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Reduced Piecewise Bivariate Hermite Interpolations

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Summary. The paper presents two methods for a piecewise Hermite interpolation of a sufficiently smooth function. The interpolation function is on each elementary rectangle, into which the given region is divided, determined by all the derivatives of the function under consideration up to a certain predetermined order. The results obtained are utilized in the solution of a general quasi-linear equation and in the solution of a non-linear integral equation.

1. Introduction

In the numerical solution of differential equations we determine as a rule only the values of the unknown solution in the nodal points of a mesh given in advance. In many physical or technical problems we are, however, often more interested in derivatives of this solution, since those have basic physical significance. It is, of course, possible to numerically compute relatively easily from the known solution of the given equation also the values of its derivatives, but in doing so the calculation is loaded with a further error. The finite element method (see [5, 6, 8, 11, 12]) facilitates the determination of the values of derivatives of the sought solution together with values of this solution already during the actual solution of the given differential equation. Certain, in the application of the method often employed interpolation methods of the unknown solution are listed in [1, 3, 12, 13].

If we employ the interpolation method of the sought function $u(x, y)$ described in [1] and if we use in doing so the interpolation polynomials of the $2m-1$ degree in each of both variables, we find in each nodal point of the given rectangular mesh m^2 parameters, i.e. all partial derivatives $D^{\varkappa}u(x, y)$, for which $0 \leqq \varkappa_1, \varkappa_2 \leqq m-1$; at the same time we denote $\varkappa = (\varkappa_1, \varkappa_2)$, $|\varkappa| = \varkappa_1 + \varkappa_2$, $\varkappa_1, \varkappa_2 \geqq 0$, $D^{\varkappa} = \frac{\partial^{|\varkappa|}}{\partial x^{\varkappa_1} \cdot \partial y^{\varkappa_2}}$. In many problems of a physical as well as technical nature, however, the knowledge of all those derivatives is not necessary. Thus e.g. in the study of a planar potential field (magnetic, electric, thermal, ...) described by a partial differential equation of the second order we are mainly interested in the first partial derivatives of the sought solution, since they give important physical magnitudes. The values of the second order mixed derivative which we would also have to determine when applying the above mentioned method do not, however, have any physical significance, and therefore their determination would be useless. Similarly in the theory of elasticity where in the course of investigating the deformations of a plate a certain partial differential equation of the fourth order is obtained, the unknown function $u(x, y)$ gives the deflection of the plate, its first partial derivatives determine the angles of slope, from the second derivatives the moments can be calculated and from the latter deformations and stress components, and finally with the aid of the third derivatives the shear forces are determined. The remaining derivatives $D^\varkappa u(x, y)$, $0 \leqq \varkappa_1, \varkappa_2 \leqq 3$, $|\varkappa| \geqq 4$, essential when applying the above cited method, are of no significance for practical purposes.

The derivatives $D^\varkappa u(x, y)$, for which $0 \leqq \varkappa_1, \varkappa_2 \leqq m-1$, $|\varkappa| \geqq m$ thus as a rule lack physical meaning. When processing the problem on a digital computer, these data would increase out of proportion the quantity of unwanted information, would reduce uselessly the capacity of the memory and their evaluation would prolong the computation. Therefore two other methods of interpolating the given function will be presented in this paper. Both the methods, similar to the method described in [1], do not contain in comparison with the cited paper the above mentioned redundant data, have the same order of accuracy in some cases, but do not guarantee the continuity of all derivatives of the interpolation function.

When estimating the error between the given function and its interpolation we shall make use of a certain lemma which will first of all be cited without proof in a similar form as given in [3]. For this purpose let us consider in the N-dimensional space E^N with a general point $x = (x_1, \ldots, x_N)$ a bounded region Ω and let us choose an arbitrary non-negative integer k. The symbol $W_p^{(k)}(\Omega)$ denotes the space of all functions $u(x)$ defined on Ω which have the generalized derivatives up to the order k inclusive and which together with those derivatives belong to the space $L_p(\Omega)$, $p \geqq 1$. Let us denote the norm of the space $W_p^{(k)}(\Omega)$

$$\|u\|_{k,p,\Omega} = \left\{ \sum_{|\varkappa| \leqq k} \int_\Omega |D^\varkappa u(x)|^p \, dx \right\}^{1/p},$$

where in agreement with the above we denote $D^\varkappa = \dfrac{\partial^{|\varkappa|}}{\partial x_1^{\varkappa_1} \ldots \partial x_N^{\varkappa_N}}$, $\varkappa = (\varkappa_1, \ldots, \varkappa_N)$, $|\varkappa| = \varkappa_1 + \cdots + \varkappa_N$, $\varkappa_1, \ldots, \varkappa_N \geqq 0$. Beside the given norm let us introduce on the above mentioned space also the semi-norm

$$|u|_{k,p,\Omega} = \left\{ \sum_{|\varkappa| = k} \int_\Omega |D^\varkappa u(x)|^p \, dx \right\}^{1/p}.$$

Lemma 1 (Bramble-Hilbert). Let a bounded region $\Omega \subset E^N$ with $\operatorname{diam} \Omega = 1$ fulfil the strong cone condition. Let $f(u)$ be a bounded linear functional defined on the space $W_p^{(k)}(\Omega)$, $p \geqq 1$, i.e. let $|f(u)| \leqq C_1 \|u\|_{k,p,\Omega}$. Let $f(\alpha) = 0$ for every polynomial α of a lesser degree than k. Then there exists a constant $C_2 > 0$, dependent only on the cone condition such that for all $u \in W_p^{(k)}(\Omega)$ it holds

$$|f(u)| \leqq C_1 C_2 |u|_{k,p,\Omega}.$$

The proof of the lemma, as well as the formulation of the strong cone condition are given in [2].

Our further considerations will be limited only to the case $N = 2$ and therefore let us note above all that the rectangle

$$R = \langle x_0, x_1 \rangle \times \langle y_0, y_1 \rangle, \tag{1}$$

which we are going to deal with in both interpolation methods first, satisfies the strong cone condition. In both cases we shall later on extend our considerations to a more general region, i.e. an arbitrary polygon Ω, the sides of which are parallel with the coordinate axes. In doing so we shall assume that on the polygon Ω there is defined a *partition* ϱ, i.e. that the polygon is expressed in the form of a union of rectangles R_τ, $\tau=1, \ldots, \tau_0$, each two of which are either disjoint or have one vertex or one side in common. If we denote a_τ, b_τ the sides of the rectangle R_τ, then each partition is characterized by two values

$$h=\max_\tau (a_\tau, b_\tau), \qquad \tilde{h}=\min_\tau (a_\tau, b_\tau). \tag{2}$$

A system C of the partitions ϱ will be called *regular*, if there exists a constant $\sigma>0$ such that for all $\varrho \in C$ it holds

$$\sigma h \leqq \tilde{h}.$$

2. First-Type Interpolation

It will be assumed in the entire section that there is given a natural number m and to this will be assigned a natural number

$$l=\left[\frac{m+1}{2}\right].$$

Limiting ourselves at the beginning to the rectangle (1), let us define above all two basic concepts.

Definition 1. The symbol $H_1^{(m)}(R)$ will be used to denote the set of all real polynomials defined on R, in each of both variables of a maximum degree $2m-1$, i.e. polynomials of the form

$$\alpha(x, y)=\sum_{i,j=0}^{2m-1} \alpha_{ij}\, x^i y^j, \tag{3}$$

such that $D^\varkappa \alpha(x_\mu, y)$ for $\varkappa=(k, 0)$ and $D^\varkappa \alpha(x, y_\nu)$ for $\varkappa=(0, k)$ form for all the $\mu, \nu=0, 1$, $k \leqq l-1$ polynomials of a maximum degree $2m-2k-1$ and such that $\alpha_{ij}=0$ for $2m-2[m/2] \leqq i, j \leqq 2m-1$.

Definition 2. The $H_1^{(m)}(R)$-*interpolate* of a function $u(x, y) \in C^{(m-1)}(R)$ will be called that element $u_m(x, y) \in H_1^{(m)}(R)$, for which it holds

$$D^\varkappa u_m(x_\mu, y_\nu)=D^\varkappa u(x_\mu, y_\nu), \qquad |\varkappa| \leqq m-1, \ \mu, \nu=0, 1. \tag{4}$$

It should be denoted that for $4m^2$ coefficients of the element u_m $2m(m+1)$ conditions in the vertices and $4l(l-1)$ conditions on the sides of the rectangle under consideration are valid and $4[m/2]^2$ coefficients are equal to zero, thus a total of $4m^2$ conditions. It will first of all be shown that those $4m^2$ conditions guarantee both the existence and the uniqueness of the defined interpolation.

Theorem 1. To each function $u(x, y) \in C^{(m-1)}(R)$ there exists exactly one $H_1^{(m)}(R)$-interpolate.

Proof. It is sufficient to show that from the conditions

$$D^{\varkappa} u_m(x_\mu, y_\nu) = 0, \quad |\varkappa| \leqq m-1, \ \mu, \nu = 0, 1 \tag{5}$$

it follows that $u_m(x, y) \equiv 0$. It is evident that for $m = 1$ this statement is satisfied. Let then $m \geqq 2$. For an arbitrary $0 \leqq k \leqq l-1$, $\varkappa = (k, 0)$, $\mu = 0, 1$ let us define the auxiliary polynomial $\beta(y) = D^{\varkappa} u_m(x_\mu, y)$ of one variable. The polynomial has according to the definition a maximum degree $2m - 2k - 1$ and there hold for it $2m - 2k$ conditions $\beta(y_\nu) = \beta'(y_\nu) = \cdots = \beta^{(m-k-1)}(y_\nu) = 0$, $\nu = 0, 1$; therefore $\beta(y) \equiv 0$ and thus also $D^{\varkappa} u_m(x_\mu, y) \equiv 0$. In a similar way it could be proved that for $\varkappa = (0, k)$, $0 \leqq k \leqq l-1$, $\nu = 0, 1$ it holds $D^{\varkappa} u_m(x, y_\nu) \equiv 0$. From the last two identities it follows that the polynomial $u_m(x, y)$ can be written in the form

$$u_m(x, y) = [(x - x_0)(x - x_1)(y - y_0)(y - y_1)]^l \cdot \alpha^{(l)}(x, y), \tag{6}$$

whereby the degree of the polynomial $\alpha^{(l)}(x, y)$ is in each of both variables maximally equal to the number $2m - 2l - 1$. It will be shown that all the coefficients of this polynomial are equal to zero. For this purpose let us define the final system of polynomials $\{\alpha^{(\lambda)}(x, y)\}_{\lambda=0}^{l}$ with the aid of the relations

$$\begin{aligned} &\alpha^{(0)}(x, y) = u_m(x, y), \\ &\alpha^{(\lambda)}(x, y) = \alpha^{(\lambda-1)}(x, y) \cdot [(x - x_0)(x - x_1)(y - y_0)(y - y_1)]^{-1}, \quad 1 \leqq \lambda \leqq l. \end{aligned} \tag{7}$$

The degree of the polynomial $\alpha^{(\lambda)}(x, y)$ is apparently in each of both variables maximally $2m - 2\lambda - 1$. It will be shown by mathematical induction that for all the indices i, j, for which $i, j \geqq 2m - 2[m/2] - 2\lambda$, its coefficients are equal to zero, i.e. $\alpha_{ij}^{(\lambda)} = 0$. If $\lambda = 0$, the statement follows directly from the definition. Let thus $0 < \lambda \leqq l$ and the polynomial $\alpha^{(\lambda)}(x, y)$ will be written in the form

$$\alpha^{(\lambda)}(x, y) = \sum_{i,j=0}^{2m-2\lambda-1} \alpha_{ij}^{(\lambda)} x^i y^j. \tag{8}$$

By substituting into (7), by rearrangement and respecting the induction assumption we obtain for $i, j \geqq 2m - 2[m/2] - 2\lambda + 2$ the equations

$$\begin{aligned} \alpha_{ij}^{(\lambda-1)} \equiv\ & \alpha_{i-2,j-2}^{(\lambda)} - (y_0 + y_1)\alpha_{i-2,j-1}^{(\lambda)} + y_0 y_1 \alpha_{i-2,j}^{(\lambda)} - (x_0 + x_1)\alpha_{i-1,j-2}^{(\lambda)} \\ & + (x_0 + x_1)(y_0 + y_1)\alpha_{i-1,j-1}^{(\lambda)} - (x_0 + x_1) y_0 y_1 \alpha_{i-1,j}^{(\lambda)} + x_0 x_1 \alpha_{i,j-2}^{(\lambda)} \\ & - x_0 x_1 (y_0 + y_1)\alpha_{i,j-1}^{(\lambda)} + x_0 x_1 y_0 y_1 \alpha_{ij}^{(\lambda)} = 0 \end{aligned}$$

from which we obtain for $j \geqq 2m - 2[m/2] - 2\lambda$, if considered that α_{ij} make sense only for $0 \leqq i, j \leqq 2m - 2\lambda - 1$ and that thus they can be considered equal zero in the other cases, gradually the relations $\alpha_{2m-2\lambda-1,j}^{(\lambda)} = 0$, $\alpha_{2m-2\lambda-2,j}^{(\lambda)} = 0, \ldots,$ $\alpha_{2m-2[m/2]-2\lambda,j}^{(\lambda)} = 0$, thus in total $\alpha_{ij}^{(\lambda)} = 0$ for $i, j \geqq 2m - 2[m/2] - 2\lambda$.

If we now apply the proven statement to the last polynomial $\alpha^{(l)}(x, y)$, we find the validity of $\alpha_{ij}^{(l)} = 0$ for all

$$i, j \geqq 2m - 2\left[\frac{m}{2}\right] - 2\left[\frac{m+1}{2}\right] = 0,$$

thus $\alpha^{(l)}(x, y) \equiv 0$ and therefore with regard to (6) also $u_m(x, y) \equiv 0$.

Let us now proceed to estimate the error between the given function and its $H_1^{(m)}(R)$-interpolate. This estimate is given by the following theorem, the proof of which is carried out in a similar way as the proof of the analogical theorem from [3].

Theorem 2. Let us arbitrarily choose a function $u(x, y) \in W_p^{(k)}(R)$, $1 < p < \infty$, $m + \left[\frac{2}{p}\right] \leqq k \leqq 2m$ and denote by $u_m(x, y)$ its $H_1^{(m)}(R)$-interpolate. Then there exists a constant $C_3 > 0$ independent of the choice of the function u such that for an arbitrary nonnegative integer $n \leqq s$, $s = \min(k, 2m - l + 1)$ it holds

$$\|u - u_m\|_{n,p,R} \leqq C_3 h^{s-n} |u|_{s,p,R},$$

where $h = \max(x_1 - x_0, y_1 - y_0)$.

Proof. With regard to $W_p^{(k)}(R) \subset C^{(k-[2/p]-1)}(R) \subset C^{(m-1)}(R)$ the element u_m is uniquely determined. In order to simplify the considerations, let us introduce the transformation

$$x = a\xi + \bar{x}, \qquad y = b\eta + \bar{y},$$

where $a = \frac{1}{2}(x_1 - x_0)$, $b = \frac{1}{2}(y_1 - y_0)$, $\bar{x} = \frac{1}{2}(x_1 + x_0)$, $\bar{y} = \frac{1}{2}(y_1 + y_0)$ which will transform the given rectangle R to the square $\tilde{R} = \langle -1, 1 \rangle \times \langle -1, 1 \rangle$ and which will transform an arbitrary function $v(x, y)$ defined on R to the function $\tilde{v}(\xi, \eta) = v(a\xi + \bar{x}, b\eta + \bar{y})$ defined on $\tilde{R}$. The derivatives of both functions are tied by formulas

$$D^{\varkappa} \tilde{v}(\xi, \eta) = a^{\varkappa_1} b^{\varkappa_2} D^{\varkappa} v(x, y). \tag{9}$$

With the aid of those relations it can easily be verified that between the norms defined on R and $\tilde{R}$ it holds

$$\|u - u_m\|_{n,p,R} \leqq \bar{C}_1 (ab)^{1/p} h^{-n} \|\tilde{u} - \tilde{u}_m\|_{n,p,\tilde{R}}, \tag{10}$$

where the constant $\bar{C}_1 > 0$ is dependent only on the ratio of the sides of the given rectangle.

In order to estimate the norm on the right hand side of the last inequality, let us define on the space $W_p^{(s)}(\tilde{R})$ for a fixed $v \in W_p^{(k)}(\tilde{R})$ the linear functional

$$f(\tilde{u}) = \sum_{|\varkappa| \leqq n} \iint_{\tilde{R}} D^{\varkappa}(\tilde{u} - \tilde{u}_m) \cdot |D^{\varkappa} v|^{p-1} \cdot d\xi \cdot d\eta.$$

If $\tilde{u}$ is a polynomial with a maximum degree $s - 1$, then $\tilde{u}_m = \tilde{u}$ and therefore $f(\tilde{u}) = 0$. It will be shown that $f(\tilde{u})$ is bounded. Let us consider that $D^{\varkappa}(\tilde{u} - \tilde{u}_m) \in L_p(\tilde{R})$, $D^{\varkappa} v \in L_p(\tilde{R})$ and thus $|D^{\varkappa} v|^{p-1} \in L_q(\tilde{R})$, $1/p + 1/q = 1$, $|\varkappa| \leqq n$. By applying the Hölder inequalities, first for integrals and finally for sums, we obtain after rearrangement

$$|f(\tilde{u})| \leqq \|\tilde{u} - \tilde{u}_m\|_{n,p,\tilde{R}} \cdot \|v\|_{n,p,\tilde{R}}^{p-1},$$

from which with regard to the inequality $n \leqq s$ it further follows that

$$|f(\tilde{u})| \leqq (\|\tilde{u}\|_{s,p,\tilde{R}} + \|\tilde{u}_m\|_{s,p,\tilde{R}}) \cdot \|v\|_{n,p,\tilde{R}}^{p-1}. \tag{11}$$

It should be noted that the element $\tilde{u}_m$ could with regard to its definition be written in the form

$$\tilde{u}_m(\xi,\eta) = \sum_{|\varkappa|\leqq m-1} \sum_{\mu,\nu=0}^{1} \omega_{\mu\nu}^{\varkappa}(\xi,\eta)\, D^{\varkappa}\tilde{u}\left((-1)^{\mu},(-1)^{\nu}\right),$$

where $\omega_{\mu\nu}^{\varkappa}(\xi,\eta)$ are polynomials in each of both variables of a maximum degree $2m-1$ and independent of the function $\tilde{u}$. Therefore

$$\|\tilde{u}_m\|_{s,p,\tilde{R}} \leqq \sum_{|\varkappa|\leqq m-1} \sum_{\mu,\nu=0}^{1} \|\omega_{\mu\nu}^{\varkappa}\|_{s,p,\tilde{R}} \left|D^{\varkappa}\tilde{u}\left((-1)^{\mu},(-1)^{\nu}\right)\right|.$$

If we furthermore consider that the Sobolev lemma guarantees the existence of a constant $\bar{C}_2>0$ dependent only on the square $\tilde{R}$ such that for all $(\xi,\mu)\in\tilde{R}$ it holds

$$|D^{\varkappa}\tilde{u}(\xi,\eta)| \leqq \bar{C}_2\|\tilde{u}\|_{s,p,\tilde{R}}, \qquad |\varkappa|\leqq m-1,$$

we arrive at the inequality

$$\|\tilde{u}_m\|_{s,p,\tilde{R}} \leqq \bar{C}_3\cdot\|\tilde{u}\|_{s,p,\tilde{R}},$$

where

$$\bar{C}_3 = C_2 \sum_{|\varkappa|\leqq m-1} \sum_{\mu,\nu=0}^{1} \|\omega_{\mu\nu}^{\varkappa}\|_{s,p,\tilde{R}}$$

is a constant independent of the choice of the element $\tilde{u}$. By substituting into (11) we obtain the inequality

$$|f(\tilde{u})| \leqq (1+\bar{C}_3)\cdot\|v\|_{n,p,\tilde{R}}^{p-1}\cdot\|\tilde{u}\|_{s,p,\tilde{R}}$$

which proves the boundedness of the functional $f(\tilde{u})$. The assumptions of the Bramble-Hilbert lemma have thus been satisfied. By the application of the latter we obtain

$$|f(\tilde{u})| \leqq C_2(1+\bar{C}_3)\cdot\|v\|_{n,p,\tilde{R}}^{p-1}\cdot|\tilde{u}|_{s,p,\tilde{R}}.$$

Let us now put specially $v=\tilde{u}-\tilde{u}_m\in W_p^{(s)}(\tilde{R})$, then after rearrangement we have

$$\|\tilde{u}-\tilde{u}_m\|_{n,p,\tilde{R}} \leqq C_2(1+\bar{C}_3)\cdot|\tilde{u}|_{s,p,\tilde{R}}.$$

This inequality will now be employed in the rearrangement of (10), thus

$$\|u-u_m\|_{n,p,R} \leqq C_3\cdot(ab)^{1/p}\cdot h^{-n}|\tilde{u}|_{s,p,\tilde{R}} \tag{12}$$

and in doing so we have put $C_3=\bar{C}_1\cdot C_2\cdot(1+\bar{C}_3)$. With the aid of the formulas (9) it can easily be verified that between the semi-norms defined on R and $\tilde{R}$ it holds

$$|\tilde{u}|_{s,p,\tilde{R}} \leqq (ab)^{-1/p}\cdot h^s\cdot|u|_{s,p,R}.$$

By substituting this inequality into (12), the statement of the theorem is arrived at.

The attained results will now be applied to the above described polygon Ω. First of all we must, of course, widen the concept of interpolation to the entire region Ω.

Definition 3. Assume that a partition ϱ of the polygon Ω is given. Then by the $H_1^{(m)}(\varrho,\Omega)$-*interpolate* of a function $u(x,y)\in C^{(m-1)}(\Omega)$ there will be understood such a function $u_{m,\varrho}(x, y)$ defined on Ω which coincides with the $H_1^{(m)}(R_\tau)$-interpolate of the given function $u(x, y)$ on each elementary rectangle R_τ, from which the polygon Ω is composed.

Theorem 3. Assume that Ω is an arbitrary polygon whose sides are parallel with the coordinate axes and ϱ is an arbitrary partition of this polygon. Then to each function $u(x, y) \in C^{(m-1)}(\Omega)$ there exists exactly one $H_1^{(m)}(\varrho, \Omega)$-interpolate $u_{m,\varrho}(x, y)$ and it holds $u_{m,\varrho}(x, y) \in C^{(l-1, l-1)}(\Omega)$.

Proof. The existence and uniqueness follow from the fact that $u_{m,\varrho}$ forms a uniquely determined $H_1^{(m)}(R_\tau)$-interpolate on each rectangle R_τ. It thus remains to prove the continuity of the derivatives $D^\varkappa u_{m,\varrho}$ for $0 \leqq \varkappa_1, \varkappa_2 \leqq l-1$. It is sufficient to show that on each side of an arbitrary elementary rectangle R_τ the $H_1^{(m)}(R_\tau)$-interpolate together with the given derivatives is uniquely determined merely by the values of the derivatives $D^\varkappa u(x, y)$, $|\varkappa| \leqq m-1$ in two vertices lying on the side under consideration. Let us consider e.g. the side $x = x_0$ of the rectangle R_τ and let us define the auxiliary polynomial $\beta(y) = D^{\bar\varkappa} u_{m,\varrho}(x_0, y)$, $\bar\varkappa = (\varkappa_1, 0)$, $0 \leqq \varkappa_1 \leqq l-1$. This polynomial has according to the definition a degree maximally $2m - 2\varkappa_1 - 1$ and at the same time is given in two vertices lying on the side under consideration by $2m - 2\varkappa_1$ values $\beta(y)$, $\beta'(y), \ldots$, $\beta^{(m-\varkappa_1-1)}(y)$; it is therefore uniquely determined by those values. It will be proved by differentiation that also the derivatives $\beta^{(\varkappa_2)}(y) = D^\varkappa u(x_0, y)$, $0 \leqq \varkappa_2 \leqq l-1$ are uniquely determined by the above mentioned values. In a similar way the continuity could be shown also on the other sides of the rectangle R_τ.

It follows from the proved theorem that $u_{m,\varrho} \in W_p^{(n)}(\Omega)$ for every integer $0 \leqq n \leqq l$. If we furthermore consider that for all $v \in W_p^{(n)}(\Omega)$ it holds

$$\|v\|_{n,p,\Omega}^p = \sum_\tau \|v\|_{n,p,R_\tau}^p, \qquad |v|_{n,p,\Omega}^p = \sum_\tau |v|_{n,p,R_\tau}^p$$

and if we limit ourselves to a regular partition system, we can formulate the following theorem.

Theorem 4. Assume that Ω is the above described polygon and C a regular system of partitions. Let us choose arbitrarily $\varrho \in C$ and $u(x, y) \in W_p^{(k)}(\Omega)$, $1 < p < \infty$, $m + \left[\frac{2}{p}\right] \leqq k \leqq 2m$. Let us denote with the symbol $u_{m,\varrho}(x, y)$ the $H_1^{(m)}(\varrho, \Omega)$-interpolate of the function $u(x, y)$. Then for an arbitrary integer $1 \leqq n \leqq l$ there exists a constant $C_4 > 0$, independent of the choice of the function u as well the partition ϱ such that the estimate

$$\|u_{m,\varrho} - u\|_{n,p,\Omega} \leqq C_4 \cdot h^{s-n} |u|_{s,p,\Omega}$$

holds, h being given by the relation (2), $s = \min(k, 2m - l + 1)$.

3. Second-Type Interpolation

In certain cases it is possible to use another, more advantageous method of interpolation, in which the interpolation polynomials have in contrast to the method described above generally a larger number of zero coefficients. On the other hand, however, this method does not guarantee the continuity of any of the derivatives of the interpolation function.

Let us again choose arbitrarily a natural number m, limit ourselves to the rectangle (1) and let us define, in a similar way as above, the basic concepts.

Definition 4. By the symbol $H_2^{(m)}(R)$ there will be understood the set of all polynomials (3) defined on R, for which $\alpha_{ij}=0$, if $[i/2]+[j/2]\geqq m$.

Definition 5. By the $H_2^{(m)}(R)$-*interpolate* of a function $u(x, y)\in C^{(m-1)}(R)$ there will be understood that element $u_m(x, y)\in H_2^{(m)}(R)$, for which (4) is valid.

The coefficients of the element u_m are determined by $2m(m+1)$ conditions in the vertices of the rectangle under consideration, $2m(m-1)$ coefficients are equal to zero. It will first of all be shown that the element u_m is uniquely determined by the given conditions.

Theorem 5. For each function $u(x, y)\in C^{(m-1)}(R)$ there exists exactly one $H_2^{(m)}(R)$-interpolate.

Proof. Analogically as in Theorem 1 it is sufficient to show that the identity $u_m(x, y)\equiv 0$ follows from (5). Since for $m=1$ the statement is apparently satisfied, it can be assumed that $m\geqq 2$. It will be shown that for all values of λ, for which

$$0\leqq\lambda\leqq l=\left[\frac{m+1}{2}\right],$$

it can be written

$$u_m(x, y)=[(x-x_0)(x-x_1)(y-y_0)(y-y_1)]^{\lambda}\cdot\alpha^{(\lambda)}(x, y), \tag{13}$$

where $\alpha^{(\lambda)}(x, y)$ is a polynomial in each of both variables maximally of the degree $2m-2\lambda-1$ which (for $\lambda<l$) in the vertices of the rectangle R satisfies the conditions $D^{\varkappa}\alpha^{(\lambda)}(x_\mu, y_\nu)=0$, $|\varkappa|\leqq m-2\lambda-1$, $\mu, \nu=0, 1$, and for whose coefficients $\alpha_{ij}^{(\lambda)}$ it holds $\alpha_{ij}^{(\lambda)}=0$, if $[i/2]+[j/2]\geqq m-2\lambda$. The proof will be carried out by mathematical induction. For $\lambda=0$ the statement is apparently satisfied, since we are putting $\alpha^{(0)}(x, y)=u_m(x, y)$. Let us therefore choose $0<\lambda\leqq l$. In accordance with the induction assumption the polynomial $\alpha^{(\lambda-1)}(x, y)$ has in each of both variables non-zero coefficients maximally at the power $2m-4\lambda+3$, since $\alpha_{ij}^{(\lambda-1)}=0$ for $[i/2]+[j/2]\geqq m-2\lambda+2$. The auxiliary polynomial $\beta(y)=\alpha^{(\lambda-1)}(x_\mu, y)$ has therefore $2m-4\lambda+4$ coefficients and in accordance with the induction assumption it must satisfy $2m-4\lambda+4$ conditions $\beta(y_\nu)=\beta'(y_\nu)=\cdots=\beta^{(m-2\lambda+1)}(y_\nu)=0$, $\nu=0, 1$. Therefore $\beta(y)\equiv 0$ and thus also $\alpha^{(\lambda-1)}(x_\mu, y)\equiv 0$. In a similar way it can be shown that $\alpha^{(\lambda-1)}(x, y_\nu)=0$ for $\nu=0, 1$, and therefore we can write

$$\alpha^{(\lambda-1)}(x, y)=(x-x_0)(x-x_1)(y-y_0)(y-y_1)\cdot\alpha^{(\lambda)}(x, y), \tag{14}$$

where $\alpha^{(\lambda)}(x, y)$ has in each of both variables the maximum degree $2m-2\lambda-1$. From the given relation the validity of (13) follows immediately. By similar considerations as were employed in the proof of Theorem 1 it can be shown that $\alpha_{ij}^{(\lambda)}=0$ for $[i/2]+[j/2]\geqq m-2\lambda$.

It remains to verify the conditions in the vertices of the rectangle R for $\lambda\leqq l$. With regard to (14) it can be written

$$D^{\varkappa}\alpha^{(\lambda-1)}(x,y)=\sum_{\omega_1=0}^{\varkappa_1}\sum_{\omega_2=0}^{\varkappa_2}\binom{\varkappa_1}{\omega_1}\binom{\varkappa_2}{\omega_2}$$

$$\cdot D^{\omega}[(x-x_0)(x-x_1)(y-y_0)(y-y_1)]\cdot D^{\varkappa-\omega}\alpha^{(\lambda)}(x,y),$$

where $\omega=(\omega_1,\omega_2)$. Respecting the induction assumption we obtain after a short rearrangement for all $\varkappa$, $|\varkappa|\leqq m-2\lambda+1$ the equations

$$\begin{aligned}\varkappa_1\varkappa_2\{&(-1)^{\mu+\nu}(x_1-x_0)(y_1-y_0)\cdot D^{(\varkappa_1-1,\,\varkappa_2-1)}\alpha^{(\lambda)}(x_\mu,y_\nu)\\ &-(-1)^{\mu}(x_1-x_0)(\varkappa_2-1)D^{(\varkappa_1-1,\,\varkappa_2-2)}\alpha^{(\lambda)}(x_\mu,y_\nu)\\ &-(-1)^{\nu}(y_1-y_0)(\varkappa_1-1)D^{(\varkappa_1-2,\,\varkappa_2-1)}\alpha^{(\lambda)}(x_\mu,y_\nu)\\ &+(\varkappa_1-1)(\varkappa_2-1)\cdot D^{(\varkappa_1-2,\,\varkappa_2-2)}\alpha^{(\lambda)}(x_\mu,y_\nu)\}=0.\end{aligned}\tag{15}$$

If we now take into consideration only those derivatives which make sense and the others will be taken as equal to zero, we obtain through a successive application of the Eq. (15) the relations $D^{\varkappa}\alpha^{(\lambda)}(x_\mu,y_\nu)=0$ for all $|\varkappa|\leqq m-2\lambda-1$, $\mu,\nu=0,1$.

It follows from the course of the proof that the relation (13) also makes sense for $\lambda=l$, thus (6) holds, where $\alpha_{ij}^{(l)}=0$ for all the i's, j's, for which

$$\left[\frac{i}{2}\right]+\left[\frac{j}{2}\right]\geqq m-2l=m-2\left[\frac{m+1}{2}\right],$$

i.e. for all $i,j\geqq 0$ and thus $u_m(x,y)\equiv 0$.

Theorem 6. Let us arbitrarily choose a function $u(x,y)\in W_p^{(k)}(R)$, $1<p<\infty$, $m+\left[\frac{2}{p}\right]\leqq k\leqq 2m$ and denote by $u_m(x,y)$ its $H_2^{(m)}(R)$-interpolate. Then there exists a constant $C_5>0$ independent of the choice of the function $u(x,y)$ such that for an arbitrarily chosen $0\leqq n\leqq k$ it holds

$$\|u-u_m\|_{n,p,R}\leqq C_5\cdot h^{k-n}\cdot|u|_{k,p,R},$$

where $h=\max(x_1-x_0,y_1-y_0)$.

The proof of this theorem is fully identical with the proof of Theorem 2 and will therefore not be carried out. Let us now proceed to extend our considerations to the above described polygon Ω.

Definition 6. Assume that some partition ϱ of the polygon Ω is given. Then the $H_2^{(m)}(\varrho,\Omega)$-*interpolate* of a function $u(x,y)\in C^{(m-1)}(\Omega)$ will be called such a function $u_{m,\varrho}(x,y)$ defined on Ω which on each above mentioned elementary rectangle R_τ coincides with the $H_2^{(m)}(R_\tau)$-interpolate of the given function $u(x,y)$.

Theorem 7. Let Ω, ϱ be the denotations of Theorem 3. Then to each function $u(x,y)\in C^{(m-1)}(\Omega)$ there exists exactly one $H_2^{(m)}(\varrho,\Omega)$-interpolate $u_{m,\varrho}(x,y)$ and it holds $u_{m,\varrho}(x,y)\in C^{(0)}(\Omega)$.

The proof of the last theorem is identical with the proof of Theorem 3 for $l=1$. As only the continuity of the $H_2^{(m)}(\varrho, \Omega)$-interpolate is guaranteed $u_{m,\varrho}(x, y) \in W_p^{(n)}(\Omega)$ holds only for $n=0, 1$ and therefore the theorem about the estimate can be formulated only in the weaker formulation.

Theorem 8. Let Ω, C, ϱ, $u(x, y)$, h, k be the denotations of Theorem 4. Let us denote the $H_2^{(m)}(\varrho, \Omega)$-interpolate of the function $u(x, y)$ with the symbol $u_{m,\varrho}(x, y)$. Then there exists a constant $C_6 > 0$ independent of the choice of the function $u(x, y)$ and the partition ϱ such that for $n=0, 1$ it holds

$$\|u_{m,\varrho} - u\|_{n,p,\Omega} \leqq C_6 \cdot h^{k-n} \cdot |u|_{k,p,\Omega}.$$

4. Quasi-Linear Differential Equations

The first-type interpolation can be utilized for solving quasi-linear partial differential equations of an arbitrary order $2n$, $n \geqq 1$, whereas the second-type interpolation only in the case $n=1$. In the solution of equations of the second order both methods of interpolation can be utilized. If the approximate solution of such an equation is not required to have continuous partial derivatives, then it is more advantageous to use the second method, since the interpolation function forms in comparison with the first method on each elementary rectangle a polynomial with generally more zero coefficients, i.e. a simpler polynomial. The number of zero coefficients will increase from $4[m/2]^2$ in the first method to $2m(m-1)$ in the second method of interpolation. The effect will occur only at $m \geqq 3$, since at $m=1, 2$ both interpolation coincide.

The results obtained in Section 2 will be applied to the solution of a general quasi-linear equation in the generalized divergence form which is studied e.g. in [4, 6, 7]. Let us have an open bounded set Ω with a sufficiently smooth boundary in the space E^N and let us consider the equation

$$\sum_{|\mu| \leqq n} (-1)^{|\mu|} \cdot D^\mu A_\mu(x, u, \ldots, D^n u) = 0, \tag{16}$$

where $n \geqq 1$. The solution of this equation will be sought on a space E, for which $\overset{\circ}{W}{}_p^{(n)}(\Omega) \subset E \subset W_p^{(n)}(\Omega)$. It will be assumed that the coefficients of the equation satisfy the following condition:

1) If $u \in W_p^{(n)}(\Omega)$, then $A_\mu(x, u, \ldots, D^n u) \in L_q(\Omega)$, $1/p + 1/q = 1$, for all $|\mu| \leqq n$.

To the Eq. (16) there will be assigned for an arbitrary $u, v \in W_p^{(n)}(\Omega)$ the generalized Dirichlet form

$$a(u, v) = \sum_{|\mu| \leqq n} \int_\Omega A_\mu(x, u, \ldots, D^n u) \cdot D^\mu v \cdot dx. \tag{17}$$

By applying the Hölder inequality to both the integrals and the sums it will be found that

$$|a(u, v)| \leqq \left\{ \sum_{|\mu| \leqq n} \int_\Omega |A_\mu(x, u, \ldots, D^n u)|^q \cdot dx \right\}^{1/q} \cdot \|v\|_{n,p,\Omega},$$

i.e. that this form is with respect to the variable v a bounded linear functional. For each element $u \in E$ thus there exists an element $F(u) \in E^*$ such that for all $v \in E$ it holds

$$(v, F(u)) = a(u, v),$$

where $(\cdot, \cdot)$ denotes the pairing between the spaces E and E^*. The element $u^* \in E$, which for all $v \in E$ satisfies the identity $a(u^*, v) = 0$ and which is therefore the solution of the equation $F(u^*) = \Theta$, will be called the *weak solution* of the Eq. (16). The Dirichlet form (17) will be required to satisfy the following two requirements:

2) For arbitrary elements $u, v \in E$ it holds

$$\alpha(\|u - v\|_{n,p,\Omega}) \leqq a(u, u - v) - a(v, u - v) \leqq \beta(\|u - v\|_{n,p,\Omega}),$$

where $\alpha(t)$, $\beta(t)$ are arbitrary non-negative functions of the non-negative variable of such a type that the functions

$$\bar{\alpha}(R) = \int_0^1 \alpha(Rt) \cdot t^{-1} \cdot dt, \quad \bar{\beta}(R) = \int_0^1 \beta(Rt) \cdot t^{-1} dt$$

are continuous, increasing and it holds

$$\bar{\beta}(0) = 0 \quad \text{and} \quad \lim_{R \to \infty} \frac{\bar{\alpha}(R)}{R} = \infty.$$

3) For arbitrary elements u, h_1, h_2 there exists the Gateaux derivative $a'_u(h_1, h_2) = \lim_{s \to 0} [a(u + s h_1, h_2) - a(u, h_2)]/s$, it is continuous according to the variable u on an arbitrary hyperplane passing through the point u and it can be written $a'_u(h_1, h_2) = a'_u(h_2, h_1)$.

Under the given assumptions it can be shown in a similar way as in [8] that the operator F is on the space E potential with the potential

$$f(u) = \int_0^1 a(tu, u)\, dt \tag{18}$$

and satisfies the assumptions of the statement in the cited paper so that the following lemma can be formulated.

Lemma 2. Assume that in the space E^N there is given a bounded open set Ω with a sufficiently smooth boundary. Then the Eq. (16), whose coefficients satisfy the condition 1) and the corresponding Dirichlet form (17) satisfies the conditions 2) and 3), has on an arbitrary space E, $\overset{\circ}{W}{}_p^{(n)}(\Omega) \subset E \subset W_p^{(n)}(\Omega)$, exactly one weak solution $u^* \in E$. If we replace this solution on an arbitrary closed (or weakly closed) convex set $M \subset E$ by the approximate solution $\bar{u} \in M$, which on the set M minimizes the functional (18), then there holds with an arbitrary $u \in M$ for the error of the solution the estimate

$$\|\bar{u} - u^*\|_{n,p,\Omega} \leqq \gamma(\|u - u^*\|_{n,p,\Omega}),$$

where $\gamma(t)$ is a certain continuous, increasing, non-negative function of the non-negative argument, independent of the choice of the set M, for which $\gamma(0) = 0$.

Let us now limit ourselves to a plane polygon Ω, whose sides are parallel with the coordinate axes; let us arbitrarily choose $m \geqq 2n-1$, $\varrho \in C$ and put $M = E \cap H_1^{(m)}(\varrho, \Omega)$, where $H_1^{(m)}(\varrho, \Omega)$ is the set of all functions defined on Ω which on each elementary rectangle R_τ, from which Ω is composed, form an element of $H_1^{(m)}(R_\tau)$. If $u^* \in W_p^{(k)}(\Omega)$, $1 < p < \infty$, $m + \left[\frac{2}{p}\right] \leqq k \leqq 2m$ then as the element u from the previous lemma the element $u^*_{m,\varrho} \in H_1^{(m)}(\varrho, \Omega)$ which forms the $H_1^{(m)}(\varrho, \Omega)$-interpolate of the element u^* can be taken. If $u^*_{m,\varrho} \in E$, then $u^*_{m,\varrho} \in M$ and for the error of the solution the estimate

$$\|\bar{u} - u^*\|_{n,p,\Omega} \leqq \delta(h^{s-n})$$

is valid, where $s = \min(k, 2m - l + 1)$ and $\delta(t) = \gamma(C_4 |u^*|_{s,p,\Omega} \cdot t)$; thus the function $\delta(t)$ is non-negative, increasing, continuous, independent of the choice of the partition $\varrho \in C$ and the set M and it holds that $\delta(0) = 0$.

In practice the case $\alpha(t) = (\alpha_0 t)^2$, $\beta(t) = (\beta_0 t)^2$ often occurs. Then the error of the solution is of the same order as the error of the chosen interpolation, i.e.

$$\|\bar{u} - u^*\|_{n,p,\Omega} \leqq \frac{\beta_0}{\alpha_0} C_4 |u^*|_{s,p,\Omega} \cdot h^{s-n}.$$

5. Non-Linear Integral Equations

The application of the second-type interpolation will be illustrated on the solution of a non-linear integral equation which is solved in [7, 9]. In the examination of this equation the results from [6] will be utilized.

Let us assume that in the space E^N there is given a set Ω of a finite measure and let us deal with the Hammerstein equation

$$u(x) = \int_\Omega K(x, y) \cdot g(y, u(y)) dy \tag{19}$$

which can be written in the equivalent operator form

$$F(u) \equiv u - A\,G(u) = \Theta, \tag{20}$$

where

$$A\,v = \int_\Omega K(x, y) \cdot v(y) \cdot dy \tag{21}$$

is the linear operator and

$$G(u) = g(x, u(x)) \tag{22}$$

is the non-linear Nemycki operator.

The solution of Eq. (19) or (20) respectively, will be sought on the space $E = L_2(\Omega)$. The scalar product defined on this space will be denoted for simplicity $(\cdot,\cdot)$, the norm corresponding to the latter $\|\cdot\|$. For the solution $u^* \in E$ of the equation under consideration apparently it holds $(u, F(u^*)) = 0$ for all $u \in E$. In the space E let us choose an arbitrary finite-dimensional subspace M. The element $\bar{u} \in M$ which satisfies the relation $(u, F(\bar{u})) = 0$ for all $u \in M$ will be called the *approximate solution* of the Eq. (20), thus also (19).

In our further considerations it will be required that the functions $K(x, y)$ and $g(x, u(x))$ satisfy the following assumptions:

4) $K(x, y)$ is such a function that the operator (21) defined on E is bounded and self-adjoint.

5) $g(x, u)$ is a real function defined for all $x \in \Omega$, $u \in (-\infty, \infty)$, continuous according to u with almost all $x \in \Omega$ and measurable on Ω for every $u \in (-\infty, \infty)$.

6) For arbitrary $u_1, u_2, v \in E$ it holds

$$|(g(x, u_1) - g(x, u_2), v| \leq \beta \|v\| \cdot \|u_1 - u_2\|, \qquad \beta \cdot \|A\| < 1.$$

Under the above mentioned assumptions the operator $F(u)$ defined on E satisfies the assumptions of the statement from [6], i.e. it is finitely continuous, strongly monotone and Lipschitz continuous. In order to facilitate the verification of this fact, let us note first of all that the expression

$$\big(F(u_1) - F(u_2), v\big) = (u_1 - u_2, v) - \big(AG(u_1) - AG(u_2), v\big)$$

can with regard to the self-adjointion of the operator A and the definition of the operator G be written in the form

$$\big(F(u_1) - F(u_2), v\big) = (u_1 - u_2, v) - \big(g(x, u_1) - g(x, u_2), Av\big). \tag{23}$$

By applying condition 6) to this identity we obtain the inequality

$$\big|\big(F(u_1) - F(u_2), v\big)\big| \leq (1 + \beta \cdot \|A\|) \|v\| \cdot \|u_1 - u_2\|,$$

from which the finite continiuty of the operator $F(u)$, i.e. that from $\{u_n\}_{n=1}^{\infty} \subset M$, $u_n \to u \in M$ there follows $\big(F(u_n), v\big) \to \big(F(u), v\big)$ for each $v \in E$, follows on one hand and by putting $v = F(u_1) - F(u_2)$ also the Lipschitz continuity of this operator, i.e. the validity of the inequality

$$\|F(u_1) - F(u_2)\| \leq (1 + \beta \cdot \|A\|) \cdot \|u_1 - u_2\|$$

for all $u_1, u_2 \in E$, on the other hand. The strong monotony of the operator $F(u)$ is guaranteed by the inequality

$$\big|\big(F(u_1) - F(u_2), u_1 - u_2\big)\big| \geq \gamma \|u_1 - u_2\|^2$$

which follows from (23) and the condition 6) and is valid for arbitrary $u_1, u_2 \in E$, the constant $\gamma > 0$ being given by the relation

$$\gamma = 1 - \beta \cdot \|A\|. \tag{24}$$

All the assumptions of the statement from [6] have been satisfied and therefore it is permissible to formulate the following lemma.

Lemma 3. Assume that on the space E^N there is given a set Ω with a finite measure. Assume further that $K(x, y)$ satisfies the condition 4) and $g(x, u(x))$ satisfies the conditions 5) and 6). Then Eq. (20) or (19) respectively, has on the space $E = L_0(\Omega)$ exactly one solution u^*. If we replace this solution on an arbitrary finite-dimensional subspace $M \subset E$ by a uniquely determined approximate solution $\bar{u} \in M$, i.e. with the element $\bar{u} \in M$ that satisfies for all $u \in M$ the relation

$$(\bar{u}, u) = \big(G(\bar{u}), A u\big), \tag{25}$$

then the following estimate holds:

$$\|\bar{u} - u^*\| \leqq \frac{1}{\gamma} (1 + \beta \cdot \|A\|) \cdot \|u - u^*\|,$$

where $u \in M$ is an arbitrary element and γ is given by the relation (24).

Assume now that on the above described plane polygon Ω there is given a partition $\varrho \in C$ and choose arbitrarily $m \geqq 1$. Let us take as the subspace M the set $M = H_2^{(m)}(\varrho, \Omega)$, i.e. the set of all functions defined on Ω which on every elementary rectangle R_τ that forms the polygon Ω form an element from $H_2^{(m)}(R_\tau)$. If $u^* \subset W_2^{(2m)}(\Omega)$, then as the element u from the Lemma 3 the $H_2^{(m)}(\varrho, \Omega)$-interpolate $u^*_{m,\varrho} \in M$ of the element u^* will be taken. For the error of the solution of the Eq. (20) or (19) respectively, thus with regard to the Theorem 8 the following estimate holds:

$$\|\bar{u} - u^*\| \leqq \frac{1}{\gamma} (1 + \beta \cdot \|A\|) \cdot C_8 \cdot |u^*|_{2m,2,\Omega} \cdot h^{2m}.$$

If we denote by $u_1, u_2, \ldots, u_n$ the base of the space M, an arbitrary element $u \in M$ can be written in the form $u = \sum_{i=1}^{n} c_i \cdot u_i$, where $c_1, c_2, \ldots, c_n$ are suitable real numbers. The coefficients $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n$ of the approximate solution $\bar{u}$ will be determined by solving the system of equations

$$\sum_{j=1}^{n} c_j (u_j, u_i) = \left(g\left(x, \sum_{k=1}^{n} c_k \cdot u_k\right), A u_i\right), \qquad i = 1, \ldots, n$$

which is a consequence of (25). This generally non-linear system has, as follows from the Lemma 3, exactly one solution.

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