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Invariants of Knot Cobordism

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1. In [2] certain abelian groups, G_+ and G_-, are introduced in order to study the cobordism groups C_n, of knotted n-spheres in $(n+2)$-space, for odd n. It is shown that $C_{2n-1} \approx G_\varepsilon$ for $\varepsilon = (-1)^n$ and $n \geqq 3$, C_3 is isomorphic to a subgroup of G_+ of index 2 and C_1 has G_- as a quotient group. In this work we shall construct a complete collection of numerical invariants on G_ε. As a consequence, for example, it will be shown that every element of G_ε has order 1, 2, 4 or ∞, and there are elements of all these orders. In fact, G_ε is a sum of an infinite number of cyclic groups of each of these orders.

We will rely heavily on techniques and results of Milnor [4].

2. We recall the definition of G_ε. Let A be a square integral matrix satisfying:

$$\text{determinant}(A + \varepsilon A^T) = \pm 1$$

where $\varepsilon = \pm 1$; such matrices will be referred to as ε-*matrices*. The Alexander polynomial of A, $\Delta_A(t)$, is defined to be determinant $(tA + A^T)$ – note this differs from [2]. We will say A is *null-cobordant* if A is congruent to a matrix of the form:

$$\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

where B, C and D are square matrices and 0 is the zero matrix. Two ε-matrices A_1, A_2 are *cobordant* if the block sum $A_1 \oplus (-A_2)$ is null-cobordant. Then cobordism is an equivalence relation and block sum induces the structure of an abelian group on the set G_ε of cobordism classes (see [2]).

3. Consider now square matrices A, with entries in a *field* F satisfying:

$$(A - A^T)(A + A^T) \quad \text{is non-singular}.$$

We will refer to such matrices as *admissible*.

Defining cobordism as in § 2, it follows similarly that cobordism is an equivalence relation among admissible matrices (see [2, Lemma 1] – the same argument works for matrices over a field) and the set G^F of cobordism classes becomes an abelian group under block sum.

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There are obvious homomorphisms $G_\varepsilon \to G^Q$, since $A+\varepsilon A^T$ unimodular implies $A-\varepsilon A^T$ has odd determinant. These are, in fact, monomorphisms, since, by the argument of [2, Lemma 8], an integral matrix A is null-cobordant over the integers if and only if it is null-cobordant over the rationals. Thus, it will suffice to construct a complete set of invariants on G^Q.

We define, for any admissible matrix A, the *Alexander polynomial* $\Delta_A(t) = \det(tA + A^T)$. Note that $\Delta_A(1)\,\Delta_A(-1) \neq 0$.

4. We will deal with several related notions of (algebraic) cobordism. From now on F will be a field of *characteristic zero* and $\langle\,,\rangle$ a non-degenerate quadratic form on a finite-dimensional vector space V over F. We will add quadratic forms by orthogonal sum $\perp$ (see [5]). We will say $\langle\,,\rangle$ is *null-cobordant* if V contains a totally isotropic subspace of half the dimension of V. Two quadratic forms $\langle\,,\rangle$ and $\langle\,,\rangle'$ are *cobordant* if $\langle\,,\rangle \perp (-\langle\,,\rangle')$ is null-cobordant. This is an equivalence relation.

Precisely the same definitions can be made for Hermitian forms over a field with a non-trivial involution.

5. Returning to $\langle\,,\rangle$ a quadratic form, *determinant* $\langle\,,\rangle$ is a well-defined element in $\dot{F}/(\dot{F})^2$, where $\dot{F}$ is the multiplicative group of non-zero elements in F (see [5]). Since determinant is multiplicative and a null-cobordant form of rank $2r$ has determinant $(-1)^r$, it follows that $d(\langle\,,\rangle) = (-1)^r \det\langle\,,\rangle$ is a cobordism invariant for $\langle\,,\rangle$ of rank $2r$.

If F is the real numbers, then the *signature* $\sigma(\langle\,,\rangle)$ is defined and is well known to be a complete invariant of the cobordism class of $\langle\,,\rangle$.

If F is a local field (see [5]), the *Hasse symbol* $S(\langle\,,\rangle) = \pm 1$ is well-defined. To convert this to a cobordism invariant we define

$$\mu(\langle\,,\rangle) = (-1,-1)^{\frac{r(r+3)}{2}} (\det\langle\,,\rangle, -1)^r\, S(\langle\,,\rangle)$$

where $(\,,)$ is the Hilbert symbol for F and $\langle\,,\rangle$ has rank $2r$. Using the additivity formula (see [5]):

$$S(\langle\,,\rangle \perp \langle\,,\rangle') = S(\langle\,,\rangle)\, S(\langle\,,\rangle')(\det\langle\,,\rangle, \det\langle\,,\rangle')$$

and properties of the Hilbert symbol, it is a straightforward exercise to show that μ is a cobordism invariant. Note that $\langle\,,\rangle$ is null-cobordant if and only if it is a sum of "hyperbolic planes" (see [5]). It follows from the classification of quadratic forms over local fields [5] that d and μ are complete invariants of cobordism class.

6. Let F be a field, $\langle\,,\rangle$ a non-degenerate quadratic form on a finite-dimensional vector space V over F, and T an isometry of V. We shall refer to the pair $(\langle\,,\rangle, T)$ as an *isometric structure*. We can add isometric structures by orthogonal sum of the forms and direct sum of the isometries.

An isometric structure $(\langle\,,\rangle, T)$ is *null-cobordant* if V contains a totally isotropic subspace, invariant under T, of half the dimension of V. Two isometric structures $(\langle\,,\rangle, T)$ and $(\langle\,,\rangle, T')$ are *cobordant* if $(\langle\,,\rangle, T)\perp(-\langle\,,\rangle', T')$ is null-cobordant. This is readily checked to be an equivalence relation; cobordism classes form an abelian group.

7. Let $(\langle\,,\rangle, T)$ be an isometric structure; let $\Delta_T(t)$ be the characteristic polynomial of T.

Lemma. (a) *If* $d=\operatorname{rank}\langle\,,\rangle=\operatorname{degree}\Delta_T(t)$, *then, for some* $a\in F$, $\Delta_T(t)=at^d\Delta_T(t^{-1})$. *If* $\Delta_T(1)\neq 0$ *then* $a=1$; *if* $\Delta_T(1)\Delta_T(-1)\neq 0$, *then* d *is even.*

(b) *If* $(\langle\,,\rangle, T)$ *is null-cobordant, then* $\Delta_T(t)=c\,t^e\,\theta(t)\,\theta(t^{-1})$, *where* $d=2e$, $\theta(t)$ *is a polynomial of degree* e *and* $c\in F$.

(c) *If* $\Delta_T(1)\,\Delta_T(-1)\neq 0$, *then* $\det\langle\,,\rangle=\Delta_T(1)\,\Delta_T(-1)\in\dot{F}/(\dot{F})^2$.

Proof. Let S, Q be matrix representatives of T and $\langle\,,\rangle$ respectively – then $S^T Q S=Q$. Now

$$\begin{aligned}\Delta_T(t)&=\det(t-S)=\det(t-S^T)=\det(t-QS^{-1}Q^{-1})\\&=\det(t-S^{-1})=\det(-t\,S^{-1}(t^{-1}-S))\\&=t^d\det(-S^{-1})\,\Delta_T(t^{-1}).\end{aligned}$$

This proves the first statement of (a). Substituting $t=1$, we have $\Delta_T(1)=a\,\Delta_T(1)$; if $\Delta_T(1)\neq 0$, then $a=1$. If we now substitute $t=-1$, we have $\Delta_T(-1)=(-1)^d\,\Delta_T(-1)$; if $\Delta_T(-1)\neq 0$, then d is even. This proves (a).

To prove (c), we first observe that, by a straightforward computation, $Q(1+S)(1-S)^{-1}$ is a skew-symmetric matrix. It follows that

$$\det(Q(1+S)(1-S)^{-1})=(\det\langle\,,\rangle)\,\Delta_T(-1)/\Delta_T(1)$$

is square, which implies (c).

We now prove (b). Suppose $(\langle\,,\rangle, T)$ is null-cobordant. Let $v_1,\dots,v_n$; $w_1,\dots,w_n$ be a "symplectic" basis of V i.e. $\langle V_i, v_j\rangle=\langle w_i, w_j\rangle=0$ and $\langle v_i, w_j\rangle=\delta_{ij}$, such that the subspace spanned by $v_1,\dots,v_n$ is invariant under T. Then T is represented by a matrix of the form

$$\begin{pmatrix}A & C\\ 0 & B\end{pmatrix},$$

where A, B, C are square matrices. If $A=(a_{ij})$, then $a_{ij}=\langle Tv_i, w_j\rangle=\langle v_i, T^{-1}w_j\rangle$, which is the (j, i)-entry of B^{-1}. Thus $B^{-1}=A^T$ and (b) follows easily.

8. Let G_F be the group of cobordism classes of isometric structures $(\langle\,,\,\rangle, T)$ satisfying $\Delta_T(1)\,\Delta_T(-1)\neq 0$. Recall G^F from § 3.

Theorem. $G^F \approx G_F$.

We first need:

Lemma. *Any admissible matrix with entries in any field F is cobordant to a non-singular admissible matrix.*

Proof. We will show that a singular matrix A is cobordant to a smaller matrix. By elementary row operations on A, we may assume the first row is zero – the corresponding column operations can then be performed and the first row is still zero. By further elementary row operations, not involving the first row, we may assume that the first column is zero, except perhaps in the second row. The corresponding column operations will not change the first row or column.

So we find A is congruent to a matrix of the form

$$\begin{pmatrix} 0 & \dots & 0 \\ a & b & M \\ 0 & & \\ \vdots & N & B \\ 0 & & \end{pmatrix}$$

where, if A is an $(n+2)\times(n+2)$ matrix, then M, N and B are, respectively, $1\times n$, $n\times 1$ and $n\times n$ matrices and a, b are scalars. If A is admissible, it is easy to see that $a\neq 0$ and B is admissible.

Claim. A is cobordant to B.

By forming the block sum with $-B$, it suffices to show that A is null-cobordant if B is null-cobordant. Suppose PBP^T has all zeroes in its upper left quadrant. Define

$$Q=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \vdots & & & \\ \vdots & 0 & & P & \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix}.$$

It is straightforward to check that QAQ^T has all zeroes in its upper left quadrant.

9. Let A be a non-singular admissible matrix. Define two new matrices $B=-A^{-1}A^T$ and $Q=A+A^T$. It is readily verified that $B^TQB=Q$ and the congruence class of A determines the congruence class of Q and the similarity class of B. It follows that (Q, B) are matrix representatives of a well-defined isometric structure $(\langle\ ,\ \rangle, T)$. Moreover if A is null-cobordant, so is $(\langle\ ,\ \rangle, T)$. Notice that

$$\Delta_A(t) = c\,\Delta_T(t) \qquad \text{where } c \in \dot{F}$$

since $tA + A^T = A(t + A^{-1}A^T) = A(t - B)$. Thus A admissible means $\Delta_T(1)\,\Delta_T(-1) \neq 0$ and $(\langle\ ,\ \rangle, T)$ defines an element of G_F. The correspondence $A \to (Q, B)$ is additive and invertible since we may solve for A by the formula:

$$A = Q(1-B)^{-1}.$$

We also find:

$$A - A^T = Q(1-B)^{-1}(1+B).$$

Thus if $\Delta_T(1)\Delta_T(-1) \neq 0$, A is admissible. This establishes the desired isomorphism.

10. We have reformulated our problem into an investigation of G_F, and especially G_Q.

Let $(\langle\ ,\ \rangle, T)$ be an isometric structure over F. Let $\Lambda = F[t, t^{-1}]$ be the ring of Laurent polynomials over F. We will consider the vector space on which $\langle\ ,\ \rangle$ and T are defined as a Λ-module, defining the action of t by T. If $\lambda(t)$ is an irreducible factor of $\Delta_T(t)$, we denote by V_λ the $\lambda(t)$-primary component of V:

$$V_\lambda = \operatorname{Ker} \lambda(t)^N, \qquad \text{for } N \text{ large}.$$

Then V is the direct sum of the $\{V_\lambda\}$.

Lemma. *Let $\lambda(t)$, $\mu(t)$ be irreducible factors of $\Delta_T(t)$. Then V_λ is orthogonal to V_μ if $\lambda(t)$ and $\mu(t^{-1})$ are relatively prime.*

See [4, Lemma 3.1] for a proof.

11. It follows from Lemma 7(a), that $\lambda(t)$ is an irreducible factor of $\Delta_T(t)$ if and only if $\lambda(t^{-1})$ is. We will say $\lambda(t)$ is *non-symmetric* or *symmetric* as $\lambda(t)$ is, or is not, relatively prime to $\lambda(t^{-1})$. Then, it follows from Lemma 10, that V splits into the orthogonal sum of two types of submodule:

(i) V_λ, where $\lambda(t)$ is symmetric, and

(ii) $V_\lambda \oplus V_{\bar{\lambda}}$, where $\lambda(t)$ is non-symmetric and $\bar{\lambda}(t)$ is defined to be $\lambda(t^{-1})$.

The restriction of $(\langle\ ,\ \rangle, T)$ to each of these summands gives an isometric structure, and it follows from Lemma 11 that those of type (ii) are null-cobordant. Furthermore $(\langle\ ,\ \rangle, T)$ is null-cobordant if and only if its restriction to each V_λ is null-cobordant, since the restrictions of $\langle\ ,\ \rangle$ are non-degenerate and any submodule of V is a direct sum of submodules of the $\{V_\lambda\}$.

We may rephrase these observations as:

Lemma. *For every irreducible symmetric polynomial $\lambda(t)$, let G_λ be the subgroup of G_F determined by $(\langle\,,\rangle, T)$ for which $\Delta_T(t)$ is a power of $\lambda(t)$ – note $\lambda(t) \neq t+1$ or $t-1$. Then G_F is the direct sum of the $\{G_\lambda\}$.*

12. We now prove:

Lemma. *Let $(\langle\,,\rangle, T)$ be an isometric structure with characteristic polynomial $\lambda(t)^e$, where $\lambda(t)$ is symmetric and irreducible, $e>0$. Then $(\langle\,,\rangle, T)$ is cobordant to an isometric structure with minimal polynomial $\lambda(t)$ – or null-cobordant.*

Proof. Suppose the minimal polynomial of T is $\lambda(t)^a$, where $a>1$. We show that $(\langle\,,\rangle, T)$ is cobordant to an isometric structure with minimal polynomial $\lambda(t)^b$, for some $b<a$. An iteration of this process will prove the lemma.

Let $W=\lambda(T)^{a-1} V \neq 0$. Now W is totally isotropic since $\langle \lambda(T)^{a-1} v, \lambda(T)^{a-1} w\rangle = \langle v, u\,\lambda(T)^{2a-2} w\rangle = 0$, where u is a unit in Λ, since $2a-2 \geqq a$ if $a>1$. Let $W^\perp$ be the orthogonal complement of W in V; then $W^\perp$ is a submodule and $W \subset W^\perp$. The quotient module $W^\perp/W$ inherits an isometric structure $(\langle\,,\rangle', T')$ from $(\langle\,,\rangle, T)$ and the minimal polynomial of T' is $\lambda(t)^b$, where $b<a$. Now the lemma follows from:

13. Lemma. *Let $(\langle\,,\rangle, T)$ be an isometric structure on V and W a totally isotropic subspace of V, invariant under T. If $(\langle\,,\rangle', T')$ is the isometric structure on $W^\perp/W$ inherited from $(\langle\,,\rangle, T)$, then $(\langle\,,\rangle, T)$ and $(\langle\,,\rangle', T')$ are cobordant.*

Proof. Consider the subspace V_0 of $V \oplus (W^\perp/W)$ consisting of all pairs (v, w), where $v \in W^\perp$ and w is the coset of v in $W^\perp/W$. It is readily checked that V_0 is a totally isotropic invariant subspace, with respect to the isometric structure $(\langle\,,\rangle, T) \perp (-\langle\,,\rangle', T')$, of half the dimension of $V \oplus (W^\perp/W)$.

14. We may immediately deal with a special case:

Proposition. *Suppose $(\langle\,,\rangle, T)$ has characteristic polynomial $\lambda(t)^e$, where $\lambda(t)$ has degree two. Then $(\langle\,,\rangle, T)$ is null-cobordant if and only if $\langle\,,\rangle$ is null-cobordant.*

Proof. Write $\lambda(t)=t^2+at+1$; by Lemma 12, we may assume $\lambda(t)$ is the minimal polynomial of T (if $\lambda(t)$ were reducible, we could assume the minimal polynomial were of degree one!). Now,

$$\begin{aligned} 0 &= \langle \lambda(T)\, v, Tv\rangle = \langle T^2 v, Tv\rangle + a\langle Tv, Tv\rangle + \langle v, Tv\rangle \\ &= 2\langle Tv, v\rangle + a\langle v, v\rangle \end{aligned}$$

for any $v \in V$. Thus

$$\langle Tv, v\rangle = \frac{-a}{2} \langle v, v\rangle .$$

Since $\lambda(t)$ has degree two, it follows that any isotropic vector generates a totally isotropic submodule of V. It follows, from Lemma 13, that $(\langle\ ,\ \rangle, T)$ is cobordant to a "smaller" isometric structure if $\langle\ ,\ \rangle$ is isotropic. The proposition follows easily from this.

15. Suppose $(\langle\ ,\ \rangle, T)$ is an isometric structure and T has minimal polynomial $\lambda'(t)$, an irreducible symmetric polynomial. If $\lambda'(1)\lambda'(-1) \neq 0$, then $\lambda'(t) = t^{2d}\lambda'(t^{-1})$ – where $2d = \text{degree}\ \lambda'(t)$ – by Lemma 7(a). Now define $\lambda(t) = t^{-d}\lambda'(t)$; then $\lambda(t) = \lambda(t^{-1})$.

Let E be the quotient field $\Lambda/(\lambda(t))$. Then E admits an involution $\xi \to \bar{\xi}$ induced by $t \to t^{-1}$; we also write $\bar{f}(t) = f(t^{-1})$, for any $f(t) \in \Lambda$. Let E_0 be the fixed field of $\xi \to \bar{\xi}$. If $\lambda_0(x)$ is the irreducible polynomial defined by $\lambda_0(t+t^{-1}) = \lambda(t)$, then E_0 is isomorphic to the quotient field $F[x]/(\lambda_0(x))$.

Milnor, in [4], associates to $(\langle\ ,\ \rangle, T)$ a Hermitian form $[\ ,\ ]$ defined on V regarded as an E-module, satisfying:

$$\langle \alpha, \beta\rangle = \text{Trace}_{E/F}[\alpha, \beta] \qquad \text{for } \alpha, \beta \in V.$$

Then $(\langle\ ,\ \rangle, T)$ is null-cobordant if and only if $[\ ,\ ]$ is null-cobordant. If V_0 is a totally isotropic (under $\langle\ ,\ \rangle$) submodule of V, then V_0 is also totally isotropic under $[\ ,\ ]$. For if $[\alpha, \beta] \neq 0$, $\alpha, \beta \in V_0$, and we set $\xi = [\alpha, \beta]^{-1}$, then $[\xi\alpha, \beta] = \xi[\alpha, \beta] = 1$ and $\langle \xi\alpha, \beta\rangle = \text{Trace}_{E/F} 1 \neq 0$ (F has characteristic zero). But $\xi\alpha$ and β are both in V_0 and so $\langle \xi\alpha, \beta\rangle = 0$.

Jacobson, in [1], defines a quadratic form $\{\ ,\ \}$ on V, regarded as an E_0-module (where $E_0 = F[t+t^{-1}]/(\lambda(t))$), by:

$$\{\alpha, \beta\} = \tfrac{1}{2}([\alpha, \beta] + [\beta, \alpha]) = \text{Trace}_{E/E_0}[\alpha, \beta].$$

Notice that $\langle \alpha, \beta\rangle = \text{Trace}_{E_0/F}\{\alpha, \beta\}$. Now the action T of t is an isometry of V with respect to $\{\ ,\ \}$ and the minimal polynomial (over E_0) of T is $t^2 - xt + 1$, where $x = t + t^{-1} \in E_0$. By Proposition 14, $(\{\ ,\ \}, T)$ is null-cobordant if and only if $\{\ ,\ \}$ is null-cobordant. But it is easy to see that $(\{\ ,\ \}, T)$ is null-cobordant if and only if $[\ ,\ ]$ is null-cobordant, since we can solve for $[\ ,\ ]$ by:

$$(t - t^{-1})[\alpha, \beta] = 2(\{t\alpha, \beta\} - t^{-1}\{\alpha, \beta\}) \qquad (\lambda(1)\lambda(-1) \neq 0).$$

16. We now apply a result proved by Milnor in [4] to obtain:

Proposition. *If F is a local field or the real numbers and $(\langle\ ,\ \rangle, T)$ an isometric structure over F with characteristic polynomial $\lambda(t)^e$, $\lambda(t)$ irreducible symmetric, then $(\langle\ ,\ \rangle, T)$ is null-cobordant if and only if $\langle\ ,\ \rangle$ is null-cobordant and e is even.*

Proof. The necessity that e be even follows from Lemma 7(b).

We may assume, by Lemma 12, that the minimal polynomial of T is $\lambda(t)$. If $\lambda(t)=t\pm 1$, the proposition is obvious. Otherwise we may assume $\lambda(t)=\lambda(t^{-1})$ – see § 15.

In this case, Milnor proves that two isometric structures with isomorphic quadratic forms and the same irreducible minimal polynomial are isomorphic. The Proposition will now follow from the assertion that any E-module V of even dimension admits a quadratic form form $\langle\, , \rangle$ such that $(\langle\, , \rangle, T)$ is null-cobordant, where T is defined by the action of t and is an isometry. Equivalently, we may construct a null-cobordant Hermitian E-form on V e.g. if $\{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\}$ is an E-basis, define $[\alpha_i, \alpha_j]=[\beta_i, \beta_j]=0$, $[\alpha_i, \beta_j]=\delta_{ij}$.

17. If $(\langle\, , \rangle, T)$ is an isometric structure over F and K is an extension field over F, then there is an obvious extension of $(\langle\, , \rangle, T)$ to an isometric structure over K.

Proposition. *An isometric structure over a global field F is null-cobordant if and only if the extension over every completion of F is null-cobordant.*

Proof. It suffices to consider an isometric structure $(\langle\, , \rangle, T)$ with minimal polynomial $\lambda(t)$ irreducible and symmetric. If $\lambda(t)=t\pm 1$, then the Proposition follows from the corresponding fact about quadratic forms (see [5]).

If $\lambda(t)=\lambda(t^{-1})$, we may consider the associated quadratic form $\{\, , \}$ (see § 15) over the field E_0. Now $\{\, , \}$ is null-cobordant if and only if the extension of $\{\, , \}$ over every completion of E_0 is null-cobordant, since E_0 is, again, a global field. The completions of E_0 are constructed as follows: let K be any completion of F and $\lambda_0(x)=\lambda_1(x)\,\lambda_2(x)\ldots\lambda_n(x)$ the decomposition of $\lambda_0(x)$ into irreducible factors over K; then each $K[x]/(\lambda_i(x))$ is a completion of E_0 (see [5, p. 34]). Let $\mu_i(t)=\lambda_i(t+t^{-1})$; then $\lambda(t)=\mu_1(t)\ldots\mu_n(t)$ and $\mu_i(t)$ is either irreducible or of the form $\theta(t)\,\theta(t^{-1})$, where $\theta(t)$ is non-symmetric. The irreducible $\{\mu_i(t)\}$ are all the irreducible symmetric factors of $\lambda(t)$ over K. It is easy to see that if we extend $(\langle\, , \rangle, T)$ over K and then restrict to the $\mu_i(t)$ – primary component, the associated quadratic form is exactly the extension of $\{\, , \}$ to $K[x]/(\lambda_i(x))$. By § 11, the extension of $(\langle\, , \rangle, T)$ over K is null-cobordant if and only if all these extensions of $\{\, , \}$ are null-cobordant. The Proposition now follows immediately.

18. We now define a collection of cobordism invariants of an isometric structure $(\langle\, , \rangle, T)$ over a global field F.

(a) Let $\lambda(t)$ be a symmetric irreducible factor of $\Delta_T(t)$ and define $\varepsilon_\lambda(\langle\, , \rangle, T)=$ exponent of $\lambda(t)$ in $\Delta_T(t)$, mod 2.

(b) Let K be a real completion of F and $\lambda(t)$ a symmetric irreducible factor of $\Delta_T(t)$ over K. Then define $\sigma_\lambda^K(\langle\,,\rangle, T)$ = signature of the restriction of $\langle\,,\rangle$, extended over K, to the $\lambda(t)$-primary component.

(c) Let K be a non-archimedean completion of F (and, therefore, a local field) and $\lambda(t)$ a symmetric irreducible factor of $\Delta_T(t)$ over K. Then define $\mu_\lambda^K(\langle\,,\rangle, T)=\mu$ (restriction of $\langle\,,\rangle$, extended over K, to the $\lambda(t)$-primary component).

Proposition. $\{\varepsilon_\lambda, \sigma_\lambda^K, \mu_\lambda^K\}$ *are cobordism invariants.*

Proof. For $\{\varepsilon_\lambda\}$, this follows from Lemma 7(b). For $\{\sigma_\lambda^K\}$ and $\{\mu_\lambda^K\}$, this follows from the cobordism invariance of σ and μ.

19. It is clear that the $\{\varepsilon_\lambda\}$ and $\{\sigma_\lambda\}$ define homomorphisms: $\varepsilon_\lambda: G_F \to Z_2$ and $\sigma_\lambda^K: G_F \to Z$, but the $\{\mu_\lambda^K\}$ are not additive. In fact they satisfy:

Lemma. $\mu_\lambda^K(\alpha+\beta)=\mu_\lambda^K(\alpha)\,\mu_\lambda^K(\beta)\big((-1)^d\lambda(1)\lambda(-1), -1\big)^{\varepsilon_\lambda(\alpha)\varepsilon_\lambda(\beta)}$ *where degree* $\lambda(t)=2d$ *and* $\varepsilon_\lambda=\varepsilon_\Phi$, *where* $\Phi(t)$ *is the symmetric irreducible polynomial over* F *which has* $\lambda(t)$ *as an irreducible factor over* K.

Proof. First observe the general formula:

$$\mu(\langle\,,\rangle \perp \langle\,,\rangle')=\mu(\langle\,,\rangle)\,\mu(\langle\,,\rangle')(-1,-1)^{rr'}(\Delta,-1)^{r'}(\Delta',-1)^{r}(\Delta,\Delta'), \quad (*)$$

where $\operatorname{rank}\langle\,,\rangle=2r$, $\operatorname{rank}\langle\,,\rangle'=2r'$, $\Delta=\det\langle\,,\rangle$, $\Delta'=\det\langle\,,\rangle'$. This follows from the definition of μ and the additivity formula for S (§ 5).

If $(\langle\,,\rangle, T)$ and $(\langle\,,\rangle', T')$ are isometric structures over F representing α and β, K is a non-archimedean extension of F and $\langle\,,\rangle_0$, $\langle\,,\rangle_0$ are the $\lambda(t)$-primary restrictions of $\langle\,,\rangle$, $\langle\,,\rangle'$ extended to K, then

(a) $\varepsilon_\lambda(\alpha)=\dfrac{\operatorname{rank}\langle\,,\rangle_0}{2d}$, $\varepsilon_\lambda(\beta)=\dfrac{\operatorname{rank}\langle\,,\rangle_0'}{2d}$ (mod 2).

(b) $\det\langle\,,\rangle_0=(\lambda(1)\lambda(-1))^{\varepsilon_\lambda(\alpha)}$, $\det\langle\,,\rangle_0'=(\lambda(1)\lambda(-1))^{\varepsilon_\lambda(\beta)}$ — see Lemma 7(c).

The lemma follows by substituting from (a) and (b) into formula (*).

Notice that $\mu_\lambda^K(2\alpha)=\big((-1)^d\lambda(1)\lambda(-1), -1\big)^{\varepsilon_\lambda(\alpha)}$, which is independent of $\mu_\lambda^K(\alpha)$.

20. If $F=Q$ and $K=R$, the only archimedean completion of Q, then the symmetric irreducible factors $\lambda(t)$ of $\Delta_T(t)$, over R, correspond to the roots of $\Delta_T(t)$ of the form $e^{i\theta}$. The invariant σ_λ coincides with the invariant σ_θ defined in [3]. It also may be verified that the invariants $\{\sigma_\lambda\}$ are equivalent to the invariant σ_A (using the isomorphism $G_Q \approx G^Q$) defined in [2].

21. Theorem. $\{\varepsilon_\lambda, \sigma_\lambda^K, \mu_\lambda^K\}$ *form a complete set of cobordism invariants for isometric structures over a global field* F, i.e., *if* $\alpha, \beta \in G_F$, *then* $\alpha=\beta$ *if and only if* $\varepsilon_\lambda(\alpha)=\varepsilon_\lambda(\beta)$, $\sigma_\lambda^K(\alpha)=\sigma_\lambda^K(\beta)$, $\mu_\lambda^K(\alpha)=\mu_\lambda^K(\beta)$ *for all* $\lambda(t)$ *for which these invariants are defined.*

Proof. We first point out that the invariants vanish on $\alpha-\beta$ if they are equal on α and β. This follows from the additivity of $\{\varepsilon_\lambda, \sigma_\lambda^K\}$ and a straightforward exercise using Lemma 19, for $\{\mu_\lambda^K\}$. Thus, it suffices to show that $\alpha=0$ if and only if all the invariants are zero on α.

By Proposition 17, we consider α on completions of F. On complex completions, every isometric structure is null-cobordant since irreducible polynomials have degree one and all quadratic forms are null-cobordant; now apply Lemma 12. On any completion, by Lemma 11, we need only look at the primary components. By Proposition 16, the $\{\sigma_\lambda^K\}$ are a complete system of invariants over the real completions. Similarly, the $\{\mu_\lambda^K\}$, together with the invariant d (see § 5) on the $\lambda(t)$-primary component, are a complete system of invariants over the non-archimedean completions. But the determinant of $\langle\,,\rangle$, on the $\lambda(t)$-primary component, is $(\lambda(1)\,\lambda(-1))^{\varepsilon_\lambda(\alpha)}$, by Lemma 7(c). Since we are assuming $\varepsilon_\lambda(\alpha)=0$, it follows that $d=(-1)^r$, where $2r=K$-dimension of $\lambda(t)$-primary component $=$ (degree $\lambda(t)$)$\cdot$(exponent of $\lambda(t)$ in $\Delta_T(t)$). Since $\lambda(t)$ has even degree and exponent of $\lambda(t)=\varepsilon_\lambda(\alpha)$ (mod 2), r is even. Thus $d=0$.

22. We now make a few general observations, based on Theorem 21, about the group G_F for F a global field.

Proposition. *Suppose* $\alpha\in G_F$. *Then*

(a) α *has finite order if and only if every* $\sigma_\lambda^K(\alpha)=0$.

(b) *If* α *has finite order, then* $4\alpha=0$; *therefore every element of* G_F *has order* 1, 2, 4 *or* ∞.

(c) α *has order* 4 *if and only if all* $\sigma_\lambda^K(\alpha)=0$, *but, for some* $\lambda(t)$ *over a non-archimedean completion* K, $\varepsilon_\lambda(\alpha)\neq 0$ *and* $((-1)^d\,\lambda(1)\,\lambda(-1), -1)\neq 0$.

Proof. Notice that $\varepsilon_\lambda(2\alpha)=0$, for any α, and $\mu_\lambda^K(2\alpha)=0$ if $\varepsilon_\lambda(\alpha)=0$ (see Lemma 19). Thus $\mu_\lambda^K(4\alpha)=0$. If $\sigma_\lambda^K(\alpha)=0$, then all the invariants vanish on 4α; if $\sigma_\lambda^K(\alpha)\neq 0$, then $\sigma_\lambda^K(k\alpha)\neq 0$, for any integer k. This proves (a) and (b). Finally (c) follows from (a), (b) and Lemma 19, since the stated conditions would imply $\mu_\lambda^K(2\alpha)\neq 0$.

23. Suppose $\alpha\in G_Q$ is represented by an isometric structure $(\langle\,,\rangle, T)$ where $\Delta_T(t)=\lambda_1(t)^{e_1}\dots\lambda_K(t)^{e_K}$ and each $\lambda_i(t)$ has degree 2. In this case many of the criteria of Proposition 22 simplify:

Corollary. (a) *If* $\lambda_i(1)\,\lambda_i(-1)<0$, *for all* i, *then* α *has finite order.*

(b) *If* $\lambda_i(1)\,\lambda_i(-1)>0$ *and* e_i *is odd, for some* i, *then* α *has infinite order.*

(c) *If* α *has finite order, then* α *has order* 4 *if and only if, for some* i, *and prime* p, *the following properties hold:*

(i) $p\equiv 3 \bmod 4$.

(ii) e_i *is odd.*

(iii) $\lambda_i(1)\,\lambda_i(-1)=p^a\cdot q$, *where* a *is odd and* q *is relatively prime to* p.

Proof. Write $\lambda_i(t)=t^2+a_i t+1$; the discriminant is $a_i^2-4=-\lambda_i(1)\lambda_i(-1)$. Thus $\lambda_i(t)$ is reducible over R if and only if $\lambda_i(1)\lambda_i(-1)<0$ (recall $\lambda_i(1)\lambda_i(-1)\neq 0$); but then $\lambda_i(t)=t\,\theta_i(t)\,\theta_i(t^{-1})$, where $\theta_i(t)$ is unsymmetric; and so $\sigma_{\lambda_i}(\alpha)=0$. Now (a) follows from Proposition 22(a).

If $\lambda_i(1)\lambda_i(-1)>0$, then the $\lambda_i(t)$-component has dimension $2e_i$ and the restriction of $\langle\,,\rangle$ has positive determinant by Lemma 7(c). But, in general, any real quadratic form with rank r and signature s satisfies:

$$\text{determinant}=(-1)^{\frac{1}{2}(r-s)}$$

an easily verified formula. Thus, if e_i is odd, the signature $\sigma_{\lambda_i}(\alpha)\neq 0$, which proves (b).

To prove (c), we apply Proposition 22(c). Since $(-\lambda_i(1)\lambda_i(-1),-1)\neq 0$, -1 must not be square in K. If K is the p-adic numbers, this means (i) (see [5, p. 159] and [1, p. 82]) – notice that p cannot be 2. Now condition (iii) implies that the discriminant of $\lambda_i(t)$ is not square in K, and, therefore, $\lambda_i(t)$ is irreducible. It remains to observe that $e_i=\varepsilon_{\lambda_i}^K(\alpha) \bmod 2$ and $(-\lambda_i(1)\lambda_i(-1),-1)\neq 0$ exactly when conditions (i) and (iii) hold (see [5, p. 166]).

24. As a consequence of Propositions 22 and 23 we prove:

Theorem. *G_ε is the direct sum of cyclic groups of orders* 2, 4 *and* ∞, *and there are an infinite number of summands of each of these orders.*

Proof. It follows from Proposition 22 that every non-zero element of G_ε has order 2, 4 or ∞. In fact the invariants $\{\sigma_\lambda^K\}$ induce a homomorphism of G_ε into a free abelian group and, by Proposition 22(a), the kernel is precisely the torsion subgroup of G_ε. This implies that G_ε is the direct sum of its torsion subgroup T and a free abelian group. By Proposition 22(b) and [6, p. 173] T is a direct sum of cyclic groups.

It was proved in [2] that G_ε has infinite rank. To complete the proof it will suffice to construct elements $\{\alpha_i,\beta_i\}$ of G_ε, $i=1,2,\ldots$, satisfying

(i) α_i is not the multiple of any other element of G_ε,

(ii) $\sum\lambda_i\alpha_i=0$ if and only if each λ_i is even,

(iii) $\sum\lambda_i\beta_i=0$ if and only if each λ_i is divisible by 4.

Recall (e.g., from [2]) the result that a polynomial $\Delta(t)$ can be realized as $\Delta_A(t)$ for some ε-matrix A (see § 2) if and only if:

(1) $\Delta(t)=t^{2\mu}\Delta(t^{-1})$, for some μ,

(2) $\Delta(-1)$ is square,

(3) $\Delta(\varepsilon)=(-\varepsilon)^\mu$.

This is Proposition 1 and 2 of [2] – note the difference in the definitions of $\Delta_A(t)$.

Set

$$\Delta_i(t) = \varepsilon\, a_i\, t^2 - (1+2a_i)\, t + \varepsilon\, a_i$$

where

$$a_i = \tfrac{1}{4}(9^i - 1).$$

It may be checked directly that $\Delta_i(t)$ satisfies (1)–(3) above and so there exists $\alpha_i \in G_\varepsilon$ with $\Delta_i(t) = \Delta_A(t)$ for a representative A of α_i. Now one may check, from Proposition 23(a), (c) that α_i has order 2. Moreover, α_i satisfies (i), since otherwise $\Delta_i(t)$ would have to be decomposable. To prove (ii), we observe that, for a representative ε-matrix of $\sum \lambda_i\, \alpha_i$, one has

$$\Delta_A(t) = \prod_i \Delta_i(t)^{\lambda_i}.$$

Since the $\{\Delta_i(t)\}$ are distinct irreducible symmetric polynomials, it follows from Lemma 7(b), that $\sum \lambda_i\, \alpha_i = 0$ if and only if each λ_i is even.

To produce the desired $\{\beta_i\}$ we proceed in a similar fashion. Set:

$$\Delta_i(t) = \Gamma_i(t)\, \Gamma_{i+1}(t) \qquad \text{for } i \geqq 0$$

where

$$\Gamma_i(t) = a_i\, t^2 + \varepsilon(1 - 2a_i)\, t + a_i$$

$$a_i = \tfrac{1}{4}(1 - 3^{2i+1}).$$

Then $\Delta_i(t)$ satisfies (1)–(3) and so admits a corresponding element β_i. It follows from Proposition 23(a), (c) that β_i has order 4. To prove (iii), we first observe, as we did in proving (ii), that $\sum \lambda_i\, \beta_i = 0$ implies

$$\prod \Delta_i(t)^{\lambda_i} = \prod \Gamma_i(t)^{\lambda_i + \lambda_{i-1}}$$

has the form prescribed in Lemma 7(b). Therefore, since the $\{\Gamma_i(t)\}$ are distinct irreducible symmetric polynomials, each $\lambda_i + \lambda_{i-1}$ is even; this readily implies each λ_i is even.

Now set $\lambda_i = 2\mu_i$ and consider $\sum \mu_i\, \beta_i = \beta$. If $\beta = 0$, it follows by the same argument that each μ_i is even and, therefore, λ_i is divisible by four. On the other hand if $\beta \neq 0$, then β has order 2, since $2\beta = \sum \lambda_i\, \beta_i$.

Now the polynomial associated with β is

$$\prod_i \Delta_i(t)^{\mu_i} = \prod_i \Gamma_i(t)^{\mu_i + \mu_{i-1}}.$$

Since $\Gamma_i(1)\, \Gamma_i(-1) = -3^{2i+1}$, it follows from Proposition 23(c) that $\mu_i + \mu_{i-1}$ must be even, and, therefore, each μ_i is even.

25. Theorem 24 also applies to C_n for n odd > 1 (see [2]). If $n = 2$, we must, in addition recall that C_3 is isomorphic to the inverse image, under a homomorphism $G_+ \to Z$, of the elements of $2Z$. This implies that the torsion subgroup of C_3 is isomorphic to the torsion subgroup of G_+, and the result follows.

We cannot use Theorem 24 to say much about C_1 except that it has infinite rank. It is known that C_1 contains elements of order 2 e.g. the figure eight knot is amphicheiral, but is not a slice knot. I do not know whether C_1 contains any element of order 4; the knot 7_7 of the Alexander-Briggs knot table is the first candidate (it gives an element of G_- of order 4).

Incidentally the knot 8_8 is the first knot determining the zero element of G_-, but which I have not been able to show is a slice knot.

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