Invent. math. 92, 243–254 (1988)

Inventiones mathematicae

$\mathbb{Z}/k$-Manifolds and families of Dirac operators

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To my teacher Isadore M. Singer

Summary. The index of a family of a family of Dirac operators is a K-Theory element in the parameter space. Sullivan's $\mathbb{Z}/k$-manifolds are used to detect this index completely. For the first Chern class this gives a topological interpretation of Witten's global anomaly. The relationship with the geometry of the index bundle is considered.

Witten's "global anomaly" formula in physics [W] has prompted some recent work related to the Atiyah-Singer Index Theorem[1]. The anomaly measures the nontriviality of the determinant line bundle $\mathscr{L}$ of a family of Dirac operators [AS2], which is given topologically by the first Chern class. Over the reals there is a local formula for the Chern class of $\mathscr{L}$ in terms of characteristic classes. The *global* anomaly captures the integral information in $c_1(\mathscr{L})$ beyond the real information. Sullivan's $\mathbb{Z}/k$-manifolds provide the appropriate topological tool to detect this information. The purpose of this paper is to explain this observation, which is encoded in (2.4). Geometrically, there is a more profound interpretation of the anomaly in terms of the geometry of $\mathscr{L}$ – the determinant bundle carries a natural connection, and the anomaly is its curvature and holonomy. That point of view has been developed elsewhere[2]; our goal here is to elucidate the connection between the geometry and the topology.

In § 1 we discuss the direct image map in K-theory for $\mathbb{Z}/k$-manifolds. Recall that for closed Spin^c manifolds the direct image (to a point) can be expressed by a cohomological formula involving characteristic classes. For $\mathbb{Z}/k$-manifolds this is not possible due to the presence of denominators in this cohomology expression. Rather, there is an analytic interpretation of the direct image involving the η-invariant of Atiyah-Patodi-Singer [APS]. Our proof that this analytic expression computes the direct image only covers special cases (where a certain

The author is partially supported by an NSF Postdoctoral Research Fellowship

[1] Lest the reader shake with unfounded trepidation, we immediately reassure him that ignorance of Physics may enhance, rather than hinder, his understanding of this paper

[2] Other geometric ramifications of Witten's global anomaly been pursued by Cheeger [C1], [C2] and Singer [S]

bordism element vanishes). The formula in general follows from an index theorem for $\mathbb{Z}/k$-manifolds, which is joint work with Richard Melrose [FM].

The multiplicative axiom for the direct image provides the link between the index bundle of a Dirac family and these $\mathbb{Z}/k$ periods of K-theory elements. Our application in § 2 is to the determinant line bundle $\mathscr{L}$ i.e., to Witten's formula. Topologically, this detects the integral first Chern class $c_1(\mathscr{L})$ completely. Geometrically, Witten's formula computes the holonomy of the canonical connection on $\mathscr{L}$ and we recover this holonomy theorem for torsion loops from the curvature formula via our topological considerations.

We make some brief remarks about the higher Chern classes in Chern classes in § 3. The direct images considered here do not directly calculate Chern classes. Rather, they completely determine a K-Theory class. On the analytic side this suggests a possible generalization of Witten's global anomaly formula to higher dimensional manifolds[3]. We indicate the statement.

This paper is dedicated to Iz Singer. As my advisor he taught me nearly all I know about mathematics. Through his lectures, writings, and private conversations he has been the primary influence on my work. The discussion here, in which various index theorems conspire to explain global anomalies, is a tribute to that influence.

Acknowledgements. The author thanks Gunnar Carlsson, Mike Freedman, and Ron Stern for directing him to Sullivan's work. He is grateful to John Morgan, Graeme Segal, and Dennis Sullivan for helpful topological counselling.

1. Direct images

To begin we review the direct image map in K-theory for closed manifolds. Let Q be a closed even dimensional smooth manifold, and suppose the tangent bundle of Q is endowed with a Spin^c structure. The Spin^c structure provides an orientation in K-theory. Thus there is a direct image map

$$\pi_!^Q\colon K(Q) \to \mathbb{Z}. \tag{1.1}$$

Let $i\colon Q \hookrightarrow S^{2N}$ be an embedding of Q in an even dimensional sphere. Choose a tubular neighborhood $v \hookrightarrow S^{2N}$ of Q in S^{2N}. Then the K-orientation yields a Thom isomorphism $K(Q) \simeq K(v)$, where $K(v)$ denotes K-theory with compact supports. Composition with extension by zero gives a map $i_!\colon K(Q) \to K(S^{2N})$. Bott periodicity implies $K(S^{2N}) = \mathbb{Z} \oplus \mathbb{Z}$; the first $\mathbb{Z}$ is the augmentation while the second $\mathbb{Z}$ is $\tilde{K}(S^{2N})$. The direct image $\pi_!^Q$ is $i_!$ followed by projection onto $\tilde{K}(S^{2N}) = \mathbb{Z}$. (Compare [AS(I), § 1].)

Suppose now that $\pi^X\colon Q \to M$ is fibration of manifolds with typical fiber an even dimensional closed manifold X. Assume that this family of manifolds (or rather the vertical tangent bundle $T(Q/M)$) has a Spin^c structure. Then the preceding construction can be carried out fiberwise to give a direct image

$$\pi_!^Q\colon K(Q) \to K(M). \tag{1.2}$$

[3] We understand that Singer has also announced such a generalization. Cheeger mentions the higher dimensional case in [C2]

Added in proof: Recently, Bismut and Cheeger [BC] proved related formulas

(Compare [AS(IV)].) If M is also an even dimensional closed Spin^c manifold, then Q inherits a Spin^c structure, and the multiplicative property of the Thom isomorphism implies that the diagram

(1.3)
$$\begin{array}{ccc} K(Q) & & \\ {\scriptstyle \pi_!^X}\downarrow & \searrow{\scriptstyle \pi_!^Q} & \\ K(M) & \xrightarrow[\pi_!^M]{} & \mathbb{Z} \end{array}$$

commutes. Briefly,

(1.4)
$$\pi_!^Q = \pi_!^M \circ \pi_!^X.$$

This is essentially the multiplicative axiom for the index [AS(I), § 4].

There is a cohomological formula for the direct image (1.1); it is derived by a standard computation comparing the Thom isomorphisms in K-theory and cohomology (e.g. [AS(III), § 1]). Let Q be a closed even dimensional Spin^c manifold and $\omega \in H^2(Q)$ the first Chern class of the Spin^c structure defined by the homomorphism Spin$^c(n) \to U(1)$. Then for $a \in K(Q)$ we have

(1.5)
$$\pi_!^Q(a) = \hat{A}(Q)\, e^{\omega/2}\, \mathrm{ch}(a)[Q],$$

where the right hand side is interpreted in rational cohomology. In the special case of a Riemann surface Σ endowed with a Spin structure ($\omega = 0$), the K-theory direct image coincides with the direct image in ordinary cohomology:

(1.6)
$$\pi_!^\Sigma(a) = c_1(a)[\Sigma].$$

The direct image construction for K-oriented closed manifolds computes $\mathbb{Z}$ periods of a K-theory class. Sullivan [S1], [S2], [MS] introduced $\mathbb{Z}/k$-manifolds to compute $\mathbb{Z}/k$ periods.

(1.7) **Definition.** A (closed) $\mathbb{Z}/k$-manifold consists of

(1) a compact manifold Q with boundary;
(2) a closed manifold P;
(3) a decomposition $\partial Q = \coprod_{i=1}^{k} (\partial Q)_i$ of the boundary of Q into k disjoint manifolds, and diffeomorphisms $\theta_i \colon P \xrightarrow{\ \sim\ } (\partial Q)_i$.

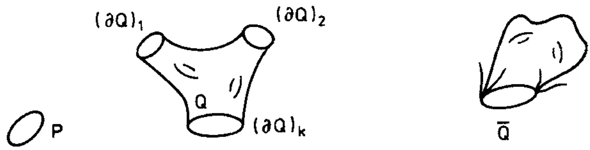

Fig. 1

The identification space $\bar{Q}$, formed by attaching Q to P via θ_i, is more properly called the $\mathbb{Z}/k$-manifold. Of course, $\bar{Q}$ is singular at the identification points. If Q and P are compatibly oriented, then $\bar{Q}$ carries a fundamental class $[\bar{Q}]\in H_*(\bar{Q};\mathbb{Z}/k)$. Similarly, compatible K-orientations determine a fundamental class in K-theory, which we proceed to describe in terms of the smooth manifolds Q and P.

First, we describe the K-theory of the collapsed space $\bar{Q}$. By general principles there is an exact sequence associated to $P\subset\bar{Q}$:

$$\begin{array}{ccccc} K(\bar{Q},P) & \longrightarrow & K(\bar{Q}) & \longrightarrow & K(P) \\ \uparrow & & & & \downarrow \\ K^{-1}(P) & \longleftarrow & K^{-1}(\bar{Q}) & \longleftarrow & K^{-1}(\bar{Q},P). \end{array} \tag{1.8}$$

We can identify the pair $(\bar{Q},P)$ with $(Q,\partial Q)$, so that $K(\bar{Q},P)$ is the K-group (with compact supports) of the interior of Q. While (1.8) gives some feel for $K(\bar{Q})$, a more explicit description can be given directly in terms of vector bundles. Define

$$\begin{aligned}\operatorname{Vect}(\bar{Q})=\{\langle E,F,\tilde{\theta}_i\rangle: & \ E\to Q,\ F\to P \text{ are complex vector bundles,}\\ & \tilde{\theta}_i\colon F\xrightarrow{\ \simeq\ } E|_{(\partial Q)_i} \text{ is an isomorphism lifting } \theta_i\}.\end{aligned}$$

This is an abelian semigroup, and the corresponding K-group is $K(\bar{Q})$.

A Spin^c structure on $\bar{Q}$ consists of Spin^c structures on Q and P compatible with θ_i. Now suppose that Q is even dimensional. Then the Spin^c structures determine a direct image map

$$\pi_!^{\bar{Q}}\colon K(\bar{Q})\to\mathbb{Z}/k \tag{1.9}$$

(cf. [S2, §6]). Let (D^{2N},S^{2N-1}) be the $2N$-ball with $\mathbb{Z}/k$ acting on S^{2N-1} via the embedding $\mathbb{Z}/k\subset U(1)\to U(1)\times U(N-1)\subset U(N)$. The quotient of (D^{2N},S^{2N-1}) by that action is the *Moore space* M_k – its only reduced homology is $\mathbb{Z}/k$ in dimension $2N$. For N sufficiently large there is an embedding $i\colon(Q,\partial Q)\hookrightarrow(D^{2N},S^{2N-1})$ respecting the $\mathbb{Z}/k$ action on the boundary. Represent an element $a\in K(\bar{Q})$ as a difference of elements in $\operatorname{Vect}(\bar{Q})$. Choose a tubular neighborhood ν of Q in D^{2N} so that $\nu|_{S^{2N-1}}$ consists of k disjoint diffeomorphic pieces. The Clifford multiplication determined by the Spin^c structure on the normal bundle defines an explicit Thom class [ABS], and allows us to define a semigroup homomorphism $\operatorname{Vect}(\bar{Q})\to\operatorname{Vect}(\bar{\nu})$. That the induced map on K-theory is an isomorphism follows from (1.8), the ordinary Thom isomorphisms, and the 5-lemma. Extension by zero is a map $K(\bar{\nu})\to K(M_k)$. Denote the image of $\bar{a}\in K(\bar{Q})$ under this process by $i_!(\bar{a})\in K(M_k)$. Finally, $K(M_k)=\mathbb{Z}\oplus\mathbb{Z}/k$, and the direct image $\pi_!^{\bar{Q}}(\bar{a})$ is the projection of $i_!(\bar{a})$ onto $\mathbb{Z}/k$.

Since this direct image is defined by the Thom isomorphism, it enjoys a multiplicative property. A fibering $\pi^X\colon\bar{Q}\to\bar{M}$ with X closed is, of course, a pair of compatible fiberings $\pi^X\colon Q\to M$ and $\pi^X\colon P\to N$. If compatible Spin^c structures are given along the fibers, then

$$\pi_!^X\colon K(\bar{Q})\to K(\bar{M}) \tag{1.10}$$

is defined. If, in addition, a Spin^c structure is given on $\bar{M}$, we have

(1.11) $$\pi_!^{\bar{Q}} = \pi_!^{\bar{M}} \circ \pi_!^X.$$

At this point the precise analogy with closed manifolds breaks down; there is no cohomological formula for the direct image. Essentially this is because $\mathbb{Z}/k$-manifolds detect *integral* information in the K-theory class, whereas (1.5) is an expression in *rational* cohomology – it involves denominators. Instead we use more precise geometric data and analytic invariants. Thus let (Q, P) be a $\mathbb{Z}/k$-manifold as in (1.7), and suppose that Q and P carry Riemannian metrics. We assume that the metric on Q is a product near the boundary, and that it restricts on each $(\partial Q)_i$ to the given metric on P (using the identification θ_i). Denote the curvature of Q by $\Omega^{(Q)}$. We also give a geometric realization of a K-theory class on $\bar{Q}$ by Hermitian vector bundles $E \to Q$ and $F \to P$ with unitary connections and identifications $\tilde{\theta}_i$ which preserve the metric and connection. The metric and connection on E are a product near the boundary. Let $\Omega^{(E)}$ be the curvature of E. The pair of bundles $\langle E, F, \tilde{\theta}_i \rangle$ represents an element $[\bar{E}] \in K(\bar{Q})$.

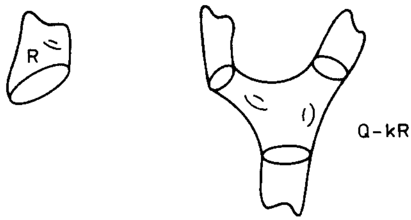

Fig. 2

Suppose that (Q, P) carries a Spin^c structure. There is an associated characteristic line bundle, the first Chern class of the Spin^c structure, and we assume that it is endowed with a metric and compatible connection. Let $\omega \in \Omega^2(Q)$ denote its curvature. Now from the induced data on P we construct a self-adjoint Dirac operator acting on spinor fields. Atiyah-Patodi-Singer [APS] define a spectral invariant ξ_P of this operator[4].

(1.12) **Theorem.** *The direct image is given by the analytic expression*

$$\pi_!^{\bar{Q}}([\bar{E}]) \equiv \frac{1}{k} \int_Q \hat{A}(\Omega^{(Q)})\, e^{\omega/2}\, \mathrm{ch}(\Omega^{(E)}) - \xi_P \pmod 1.$$

Here we view $\mathbb{Z}/k \cong \mathbb{Z}\left[\frac{1}{k}\right] \Big/ \mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. The proof we give here only covers the case where a certain torsion bordism element vanishes. Since the relevant bordism group has torsion (due to torsion in $\Omega_*^{\mathrm{Spin}^c}$), our proof does not apply in general. A different proof for the general case will be given in [FM], where we develop an index theorem for $\mathbb{Z}/k$-manifolds. Both proofs use the Atiyah-Patodi-Singer index theorem for manifolds with boundary [APS(I)].

[4] It is $\frac{1}{2}(\eta + h)$ where η is the η-invariant and h is the dimension of the kernel

Proof for special cases: The bundle $F \to P$ represents an element of order k in the bordism group $\Omega_*^{\mathrm{Spin}^c}(Gr)$ of some complex Grassmannian Gr. We only prove the proposition when this element vanishes. Suppose, therefore, that $F \to P$ bounds a bundle $G \to R$ for some Spin^c manifold R. Extend the differential geometric data to R, keeping a product structure near the boundary. Pasting k copies of R onto Q we obtain a closed Spin^c manifold $Q-kR$ and a bundle $E-kG$. Then (1.5) together with Chern-Weil Theory yield

$$(1.13)\quad \pi_!^{Q-kR}([E-kG]) = \int_Q \hat{A}(\Omega^{(Q)})\, e^{\omega/2}\, \mathrm{ch}(\Omega^{(E)}) - k \int_R \hat{A}(\Omega^{(R)})\, e^{\omega/2}\, \mathrm{ch}(\Omega^{(G)}).$$

The index theorem for manifolds with boundary implies that the integral over R is congruent (mod 1) to the ξ-invariant of the boundary P. After dividing by k the left hand side of (1.13) reduced (mod 1) is $\pi_!^{\bar{Q}}([\bar{E}])$, and the proposition follows.

Remark. This proof applies to *all* stably almost complex $\mathbb{Z}/k$-manifolds since Ω_*^U, hence also $\Omega_*^U(Gr)$, is torsion free. Stably almost complex manifolds suffice if we use the $\mathbb{Z}/k$-manifolds as probes to measure a K-Theory class. However, our application in the next section requires the more general Spin^c manifolds.

Although the K-theory direct image for $\mathbb{Z}/k$-manifolds has no cohomological interpretation in general, in two dimensions we have the following analog of (1.6).

(1.14) **Proposition.** *Let (Σ, S) be a $\mathbb{Z}/k$-manifold with Σ a Riemann surface and S a circle. Fix a spin structure on Σ and let S have the induced spin structure*[5]*. Then for $\bar{a} \in \tilde{K}(\bar{\Sigma})$*

$$\pi_!^{\bar{\Sigma}}(\bar{a}) = c_1(\bar{a})[\bar{\Sigma}],$$

where $c_1(\bar{a}) \in H^2(\bar{\Sigma})$ is the integral first Chern class of $\bar{a}$, and $[\bar{\Sigma}] \in H_2(\bar{\Sigma}; \mathbb{Z}/k)$ is the fundamental class.

Proof. Write $S = \partial R$ (R a 2-disk). Represent $\bar{a}$ by a virtual bundle which restricts to a $\mathbb{Z}/k$ invariant trivial bundle in a collar of the boundary. This determines a trivial bundle in a collar of ∂R. Extend to a trivial bundle over R, thus defining $u \in K(\Sigma - kR)$. Now (1.6) shows that

$$\pi_!^{\Sigma-kR}(u) = c_1(u)[\Sigma - kR],$$

since the Spin^c structure comes from a Spin structure. Divide by k and reduce (mod 1):

$$\pi_!^{\bar{\Sigma}}(\bar{a}) \equiv \frac{1}{k}\, \pi_!^{\Sigma-kR}(u) \equiv \frac{1}{k}\, c_1(u)[\Sigma - kR] \equiv c_1(\bar{a})[\bar{\Sigma}] \qquad (\mathrm{mod}\ 1).$$

2. Global anomalies

From a topological standpoint the anomaly is the topological equivalence class of the determinant line bundle of a family of Dirac operators. Atiyah and Singer

[5] It is the nontrivial double cover of the frame bundle

[AS2] discovered this link and computed the determinant bundle over the reals, that is, its *real* first Chern class. The physicists call this the *local* anomaly, as the result can be expressed in terms of a local differential form on the parameter space. Witten [W] introduced *global* anomalies expressed in terms of (nonlocal) η-invariants. Here we explain that Witten's invariant detects the *integral* first Chern class of the determinant bundle, or, more precisely, the integral information beyond the real information. We emphasize that global anomalies measure much more precise geometry of the determinant line bundle; this viewpoint is explored in [BF1], [BF2], [F]. On the other hand the topological interpretation suffices when the first homology group of the parameter space is torsion, as is the case Witten's original paper.

Let $Z \xrightarrow{X} Y$ be a fibering of manifolds with a Spin^c structure along the fibers. Also, suppose $E \to Z$ is a vector bundle. This data does not really specify a family of Dirac operators along the fibers; for that we need metrics and connections. However, the family of symbols is well-defined, and the topology of the *index bundle* is independent of the particular metrics and connections. Fix a realization of these symbols by a family of Dirac operators D_y. Then the index bundle is $[\ker D_y] - [\operatorname{coker} D_y]$. While in general neither the kernels nor the cokernels piece together to a vector bundle over Y (the rank may jump), their difference makes sense in the K-Theory of Y. The Atiyah-Singer Index Theorem for Families [AS(IV)] states that in K-Theory this index equals the direct image $\pi_!^X([E])$.

The determinant line of a single Dirac operator D is $(\det \ker D)^* \otimes (\det \operatorname{coker} D)$. Over the parameter space Y of a family of Dirac operators these lines patch together to form a line bundle $\mathscr{L} \to Y$, the determinant line bundle. Topologically, this corresponds to the map $BU \to \mathbb{CP}^\infty$, i.e., the determinant is minus the first Chern class of the index. (The minus sign comes because our definition of the determinant involves the dual of the kernel rather than the dual of the cokernel.) The Atiyah-Singer formula implies $c_1(\mathscr{L}) = -c_1(\pi_!^X([E]))$.

The real part of the first Chern class is detected by mapping closed Riemann surfaces Σ to Y. Such a map induces a fibering $Q \xrightarrow{X} \Sigma$. As noted in (1.6) the direct image map on a Riemann surface is $\pi_!^\Sigma(u) = c_1(u)[\Sigma]$ if the Spin^c structure comes from a Spin structure; this follows easily from (1.5). Therefore, the multiplicative axiom for the index (1.4) implies

(2.1) $$c_1(\mathscr{L})[E] = -\pi_!^\Sigma \pi_!^X([E]) = -\pi_!^Q([E]).$$

Here we still denote the pulled back bundle by E. The cohomology formula (1.5) yields

(2.2) $$c_1(\mathscr{L})[E] = -\hat{A}(Q)\, e^{\omega/2} \operatorname{ch}([E])[Q].$$

The right hand side can be expressed as an integral of a differential form over Q via Chern-Weil Theory if appropriate metrics and connections are specified. This is the local expression which was derived by Atiyah and Singer. These $\mathbb{Z}$ periods of $c_1(\mathscr{L})$ (over all choices of Σ and of maps) determine the real part of this cohomology class.

The rest of the information in $c_1(\mathscr{L})$ is determined by evaluating the Chern class on $\mathbb{Z}/k$ cycles. (This follows from the universal coefficient theorem in cohomology. That compatible $\mathbb{Z}$ periods and $\mathbb{Z}/k$ periods actually determine a cohomology class is proved in [MS, §2]. We discuss this further in §3.) Now all homology classes in two dimensions are realized by Riemann surfaces, and so it suffices to consider $\bar{\Sigma} \to Y$ over all $\mathbb{Z}/k$-surfaces $\bar{\Sigma}$ and all maps. Such a map induces a fibering $\bar{Q} \xrightarrow{X} \bar{\Sigma}$, and E pulls back to a bundle $\bar{E} \to \bar{\Sigma}$. Proposition (1.14) states that the direct image on $\bar{\Sigma}$ evaluates the first Chern class, and so the multiplicative axiom (1.11) implies

$$c_1(\mathscr{L})[\bar{E}] = -\pi_!^{\bar{\Sigma}} \pi_!^X([\bar{E}]) = -\pi_!^{\bar{Q}}([\bar{E}]). \tag{2.3}$$

The right hand side can be reexpressed analytically via Proposition 1.12:[6]

$$c_1(\mathscr{L})[\bar{\Sigma}] \equiv \xi_P - \frac{1}{k} \int_Q \hat{A}(\Omega^{(Q)})\, e^{\omega/2} \operatorname{ch}(\Omega^{(E)}) \pmod 1. \tag{2.4}$$

This is Witten's global anomaly formula.

We emphasize that our account here is topological. In particular, we necessarily consider only *torsion* loops $S^1 \to Y$, while in general $H_1(Y)$ contains elements of infinite order. From a geometric point of view Witten's formula measures the holonomy of a canonical connection on the determinant line bundle. The holonomy around nontorsion loops (say for a flat connection) is not captured by the first Chern class; it depends not only on the topology of the bundle, but also on the specific connection chosen. We now recall briefly this geometric interpretation of the anomaly and show how our considerations fit in with the holonomy theorem of [BF2].

As a preliminary, we reexpress the direct image on Riemann surfaces in terms of curvature. Suppose Σ is a closed Riemann surface and $\mathscr{L} \to \Sigma$ a Hermitian line bundle with unitary connection. Let $\Omega^{(\mathscr{L})}$ denote its curvature. Then Chern-Weil Theory implies

$$c_1(\mathscr{L})[\Sigma] = \frac{i}{2\pi} \int_\Sigma \Omega^{(\mathscr{L})}. \tag{2.5}$$

Similarly, if (Σ, S) is a $\mathbb{Z}/k$-surface (as in Proposition 1.14), and $\mathscr{L} \to \bar{\Sigma}$ a Hermitian line bundle with unitary connection, then

$$c_1(\mathscr{L})[\bar{\Sigma}] \equiv \frac{1}{k} \cdot \frac{i}{2\pi} \int_\Sigma \Omega^{(\mathscr{L})} + \frac{i}{2\pi} \ln \operatorname{hol}(S) \pmod 1. \tag{2.6}$$

Here $\operatorname{hol}(S) \in U(1)$ is the holonomy of the connection around S. Formula (2.6) is a special case of Proposition 1.12, after we observe that the ξ-invariant on a circle is the logarithm of the holonomy (up to a factor). There is a simpler, more direct proof based on the fact that the *torsionfree* space $\mathbb{CP}^\infty$ classifies line bundles with connection.

[6] Our proof in §1 suffices when X is a stably almost complex manifold. However, we want to consider the more general case where X is Spin^c, which is why we require a different proof

In the geometric situation we start with a smooth family of manifolds $Z \xrightarrow{X} Y$, a metric and Spin^c structure along the fibers, a projection $TZ \to T(Z/Y)$, and a Hermitian vector bundle $E \to Z$ with unitary connection. Let ω be the curvature of a connection on the characteristic line bundle of the Spin^c structure, and denote the curvature of E by $\Omega^{(E)}$. The given data yields a connection of $T(Z/Y)$, and naturally we use the symbol $\Omega^{(Z/Y)}$ for its curvature. Now the determinant line bundle $\mathscr{L} \to Y$ is smooth, carries a metric (constructed by Quillen), and the main result of [BF1] states that $\mathscr{L}$ also carries a compatible connection whose curvature is the 2-form

(2.7) $$\Omega^{(\mathscr{L})} = [2\pi i \int_X \hat{A}(\Omega^{(Z/Y)})\, e^{\omega/2}\, \mathrm{ch}(\Omega^{(E)})]_{(2)}.$$

Loosely stated, the index theorem for families, at least for c_1, holds at the level of differential forms. The proof of (2.7) uses Quillen's transgression formula for superconnections [Q] and Bismut's proof of the index theorem for families [B].

Taking (2.7) as a given, we derive the formula for the holonomy of $\mathscr{L}$ around loops $S \to Y$ which are torsion in $H_1(Y)$. Any loop induces a fibering $P \xrightarrow{X} S$ by pullback. Introduce an arbitrary metric $g^{(S)}$ on the circle, and fix the bounding spin structure on S.[7] This determines a metric and Spin^c structure on P, and so a self-adjoint Dirac operator. Now introduce a factor ε which scales $g^{(S)}$ to $g^{(S)}/\varepsilon^2$.

(2.8) **Theorem** [BF2, Theorem 3.18]. *We have*

$$\mathrm{hol}(S) = \lim_{\varepsilon \to 0} e^{-2\pi i \xi_\varepsilon}.$$

Here ξ_ε is the ξ-invariant of the Dirac operator for the scaled metric.

Physicists term $\varepsilon \to 0$ the *adiabatic limit.*

Proof for torsion loops. Since we assume that $S \to Y$ is torsion, we can find an integer k, a $\mathbb{Z}/k$-manifold (Σ, S), and a map $\bar{\Sigma} \to Y$ which extends the given loop $S \to Y$. Let $\bar{Q} \xrightarrow{X} \bar{\Sigma}$ be the induced family. Extend the metric $g^{(S)}$ to a metric $g^{(\Sigma)}$ which is a product near the boundary, and fix a spin structure on Σ. The $\mathbb{Z}/k$-manifold $\bar{Q} = (Q, P)$ then inherits a Spin^c structure and family of metrics parametrized by ε. Equations (2.4), (2.6), and (2.7) combine to give

$$\begin{aligned} &-\int_\Sigma \int_X \hat{A}(\Omega^{(Q/M)})\, e^{\omega/2}\, \mathrm{ch}(\Omega^{(E)}) + \frac{i}{2\pi} \ln \mathrm{hol}(S) \\ &\equiv -\int_Q \hat{A}(\Omega_\varepsilon^{(Q)})\, e^{\omega/2}\, \mathrm{ch}(\Omega^{(E)}) + \xi_\varepsilon \qquad (\mathrm{mod}\ 1). \end{aligned}$$

[7] This is the spin structure which enters in Proposition 1.14. In [BF2] we use the nonbounding spin structure, and correspondingly the holonomy formula there differs from (2.8) by a factor of $(-1)^{\mathrm{index}\, D}$. That factor can be derived by appealing to the flat index theorem of [APS(III)]

But a straightforward calculation in Riemannian geometry (cf. [BF2, §3j]) implies

$$\lim_{\varepsilon\to 0} \hat{A}(\Omega_\varepsilon^{(Q)}) = \hat{A}(\Omega^{(Q/M)})\,\hat{A}(\Omega^{(\Sigma)}) = \hat{A}(\Omega^{(Q/M)}).$$

Hence

$$\lim_{\varepsilon\to 0} \xi_\varepsilon = \frac{i}{2\pi}\ln \operatorname{hol}(S) \qquad (\text{mod } 1),$$

as desired.

3. Higher dimensions

The first Chern class is magical from many points of view. Topologically, the classifying space BU splits as a product $K(\mathbb{Z},2)\times BSU$; the first Chern class c_1 is the projection $BU\to K(\mathbb{Z},2)$. The determinant line bundle of a K-Theory element isolates c_1. By contrast, it is impossible to split off $K(\mathbb{Z},2j)$, $j\geqq 2$, so there is no natural way to isolate the higher c_j. A multiple of c_j does split off, though. Over $\mathbb{Q}$ we have

$$BU \sim K(\mathbb{Q},2)\times K(\mathbb{Q},4)\times K(\mathbb{Q},6)\times \dots,$$

the equivalence being induced by the Chern character. From another point of view, on a closed Riemann surface Σ the direct image $\pi_!^\Sigma(u)=c_1(u)[\Sigma]$ depends only on the K-Theory class u. However, for higher dimensional M the direct image $\pi_!^M(u)$ also involves the topology of M (cf. equation (1.5)) beyond the orientation class in homology. The same remarks apply to $\mathbb{Z}/k$-manifolds $\bar{\Sigma}$ and $\bar{M}$ (cf. (1.12) and (1.14)).

Consequently, when we use higher dimensional manifolds to probe a K-Theory class $u\in K(Y)$ – perhaps the index of a family of Dirac operators – then we cannot expect to compute $c_j(u)$, $j\geqq 2$, directly from direct images. On the other hand, the arguments of §2 demonstrate that direct images determine the first Chern class $c_1(u)$. The proper generalization is

(3.1) **Proposition.** *The (compatible)* $\mathbb{Z}$ *and* $\mathbb{Z}/k$ *periods over all* Spin^c *manifolds* $M\to Y$ *and* $\bar{M}\to Y$ *completely determine a K-Theory class* $u\in K(Y)$.

The same theorem holds in cohomology: a cohomology class is specified by its $\mathbb{Z}$ and $\mathbb{Z}/k$ periods [MS, §2]. Proposition 3.1 is essentially contained in [S2, §6]. We also need the fact that $\Omega_*^{\mathrm{Spin}^c}(Y;\mathbb{Z}/k)\to K_*(Y;\mathbb{Z}/k)$ is surjective; this follows from the theorem of Conner-Floyd relating bordism and K-Theory [CF]. Our interest is in the easy half of (3.1): If the $\mathbb{Z}$ and $\mathbb{Z}/k$ periods of a given K-theory class $u\in K(Y)$ vanish, then $u=0$. The converse, which states that compatible $\mathbb{Z}$ and $\mathbb{Z}/k$ periods define an element of K-theory, is more subtle.

These topological discussions have analytic analogs, due to recent developments in the geometry of the index bundle. Quillen [Q] introduced *superconnections* as the differential geometric manifestations of a K–Theory class (just as ordinary connections are the geometric realizations of an equivalence class of vector bundles). Bismut [B] constructed a superconnection to represent the index bundle $u \in K(Y)$ of a family of Dirac operators parametrized by Y.[8] He then proves that this representation renders the index theorem for families true on the level of differential forms: Up to a normalization the curvature $\Omega^{(u)}$ of his superconnection satisfies

$$\operatorname{ch}(\Omega^{(u)}) = \int_X \hat{A}(\Omega^{(Q/M)})\, e^{\omega/2} \operatorname{ch}(\Omega^{(E)}). \tag{3.2}$$

The representation of u is by an *infinite dimensional* bundle. All of the intricacy lies in this infinite dimensionality.

The analytic magic of the first Chern class is that (2.7) and (3.2) are compatible. In other words, the topological splitting $BU \to K(\mathbb{Z}, 2)$ is realized analytically by associating to Bismut's superconnection representation of u an (ordinary) connection on the determinant line bundle $\mathscr{L}$. Its curvature is then the 2-form part of (3.2). The secondary invariant corresponding to curvature is holonomy, and the formula for the holonomy is Theorem 2.8.

Our considerations in this section suggest that this holonomy formula should have an analog in higher dimensions. That higher dimensional formula should pertain to K-Theory, not cohomology, and should agree with the holonomy formula over a circle. As this formula will involve secondary invariants in K-Theory (i.e., ξ-invariants), hence differential geometric data, the index bundle will be represented by Bismut's superconnection. Therefore, rewrite (2.8) as

$$\lim_{\varepsilon \to 0} \xi_\varepsilon^{(P)} = \xi_{\mathscr{L}}^{(S)}. \tag{3.3}$$

On the left we have the adiabatic limit of ξ-invariants on P, and on the right we have ξ-invariant of the Dirac operator on the circle coupled to the connection on $\mathscr{L}$. For a general fibration $P \xrightarrow{X} N$ over an odd dimensional Spin^c manifold N, we conjecture

$$\lim_{\varepsilon \to 0} \xi_\varepsilon^{(P)} = \xi_u^{(N)}. \tag{3.4}$$

Now the right hand side is the ξ-invariant of the Dirac operator on N coupled to Bismut's superconnection. Furthermore, just as the topological expression for the direct image involves the topology of N, this expression will depend on the metric structure of N. An appropriate generalization of the index theorem for manifolds with boundary will provide a proof of (3.4) in torsion situations, along the lines of our arguments in § 2; the general proof will involve analysis modeled after either [BF2] or [ADS], [C1].

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Oblatum 14-X-1986 & 29-X-1987