathscr{C}_{d-1}$. Note that every homology 2-sphere is a triangulated 2-sphere. Therefore for $d \geqq 4$, $\mathscr{C}_d$ includes all homology $(d-1)$-manifolds (and, in particular, all triangulated $(d-1)$-manifolds). $\mathscr{C}_4$ is exactly the class of homology 3-manifolds.

Theorem 6.1. *If $C \in \mathscr{C}_d$ then C is generically d-rigid.*

Lemma 6.2. *Let C be a strongly connected d-dimensional simplicial complex. Then C is generically d-rigid.*

Proof. (Compare [36].) If every two vertices of C are adjacent then C is clearly generically d-rigid. Otherwise, since C is strongly connected, there are two non-adjacent vertices u, v of C, and two adjacent d-faces S and T, such that $u \in S$ and $v \in T$. Let $\bar{C}$ be the simplicial complex obtained from C by adding to C all d-faces of $\overline{S \cup T}$. The affect of the operation $C \to \bar{C}$ on $G(C)$ is just adding one new edge $\{u,v\}$. The graph induced by $G(C)$ on the vertices of $S \cup T$ is a complete graph on $d+2$ vertices minus an edge ("{"u,v"}"). This graph is clearly generically d-rigid and by the Replacement Lemma (Sect. 3) if $\bar{C}$ is generically d-rigid so is C. Repeated application of this operation will terminate with a complex $\hat{C}$ whose graph is complete. $\hat{C}$ is clearly generically d-rigid.

Proof of Theorem 6.1. By induction on d. For $d=3$, $\mathscr{C}_d$ is the class of triangulated 2-spheres which are generically 3-rigid by Gluck's theorem, we assume the truth of the theorem for $d-1$ and prove it for d. Let $C \in \mathscr{C}_d$. For a vertex $v \in C$, the neighborhood $N(v)$ of v is defined by $N(v)=\{v\} \cup \{u \in V(C)$: $\{u,v\} \in C\}$. For a vertex $v \in C$, $\mathrm{lk}(v,C) \in \mathscr{C}_{d-1}$ and by the induction hypothesis $\mathrm{lk}(v,C)$ is generically $(d-1)$-rigid. By the cone lemma (Sect. 3), $st(v,C) = \{v\} * \mathrm{lk}(v,C)$ is generically d-rigid. Let $K_d(N(v))$ denote the complete d-dimensional complex on $N(v)$. By the replacement lemma (Sect. 3), C is generically d-rigid iff $C \cup K_d(N(v))$ is generically d-rigid. Repeated application of this argument shows that C is generically d-rigid iff $\bar{C} = \bigcup \{K_d(N(v)): v \in V(C)\}$ is generically d-rigid. But $\bar{C}$ easily seen to be a strongly connected d-dimensional complex, hence generically d-rigid.

Theorem 6.1 and the MPW reduction give:

Theorem 6.2. *If $C \in \mathscr{C}_d$ and C has n vertices then $f_k(C) \geqq \varphi_k(n,d)$ for all $d \geqq k \geqq 1$.*

Remarks. 1. The inductive argument in the proof of Theorem 6.1 seems to be quite old. It is hinted in [60, foornote p. 119] and perhaps goes back to the works of Alexandrov and Pogorelov. Whiteley's proof of Theorem W uses similar (but more delicate) inductive argument.

2. Theorem 6.1 strengthened the fact that the graph of a triangulated $(d-1)$-manifold is d-connected. Barnette [11] proved that the graph of every polyhedral $(d-1)$-manifold is d-connected, thus extending a result of Balinski [7] which asserts that the graph of every d-polytope is d-connected.

Let G be a graph with n vertices and e edges, $n \geqq d$. Recall that $b_d(G)$ is the dimension of the space of stresses of G w.r.t. a generic d-embedding. $b_d(G) \geqq e - dn + \binom{d+1}{2}$ and equality holds iff G is generically d-rigid. Theorem 6.1 thus implies that for $C \in \mathscr{C}_d$, $\gamma(C)$ is the dimension of the space of stresses of a generic d-embedding of $G(C)$.

Theorem 6.1 implies also an upper bound for the number of edges of subgraphs of graphs of triangulated manifolds.

Theorem 6.3. *Let $C \in \mathscr{C}_d$ and let H be a subgraph of $G(C)$. Then $f_1(H) \leqq \mathrm{d} f_0(H) - \binom{d+1}{2} + \gamma(C)$.*

Proof. Let H be a subgraph of $G(C)$. (We may assume that H has at least d vertices.) Denote $\gamma(H) = f_1(H) - \mathrm{d} f_0(H) + \binom{d+1}{2}$. Note that if H is a subgraph of G then $b_d(H) \leqq b_d(G)$. Therefore,

$$\gamma(C) = b_d(G(C)) \geqq b_d(H) \geqq \gamma(H).$$

We conclude this section by showing that the proof of Theorem 6.1 applies in a slightly more general situation. (We use this fact in Sects. 9 and 11.) Let C be a strongly connected $(d-1)$-dimensional simplicial complex and let T be a tree in $G(C)$. It is easy to see that $\bigcup \{K_d(N(v))$: v a vertex of $T\}$ is a strongly connected d-dimensional simplicial complex. Therefore, the proof of Theorem 6.1 gives.

Proposition 6.4. *Let C be a strongly connected $(d-1)$-dimensional simplicial complex. Let T be a tree in $G(C)$ which satisfy:* (i) *Every vertex u of C is adjacent to some vertex of T,* (ii) $\mathrm{lk}(v, C)$ *is generically $(d-1)$-rigid for every vertex v of T. Then C is generically d-rigid.*

7. The extremal cases in the lower bound theorem

Recall that a stacked $(d-1)$-sphere is a triangulated $(d-1)$-sphere which is isomorphic to the boundary complex of a stacked d-polytope. As easily seen, C is a stacked $(d-1)$-sphere iff C can be obtained from the boundary complex of a d-simplex by repeated applications of stellar subdivisions of facets.

Theorem 7.1. *Let d, k be fixed integers $d > 3$, $d > k \geqq 1$. Let C be a simplicial complex in $\mathscr{C}_d$ with n vertices and $\varphi_k(n, d)$ k-faces. Then C is a stacked $(d-1)$-sphere.*

Proof. The MPW reduction shows that for $C \in \mathscr{C}_d$, if $f_0(C)=n$ and $f_k(C) = \varphi_k(n,d)$ for some $1<k<d$, then $f_1(C)=\varphi_1(n,d)$, i.e., $\gamma(C)=0$. Define: $\mathscr{C}_d^0 = \{C \in \mathscr{C}_d: \gamma(C)=0\}$. By Theorem 6.1 every $C \in \mathscr{C}_d$ is generically d-rigid. Therefore, for $C \in \mathscr{C}_d$, $C \in \mathscr{C}_d^0$ iff C is d-acyclic. (See Sect. 3.)

Lemma 7.2. *If $C \in \mathscr{C}_d^0$, $d \geqq 4$, then for every vertex $v \in C$, $\mathrm{lk}(v, C) \in \mathscr{C}_{d-1}^0$.*

Proof. Assume to the contrary that $C \in \mathscr{C}_d^0$, v is a vertex of C, and $\mathrm{lk}(v, C) \notin \mathscr{C}_{d-1}^0$. Thus, $\mathrm{lk}(v, C)$ is not $(d-1)$-acyclic and from the Cone Lemma (Sect. 3) it follows that $st(v, C) = v^* \mathrm{lk}(v, C)$ is not d-acyclic. Since $C \supset st(v, C)$, C is not d-acyclic as well. A contradiction.

Proof of Theorem 7.1 (end). The case $d=4$ of Theorem 7.1 was proved already by Walkup ([63, Th. 1]). (Barnette's result mentioned below also covers this case.) Assume now that for $d \geqq 5$, if $C \in \mathscr{C}_{d-1}^0$ then C is a stacked $(d-1)$-sphere. Let $C \in \mathscr{C}_d^0$, $d \geqq 5$. Recall that $\gamma^k(C) = \sum \{\gamma(\mathrm{lk}(S, C)): S \in C, |S|=k\}$. (See Sect. 5.) Lemma 7.2 implies that for every $S \in C$, $\gamma(\mathrm{lk}(S, C))=0$. Therefore, for every $k \geqq 1$, $\gamma^k(C)=0$. By Proposition 5.1, $f_{d-1}(C) = \varphi_{d-1}(n,d)$. By Lemma 7.2 for every vertex $v \in C$, $\mathrm{lk}(v, C) \in \mathscr{C}_{d-1}^0$. By the induction hypothesis $\mathrm{lk}(v, C)$ is a stacked sphere, and therefore C is a triangulated $(d-1)$-manifold. Barnette proved ([9, 11]) that if a triangulated $(d-1)$-manifold C with n vertices satisfies $f_{d-1}(C) = \varphi_{d-1}(n,d)$ then C is a stacked $(d-1)$-sphere. This completes the proof of Theorem 7.1.

A direct proof of Theorem 7.1 is given in Sect. 9. We use there a characterization of stacked spheres which is proved in the next section.

The proof of Lemma 7.2 gives more:

Theorem 7.3. *Let C be a generically d-rigid pure $(d-1)$-dimensional simplicial complex. Then for every vertex v of C, $\gamma(\mathrm{lk}(v, C)) \leqq \gamma(C)$.*

Proof. Define $G_1 = G(\mathrm{lk}(v, C))$, $G_2 = G(st(v, C))$ $(= G_1 * \{v\})$. Let H be a maximal $(d-1)$-acyclic subgraph of G_1. By the cone Lemma, $H^* \{v\}$ is a maximum d-acyclic subgraph of G_2. Therefore

$$\gamma(\mathrm{lk}(v, C)) \leqq b_{d-1}(G_1) = b_d(G_2) \leqq b_d(G(C)) = \gamma(C).$$

8. Triangulated manifolds with stacked links

In this section we study triangulated manifolds C such that $\mathrm{lk}(v, C)$ is a stacked sphere for every vertex v of C. For manifolds of dimensions greater than 3 this condition implies a severe topological restriction. We also derive a characterization of stacked spheres among pseudomanifolds in $\mathscr{C}_d$ which is used in the next sections.

Consider the following two operations on triangulated manifolds. Let C and D be pure simplicial complexes with disjoint sets of vertices, S be a facet of C and T be a facet of D. Let ψ be a bijection between $V(S)$ and $V(T)$. The *connected sum* $C \#_\psi D$ of C and D is the simplicial complex obtained by identifying the vertices of S with the vertices of T by ψ and deleting the facet S

$(=T)$. Connected sums of two triangulated manifolds is a triangulated manifold. Note that if $E = C \#_{\psi} D$ then for $v \in S, \mathrm{lk}(v, E)$ is a connected sum of $\mathrm{lk}(v, C)$ and $\mathrm{lk}(\psi(v), D)$. All other links are unchanged.

Let C be a pure $(d-1)$-dimensional simplicial complex, S and T be two disjoint facets of C, and ψ be a bijection from $V(S)$ to $V(T)$. Assume further that no vertex of S is adjacent to a vertex of T and that no vertex in C is adjacent to both a vertex v in S and to its image $\psi(v)$ in T. Let C^{ψ} be the simplicial complex obtained from C by identifying the vertices of S to the vertices of T via ψ and deleting the facet $S(=T)$. We say that C^{ψ} is obtained by *forming a handle* over C. Note that $\mathrm{lk}(v, C^{\psi}) = \mathrm{lk}(v, C)$ unless $v \in S$ $(=T)$, and then $\mathrm{lk}(v, C^{\psi}) = \mathrm{lk}(v, C) \# \mathrm{lk}(\psi(v), C)$.

Note also that

$$\gamma(C \#_{\psi} D) = \gamma(C) + \gamma(D), \tag{8.1}$$

$$\gamma(C^{\psi}) = \gamma(C) + \binom{d+1}{2} \quad (d = \dim C - 1). \tag{8.2}$$

Walkup defined the class $\mathscr{H}^d(k)$ of $(d-1)$-dimensional simplicial complexes as follows: $\mathscr{H}^d(0)$ is the class of stacked $(d-1)$-spheres. $C \in \mathscr{H}^d(k)$ if $C = D^{\psi}$ for some $D \in \mathscr{H}^d(k-1)$. Define $\mathscr{H}^d = \bigcup \{\mathscr{H}^d(k) \colon k \geqq 0\}$. Note that a connected sum of two complexes in $\mathscr{H}^d$ is in $\mathscr{H}^d$. In fact, $\mathscr{H}^d$ is exactly the class of simplicial complexes obtained from boundary complexes of d-simplices by successively applying the operations $C \#_{\psi} D$ and C^{ψ}. For $d \geqq 4$, if $C \in \mathscr{H}^d(k)$ then rank $H_1(C) = k$ and $\gamma(C) = k\binom{d+1}{2}$. From the description of links of vertices of $C \#_{\psi} D$ and C^{ψ}, it follows that if $C \in \mathscr{H}^d$, then $\mathrm{lk}(v, C)$ is a stacked $(d-2)$-sphere for every vertex v of C.

The notion of a *missing face* (see [3],) will play an important role from now on.

Definition 8.1. *Let C be a simplicial complex on the vertex set V. A subset S of V is a missing face of C, if $S \notin C$ but for every proper subset R of S, $R \in C$. A k-missing face is a missing face with $k+1$ vertices.*

Theorem 8.2. *Let C be a $(d-1)$-pseudomanifold, $d \geqq 4$. If for every vertex $v \in C, \mathrm{lk}(v, C) \in \mathscr{H}^{d-1}(0)$ and C has no $(d-2)$-missing faces, then $C \in \mathscr{H}^d$.*

Lemma 8.3. *Let P be a stacked d-polytope.* (i) *P has no k-missing faces for $1 < k < d-1$.* (ii) *If P is not a d-simplex then P has a missing $(d-1)$-face.*

Proof. Let P and Q be two simplicial d-polytopes such that Q is obtained from P by adding a pyramid over a facet T of P. (The boundary complex of Q is obtained from the boundary complex of P by a stellar subdivision of T.) It is easy to see that every missing face of P is a missing face of Q and, in addition, Q has one new $(d-1)$-missing face T and $f_0(P) - d$ new 1-missing faces of the form $\{u, v\}$ where u is the new vertex of Q and $v \notin T$. Lemma 8.3 follows by induction from the definition of stacked polytopes.

Proof of Theorem 8.2. Let $v \in C$ and let S be a $(d-2)$-missing face in $\mathrm{lk}(v, C)$. (Unless C is a simplex there is a vertex v in C whose degree is more than d

and therefore $\mathrm{lk}(v, C)$ has a $(d-2)$-missing face.) Since C has no $(d-2)$-missing faces, S must be a face of C and therefore $T=S\cup\{v\}$ is a $(d-1)$-missing face of C. Cut C along ∂T and patch with two $(d-1)$-simplices. (As was shown by Walkup, [63, Lemma 4.2], this operation can always be performed.) The resulting complex is a (possibly not connected) triangulated $(d-1)$-manifold $\bar{C}$. If $\bar{C}$ is connected then C is obtained from $\bar{C}$ by forming a handle. If $\bar{C}$ is not connected it has two connected components and C is their connected sum. Theorem 8.2 follows by double induction on $\gamma(C)$ and $f_0(C)$.

Corollary 8.4. *Let C be a $(d-1)$-pseudomanifold, $d\geqq 5$. If for every vertex $v\in C$, $\mathrm{lk}(v, C)$ is a stacked $(d-2)$-sphere, then $C\in\mathscr{H}^d$.*

Proof. It is enough to show that C does not have $(d-2)$-missing faces. Indeed, if S is a $(d-2)$-missing face of C and v is a vertex of S then $S\setminus\{v\}$ is a $(d-3)$-missing face of $\mathrm{lk}(v, C)$. This is impossible by Lemma 8.2(i) since $\mathrm{lk}(v, C)$ is a stacked $(d-2)$-sphere and $(d-3)>1$.

Remark 8.5. Perles proved (see [4]) that if P is a neighborly 4-polytope then every link of a vertex of P is stacked. Thus, the class of triangulated 3-manifolds with stacked 2-spheres as the only links of vertices, is much larger than $\mathscr{H}^4$. Having only stacked spheres as links impose a severe topological restriction on d-manifolds for $d\geqq 4$. *Problem:* Which 3-manifolds admit a triangulation with only stacked 2-spheres as links of vertices? (Compare [23].)

We derive now from Theorem 8.2 a useful characterization of stacked spheres. Recall that a cycle M in a graph G is *chordless* if M is an induced subgraph of G. (Thus, M is a subgraph of G with a set of vertices $V(M)=\{v_1,\ldots,v_m\}$, $m\geqq 3$ and edges $\{v_1,v_2\},\ldots,\{v_{m-1},v_m\},\{v_m,v_1\}$ and the only edges of G with endpoints in $V(M)$ are edges of M.) A graph is *chordal* if it does not contain a chordless m-cycles for $m\geqq 4$.

Theorem 8.5. *Let $C\in\mathscr{C}_d$, $d\geqq 3$. The following are equivalent:*

(i) *C is a stacked $(d-1)$-sphere,*
(ii) *$G(C)$ is chordal and C has no k-missing faces for $1<k<d-1$.*

Proof. (i)$\to$(ii). Let C be a stacked $(d-1)$-sphere, $d\geqq 3$. By Lemma 8.2, C has no k-missing faces for $1<k<d-1$. It is left to show that $G(C)$ is chordal. Let P and Q be two simplicial d-polytopes such that Q is obtained from P by adding a pyramid over a facet T of P. $G(Q)$ is obtained from $G(P)$ by adding a new vertex u and connecting it to all vertices of T. From this description it is clear that if $G(P)$ is chordal then so is $G(Q)$. Therefore, graphs of stacked $(d-1)$-spheres are chordal.

(ii)$\to$(i). The proof will proceed by induction on d. For $d=3$ we have to prove that every triangulated 2-sphere C with a chordal graph, is a stacked 2-sphere. Assume to the contrary, that C is a counterexample with a minimal number of vertices. If C has a 2-missing face then C is the connected sum of two smaller triangulated 2-spheres C_1 and C_2. $G(C_1)$ and $G(C_2)$ are chordal and by the minimality of C, C_1 and C_2 are stacked and therefore so is C. Thus, C does not have a 2-missing face. Let v be a vertex of degree 4 or 5 in C. (Such a vertex always exists unless C is the boundary of a 3-simplex.) If v

has 4 neighbors they form a 4-cycle (with the edges of $\mathrm{lk}(v, C)$) and this 4-cycle must have a diagonal. Since C has no 2-missing faces C is a stacked 2-sphere with 5 vertices. If v has 5 neighbors then by the same argument C is a stacked 2-sphere with 6 vertices. A contradiction.

Let $d \geqq 4$, and assume that the implication (ii)$\to$(i) holds for every $d' < d$. Let C be a member of $\mathscr{C}_d$ with a chordal graph and no k-missing faces for $1 < k < d-1$. First we show that C is simply-connected. Otherwise, let M be a minimal cycle in $G(C)$ which is not null-homotopic in C. M must be chordless and if M is a triangle it must be a 2-missing face. Let v be a vertex of C. If S is a k-missing face of $\mathrm{lk}(v, C)$, $1 < k < d-2$ then either S itself or $S \cup \{v\}$ is a missing face of C. This is impossible by the assumption on C. If M is a chordless cycle in $\mathrm{lk}(v, C)$ then since C has no 2-missing faces, M is chordless in C as well. Thus, by the induction hypothesis, $\mathrm{lk}(v, C)$ is a stacked $(d-2)$-sphere for every vertex v of C. Since C does not have $(d-2)$-missing faces and is simply-connected, by Theorem 8.2, C is a stacked $(d-1)$-sphere.

Both conditions of Theorem 8.5(ii) are necessary. The graph of every 2-neighborly d-polytope is chordal. The d-cross polytope has k-missing faces only for $k=1$. The implication (ii)$\to$(i) does not hold for arbitrary $(d-1)$-pseudomanifolds as shown by the 3-neighborly 3-pseudomanifolds of Altshuler ([2]).

9. Direct proof of Theorem 7.1

Lemma 9.1. *If $C \in \mathscr{C}_d^0$, S is a missing face of C then either* $\dim S = 1$ *or* $\dim S = d - 1$.

Proof. The lemma says nothing for $d=3$. Let $C \in \mathscr{C}_d^0$, $d \geqq 4$. Let us first show that C has no 2-missing faces. Assume to the contrary that T is a 2-missing face of C. Let v be a vertex of T and let $E = T \setminus \{v\}$. E is an edge of C, the vertices of E are adjacent to v and are therefore vertices of $st(v, C)$. But E itself does not belong to $\mathrm{lk}(v, C)(E \cup \{v\} \notin C)$, and therefore E does not belong to $st(v, C)$. Since $st(v, C)$ is generically d-rigid, E depends on $st(v, C)$ w.r.t. a generic d-embedding. However, $E \notin st(v, C)$ and therefore $st(v, C) \cup E$ is not d-acyclic. Since $st(v, C) \cup E \subset C$, C is not d-acyclic. A contradiction.

If T is a k-missing face of C, $2 < k < d-1$ then for every subset S of T of size $k-2$, $T \setminus S$ is a 2-missing face of $\mathrm{lk}(S, C)$. By Lemma 7.2, $\mathrm{lk}(S, C) \in \mathscr{C}_{d-k+2}^0$. But $d-k+2 \geqq 4$ and therefore $\mathrm{lk}(S, C)$ does not have a 2-missing face. A contradiction.

Lemma 9.2. *If $C \in \mathscr{C}_d^0$ then $G(C)$ is chordal.*

Proof. Assume to the contrary that $C \in \mathscr{C}_d^0$ and M is a chordless m-gon in C, $m \geqq 4$. Let $E = \{v_1, v_2\}$ be an edge in M. Let U be the set of vertices of M which are not in E, and let H be the induced subgraph of M on U. (H is a path.) Let W be the set of vertices of C which are adjacent to some vertex of U. Clearly $v_1, v_2 \in W$. Define a simplicial complex D on W by $D = \bigcup \{st(u, C) : u \in U\}$. Since M is chordless $E \notin D$. By Proposition 6.4, D is generally d-rigid. But the vertices of

E belongs to D, therefore $D \cup E$ is not d-acyclic and since $D \cup E \subseteq C$. C is not d-acyclic. A contradiction.

Direct proof of Theorem 7.1 (end). Let $C \in \mathscr{C}_d^0$, by Lemmas 9.1 and 9.2, C has no k-missing faces for $1<k<d-1$ and no chordless m-gons for $m \geqq 4$. By Theorem 8.5, C is a stacked $(d-1)$-sphere.

Remark. Most of the work is needed just for the case $d=4$. If one assumes the assertion of Theorem 7.1 for $d=4$, then the general case follows easily by induction, from Lemma 7.2 and Theorem 8.2.

Corollary 8.4 and Theorem 7.1 imply:

Theorem 9.3. *If C is a simply-connected triangulated $(d-1)$-manifold, $d \geqq 5$, and for every vertex $v \in C$, $\gamma(\mathrm{lk}(v, C))=0$ then $\gamma(C)=0$.*

Second proof of Theorem 9.3 (hint). In order to show that $\gamma(C)=0$ it is enough to prove that for every edge $E \in C$, a generic d-embedding ρ of $C \backslash E$ has a non-trivial infinitesimal flex v. (See Sect. 3). Let $E=\{v_1, v_2\}$ be an edge of C. Since $\mathrm{lk}(v_1, C)$ is acyclic, $st(v_1, C) \backslash E$ has a non-trivial infinitesimal flex. Choose such a flex v_0. We will extend this infinitesimal flex to an infinitesimal flex of $C \backslash E$. Let $r=d(v_0(v_1), v_0(v_2))$.

Let $\{w, u\}$ be an edge in C, and let ξ be an infinitesimal flex of $st(w, C) \backslash E$. Consider the restriction of ξ to $D_0=st(\{u, w\}, C) \backslash E$ and extend it to an infinitesimal flex $\bar{\xi}$ of $D_1=st(u, C) \backslash E$. This can always be done (here we use the fact that $d \geqq 5$). The extension is unique unless $v_1, v_2 \in D_1$, but either v_1 or v_2 are not in D_0. In this case extend ξ under the condition that $d(\bar{\xi}(v_1), \bar{\xi}(v_2))=r$.

Apply this operation to extend v_0 to stars of all the vertices in C. It can be shown that if an infinitesimal flex is defined on $st(v, C)$ using this procedure via a path l from v_1 to v, then it depends only on the homotopy class of the path l. Therefore, if C is simply-connected one gets a well-defined non-trivial infinitesimal flex on $C \backslash E$.

Third proof of Theorem 9.3 for boundary complexes of simplicial polytopes. Let $\delta(C)=f_2(C)-(d-1)f_1(C)+\binom{d}{2}f_1(C)-\binom{d+1}{3}$ $(=h_3(C)-h_2(C)$, see Sect. 14). It is easy to check that

$$\sum\{\gamma(\mathrm{lk}(v, C)) \colon v \in V(C)\}=3\,\delta(C)+(d-1)\,\gamma(C). \tag{9.1}$$

It is plausible that $\delta(C) \geqq 0$ holds for every simply-connected triangulated $(d-1)$-manifold C, $d \geqq 5$. This is known only when $d=5$ and when $d>5$ and C is the boundary complex of a d-polytope. Clearly if $\delta(C) \geqq 0$ and the left hand side of Eq. (9.1) is equal to zero then: $\gamma(C)=\delta(C)=0$.

For a triangulated 4-manifold C, the Dehn-Sommerville equations assert that $\delta(C)=10(\chi(C)-2)$ where $\chi(C)$ is the Euler characteristic of C. In particular, if C is simply-connected then $\delta(C)=10\,b_2 \geqq 0$ where $b_2=\operatorname{rank} H_2(C) \geqq 0$ is the second Betti-number of C.

The inequality $\delta(P) \geqq 0$ for a simplicial d-polytope P, $d \geqq 5$, is a special case of the "generalized lower bound inequalities" [49, 55] (see Sect. 14). (In fact, the "g-theorem" in its full strength implies that if $\gamma(P)=0$ then $\delta(P)=0$. This implies also, by (9.1), Lemma 7.2 for polytopes.)

10. The lower bound conjecture for pseudomanifolds

The lower bound conjecture for pseudomanifolds. (a) If C is a $(d-1)$-pseudomanifold with n vertices, then $f_k(C) \geqq \varphi_k(n,d)$ for $1 \leqq k \leqq d-1$. (b) If equality holds for some k, $d>k>1$, then C is a stacked sphere.

The case $k=d-1$ of part (a) of this conjecture was proved by Klee [42]. The remaining cases are still open.

Definition 10.1. A $(d-1)$-pseudomanifold is *normal* if every face $S \in \mathrm{skel}_{d-3} C$ has a connected link.

Note that the class $\mathscr{C}_d$ of $(d-1)$-pseudomanifolds defined in Sect. 6 is the class of normal $(d-1)$-pseudomanifolds whose singular part has codimension greater than 2. (If $C \in \mathscr{C}_d$ and S is a face of C of size $d-3$, then $\mathrm{lk}(S,C)$ is a triangulated 2-sphere.)

The class of normal pseudomanifolds is closed under taking links of faces. Therefore, the LBT for normal pseudomanifolds reduces by the MPW-reduction to the case $k=1$. As in the proof of Theorem 6.1 the generic d-rigidity of normal $(d-1)$-pseudomanifolds follows from the generic 3-rigidity of normal 2-pseudomanifolds, which are just triangulated 2-manifolds. Part (a) of the LBC for normal pseudomanifolds would thus follow from the following old standing conjecture:

Conjecture G [28, 25]. *The graph of every triangulated* 2*-manifold is generically* 3*-rigid*[2].

Remark. Connelly gave in [24] an example of a flexible embedding of a triangulated 2-sphere, and thus refuted the old conjecture (going back to Euler,) that *every* triangulated 2-manifold embedded in R^3 is rigid.

Conjecture G would also imply part (b) of the LBC for normal pseudomanifolds as follows: It is enough to show it for normal 3-pseudomanifolds and then to proceed as in Sect. 7. Conjecture G implies that a 3-pseudomanifold C is generically 4-rigid. Thus if $\gamma(C)=0$ then C must be 4-acyclic, and every link of a vertex of C must be 3-acyclic hence a triangulated 2-sphere.

In order to reduce the LBC for arbitrary pseudomanifolds to the normal case, and also to extend Theorem 1.1 to arbitrary pseudomanifolds with singular set of codimension greater than two, we need the following normalization process [57, p. 83] (compare [29, p. 151, 17]).

Let C be a $(d-1)$-pseudomanifold. Choose a non-empty face S of C of smallest possible dimension k, $k<d-2$ with a non-connected link. "Pull apart" C at S to get a new complex $N_S(C)$ as follows: Create a copy F_i of F for each component K_i of $\mathrm{lk}(F,C)$ so that the link of F_i in the new complex $N_S(C)$ is K_i. Repeated applications of this operation will terminate with a normal $(d-1)$-pseudomanifold $N(C)$.

Direct computation gives:

$$(f_k(C)-\varphi_k(n,d))>(f_k(N_S(C))-\varphi_k(n,d)) \quad \text{for every } 1 \leqq k<d. \tag{10.1}$$

---

[2] Whiteley and Graver have recently proved (independently) that all triangulations of the torus are generically 2-rigid. Connelly proved (private communication) that every triangulated 2-manifold admits a generically 3-rigid subdivision

It is likely but unknown that if $N_S(C)$ is generically d-rigid so is C.

Remarks 1. Altshuler constructed in [2] 3-pseudomanifolds such that *none* of their 2-dimensional links are spheres.

2. Note that the lower bound inequalities need not hold for a strongly connected $(d-1)$-dimensional complex, in which every $(d-2)$-face is included in *at least* two facets. A counterexample is two tetrahedra identified along an edge.

11. Manifolds with boundary

In this section we prove a lower bound, conjectured by Björner [17], for the number of k-faces of a triangulated $(d-1)$-manifold *with* boundary as a function of the numbers of interior vertices and boundary vertices. The problem was originated in the study of polytope pairs, see [40, 41, 17, 16]. We first need a few definitions.

A *d-tree* ([34]) is defined inductively as follows: A complete simplicial complex on $d+1$ vertices is a d-tree. If C is a d-tree on the vertex set V, $u \notin V(C)$, and S is any $(d-1)$-face of C, then the simplicial complex obtained from C by adding u to the vertex set V and adding the new facet $S \cup \{u\}$, is a d-tree. A *simple d-tree* ([63]) is a d-tree in which every $(d-1)$-face is included in at most two facets. (I.e., it is a pseudomanifold with boundary.) A simple d-tree is actually a triangulated d-ball. In fact, given a stacked d-polytope P, $d \geqq 3$, P can be divided uniquely into d-simplices $S_1, \dots, S_m$, such that every $(d-2)$-face of any of these simplices is a face of P. The sets of vertices of these simplices form the set of facets of a simple d-tree. This gives a $1-1$ correspondence between simple d-trees and stacked d-polytopes, $d \geqq 3$.

A d-tree on n-vertices has $\psi_k(n,d) = \binom{d}{k} n - \binom{d+1}{k+1} k$ k-faces ([34]). A simple result of Beineke and Pippert [19] and Björner [17], asserts that every strongly connected d-dimensional simplicial complex C with n vertices has at least $\psi_k(n,d)$ k-faces. This bound applies, in particular, to $(d-1)$-pseudomanifolds with boundary. Beineke and Pippert showed that equality holds only for k-trees. (The earliest result of this type was proved by Klee [40].)

Define a stacked $(d-1)$-ball to be a triangulated $(d-1)$-ball C which is obtained from a simple $(d-1)$-tree by repeated stellar subdivisions of facets. Equivalently, C is a stacked $(d-1)$-ball if C is the antistar of a vertex of a stacked $(d-1)$-sphere.

Let C be a stacked $(d-1)$-ball with n vertices, n_b of them on the boundary and n_i in the interior. (n_b is always at least d.) Thus, C is obtained from a simple $(d-1)$-tree with n_b vertices by n_i applications of stellar subdivisions of facets. Let $\varphi_k^{\mathbf{b}}(n_i, n_b, d)$ be the number of k-faces of C. As easily seen this number depends only on n_i, n_b and d, and is given by formula (1.2):

Theorem 11.1. *Let C be a triangulated $(d-1)$-manifold, $d \geqq 4$, with non-empty boundary. If C has n_i vertices in the interior and n_b vertices in the boundary then*

(i) $f_k(C) \geqq \varphi_k^{\mathbf{b}}(n_i, n_b, d)$, *for every* k, $1 \leqq k \leqq d-1$.

(ii) *If* $f_k(C) = \varphi_k^{\mathbf{b}}(n_i, n_b, d)$ *for some* k, $1 \leqq k \leqq d-1$ *then* C *is a stacked* $(d-1)$-*ball.*

Proof. Let u be a vertex not in C and $D = C \cup (\{u\} * \partial C)$. D is a $(d-1)$-pseudomanifold (without boundary).

Claim 11.2. D is generically d-rigid.

Proof. Note that for every vertex $v \in D$, different from u, $\mathrm{lk}(v, C)$ is a homology $(d-2)$-sphere, and is generically $(d-1)$-rigid by Theorem 6.1. Choose any tree T in $G(D)$ which contains all vertices of D except u. The conditions of Proposition 6.4 hold and therefore C is generically d-rigid.

Proof of Theorem 11.1 (continued). Put $n = f_0(D)$ $(= n_i + n_b + 1)$. Recall that for $k \geqq 1$, $\gamma_k(D) = f_k(D) - \varphi_k(n, d)$, (Sect. 5.) Put $\gamma_0(D) = 0$. A simple inspection shows that

$$f_k(C) - \varphi_k^{\mathbf{b}}(n_i, n_b, d) = \gamma_k(D) - \gamma_{k-1}(\mathrm{lk}(u, D)). \tag{11.1}$$

Proposition 11.3. *Let D be a generically d-rigid $(d-1)$-pseudomanifold. Then for every vertex v of D, $\gamma_k(D) \geqq \gamma_{k-1}(\mathrm{lk}(v, D))$. If equality holds then $\gamma(D) = 0$.*

Proof. Recall that $\gamma^i(D) = \sum \{\gamma(\mathrm{lk}(S, D)) : S \in D, |S| = i\}$. Proposition 5.1 asserts that $\gamma_k(D) = \sum_{i=0}^{k-1} w_i(k, d)\, \gamma^i(D)$. The coefficients $w_i(k, d)$ are given by formula (5.6).

We need the following two inequalities:

$$\gamma^i(D) \geqq \gamma^{i-1}(\mathrm{lk}(v, D)) + \gamma^i(\mathrm{lk}(v, D)). \tag{11.2}$$

$$w_i(k, d) + w_{i+1}(k, d) > w_i(k-1, d-1) \quad \text{for every } 1 \leqq i \leqq k-1. \tag{11.3}$$

To prove (11.2) divide the set of $(i-1)$-faces of D into three parts. (a) Those faces S which contain the vertex v, (b) Those faces S which do not contain v but $S \cup \{v\} \in D$ and (c) the remaining $(i-1)$-faces of D. Note that the sum of $\gamma(\mathrm{lk}(S, C)$ over faces in the first family is exactly $\gamma^{i-1}(\mathrm{lk}(v, C)$. If S belongs to the second family and $T = S \cup \{v\}$ then by Theorem 7.3, $\gamma(\mathrm{lk}(S, D)) \geqq \gamma(\mathrm{lk}(T, D)$. But $\mathrm{lk}(T, D) = \mathrm{lk}(S, \mathrm{lk}(v, D))$, and therefore the sum of $\mathrm{lk}(S, C)$ over all faces in the second family is at least $\gamma^i(\mathrm{lk}(v, D))$.

To prove (11.3) use formula (5.6) and note that always $a_k(d) > a_{k-1}(d-1)$ and $1/(k+1)\binom{k}{i-1} + 1/(k+1)\binom{k}{i} = 1/k\binom{k-1}{i-1}$.)

By Proposition 4.1, (11.2) and (11.3),

$$\begin{aligned}
\gamma_k(D) &= \sum_{i=0}^{k-1} w_i(k, d)\, \gamma^i(D) \geqq w_0(k, d)\, \gamma(\mathrm{lk}(v, D)) + \sum_{i=1}^{k-1} w_i(k, d)(\gamma^{i-1}(\mathrm{lk}(v, D)) + \gamma^i(\mathrm{lk}(v, D))\\
&\geqq \sum_{i=0}^{k-2} (w_i(k, d) + w_{i+1}(k, d))\, \gamma^i(\mathrm{lk}(v, D)) \geqq \sum_{i=0}^{k-2} w_i(k-1, d-1)\, \gamma^i(\mathrm{lk}(v, D))\\
&= \gamma_{k-1}(\mathrm{lk}(v, D)).
\end{aligned}$$

This gives the required inequality. Since (11.2) is a strict inequality if $\gamma_k(D) = \gamma_{k-1}(\mathrm{lk}(v, D))$ then $\gamma(D) = 0$.

Back to the proof of Theorem 11.1. By Claim 11.2, D is generically d-rigid. Formula (11.1) and Proposition 11.3 give part (i) and show that in case of equality $\gamma(D)=0$. In order to prove part (ii) we need

Claim 11.4. If $\gamma(D)=0$ then D is a stacked $(d-1)$-sphere.

Proof. Let $E=\mathrm{lk}(u,D)$. If E is not connected, apply the normalization procedure described in Sect. 10 to the vertex u. The proof of Claim 11.2 apply for the resulting complex $\hat{D}$ and by formula (10.1), $\gamma(D)>\gamma(\hat{D})\geqq 0$. A contradiction. If E is connected then for $d\geqq 5$, $D\in\mathscr{C}_d$ and by Theorem 7.1, D is a stacked $(d-1)$-sphere. For $d=4$, $\mathrm{lk}(u,D)$ may be any triangulated 2-manifold. However, since $\gamma(D)=0$, $G(D)$ is 4-acyclic and $\mathrm{lk}(u,D)$ must be 3-acyclic. Therefore, $\mathrm{lk}(u,D)$ is a triangulated 2-sphere, $D\in\mathscr{C}_4$ and by Theorem 7.1, D is a stacked 3-sphere.

Proof of Theorem 11.1 (ii) (end). By Claim 11.4, D is a stacked $(d-1)$-sphere, hence C is a stacked $(d-1)$-ball.

Remark 11.5. Björner conjectured in [17] that Theorem 11.1(i) holds for every $(d-1)$-pseudomanifold with boundary. Björner proved the case $d=3$ of this conjecture, and showed that the conjecture imply the lower bound inequalities for pseudomanifolds without boundary. It can be shown that the assertion of Theorem 11.1 for arbitrary pseudomanifolds with boundary would also follow from the generic 3-rigidity of all triangulated 2-manifolds (Conjecture G). Our proof can be applied to all normal $(d-1)$-pseudomanifolds with boundary with singular part of codimension 3 or more.

12. A lower bound conjecture for polyhedral manifolds

For a polyhedral complex C, $f_2^k(C)$ is the number of 2-faces of C which are k-gons. For a polyhedral $(d-1)$-dimensional complex C define:

$$\gamma(C)=f_1(P)+\sum_{k\geqq 3}(k-3)f_2^k(P)-dn+\binom{d+1}{2} \tag{12.1}$$

For a d-polytope P, with boundary complex $\mathscr{B}(P)$, $\gamma(P)$ stands for $\gamma(\mathscr{B}(P))$.

Conjecture 12.1. *If P is a polyhedral $(d-1)$-manifold then* $\gamma(C)\geqq 0$.

Perhaps the ultimate generality for conjecture 12.1 (and a convenient context to study this conjecture,) is for "graph manifolds" which are defined in [11]. (See also [12].)

As we already mentioned in Sect. 5, Whiteley's theorem implies the truth of Conjecture 12.1 for boundary complexes of d-polytopes (Theorem 1.4). Previously, it was proved for rational polytopes as a consequence of some deep results in algebraic-geometry. In fact, for such a polytope P, $\gamma(P)$ is the dimension of the second primitive intersection homology group ([29]) of the toric variety associated with P. (See [46], [58, Ch. 4], [59].) However, as was shown by Perles [31, pp. 92–95], there are polytopes which are *not* combinatorially equivalent to rational polytopes.

One difficulty in dealing with Conjecture 12.1 is the fact that the generic d-rigidity of d-polytopal frameworks is not a local property as in the simplicial case (for $d \geqq 4$). In the case of a simplicial d-polytope, $d \geqq 4$, (or a triangulated (d – 1)-manifold,) the graph induced on a neighborhood of any vertex is already generically d-rigid. This is not the case for d-polytopal frameworks. As an example, let P be a pyramid over the octahedron Q, and consider the neighborhood of any vertex of Q.

For a d-polytopal framework $\mathscr{F}$ based on a d-polytope P it is only for the (highly non-generic) embeddings which realize P as a convex polytope that it is possible to prove "local" infinitesimal rigidity at any vertex [66, p. 456]. This in turn, implies the infinitesimal rigidity and hence the generic rigidity of $\mathscr{F}$. We do not know how to find such a pleasant embedding for arbitrary polyhedral $(d-1)$-manifolds (or even polyhedral $(d-1)$-spheres).

We mention now two corollaries of Theorem 1.4. A polyhedral complex P is *k-simplicial* if every j-face S of P, $j \leqq k$ is a simplex. Theorem 1.4 and the MPW-reduction imply:

Theorem 12.2. *Let P be a k-simplicial d-polytope with n vertices then $f_i(P) \geqq \varphi_i(n, d)$ for $1 \leqq i \leqq k$.*

Let us check now what does Theorem 1.4 says for simple polytopes. If P is a simple polytope with n vertices then $f_1(P) = \frac{dn}{2}$ and $\sum k f_2^k = f_1(P)(d-1)$. The inequality $\gamma(P) \geqq 0$ reduces in this case to:

$$f_2(P) - (d-2) f_1(P) \leqq \binom{d+1}{2}.$$

A posteriori this follows, of course, from Billera, Lee and Stanley's complete characterization of f-vectors of simplicial polytopes.

A d-polytope P is *elementary* if $\gamma(P) = 0$. In [38] we study the function $\gamma(P)$ for polytopes and especially the class of elementary polytopes. We prove there that quotients and faces of elementary polytopes are elementary and that for every face S of an elementary polytope P either S or $\mathrm{lk}(S, P)$ is a simplex. We prove also that the class of elementary polytopes is self-dual. The starting point for the proof is the fairly simple identity: For a 4-polytope P, $\gamma(P) = \gamma(P^*)$. It would be interesting to find a natural isomorphism between the spaces of stresses of the polytopal frameworks based on a 4-polytope P and its dual P^*.

13. Topological subgraphs of triangulated manifolds

In this section we diverge from lower bound theorems. We prove using some of our previous results a property of graphs of traingulated manifolds of a different nature.

A graph H is embeddable in a graph G if G contains some subgraph homeomorphic to H. Grünbaum proved ([31, p. 200]) that K_{d+1}, the complete graph on $d+1$ vertices, is embeddable in the graph of every d-polytope. Barnette proved in [11] that K_{d+1} is embeddable in the graph of every polyhedral $(d-1)$-manifold. (These results are immediate in the simplicial case.) For a graph G with no vertices of degree 2, TG stands for any graph homeomorphic to G.

Theorem 13.1. Let $d \geqq 4$ be a fixed integer. K_{d+2} is embeddable in the graph of a triangulated $(d-1)$-manifold C iff C is not a stacked $(d-1)$-sphere.

Proof. It is well-known and easy that if C is isomorphic to a stacked d-polytope then K_{d+2} is not embeddable in $G(C)$. In fact, $G(C)$ does not contain K_{d+2} even as a minor.

Let K_d^- denotes a K_d minus an edge. The two vertices of a TK_d^- $(d>4)$ of degree $d-2$ are called *special.*

Lemma 13.2. *Every two non-adjacent vertices of a simplicial* 3*-polytope serve as special vertices of a* TK_5^-; *every two non-adjacent vertices of a stacked* d*-polytope* $(d>3)$ *serve as the special vertices of some* TK_{d+2}^-.

Proof. The first part follows from the 3-connectivity of C, the second part can easily be checked directly.

Proof of Theorem 13.1 (end). Let C be a triangulated $(d-1)$-manifold, and assume that C does not contain a TK_{d+2}. We can assume that C has no vertices of degree d (otherwise we delete them successively). We apply induction on d. Let v be a vertex of C and u, w be a pair of non-adjacent vertices in $\mathrm{lk}(v, C)$. By Lemma 13.2, (and the induction hypothesis if $d>4$,) u and w are the two special vertices of some TK_{d+1}^- in $\mathrm{lk}(v, C)$. Therefore u and w are not adjacent in C nor they are connected in a path that avoids $st(v, C)$. This directly implies that C has no 2-missing faces and no chordless m-gons for $m \geqq 4$. For $d>4$ the induction hypothesis implies that C does not contain missing k-faces for $2<k<d-1$ as well. By Theorem 8.5, C is isomorphic to a stacked sphere.

Remarks. (1) For triangulated 2-manifolds the situation is this. K_5 is not embeddable in any triangulated 2-sphere (stacked or not) by (the easy part of) Kuratowski's Theorem. It is plausible but unknown that K_5 is embeddable in every triangulated 2-manifold which is not a sphere. This will follow from an oldstanding conjecture of Dirac [27] which asserts that K_5 is embeddable in every graph with n vertices and more than $3n-6$ edges. Assuming the the truth of Dirac's conjecture it can be shown that Theorem 13.1 holds for arbitrary $(d-1)$-pseudomanifolds. (While our proof applies only for pseudomanifolds in $\mathscr{C}_d$.)

(2) Grünbaum proved ([31, p. 200]) that for every d-polytope P, $\mathrm{skel}_i(\Delta_d)$ is embeddable in $\mathrm{skel}_i(P)$. *Problem:* For which simplicial d-polytopes is $\mathrm{skel}_i(\Delta_{d+1})$ embeddable in $\mathrm{skel}_i(P)$? By van Kampen-Flores theorem ([31, Ch. 11]) this may never occur if $i \geqq [d+1/2]-1$.

14. Concluding remarks and open problems

14.1. $\gamma(M)$ *and the topology of* M. For a manifold M, (of dimension at least 2,) define $\gamma(M)=\min\{\gamma(C)\colon C$ is a triangulation of $M\}$. For every manifold

M, which admits some finite triangulation, $\gamma(M)$ is a non-negative integer, and we have proved that $\gamma(M)=0$ only if M is a sphere. If M is two dimensional $\gamma(M)=3(2-\chi(M))$, where $\chi(M)$ is the Euler characteristics of M.

Walkup proved [63] that (i). For a 3-manifold M which is not a sphere, $\gamma(M)\geqq 10$, and $\gamma(M)=10$ iff M is $S^1\times S^2$ or the corresponding non-orientable "handle". (He also showed that the only triangulations C of these manifolds which satisfy $\gamma(C)=10$ are in $\mathscr{H}^4(1)$.) (ii) For all other 3-manifolds M, $\gamma(M)\geqq 17$ and $\gamma(M)=17$ only when M is the three dimensional projective space.

In [39] we show that for every fixed non-negative integers $d, c, d\geqq 2$, there are only finitely many d-manifolds M for which $\gamma(M)<c$.

We would like to understand how the topology of M affects the invariant $\gamma(M)$. Let $b_i(M)$ denotes the i-th (reduced) Betti number of M. (Thus, $b_i(M) = \operatorname{rank} \tilde{H}_i(M, \mathbb{Z})$.)

Conjecture 14.1. *For a $(d-1)$-manifold M, $d\geqq 4$, $\gamma(M)\geqq b_1(M)\binom{d+1}{2}$.*

If $C\in\mathscr{H}^d(k)$ then $\gamma(C)=b_1(C)\binom{d+1}{2}$. (Are these the only cases of equality?) Walkup proved ([63]) that for every 4-manifold M, (and even every 4-pseudomanifolds in $\mathscr{C}_5$,) $\gamma(M)\geqq\frac{15}{2}(2-\chi(M))$ and equality holds iff $C\in\mathscr{H}^5\left(1-\frac{\chi(M)}{2}\right)$.

The problem of finding $\gamma(M)$ for a $(d-1)$-manifold M resembles the well-known problem of finding $\alpha(M)$ the minimal number of vertices in a triangulation of M (see [50]). Let i, d be fixed integers, $d\geqq 3$, $0<i<\left[\frac{d-1}{2}\right]$. It can be shown quite easily that $\alpha(M)\geqq C(i,d)\,b_i(M)^{\frac{1}{i+1}}$, where $C(i,d)$ is a positive constant depending on i and d (compare [18]). We conjecture that similarly (for $i>0$,) $\gamma(M)\geqq D(i,d)\,b_i(M)^{\frac{1}{i}}$, where $D(i,d)$ is another positive constant depending on i and d.

We would like to know the exact values of $\gamma(S^1\times S^1\times S^1)$, $\gamma(S^2\times S^2)$ and $\gamma(\mathbb{C}P^2)$. Kühnel's 3-neighborly complex projective plane with 9 vertices ([43, 44]) shows that $\gamma(\mathbb{C}P^2)\leqq 6$.

14.2. *The generalized lower bound conjecture*

Let d be a fixed integer, $d\geqq 1$. For a vector of non-negative integers $f=(f_{-1}, f_0, f_1, \ldots, f_{d-1})$, $f_{-1}=1$ define $h[f]=(h_0, h_1, \ldots, h_d)$ where

$$h_k=\sum_{i=0}^{k}(-1)^i\binom{d-k+i}{i}f_{k-i-1}.$$

If f is the f-vector of a simplicial d-polytope or a $(d-1)$-dimensional complex C, $h[f]$ is called the *h-vector* of C. h-vectors of simplicial polytopes were introduced by McMullen and Walkup [49]. This concept plays a crucial part in the combinatorial theory of simplicial polytopes and in several other areas of combinatorics ([48, 54, 55]). (The original notation was $g_i^{(d)}$ for h_i and $g_i^{(d+1)}$ for $h_{i+1}-h_i$.)

A simplicial d-polytope P is *k-stacked* if P can be triangulated without introducing new j-faces for $j \leqq d-k-1$. A triangulated $(d-1)$-sphere C is k-stacked if it is the boundary of a triangulated d-ball B with the same $(d-k-1)$-skeleton.

McMullen and Walkup suggested in [49] the following far reaching generalization of the lower bound conjecture.

The generalized lower bound conjecture. (i) (The generalized lower bound inequalities.) If P is a simplicial d-polytope and $0 \leqq k \leqq \left[\frac{d}{2}\right]-1$ then $h_{k+1}(P) - h_k(P) \geqq 0$. (ii) If $h_{k+1}(P)-h_k(P)=0$ then P is a k-stacked polytope.

The generalized lower bound inequalities were proved by Stanley [55] as part of his proof of the necessity part of the "g-theorem". Part (ii) is still open.

Note that $\gamma(P)=h_2(P)-h_1(P)$. The Dehn-Sommerville equations (see [48, 56]) assert that $h_i=h_{d-i}$, $0 \leqq i \leqq d$. In particular, if $d=2k+1$ then $h_{k+1}-h_k=0$ for every simplicial d-polytope.

It is widely believed that the assertions of the GLBC and the "g-theorem" are true for arbitrary triangulated spheres. (See [32, 56].) In part (ii), "a k-stacked polytope" should be replaced by "a k-stacked sphere"[3].

For a triangulated $(d-1)$-manifold C define

$$\hat{h}_k(C)=h_k(C)-\binom{d}{k}\sum_{i=0}^{k-1}(-1)^i b_{k-i-1}(C).$$

Schenzel proved ([52], see also [57, pp. 84–85]) that every triangulated $(d-1)$-manifold with boundary C satisfies $\hat{h}_k(C) \geqq 0$, for every $k \geqq 0$.

Conjecture 14.2. Let C be a triangulated $(d-1)$-manifold (without boundary). Then for every k, $0 \leqq k \leqq \left[\frac{d}{2}\right]-1$, $\hat{h}_{k+1}(C)-\hat{h}_k(C) \geqq \binom{d}{k-1} b_k(C)$.

Note that Conjecture 14.1 is a special case of conjecture 14.2. The Dehn-Sommerville equations assert that $\hat{h}_d(C)=0$ and $\hat{h}_i(C)=\hat{h}_{d-i}(C)$, $1 \leqq i < d$.

Many of the results of this paper have obvious analogs in the context of the generalized lower bound inequalities. Proving them seems hard. Only the third proof of Theorem 9.3 and the proof hinted there for Lemma 7.2 extend directly.

14.3. *Flexible weak embeddings.* Let C be a pure simplicial complex. An embedding of the vertices of C into $\mathbb{R}^d$ is a *weak embedding* of C if the images of the vertices of every facet of C are affinely independent. If C is the boundary complex of a stacked d-polytope then every weak embedding of C into $\mathbb{R}^d$ is rigid. Bricard constructed in 1897 ([20]) a flexible weak embedding of the octahedron into $\mathbb{R}^3$.

---

[3] The conjectured equality cases for spheres do not imply the conjecture for polytopes. One consequence from the GLBC would be that for $1 \leqq k \leqq \left[\frac{d}{2}\right]-1$ every d-polytope whose boundary complex is k-stacked (as a sphere) is a k-stacked polytope. We doubt if this is true for $k \geqq \left[\frac{d}{2}\right]$

Conjecture 14.3. *Every non-stacked 3-polytope have a flexible weak embedding.*

14.4. *Rigidity of spaces and separations properties.* All topological spaces mentioned here admit finite triangulations. A topological space X is d-rigid if every triangulation of X is generically d-rigid. A simple sufficient condition for d-rigidity follows from the generic d-rigidity of strongly connected d-dimensional simplicial complexes.

Theorem 14.4. *Let X be a topological space. If for every $Y \subset X$ which separates X, $\dim Y \geqq d-1$ then X is d-rigid.*

Conjecture 14.5. *Let $d \geqq 3$. Let X be a topological space. If for every $Y \subset X$ which separates X, $H_{d-2}(Y) \neq 0$ then X is d-rigid.*

14.5. *Rigidity of tight manifolds.* A triangulated 2-manifold M enbedded in R^3 is *tight* (see [45, 8]) if $M \cap H$ is connected for every half space H of $\mathbb{R}^3$. (This property is known as Banchoff's two piece property and is weaker then tightness in more general contexts.) M is *strictly tight* if it is tight and no two adjacent facets of M are in the same plane (in particular if the vertices of C are in general position).

Strictly tight embeddings of a triangulated 2-sphere C are just realizations of C as the boundary of simplicial polytopes. All these embeddings are rigid by Cauchy's theorem. Connelly proved in [26] that all tight embeddings of a 2-sphere, i.e., embeddings as convex surfaces, are rigid.

Conjecture 14.6. *A tight embedding of a triangulated 2-manifold in $\mathbb{R}^3$ is rigid.*

Conjecture 14.6 implies that triangulated 2-manifolds which admits strictly tight embeddings are generically 3-rigid. Yet, it is hard to suggest this approach for proving Conjecture G (Sect. 10) for orientable 2-manifolds. It is not even known whether every orientable triangulated 2-manifold can be geometrically embedded in $\mathbb{R}^3$ (See [32, 13]).

References

1. Alexandrov, A.D.: Convex polyhedra. Moscow, 1950 (Russian); German transl., Konvexe Polyeder. Berlin: Akademie-Verlag 1958
2. Altshuler, A.: 3-pseudomanifolds with preassigned links. Trans. Am. Math. Soc. **241**, 213–237 (1978)
3. Altshuler, A., Perles, M.A.: Quotient polytopes of cyclic polytopes, Part I: Structure and characterization. Isr. J. Math. **36**, 97–125 (1980)
4. Altshuler, A., Steinberg, L.: Neighborly 4-polytopes with 9 vertices. J. Comb. Theory, Ser. A **15**, 270–287 (1973)
5. Asimow, L., Roth, B.: The rigidity of graphs I. Trans. Am. Math. Soc. **245**, 279–289 (1978)
6. Asimow, L., Roth, B.: The rigidity of graphs II. J. Math. Anal. Appl. **68**, 171–190 (1979)
7. Balinski, M.: On the graph structure of convex polyhedra in n-space. Pac. J. Math. **11**, 431–434 (1961)
8. Banchoff, T.F.: Tightly embedded 2-dimensional polyhedral manifolds. Am. J. Math. **87**, 462–472 (1975)
9. Barnette, D.: The minimum number of vertices of a simple polytope. Isr. J. Math. **10**, 121–125 (1971)
10. Barnette, D.: A proof of the lower bound conjecture for convex polytopes. Pac. J. Math. **46**, 349–354 (1973)

11. Barnette, D.: Graph theorems for manifolds. Isr. J. Math. **16**, 62–72 (1973)
12. Barnette, D.: Generalized combinatorial cells and facet splitting. Pac. J. Math. **46**, 349–354 (1973)
13. Barnette, D.: Polyhedral maps on 2-manifolds. In: Convexity and related combinatorial geometry. Kay, D.C., Breen, M. (eds.), pp. 7–19. New York: Marcel Dekker Inc. 1982
14. Barnette, D., Grünbaum, B.: On Steinitz's theorem concerning convex polytopes and on some properties of planar graphs. The many facets of graph theory. Lecture Notes in Math., vol. 110, pp. 27–40. Berlin-Heidelberg-New York: Springer 1969
15. Billera, L.J., Lee, C.W.: A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes. J. Comb. Theory, Ser. A **31**, 237–255 (1981)
16. Billera, L.J., Lee, C.W.: The number of faces of polytopes pairs and unbounded polyhedra. Eur. J. Comb. **2**, 307–322 (1981)
17. Björner, A.: The minimum number of faces of a simple polyhedron. Eur. J. Comb. **1**, 27–31 (1980)
18. Björner, A., Kalai, G.: An extended Euler-Poincaré formula. (Preprint)
19. Beineke, L.W., Pippert, R.E.: Properties and characterizations of k-trees. Mathematica **18**, 141–151 (1971)
20. Bricard, R.: Mémoire sur la théorie de l'octa dre articulé. J. Math. Pures Appl., IX. Ser. 3, 113–148 (1897)
21. Brönsted, A.: An introduction to convex polytopes. Graduate text in Mathematics, vol. 90. Berlin-Heidelberg-New York: Springer 1983
22. Cauchy, L.: Sur les polygones et les polyèdres, Second Memoire, I. Ecole Polytechnique 9 (1813), 87 (= Oeuvres complètes d'Augustin Cauchy, 2nd sér., Tome 1, 1905, pp. 26–38)
23. Cooper, D., Thurston, W.P.: Triangulating 3-manifolds using 5 vertex link types. Preprint
24. Connelly, R.: A counterexample to the rigidity conjecture for polyhedra. Publ. Math., Inst. Hautes Étud. Sci. **47**, 333–338 (1978)
25. Connelly, R.: Conjectures and open problems in rigidity. Proc. International Congress of Math., Helsinki, 1978
26. Connelly, R.: The rigidity of certain cabled frameworks and the second order rigidity of arbitrarily triangulated convex surfaces. Adv. Math. **37**, 272–299 (1980)
27. Dirac, G.A.: Homomorphism theorems for graphs. Math. Ann. **153**, 69–80 (1964)
28. Gluck, H.: Almost all simply connected closed surfaces are rigid. In: Geometric topology. Lecture Notes in Math., vol. 438, pp. 225–239. Berlin-Heidelberg-New York: Springer 1975
29. Goresky, M., MacPherson, R.: Intersection homology theory. Topology 19, 135–162 (1980)
30. Graver, J.: A combinatorial approach to infinitesimal rigidity. (Preprint)
31. Grünbaum, B.: Convex polytopes. New York: Wiley Interscience 1967
32. Grünbaum, B.: Polytopes, graphs and complexes. Bull. Am. Math. Soc. **76**, 1131–1201 (1970)
33. Grünbaum, B., Shephard, G.C.: Lecture on lost mathematics. Preprint Syracuse University, 1978
34. Harary, F., Palmer, E.M.: On acyclic simplicial complexes. Mathematika **15**, 115–122 (1968)
35. Kalai, G.: Combinatorial problems in convexity and the combinatorics of simplicial complexes. Ph.D. Thesis, Jerusalem, 1983
36. Kalai, G.: Weakly saturated graphs are rigid. Ann. Discrete Math. **20**, 189–190 (1984)
37. Kalai, G.: Heperconnectivity of graphs. Graphs Comb. **1**, 65–80 (1985)
38. Kalai, G.: Rigidity and the lower bound theorem II: Elementary polytopes. (Preprint)
39. Kalai, G.: Rigidity and the lower bound theorem III: Triangulated manifolds with "few edges".(Preprint)
40. Klee, V.: A comparison of primal and dual method of linear programming Num. Math. **9**, 227–235 (1966)
41. Klee, V.: Polytope pairs and their relations to linear programming. Acta Math. **113**, 1–25 (1974)
42. Klee, V.: A d-pseudomanifold with f_0 vertices has at least $df_0-(d-1)(d+2)$ d-simplices. Houston J. Math. **1**, 81–86 (1975)
43. Kühnel, W., Banchoff, T.F.: The 9 vertex complex projective plane. The Math. Intell. **5**, issue 3, 11–22 (1983)
44. Kühnel, W., Lassmann, G.: The unique 3-neighborly 4-manifold with few vertices. J. Comb. Theory, Ser. A **35**, 173–184 (1983)
45. Kuiper, N.H.: Tight embeddings and maps. Submanifolds of geometric type tree in E^N, The Chern Symposium. Singer, I.M. (ed.), pp. 97–145, Berlin Heidelberg New York: Springer 1980

46. MacPherson, R.: Intersection homology. Herman Weyl lectures. (in press) (1987)
47. McMullen, P.: The numbers of faces of simplicial polytopes. Isr. J. Math. **9**, 559–570 (1971)
48. McMullen, P., Shephard, G.C.: Convex polytopes and the upper bound conjecture. London Math. Soc. Lecture Notes Series, vol. 3. London/New York: Cambridge Univ. Press 1971
49. McMullen, P., Walkup, D.W.: A generalized lower bound conjecture for simplicial polytopes. Mathematika **18**, 264–273 (1971)
50. Ringel, G.: Map color theorem. Berlin-Heidelberg-New York: Springer 1974
51. Roth, B.: Rigid and flexible frameworks. Am. Math. Mon. **88**, 6–21 (1981)
52. Schenzel, P.: On the number of faces of simplicial complexes and the purity of Frobenius. Math. Z. **178**, 125–142 (1981)
53. Spanier, E.H.: Algebraic topology. New York: McGraw Hill 1966
54. Stanley R.: The upper bound conjecture and Cohen-Macaulay rings. Stud. Appl. Math. **54**, 135–142 (1975)
55. Stanley, R.: The number of faces of simplicial convex polytopes. Adv. Math. **35**, 236–238 (1980)
56. Stanley, R.: The number of faces of simplicial polytopes and spheres. Discrete geometry and convexity, Goodman, J.E., Lutwak, E., Malkevitch, J., Pollack, R. (eds.), pp. 212-223. New York: Ann. NY Acad Sci 1985
57. Stanley, R.: Interactions between commutative algebra and combinatorics. Boston: Birkhäuser 1983
58. Stanley, R.: Enumerative combinatorics. Vol. I, Wadsworth, Monterey, 1986
59. Stanley, R.: Generalized h-vectors, intersection cohomology of toric varieties, and related results. Proc. Japan-USA workshop on commutative algebra and combinatorics (to appear)
60. Stoker, J.J.: Geometric problems concerning polyhedra in the large. Commun. Pure Appl. Math. **21**, 119–168 (1968)
61. Steinitz, E., Rademacher, H.: Vorlesungen über die Theorie der Polyeder. Berlin-Göttingen: Springer 1934
62. Tay, T.S., Whiteley, W.: Generating all isostatic frameworks. Struct. Topology **11**, 21–70 (1985)
63. Walkup, D.: The lower bound conjecture for 3- and 4-manifolds. Acta Math. **125**, 75–107 (1970)
64. Welsh, D.: Matroid theory. London: Academic Press 1976
65. Whiteley, W.: Cones, infinity and 1-story buildings. Struct. Topology **8**, 53–70 (1983)
66. Whiteley, W.: Infinitesimally rigid polyhedra I. Statics of Frameworks. Trans. Am. Math. Soc. **285**, 431–465 (1984)
67. Gromov, M.: Partial differential relations. Berlin Heidelberg New York: Springer 1986

Oblatum 26-VI-1985 & 3-IX-1985 & 3-III-1986

Note added in proof

The basic relation between the LBT and rigidity is observed independently by M. Gromov in [67, Ch. 2.4.10]. Moreover, Gromov presents a purly combinatorial "substitute" for rigidity. Using Gromov's "rigidity" concept combined with the results of Sections 7–11, it is possible to prove Theorems 1.1 and 11.1 for arbitrary pseudomanifolds.