Numer. Math. 53, 299–314 (1988)

Numerische Mathematik

On the Boundary Element Method for Some Nonlinear Boundary Value Problems

K. Ruotsalainen [1,*] and W. Wendland [2]

[1] University of Oulu, Faculty of Technology, Section of Mathematics, SF-90570 Oulu, Finland
[2] University of Stuttgart, Mathematisches Institut A, Pfaffenwaldring 57, D-7000 Stuttgart 80, Federal Republic of Germany

Summary. Here we analyse the boundary element Galerkin method for two-dimensional nonlinear boundary value problems governed by the Laplacian in an interior (or exterior) domain and by highly nonlinear boundary conditions. The underlying boundary integral operator here can be decomposed into the sum of a monotoneous Hammerstein operator and a compact mapping. We show stability and convergence by using Leray-Schauder fixed-point arguments due to Petryshyn and Nečas.

Using properties of the linearised equations, we can also prove quasioptimal convergence of the spline Galerkin approximations.

Subject Classifications: AMS(MOS): 65N30; CR: G 1.8.

1. Introduction

During the last decade boundary element methods have become a well established computational method solving boundary value problems in applications. Based on corresponding fundamental solutions, a large class of both exterior and interior elliptic boundary value problems can be reduced to equivalent integral equations on the boundary of the given domain. Equations of this kind arise in various applications such as boundary value problems in acoustics, elasticity, electromagnetics as well as in fluid dynamics [17, 20, 30, 35, 40] and references given there in) and their boundary element approximations are used in many corresponding engineering computations.

Up to now the analysis of boundary element methods is mainly restricted to linear problems. In particular, the two-dimensional Galerkin and collocation methods are studied rather extensively and the asymptotic error analysis of these methods now is quite well established [5, 6, 18, 19, 31, 36–38, 40].

In various applications, however, the problems involve nonlinearities. Among these is the steady-state heat transfer, which was studied already by Carleman

* This work was carried out while the first author was visiting the University of Stuttgart

[12] and where the boundary conditions are highly nonlinear. Also some electromagnetic problems contain nonlinearities in the boundary conditions, for instance problems, where the electrical conductivity of the boundary is variable [10, 22]. Further applications arise in heat radiation and heat transfer [9, 10].

Motivated by the above applications we consider here the potential problem

$$\Delta u = 0, \qquad \text{in } \Omega \tag{1}$$

$$\frac{\partial u}{\partial n} = -g(x, u) + f \qquad \text{on } \Gamma, \tag{2}$$

where Γ denotes a smooth simply closed curve in the plane, f and g are given real valued functions on Γ. Here Δ stands for the two-dimensional Laplacian. By the Green representation formula we formulate a nonlinear integral equation on the boundary of the domain Ω, as was already done by Lichtenstein [23]. This integral equation is of Hammerstein type with a compact perturbation. Under relatively general assumptions on the nonlinearity $g(x, u)$ we prove the equivalence of these two formulations. Further, more restrictive assumptions on g will imply strong monotonicity and for this case we discuss existence and uniqueness of the solution.

The purpose of this paper is to analyse the Galerkin method for solving the nonlinear integral equation in question. We shall also discuss the rate of convergence of the approximate solutions.

For guaranteeing the existence and boundedness of the Galerkin solution we apply the techniques of Necas [29] and Petryshyn [33, 34] (see also [16] and [41]). From the boundedness of the Gakerkin solutions and the uniqueness of the continuous solution together with the mapping properties of the integral operator we finally obtain the convergence of the BEM-Galerkin scheme.

In the last chapter we consider the rate of convergence of the Galerkin boundary element method. Here we use essentially the linearization technique that also was fruitful in the analysis of the finite element method for quasilinear elliptic equations [15, 21, 27, 39]. Our analysis is also close to some work of Amann, who used the linearization technique for obtaining the rate of convergence of a special projection scheme for a Hammerstein integral equation [2, 3]. The linearization defines a completely continuous family of integral equations. Then the results by Anselone [4] yield the optimal order of convergence for the BEM-Galerkin solutions in the $H^{\frac{1}{2}}(\Gamma)$-norm.

2. The Boundary Integral Formulation

Here we investigate the nonlinear boundary value problem

$$\Delta u = 0, \qquad \text{in } \Omega \tag{3}$$

$$-\frac{\partial u}{\partial n} = G(u) - f \qquad \text{on } \Gamma. \tag{4}$$

We assume that Γ is a smooth Jordan curve in the plane. Given are the function $f \in H^{-\frac{1}{2}}(\Gamma)$ and $G\colon H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$, in general, is some nonlinear mapping. Here $H^{\sigma}(\Gamma)$ for $\sigma \in R$ denote the usual Sobolev-Slobodetski-spaces on the closed boundary curve Γ (see [1]).

First we reformulate this problem as a nonlinear integral equation. To this end we use the well-known representation formula for harmonic functions

$$u(x) = \frac{1}{2\pi} \int_{\Gamma} u(x) \frac{\partial}{\partial n_y} \log|x-y|\, ds_y - \frac{1}{2\pi} \int_{\Gamma} \frac{\partial u}{\partial n_y} \log|x-y|\, ds_y, \tag{5}$$

for $x \in \Omega$. The classical jump relations of the potential theory (see e.g. [13]) imply the relation

$$u(x) - \frac{1}{\pi} \int_{\Gamma} u(y) \frac{\partial}{\partial n_y} \log|x-y|\, ds_y = -\frac{1}{\pi} \int_{\Gamma} \frac{\partial u}{\partial n_y} \log|x-y|\, ds_y \tag{6}$$

on the boundary Γ.

Introducing the notations

$$K u(x) \overset{\text{def}}{=} \frac{1}{\pi} \int_{\Gamma} u(y) \frac{\partial}{\partial n_y} \log|x-y|\, ds_y$$

and

$$S\psi(x) \overset{\text{def}}{=} -\int_{\Gamma} \psi(y) \log|x-y|\, ds_y$$

the Eq. (6) can be written as

$$(I-K)u = S\left(\frac{\partial u}{\partial n}\right). \tag{7}$$

Clearly, if $u \in H^1(\Omega)$ is the solution of (3) and (4), then the Cauchy-data

$$\left(u|_{\Gamma}, \left.\frac{\partial u}{\partial n}\right|_{\Gamma}\right)$$

satisfies the integral Eq. (7). Then the boundary condition $-\frac{\partial u}{\partial n} = G(u) - f$ yields

$$u - Ku + S(G(u)) = Sf. \tag{8}$$

Conversely, if $(u|_{\Gamma})$ solves (8), then the solution of the boundary value problem can be given by the representation formula (5) and will satisfy

$$-\frac{\partial u}{\partial n} = G(u) - f$$

due to (8).

For studying the solvability of the nonlinear integral Eq. (8) we collect some basic assumptions to be made here. Since the operator S may have eigensolutions [18] we assume a scaling with $\operatorname{diam}(\Omega)<1$. Then the integral operator S: $H^s(\Gamma)\to H^{s+1}(\Gamma)$ is an isomorphism for every $s\in R$ and

$$(S\psi\,|\,\psi)\geqq c\,\|\psi\|^2_{-\frac{1}{2}} \tag{9}$$

for all $\psi\in H^{-\frac{1}{2}}(\Omega)$ with some positive constant $c>0$, [18]. By $(\cdot|\cdot)$ we denote the $L^2(\Gamma)$ scalar product.

The double layer potential operator on Γ has a smooth kernel defining a compact integral operator K: $H^s(\Gamma)\to H^s(\Gamma)$ for all $s\in R$. Hence $I-K$: $H^s(\Gamma)\to H^s(\Gamma)$ is a classical Fredholm-operator with index zero.

Our second basic assumption is that the function g: $\Gamma\times R\to R$ is a Caratheodory-function i.e. $g(\cdot,u)$ is measurable for all $u\in R$ and $g(x,\cdot)$ is continuous for almost all $x\in\Gamma$. Further we assume that $\frac{\partial}{\partial u}g(x,u)$ is Borel-measurable satisfying

$$0<l\leqq\frac{\partial}{\partial u}g(x,u)\leqq L<\infty,\qquad u\in R. \tag{10}$$

Condition (10) implies that the Nemytski operator $G(\cdot)$: $L^2(\Gamma)\to L^2(\Gamma)$, defined by $G(u)(x)=g(x,u(x))$, is Lipschitz continuous and strongly monotonous, e.g.

$$(G(u)-G(v)\,|\,u-v)\geqq c\,\|u-v\|_0^2 \tag{11}$$

for all u, $v\in L^2(\Gamma)$ [32].

The mapping properties of the integral operator defined by (8) are collected in the following theorem.

Theorem 1. *For all* $s\in[0,1]$ *the operator* $u\to -Ku+SG(u)$ *is Lipschitz continuous from* $H^s(\Gamma)$ *into* $H^{s+1}(\Gamma)$.

For the solvability of the Eq. (8) we conclude from assumptions (9) and (10):

Theorem 2. *For every* $f\in H^{-\frac{1}{2}}(\Gamma)$ *there exists a unique* $u\in H^{\frac{1}{2}}$ *such that*

$$u-Ku+SG(u)=Sf. \tag{12}$$

Remark. With the theory of monotone operators we can generalize our results. We can prove the existence of the solution when the boundary condition in (4) is replaced by

$$f-\frac{\partial u}{\partial n}\in\beta(u),$$

where β: $H^{\frac{1}{2}}(\Gamma)\to 2^{H^{-\frac{1}{2}}}(\Gamma)$ is a maximally monotone, coercive, set-valued mapping (for the definition of monotone mappings see e.g. [8, 11, 24, 25, 32] and

[41]). In this case the boundary value problem is equivalent to the boundary integral inclusion problem: Find $u \in H^{\frac{1}{2}}(\Gamma)$ such that

$$Sf \in (I-K)u + S(\beta(u)),$$

where $f \in H^{-\frac{1}{2}}$. As in the following proof of Theorem 2 it suffices to consider the inclusion problem

$$f \in S^{-1}(I-K)u + \beta(u).$$

Here the linear operator $S^{-1}(I-K)$: $H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is continuous and monotone. Since β is maximally monotone and coercive we get by [8: Chapter II, Corol. 1.3] that the range of $S^{-1}(I-K)+\beta$ is $H^{-\frac{1}{2}}(\Gamma)$, which proves our statement. Further, if β is *strictly* monotone, we obtain the uniqueness of the solution.

Proof of Theorem 2. Since S: $H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is an isomorphism it is sufficient to consider the unique solvability of equation

$$D(u) \overset{\text{def}}{=} S^{-1}(I-K) + G(u) = f. \tag{13}$$

We shall prove that the operator D: $H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is continuous and strongly monotonous. Then the statement follows from the well-known theorem by Browder and Minty on monotone operators [11, 24, 26].

The continuity is clear from the mapping properties of S, K and G. Hence it remains to prove strong monotonicity of D.

The function $\psi \in H^{-\frac{1}{2}}(\Gamma)$ defined by

$$\psi \overset{\text{def}}{=} S^{-1}(I-K)u,$$

for $u \in H^{\frac{1}{2}}(\Gamma)$, is the normal derivative of the harmonic function

$$\varphi(x) = \frac{1}{2\pi} \int_\Gamma u(y) \frac{\partial}{\partial n_y} \log|x-y|\, ds_y - \frac{1}{2\pi} \int_\Gamma \psi(y) \log|x-y|\, ds_y. \tag{14}$$

Then Green's theorem yields

$$(S^{-1}(I-K)u|u) = \int_\Gamma \frac{\partial \varphi}{\partial n} u\, ds = \int_\Omega (\nabla \varphi)^2\, dx.$$

Hence, the linearity implies that for all $u, v \in H^{\frac{1}{2}}(\Gamma)$

$$(S^{-1}(I-K)(u-v)|u-v) = \int_\Omega (\nabla F)^2\, dx, \tag{15}$$

where F denotes the harmonic function corresponding to the Cauchy-data $u-v$ and $S^{-1}(I-K)(u-v)$.

On the other hand by (11) we have

$$(G(u)-G(v)|u-v) \geqq l\,\|u-v\|_0^2. \tag{16}$$

It remains to prove that

$$\|u-v\|_0 \geqq c\,\|F\|_{L^2(\Omega)}. \tag{17}$$

Then with (15) we will get the proposed inequality

$$(D(u)-D(v)\,|\,u-v) \geqq c\,\|F\|^2_{H^1(\Omega)} \geqq c'\,\|u-v\|^2_{\frac{1}{2}}$$

by the trace theorem [1].

To prove (17) we note that there exists $\chi \in H^{-\frac{1}{2}}(\Gamma)$ such that $S\chi = u-v$ on Γ [18]. Hence for all $x \in \Omega$ we have

$$-\frac{1}{2\pi}\int_\Gamma \chi(y)\log|x-y|\,ds_y = F(x). \tag{18}$$

The simple layer potential $\chi \to -\frac{1}{2\pi}\int_\Gamma \chi(y)\log|x-y|\,ds_y$ defines a continuous mapping from $H^s(\Gamma)$ into $H^{s+\frac{3}{2}}(\Omega)$ for all $s \in R$ [13, 18]. Hence for $s=0$ we find

$$\|F\|_{L^2(\Omega)} \leqq c\,\|\chi\|_{H^{-\frac{3}{2}}(\Gamma)} \leqq c'\,\|u-v\|_{H^{-\frac{1}{2}}(\Gamma)} \leqq c'\,\|u-v\|_0,$$

i.e. inequality (17), which completes the proof.

Later on we need also the regularity properties of the solution of (8).

Theorem 3. *For all $Sf \in H^s(\Gamma)$, $\frac{1}{2} \leqq s \leqq 2$, the unique solution of the Eq.* (8) *belongs to the space $H^s(\Gamma)$.*

In the proof of Theorem 3 we shall need the following lemma.

Lemma 1. *For every $u \in H^t(\Gamma)$, $0 \leqq t \leqq 1$, we have $G(u) \in H^t(\Gamma)$ and the mapping $u \to G(u)$ is bounded.*

Proof. The case $t=0$ has already been proved. For $t=1$, u is an absolutely continuous function. By assumption (10) the function $g(r)$, $r \in R$, is Lipschitz-continuous on every finite interval. Hence the composite function is also absolutely continuous [28; pp. 272].

It remains to prove the cases $0<t<1$. Due to the definition of the Sobolev spaces $H^t(\Gamma)$ it is sufficient to prove the finiteness of the double integral

$$\int_\Gamma\int_\Gamma \frac{|G(u)(x)-G(u)(y)|^2}{|x-y|^{1+2t}}\,ds_y\,ds_x. \tag{19}$$

The Lipschitz-condition implies that (19) is bounded by

$$\int_\Gamma\int_\Gamma \frac{|G(u)(x)-G(u)(y)|^2}{|x-y|^{1+2t}}\,ds_x\,ds_y \leqq L^2\,\|u\|_t^2,$$

which completes the proof of Lemma 1.

Proof of Theorem 3. Let $Sf \in H^s(\Gamma)$, $\frac{1}{2}<s\leqq\frac{3}{2}$, be given. By Theorem 1 there exists a unique solution $u \in H^{\frac{1}{2}}$ of (8). Because of Lemma 1, $Sf-SG(u) \in H^s(\Gamma)$, and

therefore $(I-K)u \in H^s(\Gamma)$. This implies together with the Fredholm-property of the operator $I-K$ that $u \in H^s(\Gamma)$, $\frac{1}{2}<s\leqq\frac{3}{2}$. If $\frac{3}{2}\leqq s\leqq 2$, then $Sf-SG(u)\in H^s(\Gamma)$. With the previous argumentations we get: $u \in H^s(\Gamma)$.

3. The Boundary Element Method

Here we consider the numerical approximation scheme for finding an approximate solution to (8). To this end we select a sequence of mesh points $\Xi=\{t_i \mid i=0, \ldots, N\}$ on Γ satisfying $t_{i+N}=t_i$ and denote by $S_h^d(\Xi)$ the boundary element space transplanted from the space of 1-periodic, $(d-1)$-times continuously differentiable splines of degree d onto Γ. Further we assume that the family of partitions is quasiuniform, e.g. $(\max h_i/\min h_i)\in[\gamma, \gamma^{-1}]$ for all partitions with some positive constant $\gamma>0$, where $h_i=|t_{i+1}-t_i|$. With these assumptions the approximation spaces have the following approximation property [5, 7, 14]:

Approximation Property. *For every $u\in H^s(\Gamma)$, $s\leqq d+1$, there exists $\psi\in S_h^d(\Xi)$ such that*

$$\|u-\psi\|_t \leqq c h^{s-t}\|u\|_s, \tag{20}$$

where $t\leqq s$, $t\leqq r<d+\frac{1}{2}$, and r is arbitrary.

Besides the approximation property the quasiuniformity provides the inverse assumption [5, 7, 14]:

Inverse Assumption. *For all $\psi\in S_h^d(\Xi)$ there holds the estimate*

$$\|\psi\|_s \leqq c h^{t-s}\|\psi\|_t \tag{21}$$

where $t\leqq s<d+\frac{1}{2}$.

Then the standard Galerkin method for Eq. (8) reads as to *find $u_h\in S_h^d(\Xi)$ such that*

$$(u_h-K u_h+SG(u_h)\mid\psi)=(Sf\mid\psi) \tag{22}$$

for all $\psi\in S_h^d(\Xi)$.

Clearly, the Eqs. (22) are equivalent to N nonlinear equations for the N coefficients of the trial function u_h spanned by a suitable basis of the boundary element space $S_h^d(\Xi)$ of dimension N.

In what follows we analyse the existence of Galerkin solutions for a family of deminishing meshwidth $h=\max|t_{i+1}-t_i|$.

The following mapping property is crucial for our analysis. A mapping A: $X\to X^*$, where X is a reflexive Banach space, is called *of type* (*S*), if it satisfies the condition:

Condition 1. *If the sequence $\{u_n\}$ has the properties*

$$u_n \rightharpoonup u$$
$$A u_n \rightharpoonup g$$
$$(A u_n\mid u_n)\to(g\mid u),$$

then there exists a subsequence $\{u_{n_j}\}$ converging to u in X.

This or analogous concepts have played an important role in nonlinear functional analysis [29, 33, 34, 41]. It is an easy task to prove that a mapping of type (S) is pseudomonotone.

Lemma 2. *The nonlinear integral operator* $u \to u - Ku + SG(u)$ *in* $L^2(\Gamma)$ *is of type* (S).

Proof. Let us suppose that $\{u_n\}$ has the required properties:

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } L^2(\Gamma) \\ Au_n &\rightharpoonup g \quad \text{in } L^2(\Gamma) \\ (Au_n | u_n) &\to (g|u). \end{aligned} \tag{23}$$

Since every weakly convergent sequence is bounded and G is continuous and bounded in $L^2(\Gamma)$, the sequence $\{G(u_n)\}$ is also bounded in $L^2(\Gamma)$ and has a weakly convergent subsequence. Let us denote by v the weak limit of $\{G(u_{n_j}\}$. S is a compact linear operator from $L^2(\Gamma)$ into $L^2(\Gamma)$, thus $SG(u_{n_j}) \to Sv$ in $L^2(\Gamma)$. Since $K: L^2(\Gamma) \to L^2(\Gamma)$ is also compact we have $Ku_{n_j} \to Ku$ in $L^2(\Gamma)$.

Now we show that

$$\lim_{j\to\infty} (u_{n_j} - Ku_{n_j} + SG(u_{n_j}) - u + Ku - Sv | u_{n_j} - u) = 0. \tag{24}$$

From the assumptions made on the sequence we have

$$(Au_{n_j} | u_{n_j} - u) = (Au_{n_j} | u_{n_j}) - (Au_{n_j} | u) \to 0 \tag{25}$$

as $j \to \infty$. Combining this with the property

$$(u - Ku + Sv | u_{n_j} - u) \to 0,$$

for $j \to \infty$, we get (24).

Since $SG(u_{n_j}) \to Sv$ and $Ku_{n_j} \to Ku$ we finally get

$$\lim_{j\to\infty} (u_{n_j} - u | u_{n_j} - u) = 0,$$

which completes the proof.

Lemma 3. *Let* $f \in H^{-\frac{1}{2}}(\Gamma)$ *be given. Then there exists* $R(f) > 0$ *such that the equations*

$$A_t u \stackrel{\text{def}}{=} u + t(-Ku + SG(u) - Sf) = 0 \tag{26}$$

do not admit any solution for any $t \in [0, 1]$ *whenever* $\|u\|_0 \geqq R(f)$.

Proof. If $u \in L^2(\Gamma)$ is a solution of the equation

$$u + t(-Ku + SG(u) - Sf) = 0,$$

then by the assumption, $f\in H^{-\frac{1}{2}}(\Gamma)$, and the mapping properties of S and G we have $u-tKu=tSf-tSG(u)\in H^{\frac{1}{2}}(\Gamma)$. This implies as in the proof of Theorem 3 that $u\in H^{\frac{1}{2}}(\Gamma)$.

As in the previous chapter we consider the equation

$$D_t u \overset{\text{def}}{=} (1-t)S^{-1}u+t[S^{-1}(I-K)u+G(u)]=tS^{-1}. \tag{27}$$

From the properties of S^{-1} and D: $H^{\frac{1}{2}}(\Gamma)\to H^{-\frac{1}{2}}(\Gamma)$ (see Chapter 2) it follows that D_t is strongly monotone, coercive, continuous and bounded from $H^{\frac{1}{2}}(\Gamma)$ $\to H^{-\frac{1}{2}}(\Gamma)$. Hence there exists a constant $R(f)>0$ such that

$$D_t u \neq tf$$

for all $u\in H^{\frac{1}{2}}(\Gamma)$ with $\|u\|_{\frac{1}{2}}\geqq R(f)$.

Thus, if $u\in L^2(\Gamma)$ and $\|u\|_0\geqq R(f)$, then u cannot be a solution of (26). The statement is proved.

From the mapping properties of A_t defined by (26) and Lemma 2 we obtain an analogous statement to that of Lemma 3.

Lemma 4. *Let us denote $U_h=\{\phi\in S_h^d(\Xi)\,|\,\|\phi\|_0\leqq R(f)\}$. Then there exists $h_0>0$ such that for all $0<h\leqq h_0$ and $t\in[0,1]$ and for some $i\in\{1,\dots,N(h)\}$*

$$a_i(\phi,t)\overset{\text{def}}{=}(A_t\phi\,|\,\mu_i)\neq 0 \tag{28}$$

holds for all $\phi\in\partial U_h$, where $\{\mu_k\,|\,k=1,\dots,N(h)\}$ forms the basis of $S_h^d(\Xi)$.

Proof. Let us assume that the statement were false. Then we are able to construct sequence $\{t_j\}$, $t_j\in[0,1]$ and $\{\phi_j\,|\,\phi_j\in S_{h_j}^d(\Xi)\}$ $(j\to\infty)$ having the following properties:

$$\|\phi_j\|_0=R(f) \tag{29}$$

$$(A_{t_j}\phi_j\,|\,\mu_i)=0, \qquad i=1,\dots,N(h_j). \tag{30}$$

Since $[0,1]$ is compact we have a subsequence $\{t_j\}$ converging to some $t_0\in[0,1]$. Hence by the definition of A_t we get for all $u\in L^2(\Gamma)$, $\|u\|_0=R(f)$,

$$\|A_{t_j}u-A_{t_0}u\|_0\leqq|t_j-t_0|\,C(R(f))\to 0, \tag{31}$$

as $j\to 0$.

From the bounded sequence $\{\phi_j\}$ we can select a weakly converging subsequence with the weak limit $u\in L^2(\Gamma)$.

According to (30) there holds for all $\chi\in S_{h_j}^d(\Xi)$

$$(A_{t_j}\phi_j\,|\,\chi)=0. \tag{32}$$

Now for an arbitrary $w\in L^2(\Gamma)$ the Schwarz inequality yields

$$|(A_{t_i}\phi_j\,|\,w)|=|(A_{t_j}\phi_j\,|\,w-P_{h_j}w)|\leqq\|A_{t_j}\phi_j\|_0\,\|w-P_{h_j}w\|_0, \tag{33}$$

where P_{h_j}: $L^2(\Gamma) \to S^d_{h_j}(\Xi)$ is the orthogonal projection. The boundedness of the sequence $\{\phi_j\}$ and the operators A_{t_j} yields

$$|(A_{t_j}\phi_j | w)| \leqq C \|w - P_{h_j} w\|_0. \tag{34}$$

By the approximation properties of the approximation spaces $S^d_h(\Xi)$ we finally obtain from (34) that

$$(A_{t_j}\phi_j | w) \to 0, \quad j \to \infty.$$

Combining this with (30) we get

$$(A_{t_0}\phi_j | w) \to 0, \quad j \to \infty. \tag{35}$$

Finally with (30) and (31)

$$\begin{aligned}(A_{t_0}\phi_j | \phi_j) &= (A_{t_0}\phi_j - A_{t_j}\phi_j | \phi_j)\\ &\leqq C(R(f))\,|t_0 - t_j| \to 0\end{aligned}$$

as $j \to \infty$.

Thus we have constructed a sequence $\{\phi_j\}$ such that

$$\phi_j \rightharpoonup u$$

$$A_{t_0}\phi_j \rightharpoonup 0$$

$$(A_{t_0}\phi_j | \phi_j) \to 0.$$

As in Lemma 2 one can prove that A_{t_0} is of type (S). Hence we can select a subsequence $\{\phi_j\}$ converging strongly to $u \in L^2(\Gamma)$, and $\|u\|_0 = R(f)$. Because A_{t_0} is continuous we have $A_{t_0}\phi_j \to A_{t_0}u$. On the other hand $A_{t_0}\phi_j \rightharpoonup 0$. By the uniqueness of the weak limit we must have $A_{t_0}u = 0$. This is a contradiction to the statement of Lemma 3.

Now we are able to prove the main theorem of this chapter.

Theorem 4. *Let $f \in H^{-\frac{1}{2}}(\Gamma)$. Then there exists $h_0 > 0$ such that for all $0 < h \leqq h_0$ the Galerkin equations*

$$(u_h - K u_h + SG(u_h) | \xi) = (Sf | \xi), \tag{36}$$

for all $\xi \in S^d_h(\Xi)$, have a solution $u_h \in S^d_h(\Xi)$. Furthermore the sequence of Galerkin solutions converges strongly to the unique solution $u \in H^{\frac{1}{2}}(\Gamma)$ of (8).

Proof. Let us consider the mapping F_h: $[0, 1] \times S^d_h(\Xi) \to S^d_h(\Xi)$ defined by

$$F_h(\phi, t) \stackrel{\text{def}}{=} \phi - \sum_{i=1}^{N(h)} a_i(\phi, t)\mu_i. \tag{37}$$

Due to the definition of $a_i(\cdot, \cdot)$, this function is continuous and bounded. Furthermore by Lemma 4 $F_h(\Phi, t) \neq \Phi$ for all $t \in [0, 1]$ and $\Phi \in \partial U_h$. $F_h(\cdot, \cdot)$ is a homotopy.

Hence the Leray-Schauder fixed-point-index is constant with respect to $t \in [0, 1]$. The function $F_h(\cdot, 0)$ is an odd function

$$F_h(-\phi, 0) = -F_h(\phi, 0)$$

for all $\phi \in \partial U_h$. Then the fixed-point-index of $F_h(\cdot, 0)$, i.e. $i(F_h(\cdot, 0), U_h) \neq 0$ [41; Theorem 15.2]. This in turn implies that $i(F_h(\cdot, 1), U_h) \neq 0$. By the Kronecker existence principle, $F_h(\cdot, 1)$ has a fixed-point $u_h \in U_h$, $\|u_h\|_0 \leqq R(f)$. This fixed-point satisfies the equation

$$\sum_{i=1}^{N(h)} a_i(u_h, 1)\mu_i = 0.$$

Because $\{\mu_i\}$ is the basis of $S_h^d(\Xi)$ be must have

$$a_i(u_h, 1) = (u_h - K u_h + SG(u_h) | \mu_i) = 0, \qquad i = 1, \ldots, N(h).$$

In other words, u_h solves the Galerkin equations.

To prove the convergence we proceed as in [41] and [34]. The uniform boundedness of the Galerkin solutions allows to select a weakly convergent subsequence $\{u_{h_j}\}$ with the weak limit v from every bounded subsequence. By the orthogonality of the Galerkin solutions and the Schwarz inequality we obtain for all $w \in L^2(\Gamma)$

$$\begin{aligned}(A u_{h_j} - Sf | w) &= (A u_{h_j} - Sf | w - P_{h_j} w) \\ &\leqq \|A u_{h_j} - Sf\|_0 \, \|w - P_{h_j} w\|_0 \\ &\leqq C(R(f)) \, \|w - P_{h_j} w\|_0 \end{aligned} \tag{38}$$

For the last inequality we have used the boundedness of A. The approximation properties of the splines imply that the right hand side in (38) converges to zero. This means that $A u_{h_j} \rightharpoonup Sf$ in $L^2(\Gamma)$. From (36) we get

$$(A u_{h_j} | u_{h_j}) = (Sf | u_{h_j}) \to (Sf | v).$$

Now by Lemma 2 we find $u_{h_j} \to v$ in $L^2(\Gamma)$, and $Av = Sf$. Since $f \in H^{-\frac{1}{2}}(\Gamma)$, v must be the *unique* solution of (8) and $v \in H^{\frac{1}{2}}(\Gamma)$.

The above arguments are valid for *every* subsequence. Hence, the complete sequence of Galerkin solutions u_h converges to u in $L^2(\Gamma)$ as $h \to 0$.

4. The Rate of Convergence

In this section we consider the asymptotic error analysis of the Galerkin procedure. We apply linearization techniques which have been fruitful in the corresponding analysis of finite element methods for nonlinear problems [15, 21, 27, 39].

Here we need first differentiability properties of the mapping

$$u \to u - K u + SG(u)$$

in the Sobolev spaces $u\in H^s(\Gamma)$, $0\leqq s\leqq 1$. The properties of the nonlinearity G guarantee that this mapping is differentiable. Namely for every $u\in H^s(\Gamma)$, $s\in[0, 1]$, there holds

$$\lim_{t\to 0}\left\|\frac{SG(u+tv)-SG(u)}{t}-S(G'(u)v)\right\|_s=0. \tag{39}$$

The differentiability of the linear part is trivial. Further, for every $u\in H^s(\Gamma)$, the linear operator

$$\psi\to\psi-K\psi+S(G'(u)\psi)$$

in $H^s(\Gamma)$ is uniformly bounded by (10). Thus the following theorem is valid.

Theorem 5. *The mapping* A: $H^s(\Gamma)\to H^S(\Gamma)$ *is differentiable at every* $u\in H^s(\Gamma)$, $0\leqq s\leqq 1$, *and*

$$A'(u)v\overset{\text{def}}{=}v-Kv+S(G'(u)v) \tag{40}$$

for all $v\in H^s(\Gamma)$. *Moreover*

$$\|A'(u)\|_{s;s}\leqq c$$

uniformly for all u.

Before we state the main theorem of this chapter we need an additional restriction by assuming that $S_h^d(\Xi)\subset H^{\frac{1}{2}}(\Gamma)$. In other words we require here $d\geqq 1$, i.e. the spline spaces are at least piecewise linear. Note that this restriction was not required in the previous section.

The Galerkin equations can be written as

$$(A(u_h)-A(u)|\phi)=0, \qquad \phi\in S_h^d(\Xi) \tag{41}$$

The Galerkin solution, which exists for all sufficiently small $h>0$, also satisfies the linearized equations

$$(A'(\xi_h)(u-u_h|\phi)=0, \qquad \phi\in S_h^d(\Xi), \tag{42}$$

where $\xi_h=\theta u+(1-\theta)u$ with $\theta\in]0, 1[$.

Equations (42) can be written equivalently as

$$[I-P_hK+P_hSG'(\xi_h)]u_h=P_hA'(\xi_h)u. \tag{43}$$

Hence it suffices to consider the properties of the operator defined by the left hand side in (43).

Lemma 5. *The operator* $A'(\chi)$: $H^{\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma)$ *is an isomorphism and there exist constants* c', $c''>0$ *such that*

$$\|A'(\chi)\|_{\frac{1}{2};\frac{1}{2}}\leqq c' \tag{44}$$

and

$$\|A'(\chi)^{-1}\|_{\frac{1}{2};\frac{1}{2}}\leqq c'' \tag{45}$$

for all $\chi\in H^{\frac{1}{2}}(\Gamma)$.

Proof. The first statement was already proved in Theorem 5.

For proving the uniform boundedness of the inverse operator we proceed as in Chapter 2. Consider the linear operator $B(\chi)\colon H^{\frac{1}{2}}(\Gamma)\to H^{-\frac{1}{2}}(\Gamma)$ defined by

$$B(\chi)v=S^{-1}(I-K)v+G'(\chi)v.$$

According to the assumptions (9) and (10) we can show as in the proof of Theorem 2 that $B(\chi)$ is continuous and coercive

$$(B(\chi)v\,|\,v)\geqq C\,\|v\|_{\frac{1}{2}}^{2},$$

where $C>0$ is independent of χ. Since $S\colon H^{-\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma)$ is an isomorphism, the composite operator $A'(\chi)=SB(\chi)\colon H^{\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma)$ is also isomorphic. The uniform boundedness of the inverse follows from the coerciveness of $B(\chi)$ with the uniform constant c.

Lemma 6. *The operator $I-P_hK+P_hSG'(\xi_h)\colon H^{\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma)$ is an isomorphism satisfying*

$$\|[I-P_hK+P_hSG'(\xi_h)]^{-1}\|_{\frac{1}{2};\frac{1}{2}}\leqq c \tag{46}$$

with $c>0$ independent of h and ξ_h.

Proof. Let us denote

$$\tilde{A}'(\xi_h):=I-P_hK+P_hSG'(\xi_h).$$

Because of assumption $S_h^d(\Xi)\subset H^{\frac{1}{2}}(\Gamma)$ made at the beginning of this section, $\tilde{A}'(\xi_h)$ maps $H^{\frac{1}{2}}(\Gamma)$ into itself. Since the partition is quasiuniform, the orthogonal projections $P_h\colon H^{\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma)$ are uniformly bounded for all $h>0$ [7, 18]. The continuity then follows from the mapping properties of K and S.

To prove the existence of the inverse operator and the uniform estimate (45) it is sufficient to show that

$$\lim_{h\to 0}\|\tilde{A}'(\xi_h)-A'(\xi_h)\|_{\frac{1}{2};\frac{1}{2}}=0 \tag{47}$$

uniformly for all ξ_h. The statement follows then from Theorem (3.1) in [4].

For showing (47) we observe that the difference can be estimated as

$$\|\tilde{A}'(\xi_h)-A'(\xi_h)\|_{\frac{1}{2};\frac{1}{2}}\leqq\|(I-P_h)K\|_{\frac{1}{2};\frac{1}{2}}+\|(I-P_h)SG'(\xi_h)\|_{\frac{1}{2};\frac{1}{2}}.$$

K maps $H^{\frac{1}{2}}(\Gamma)$ continuously into $H^s(\Gamma)$ for every $s\in R$, in particular for $s=1$. On the other hand, $I-P_h\colon H^1(\Gamma)\to H^{\frac{1}{2}}(\Gamma)$ is bounded by

$$\|I-P_h\|_{1;\frac{1}{2}}\leqq c\,h^{\frac{1}{2}}.$$

Therefore

$$\|(I-P_h)K\|_{\frac{1}{2};\frac{1}{2}}\leqq c\,\|I-P_h\|_{1;\frac{1}{2}}\leqq c\,h^{\frac{1}{2}}.$$

For the second term note that for all $v\in H^{\frac{1}{2}}(\Gamma)$ we have $G'(\xi_h)v\in L^2(\Gamma)$ and $S(G'(\xi_h)v)\in H^1(\Gamma)$ due to the mapping properties of S. Now just by the same argument as above we get

$$\|(I-P_h)SG'(\xi_h)\|_{\frac{1}{2};\frac{1}{2}}\leqq c\,h^{\frac{1}{2}}$$

and the constant is independent of ξ_h due to (10). Hence, (47) is valid, which completes the proof.

Now Lemma 5 yields immediately the desired convergence result by standard arguments.

Theorem 6. *Let $f \in H^{-\frac{1}{2}}(\Gamma)$ and $u \in H^{\frac{1}{2}}(\Gamma)$ be the solution of* (8). *Then the Galerkin solution exists uniquely for all $0 < h \leqq h_0$ and it furnishes the quasioptimality estimate*

$$\|u - u_h\|_{\frac{1}{2}} \leqq c \inf_{\psi \in S_h^d(\Xi)} \|u - \psi\|_{\frac{1}{2}}. \tag{48}$$

Further, for $f \in H^{s-1}(\Gamma)$, $\frac{1}{2} \leqq s \leqq 2$, we get by the approximation properties

$$\|u - u_h\|_t \leqq c h^{s-t} \|u\|_s, \tag{49}$$

where $t \leqq s$ and $t < d + \frac{1}{2}$.

Proof. The uniqueness and the quasioptimality estimate (48) are direct consequences of Lemma 5. The asymptotic error estimate (49) follows from the regularity of the solution (Theorem 3) and from the approximation and inverse properties of the spline spaces.

5. A Numerical Example

Here we present some numerical results to illustrate the theoretical error analysis. We consider the potential problems

$$\Delta u = 0, \qquad \text{in } \Omega \tag{50}$$

$$-\frac{\partial u}{\partial n} = u + \sin(u) - f \qquad \text{on } \Gamma, \tag{51}$$

or

$$-\frac{\partial u}{\partial n} = |u| u^3 - f \qquad \text{on } \Gamma, \tag{52}$$

where the domain is $\Omega = \{x \in \mathbf{R}^2 \,|\, |x| < 0.4\}$. Clearly, the first nonlinearity satisfies our assumption (10) in Chapter 2, whereas the second is unbounded. Hence, for any $f \in H^{\frac{1}{2}}(\Gamma)$, the nonlinear boundary integral Eq. (12),

$$(I - K)u + S(u + \sin u) = Sf \tag{53}$$

admits a unique solution $u \in H^{\frac{1}{2}}(\Gamma)$ as well as

$$(I - K)u + S(\chi(u)|u|u^3) = Sf \tag{54}$$

where $\chi(u)$ denotes an even C_0^∞ cut-off function which is nonnegative and identical 1 on a sufficiently large symmetric interval. In the Galerkin approximation we have used piecewise constants as trial functions. The discrete nonlinear system of Eqs. (36) is then solved by Newton's iteration.

In our computational examples we choose

$$Sf(x) \equiv 1 - \pi + 0.8\,\pi \log(0.4)(1 + \sin(1)). \tag{55}$$

	N	Iterations	L^2 error	Order
Example 1	10	3	$4.5 \cdot 10^{-3}$	
				1.4
	20	7	$1.7 \cdot 10^{-3}$	
				1.9
	40	5	$4.5 \cdot 10^{-4}$	
Example 2	10	3	$1.9 \cdot 10^{-3}$	
				2.0
	20	7	$4.8 \cdot 10^{-4}$	
				2.0
	40	19	$1.2 \cdot 10^{-4}$	

The corresponding exact solution is then for both examples $u \equiv 1$.

In the above two tables we list the number of grid points, of iterations, the L_2 errors and approximate orders of convergence which are in excellent agreement with (49) for $s=2$ and $t=0$.

Acknowledgements. The authors wish to express their gratitude to Dr. B. Kawohl for his advice and helpful remarks to the initiation of this work and to Mr. G. Schneider for programming and carrying out the numerical experiments. The first of the authors wants to express his gratitude to the University Stuttgart and the German Academic Exchange Service for their support.

References

1. Adams, R.A.: Sobolev Spaces. New York: Academic Press 1975
2. Amann, H.: Zum Galerkin-Verfahren für die Hammersteinsche Gleichung. Arch. Rat. Mech. Anal. **35**, 114–121 (1969)
3. Amann, H.: Über die Konvergenzgeschwindigkeit des Galerkin-Verfahrens für die Hammersteinsche Gleichung. Arch. Rat. Mech. Anal. **37**, 143–153 (1970)
4. Anselone, P.M.: Collectively Compact Operator Approximation Theory. Englewood Cliffs, New Jersey: Prentice Hall 1971
5. Arnold, D.N., Wendland. W.L.: On the asymptotic convergence of collocation methods. Math. Comput. **41**, 349–381 (1983)
6. Arnold, D.N., Wendland, W.L.: The convergence of spline collocation for strongly elliptic equations on curves. Numer. Math. **47**, 317–343 (1985)
7. Babuška, I., Aziz, A.K.: Survey lectures on the mathematical foundation of the finite element method. In: Aziz, A.K. (ed.) The Mathematical Foundation of the Finite Element Method with Applications to Partial Differential Equations, pp. 3–359. New York: Academic Press 1972
8. Barbu, V.: Nonlinear Semigroups and Differential Equations in Banach Spaces. Leiden: Noordhoff International Publishing 1976
9. Bialecki, R., Nowak, A.J.: Boundary value problems in heat conduction with nonlinear material and nonlinear boundary conditions. Appl. Math. Model. **5**, 417–421 (1981)
10. Brebbia, C.A., Telles, J.C.F., Wrobel, L.C.: Boundary Element Technique. Berlin Heidelberg New York: Springer 1984
11. Browder, F.E.: Nonlinear elliptic boundary value problems I. Bull. Am. Math. Soc. **69**, 862–875 (1963)
12. Carleman, T.: Über eine nichtlineare Randwertaufgabe bei der Gleichung $\Delta u = 0$. Math. Z. **9**, 35–43 (1921)
13. Costabel, M.: Boundary integral operators on Lipschitz domains: elementary results. TH Darmstadt Preprint No. 898, (1985) (to appear in SIAM J. Math. Anal.)
14. Elschner, J., Schmidt, G.: On spline interpolation in periodic Sobolev spaces. P-Math-01/83, Akad. Wiss. DDR, Inst. Math., Berlin 1983
15. Frehse, J., Rannacher, R.: Asymptotic L^∞-estimates for linear finite element approximation of quasilinear boundary value problems. SIAM J. Numer. Anal. **15**, 418–431 (1978)

16. Fučik, S., Nečas, J., Souček, J., Souček, V.: Spectral Analysis of Nonlinear Operators. (Lect. Notes Math., vol. 346) Berlin Heidelberg New York: Springer 1973
17. Haduong, T.: La methode de Schenck pour la resolution numerique du probleme de radiation acoustique. Bull. Dir. Etudes Recherces, Ser. C, Math. Inf. Service Inf. Math. Appl. **2**, 15–50 (1979)
18. Hsiao, G.C., Wendland, W.L.: A finite element method for some integral equations of the first kind. J. Math. Anal. Appl. **58**, 449–481 (1977)
19. Hsiao, G.C., Wendland, W.L.: The Aubin-Nitsche lemma for integral equations. J. Integral Eqs. **3**, 299–315 (1981)
20. Jaswon, M.A., Symm, G.T.: Integral Equation Methods in Potential Theory and Elastostatics. London: Academic Press 1977
21. Johnson, C., Thomee, V.: Error estimates for a finite element approximation of a minimal surface. Math. Comput. **29**, 343–349 (1975)
22. Kelmanson, M.A.: Solution of nonlinear elliptic equations with boundary singularities by an integral equation method. J. Comput. Phys. **56**, 244–283 (1984)
23. Lichtenstein, L.: Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integrodifferentialgleichungen. Berlin Heidelberg New York: Springer 1931
24. Minty, G.: Monotone (nonlinear) operators in Hilbert spaces. Duke Math. J. **29**, 341–346 (1962)
25. Minty, G.: On a "monotonicity" method for the solution of nonlinear equations in Banach spaces. Proc. Natl. Acad. Sci. **50**, 1041–1083 (1964)
26. Minty, G.: On the solvability of nonlinear functional equation of "monotonic" type. Pacific J. Math. **14**, 249–255 (1964)
27. Mittelmann, H.D.: Die Methode der Finiten Elemente zur numerischen Lösung von Randwertproblemen quasilinearer elliptischer Differentialgleichungen. Habilitationsschrift, Technische Hochschule Darmstadt (1976)
28. Natanson, I.P.: Theorie der Funktionen einer reellen Veränderlichen. Berlin: Akademie-Verlag 1969
29. Nečas, J.: Remark on the Fredholm alternative for nonlinear operators with application to nonlinear integral equations of generalized Hammerstein type. Commentationes Mathematicae Universitatis Carlolinae **13**, 109–120 (1972)
30. Nedelec, J.C.: Approximation des Equations Integrales en Mecanique et en Physique. Lecture Notes, Centre Math. Appl., Ecole Polytechnique, Palaiseau, France (1977)
31. Nedelec, J.C., Planchard, J.: Une methode variationelle d'elements finis pour la resolution numerique d'un probleme exterieur dans R^3. R.A.I.R.O. **7**, 105–129 (1973)
32. Pascali, D., Sburlan, S.: Nonlinear Mappings of Monotone Type. Bucarest: Siijthoff and Noordhoff 1978
33. Petryshyn, W.V.: Projection methods in nonlinear numerical functional analysis. J. Math. Mech. **17**, 353–372 (1967)
34. Petryshyn, W.V.: On the approximation-solvability of equations involving a-proper and pseudo-a-proper mappings. Bull. Am. Math. Soc. **81**, 223–312 (1975)
35. Rizzo, F.J.: An integral equation approach to boundary value problems of classical elastostatics. Quart. Appl. Math. **25**, 83–95 (1967)
36. Saranen, J., Wendland, W.L.: On the asymptotic convergence of collocation methods with spline functions of even degree. Math. Comput. **45**, 91–108 (1985)
37. Schmidt, G.: On spline collocation methods for boundary integral equations in the plane. Math. Methods Appl. Sci. **7**, 74–89 (1985)
38. Schmidt, G.: The convergence of Galerkin and collocation methods with splines for pseudodifferential equations on closed curves. ZAA **3**, 371–384 (1984)
39. Wendland, W.L.: On the embedding method for semilinear first order elliptic systems and related topics finite element methods. In: Wacker, H. (ed.) Continuation Methods, pp. 277–336. New York: Academic Press 1978
40. Wendland, W.L.: On some mathematical aspects of boundary element methods for elliptic problems. In: Whiteman, J.R. (ed.) The Mathematics of Finite Elements and Applications V, pp. 230–257. London: Academic Press 1985
41. Zeidler, E.: Vorlesungen über nichtlineare Funktionalanalysis I and II. Teubner-Texte zur Mathematik, Leipzig (1977)

Received September 17, 1987/December 16, 1987