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Milnor numbers and the topology of polynomial hypersurfaces

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Summary. Let F: $\mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial. The problem of determining the homology groups $H_q(F^{-1}(c))$, $c \in \mathbb{C}$, in terms of the critical points of F is considered. In the "best case" it is shown, for a certain generic class of polynomials (tame polynomials), that for all $c \in \mathbb{C}$, $F^{-1}(c)$ has the homotopy type of a bouquet of $\mu - \mu^c$ n-spheres. Here μ is the sum of all the Milnor numbers of F at critical points of F and μ^c is the corresponding sum for critical points lying on $F^{-1}(c)$. A "second best" case is also discussed and the homology groups $H_q(F^{-1}(c))$ are calculated for generic $c \in \mathbb{C}$. This case gives an example in which the critical points "at infinity" of F must be considered in order to determine the homology groups $H_q(F^{-1}(c))$.

§1. Introduction

Let F: $\mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function. According to the articles of Malgrange [M] and Pham [Ph] the knowledge of the variation of the topology of the family of hypersurfaces $\{F^{-1}(c), c \in \mathbb{C})$ is useful indetermining the asymptotics of oscillatory integrals of the form:

$$I(t) = \int_{\mathbb{R}^n} e^{itF(x)} G(x)\, dx \qquad (dx = dx_1 \ldots dx_n).$$

In this paper we consider the more basic problem of determining the homology groups $H_q(F^{-1}(c))$, $c \in \mathbb{C}$, in particular we consider the following question:

(1.1) To what extent is the topology of $F^{-1}(c)$, $c \in \mathbb{C}$ determined by the local Milnor fibrations at the critical points of F.

Recall that if P is an isolated critical point of F, then the Milnor Fibration Theorem (cf. [Mil]) asserts there is a small ball B in $\mathbb{C}^{n+1}$, centered at P, such that for all sufficiently small discs Δ^*, punctured at $b = F(P)$, the map $F: B \cap F^{-1}(\Delta^*) \to \Delta^*$ is a locally trivial fibration. Moreover, the local Milnor fibres $F^{-1}(c)\cap B$, $c\in\Delta^*$ have the homotopy type of a bouquet of μ_P n-spheres, where μ_P is the local Milnor number of F.

Let us sketch the relationship between the local Milnor fibres and the homology groups $H_q(F^{-1}(c))$. If $F:\mathbb{C}^{n+1}\to\mathbb{C}$ is an arbitrary polynomial then there is a finite set $\Gamma\subseteq\mathbb{C}$ such that $F:\mathbb{C}^{n+1}-F^{-1}(\Gamma)\to\mathbb{C}-\Gamma$ is a locally trivial fibration ([V], Cor. 5.1, p. 312 or [Ph] Appendix A 1). The smallest such set Γ we call the set of *atypical values*, denoted by α_F; the elements of $\mathbb{C}-\alpha_F$ are called *typical values* (in [Ph] α_F is called the *bifurcation set*). To each $b_i\in\alpha_F=\{b_1,\dots,b_s\}$ and $q=0,1,\dots$ there may be associated (not canonically) a corresponding "vanishing homology group" $V_q^i\subseteq\tilde{H}_q(F^{-1}(c))$, for any typical value $c\in\mathbb{C}-\alpha_F$ ([Br1] Prop. 1), and:

$$\tilde{H}_q(F^{-1}(c))=\bigoplus_{i=1}^{s}V_q^i \qquad q=0,1,\dots. \tag{1.2}$$

Suppose that F has an isolated critical point at $P\in\mathbb{C}^{n+1}$ and b, B, Δ^* are as above. If $c'\in\Delta^*$ then by trivializing $F:\mathbb{C}^{n+1}-F^{-1}(\Gamma)\to\mathbb{C}-\Gamma$ along a path from c' to c in $\mathbb{C}-\Gamma$ we obtain a homeomorphism $h: F^{-1}(c')\to F^{-1}(c)$, and thus a map:

$$h_*:\ \tilde{H}_q(F^{-1}(c')\cap B)\to\tilde{H}_q(F^{-1}(c)). \tag{1.3}$$

It will be shown that we may choose the paths above so that direct sum of the groups $\tilde{H}_q(F^{-1}(c')\cap B)$ taken over the isolated critical points of F is a direct summand of $\tilde{H}_q(F^{-1}(c))$. The isolated critical points of F are finite in number, say they are $P_1,\dots,P_l$. Let $\bar{\mu}=\sum_{i=1}^{l}\mu_{P_i}$, where μ_{P_i} is the Milnor number of F at P_i. The relationship the local Milnor fibres and $F^{-1}(c)$ may then be stated:

Theorem 1.1. *Let $F:\mathbb{C}^{n+1}\to\mathbb{C}$ be a polynomial, α_F its set of atypical values, let $\bar{\mu}$ be the sum of all the Milnor numbers of F at isolated critical points of F. Then for all $c\in\mathbb{C}-\alpha_F$*

$$H_n(F^{-1}(c))\approx\mathbb{Z}^{\bar{\mu}}\oplus A \tag{1.4}$$

where A is some finitely generated abelian group.

If F has only isolated critical points, it will be shown that the subgroup $\mathbb{Z}^{\bar{\mu}}$ is "concentrated" at the critical points of F and that the subgroup A is "concentrated at infinity".

The atypical values $b\in\alpha_F$ may arise for one or both of the following reasons:

(1.5) F has critical points on $F^{-1}(b)$,

(1.6) F has "critical points at infinity" associated to $F^{-1}(b)$.

We refer the reader to [Ph], Appendix A 1, for a description of critical points at infinity. The most well-behaved polynomials will be those where no atypical values arise as in (1.6) above. Our main theorem, Theorem 1.2 below, applies to a class of these polynomials which we call "tame polynomials". Let F be a polynomial with only isolated critical points and $c\in\mathbb{C}$, let $P_1,\dots,P_r$ be the critical points of F lying on $F^{-1}(c)$ and set

$$\mu^c = \mu^c(F) = \sum_{i=1}^{r} \mu_{P_i}(F), \qquad \mu = \mu(F) = \sum_{c \in \mathbb{C}} \mu^c(F).$$

These latter two integers are called the *fibre Milnor number* of F at c and the *total Milnor number* of F, respectively ($\bar{\mu}(F) = \mu(F)$ in this case, the distinction will be apparent later). We have:

Theorem 1.2. *Let $F: \mathbb{C}^{n+1} \to \mathbb{C}$ be a tame polynomial and let μ, μ^c, $c \in \mathbb{C}$ be the total and fibre Milnor numbers of F respectively. Then for any $c \in \mathbb{C}$, $F^{-1}(c)$ has the homotopy type of a bouquet of $\mu - \mu^c$ spheres of dimension n.*

Kouchnirenko, in an addendum to his article, [K], on Milnor numbers, states a slightly weaker result, without proof. His result is formulated for "convenient" polynomials which are non-degenerate with respect to their Newton boundaries. In §3 we show that Theorem 1.2 implies the result in [K] and give an example (Example 3.3) of a polynomial to which Theorem 1.2 applies but [K] does not.

We shall also consider a class of polynomials for which there are critical points at infinity and for which all of the homology groups $H_q(F^{-1}(c))$, $q \geqq 0$, $c \in \mathbb{C} - \alpha_F$ may be determined in terms of Milnor numbers of associated polynomials (Thm. 5.3, §5). In particular, we give an example where the group A in (1.3) is not trivial. This class of polynomials includes all polynomials of two variables, and polynomials of three variables having only isolated critical points and whose homogeneous term of highest degree is square-free.

The organization of the paper is as follows: In §2 we recall some results on multiplicities and Milnor numbers. In §3 we introduce tame polynomials. In §4 we prove Theorem 1.2, our main result. In §5 we prove Theorem 1.1 and discuss the class of polynomials referred to immediately above. In §6 we prove two technical propositions used in previous sections but whose proofs were deferred so as not to interrupt the exposition. Some of the results in §2, §5, and §6 are undoubtedly known to specialists in some form, but since no proofs were found in the literature we have given complete proofs here. The major new contributions are in §3–§4 on tame polynomials, where we have shown that there is an easy characterization of a vast array of polynomials whose "fibre homology groups" are under good control.

This article presents the main results of the author's doctoral dissertation [Br1] as well as supplying some deferred details of a previous article [Br2].

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§2. Multiplicities and Milnor numbers

We recall the definition of multiplicities and Milnor numbers (see Appendix B of [Mil], or [Or]). Let $g: U \to \mathbb{C}^m$ be a holomorphic map defined on an open subset of $\mathbb{C}^m$, $g=(g_1, \ldots, g_m)$. Let $P \in U$ be an isolated zero of g, and B a small closed ball, centered at P, such that P is the only zero of g in B. Let $\mu_P(g)$ be the topological degree of the map $\partial B \to S^{2m-1}$, $y \to g(y)/\|g(y)\|$. According to Milnor ([Mil], Appendix B) it is a finite positive integer. We say that g has a *zero of multiplicity* $\mu_P(g)$ *at* P. If $g(P) \neq 0$ we set $\mu_P(g)=0$; and if P is a non-isolated zero of g, we set $\mu_P(g)=\infty$. With this definition we always have, according to Palamadov [Pa] (see also [Or], Thm. 5.11):

$$\mu_P(g)=\dim_{\mathbb{C}} \mathcal{O}_P/(g_1, \ldots, g_m), \tag{2.1}$$

where $\mathcal{O}_P$ is any of the local rings of i) rational functions on $\mathbb{C}^m$ defined at P, ii) germs of holomorphic functions defined at P, or iii) formal power series expansions at P. Now let $g: \mathbb{C}^m \to \mathbb{C}^m$ be a polynomial map, the *total multiplicity of the zeros of* g *in* $\mathbb{C}^m$ is

$$\mu(g)=\sum_{P \in g^{-1}(0)} \mu_P(g); \qquad \mu(g)=\infty$$

if and only if $g^{-1}(0)$ has components of positive dimension. Equation (2.1) implies that $\mu(g)=\dim_{\mathbb{C}} \mathbb{C}[X_1, \ldots, X_m]/(g_1, \ldots, g_m)$. We also define the *reduced total multiplicity* $\bar{\mu}(g)$ by $\bar{\mu}(g)=\sum_P \mu_P(g)$ where the sum is taken over the isolated zeros of g.

If $F: U \to \mathbb{C}$ is a holomorphic function, define the complex gradient of F by $\partial F(x)=\left(\frac{\partial F}{\partial X_1}(x), \ldots, \frac{\partial F}{\partial X_m}(x)\right)$. The *Milnor number* of F at P, $\mu_P(F)$, is defined to be the multiplicity $\mu_P(\partial F)$. If $U=\mathbb{C}^m$, and F is a polynomial, then the *total Milnor number* $\mu(F)$, and the *reduced total Milnor number* $\bar{\mu}(F)$ are defined by $\mu(F)=\mu(\partial F)$ and $\bar{\mu}(F)=\bar{\mu}(\partial F)$. From Appendix B of [Mil], $\mu_P(F)=1$ if and only if P is a non-degenerate critical point of F, i.e., $\left(\frac{\partial^2 F}{\partial X_i \partial X_j}(P)\right)$ is a non-singular matrix. We recall several properties of multiplicities, whose proofs may be found in [Mil], Appendix B.

Proposition 2.1. *Let $V \subseteq \mathbb{C}^m$ be an open set and $g: V \to \mathbb{C}^m$ a holomorphic map. Let $U \subseteq V$ be an open subset with compact closure in V, whose boundary ∂U is a smooth compact manifold, and such that $g(x) \neq 0$, $x \in \partial U$. Then*

i) *g has finitely many zeros $P_1, \ldots, P_s$ in U and*

$$\mu_{P_1}(g)+\ldots+\mu_{P_s}(g)=d$$

where d is the topological degree of the map

$$\partial U \to S^{2m-1}, \qquad y \to g(y)/\|g(g)\|.$$

ii) *Let V, U be as above and $G: V \times I \to \mathbb{C}^m$ a continuous map such that g_t: $V \to \mathbb{C}^m$, $g_t(x)=G(x, t)$ is holomorphic for all t and such that $g_t(x) \neq 0$, $t \in I$, $x \in \partial U$. Let $P_1, \ldots, P_s$ be the zeros of g_0 in U, and $Q_1, \ldots, Q_r$ the zeros of g_1 in U. Then:*

$$\mu_{P_1}(g_0)+\ldots+\mu_{P_s}(g_0)=\mu_{Q_1}(g_1)+\ldots+\mu_{Q_r}(g_1). \tag{2.2}$$

Remark. Suppose: $g, h\colon V \to \mathbb{C}^m$ are holomorphic

$$\|g(x)-h(x)\| < \|g(x)\|, \qquad x \in \partial U,$$

then we may take $g_t(x)=(1-t)\,g(x)+t\,h(x)$ in ii) above. If $U=B$, a ball centered at P, g has no zeros in $B-\{P\}$ and $P_1, \ldots, P_s$ are the zeros of h in $B-\{P\}$ then (2.2) becomes

$$\mu_P(g)=\mu_P(h)+\sum_{i=1}^{s}\mu_{P_i}(h). \tag{2.3}$$

For $F\colon \mathbb{C}^m \to \mathbb{C}$ and $w=(w_1, \ldots, w_m)\in\mathbb{C}^m$ set

$$F^w(X_1, \ldots, X_m)=F(X_1, \ldots, X_m)-(w_1 X_1+\ldots+w_m X_m);$$

observe that $\partial(F^w)=\partial F-w$. The following proposition follows from [Mi1], Appendix B.

Proposition 2.2. *Let P be an isolated critical point of $F\colon \mathbb{C}^m \to \mathbb{C}$ and let B be a small closed ball, centered at P, such that P is the only critical point of F in B. Then there is a small open ball B', centered at 0 and a proper closed analytic subset Z of B' such that for all $w\in B'-Z$, F^w has exactly μ_P non-degenerate critical points in B.*

Semi-continuity of multiplicities. In all that follows, a (locally closed) variety shall be a subset of $\mathbb{P}^m=\mathbb{P}^m(\mathbb{C})$ of the form $V-W$ where V, W are Zariski-closed subsets of $\mathbb{P}^m$. The dimension of a variety is taken to be the maximum of the dimensions of its irreducible components.

Proposition 2.3. *Let $g_t\colon \mathbb{C}^m \to \mathbb{C}^m$, $t\in T$ be a family of polynomial self-maps of $\mathbb{C}^m$, whose coefficients depend rationally on t in the algebraic variety T, and let $P\in\mathbb{C}^m$. Then:*

i) *$\mu_P(g_t)$ is an upper semi-continuous function of t for the Zariski topology on T, i.e. the sets*

$$T^n=\{t\in T\colon \mu_P(g_t)\geqq n\}, \qquad n=0, 1, \ldots, \infty$$

are Zariski closed subsets of T, and

ii) *$\bar\mu(g_t)$ is lower semi-continuous for the Zariski topology, i.e. the sets*

$$T_n=\{t\in T\colon \bar\mu(g_t)\leqq n\}, \qquad n=0, 1, 2, \ldots$$

are Zariski closed subsets of T. If $\mu(g_t)<\infty$ for all $t\in T$, then $\mu(g_t)=\bar\mu(g_t)$ is lower semicontinuous.

Proof. The idea of the main part of the proof ($n<\infty$) is to show that T_n, T^n are closed in the ordinary topology and then to use induction on $\dim(T)$ and the theory of branched coverings to show that T_n, T^n are also Zariski closed.

Let $q\colon \mathbb{C}^m\times T \to \mathbb{C}^m\times T$ be the map $(x,t)\to(g_t(x),t)$. The set

$$S=\{(x,t)\colon \mu_x(g_t)=\infty\}=\{(x,t)\colon \dim_x g_t^{-1}(0)\geqq 1\}$$

equals the set

$$\{(x,t)\colon \dim_{(x,t)} q^{-1}(q(x,t))\geqq 1\}\cap\{(x,t)\colon g_t(x)=0\}$$

and $\{P\} \times T^\infty = (\{P\} \times T) \cap S$. The set $\{(x, t): \dim_{(x,t)} q^{-1}(q(x, t)) \geqq 1\}$ is a Zariski closed set by Chevalley's theorem ([G], p. 189) so that T^∞ must also be closed. The set T^n is a Zariski closed subset of T if and only if $T^n - T^\infty$ is a Zariski closed subset of $T - T^\infty$, so by replacing T by $T - T^\infty$ we may assume T^∞ is empty.

Now we show that T_n and T^n are both closed in the ordinary topology. Let $t_0 \in \overline{T^n}$ and let B be an open ball centered at P such that g_{t_0} has no zero in $\bar{B} - \{P\}$. We may pick $t \in T^n$ sufficiently close to t_0 so that

$$\|g_{t_0}(x) - g_t(x)\| < \|g_{t_0}(x)\|, \quad x \in \partial B \tag{2.4}$$

and by (2.3) we conclude $\mu_P(g_{t_0}) \geqq \mu_P(g_t) \geqq n$. Thus $t_0 \in T^n$, and $\overline{T^n} = T^n$. To show T_n is closed, again pick $t_0 \in \overline{T_n}$ and let $P_1, \ldots, P_l$ be the isolated zeros of g_{t_0}. Let B_i be a small ball centered at P_i such that $g_{t_0}(x) \neq 0$ $x \in \partial B_i$ set $B = \bigcup_i B_i$. Again (2.4) holds with this new B and for all $t \in T_n$ sufficiently close to t. By (ii) of Proposition 2.1 and the remark following the proposition we conclude that $\bar{\mu}(g_{t_0}) \leqq \bar{\mu}(g_t) \leqq n$.

Next we argue that the sets T_n, T^n are Zariski closed by induction on $\dim(T)$. The case $\dim T = 0$ is trivial so we may assume $\dim T > 0$. We may also assume that T is irreducible since T_n, T^n are closed iff their intersections with every irreducible component of T is closed. It suffices to find a proper closed subset T' of T such that $\mu_P(g_t)$ (or $\bar{\mu}(g_t)$) is constant on $T - T'$, say

$$\mu_P(g_t) = d, \quad t \in T - T' \qquad (\bar{\mu}(g_t) = d, t \in T - T') \tag{2.5}$$

for some $d \geqq 0$. Assume that this holds. The set $T - T'$ is dense in T, in the ordinary topology, because T is irreducible. Then by (2.5) and the semi-continuity of $\mu_P(g_t)$ and $\bar{\mu}(g_t)$ in the ordinary topology we have $\mu_P(g_t) \leqq d$, $t \in T$, ($\bar{\mu}(g_t) \geqq d$, $t \in T$). Thus:

$$T^n = T, \quad n \leqq d; \qquad T^n = (T')^n, \quad n \geqq d$$

and

$$T_n = (T')_n, \quad n \leqq d; \qquad T_n = T, \quad n \geqq d.$$

By induction these are all Zariski closed in T.

We now produce the required T' in the two different cases.

Case 1. $\mu_P(g_t)$. If $g_t(P) \neq 0$ for some t then set $T' = \{t \in T \mid g_t(P) = 0\}$. Otherwise let $G = \{(x, t) \mid g_t(x) = 0\}$. The set $S = \{(x, t) \in G \mid \dim_x g_t^{-1}(0) \geqq 1\}$, as previously noted, is closed in G and by hypothesis does not meet $\{P\} \times T$. Let $G' = G - S$ and $q: G' \to T$ the map $(x, t) \to t$. By construction q has finite fibres and is surjective since $\{P\} \times T \subseteq G'$. There is a proper closed subvariety $T'' \subseteq T$ such that the components of $G' - q^{-1}(T'')$ are pairwise disjoint and have dimension equal to $\dim T$. We may enlarge T'' to Zariski closed set T' such that $q: G' - q^{-1}(T') \to T - T'$ is an r-sheeted covering space. (This follows from Example 3.7, p. 91 of [H], Thm. 7, p. 117 of [Sh] and the Lemma on p. 320 of [Sh].) Since T is irreducible then $T - T'$ is connected, therefore it suffices to show that $\mu_P(g_t)$ is locally constant on $T - T'$. Let $t_0 \in T - T'$, then $q^{-1}(t_0) = g_{t_0}^{-1}(0) \cap G' = \{P, P_1, \ldots, P_{r-1}\}$ for some $P, \ldots, P_{r-1}$. Let B_i be a small ball centered at P_i,

B_0 centered at P, such that the B_i's are pairwise disjoint and $g_{t_0}(x) \neq 0$, $x \in \bar{B}_i - \{P_i\}$. Let B be the union of the B_i's. Since $q: G' - q^{-1}(T') \to T - T'$ is a finitely sheeted covering space and since $G' - q^{-1}(T')$ is closed in $\mathbb{C}^m \times (T - T')$ it follows that $q^{-1}(t) \subseteq B \times \{t\}$ for all $t \in T - T'$ sufficiently close to t_0. Furthermore, for each B_i and for t sufficiently close to t_0, (2.3) will hold with B replaced by B_i, and then by (2.2) each $B_i \times \{t\}$ must contain at least one of the points of $q^{-1}(t)$. But $q^{-1}(t)$ has exactly r points, so each $B_i \times \{t\}$ contains exactly one point and $B_0 \times \{t\} \cap q^{-1}(t) = \{(P, t)\}$. Again by (2.3) $\mu_P(g_{t_0}) = \mu_P(g_t)$, as required.

Case 2. $\bar{\mu}(g_t)$. Let G, G' and $q: G' \to T$ be as defined above. If $\overline{q(G')} \neq T$ then $\bar{\mu}(g_t) = 0$ for $t \in T - \overline{q(G')}$, and by a density and semi-continuity argument $\bar{\mu}(g_t) = 0$ for all $t \in T$. Otherwise we argue as in Case 1 except that we use (2.2) instead of (2.3). All is now proven.

Remark. The upper semi-continuity $\mu_P(g_t)$ also follows directly from the semi-continuity of the Hilbert-Samuel function, which in turn follows from the semi-continuity theorem on the dimension of the fibres of a coherent sheaf, applied to an appropriate sheaf of relative jets. A proof of the semi-continuity theorem of the Hilbert-Samuel function is given in the thesis of Lejeune and Teissier [L-T], though this is somewhat inaccessible.

Calculation of Milnor numbers. Bezout's theorem may be used to calculate the total Milnor number in certain cases. (Kouchnirenko shows how Newton polyhedra may be used to compute Milnor numbers in [K], see § 3.) For $F: \mathbb{C}^m \to \mathbb{C}$, let $d_i = \deg\left(\frac{\partial F}{\partial X_i}\right)$ be the highest degree of a monomial occuring in $\frac{\partial F}{\partial X_i}$. Let $G_i = X_0^{d_i} \frac{\partial F}{\partial X_i}(X_1/X_0, \ldots, X_m/X_0)$ be the homogenization of $\frac{\partial F}{\partial X_i}$ with respect to X_0. Suppose P is an isolated, common, projective zero of the G_i lying in the open subvariety of $\mathbb{P}^m = \mathbb{P}^m(\mathbb{C})$ defined by $X_j \neq 0$. Let $P = (y_0 : y_1 : \ldots y_{j-1} : 1 : y_{j+1} : \ldots : y_m)$, in homogeneous coordinates, let $\bar{P} = (y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m)$. The point $\bar{P}$ is an isolated zero of the map $g: \mathbb{C}^m \to \mathbb{C}^m$, defined by:

$$\begin{aligned}(X_1, \ldots, X_m) \to (&G_1(X_1, \ldots, X_j, 1, X_{j+1}, \ldots, X_m),\\ &\ldots, G_m(X_1, \ldots, X_j, 1, X_{j+1}, \ldots, X_m));\end{aligned}$$

define $\mu_P(G_1, \ldots, G_m) = \mu_{\bar{P}}(g)$. The integer $\mu_P(G_1, \ldots, G_m)$ is independent of j and if the common zeros of the G_i are finite in number, say, $P_1, \ldots, P_s$, then by Bezout's Theorem:

$$\sum_{i=1}^{s} \mu_{P_i}(G_1, \ldots, G_m) = d_1 \cdot d_2 \cdot \ldots \cdot d_m. \tag{2.6}$$

Also, if P is any common zero of the G_i in the set defined by $X_0 \neq 0$ then $\bar{P}$ is a critical point of F and $\mu_{\bar{P}}(F) = \mu_{\bar{P}}(\partial F) = \mu_P(G_1, \ldots, G_m)$. Let $R_1, \ldots, R_s$ be those P_i which lie in the set determined by $X_0 \neq 0$, and $Q_1, \ldots, Q_r$ be those

P_i lying on the set determined by $X_0=0$, and define

$$\mu^\infty(F)=\sum_{i=1}^{r}\mu_{Q_i}(G_1, \ldots, G_m).$$

We think of the Q_i as "critical points at infinity" and call them *extraneous critical points.* We have from (2.6):

$$\mu(F)=d_1\cdot d_2\cdot\ldots\cdot d_m-\mu^\infty(F).$$

In general the G_i will have non-isolated zeros but we can still obtain a formula as above. For $1\leqq i\leqq m$ let K_i be a homogeneous polynomial in $X_0, \ldots, X_m$ of degree d_i such that the K_i have only finitely many common zeros. Let $G_i^t=(1-t)G_i-tK_i$. For all but finitely many $t\in\mathbb{C}$, the G_i^t $i=1, \ldots, m$ have only finitely many common zeros. Let $\bar{R}_1, \ldots, \bar{R}_s$ be the isolated zeros of ∂F, let B be a disjoint union of closed balls centered at the R_i, such that B does not meet any non-isolated critical point of F; B determines a compact set $\mathbb{P}^m$ which we also denote by B. By a semi-continuity argument $\sum_{P\in B}\mu_P(G_1^t, \ldots, G_m^t)$ (where the sum is over the common zeros of the G_i in B) is the same for all small non-zero t, and equals $\bar{\mu}(F)$. Define $\mu^\infty(F)$ to be the integer $\sum_{P\notin B}\mu_P(G_1^t, \ldots, G_m^t)$ where $t\neq 0$ is small and the sum is over common zeros not in B. If the G_i themselves have only isolated zeros then both this and the previous definition of $\mu^\infty(F)$ give the same value for $\mu^\infty(F)$. Again by Bezout's Theorem we have:

Proposition 2.4. *Let $F: \mathbb{C}^m\to\mathbb{C}$ be a polynomial and let $d_i=\deg(\partial F/\partial X_i)$ and $\mu^\infty(F)$ be as defined above. Then:*

$$\bar{\mu}(F)=d_1\cdot d_2\cdot\ldots\cdot d_m-\mu^\infty(F).$$

Example 2.1. Let $F(X, Y, Z)=X^p+Y^q+Z^r-tXYZ$, $p, q, r\geqq 3$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ (T_{pqr} singularity). Then F has no extraneous critical points and $\mu(F)=(p-1)(q-1)(r-1)$. For $t\neq 0$ the fibre Milnor numbers are quite interesting. Let $s=pqr-(qr+pr+pq)$, $l=lcm(p, q, r)$, $h=sl/pqr$. The critical values of F are 0 and $R\theta$, θ an h'th root of unity; R is a non-zero complex number, depending only on t, p, q, r. The only critical point of F on $F^{-1}(0)$ is 0. It is well-known that $\mu_0(F)=p+q+r-1$ so:

$$\mu^0(F)=p+q+r-1.$$

There are pqr/l non-degenerate critical points on each $F^{-1}(R\theta)$, so

$$\mu^{R\theta}(F)=pqr/l.$$

§ 3. Tame polynomials

In this section we define the class of polynomials to which Theorem 1.2 of the introduction applies.

Definition 3.1. A polynomial $F: \mathbb{C}^m \to \mathbb{C}$ is called a "tame" polynomial if there is a compact neighborhood U of the critical points of F such that $\|\partial F(x)\|$ is bounded away from 0 on $\mathbb{C}^m - U$.

Remarks. 1) The word "tame" is used to describe these polynomials because the integral curves of the vector-field $\overline{\partial F(x)}/\|\partial F(x)\|^2$ have a tame or moderate behaviour. This fact will be used in the proof of Theorem 1.2.

2) A tame polynomial has a finite number of critical points, since the critical point set is a compact, affine algebraic variety.

3) It turns out that none of the atypical values of a tame polynomial arise from critical points at infinity and therefore the theory for the asymptotics of oscillatory integrals outlined in [M] and [Ph] applies to tame polynomials.

Tame polynomials may be characterized in terms of Milnor numbers.

Proposition 3.1. *A polynomial is tame if and only if* $\mu(F) < \infty$ *and* $\mu(F^w) = \mu(F)$ *for all sufficiently small* $w \in \mathbb{C}^m$.

Proof. Let us rephrase the definition above in terms of proper maps. Let us say that the map $p: X \to Y$ is proper above y if there is a neighborhood U of y such that $p: p^{-1}(U) \to U$ is proper. Thus F is tame if and only if ∂F: $\mathbb{C}^m \to \mathbb{C}^m$ is proper above zero. Suppose that $\mu(F^w) = \mu(F) < \infty$ for all sufficiently small w. Surround the points of $\partial F^{-1}(0)$ by small pairwise disjoint balls, let B denote the union of these balls.

For small w we have $\|\partial F^w(x) - \partial F(x)\| = \|w\| < \|\partial F(x)\|$, $x \in \partial B$. Since $\mu(\partial F^w) = \mu(F^w) = \mu(F)$ for all sufficiently small w we see by Proposition 2.1 that $\partial F^{-1}(w) \subseteq \bar{B}$ for small w and hence ∂F is proper above 0. If F is tame, then since ∂F is proper above 0 we have $\partial F^{-1}(w) \subseteq B^0$ for all sufficiently small w and hence $\mu(F^w) = \mu(F)$ by Proposition 2.1. All is now proven.

Example 3.1. Let $F(X, Y, Z) = X^p + Y^q + Z^r - tXYZ$, p, q and r as in Example 2.1. For each $w \in \mathbb{C}^3$, F^w has no extraneous critical points so $\mu(F^w) = (p-1)(q-1)(r-1)$, by Proposition 2.4 and hence is a tame polynomial.

Example 3.2. Let $F(X, Y) = X^2 Y - X$, $x_t = (t, t + 1/2t)$ then $\|x_t\| \to \infty$ and $\|\partial F(x_t)\| \to 0$ as $t \to 0$. Thus F is not tame even though F has no critical points. If $w = (a, b)$ then

$$\mu(F^w) = \infty \qquad a = -1,\ b = 0$$
$$\mu(F^w) = 0 \qquad a \neq -1,\ b = 0$$
$$\mu(F^w) = 2 \qquad b \neq 0.$$

Consequently $\mu(F) = 0$, but there are small w for which $\mu(F^w) = 2$.

Tame polynomials are generic. Let $V_{m,d}$ be the vector space of polynomials in m variables and degree less than or equal to d. By a variety of polynomials we mean a subvariety of some $V_{m,d}$. If T is a variety of polynomials and $t \in T$ we denote by F_t the polynomial corresponding to t and identify F_t with t so that the notation $F_t \in T$ makes sense.

Proposition 3.2. a) *Let T be a variety of polynomials. Then the subset of T, consisting of tame polynomials, is a constructible subset of T.*

b) *Each $V_{m,d}$ contains a dense, constructible subset of tame polynomials.*

Proof. To prove b), assuming a), it suffices to show that there is an open set of tame polynomials in $V_{m,d}$. For $t \in V_{m,d}$ let F_t^d the homogeneous piece of F_t of degree d. The set of polynomials F_t for which F_t^d has an isolated singularity at 0 is $V_{m,d} - \{t \in V_{m,d}: \mu(F_t^d) = \infty\}$, a non-empty open set. If $d > 1$, then for t in this open subset and all $w \in \mathbb{C}^m$, F_t^w has no extraneous critical points at infinity, and $\mu(F_t^w) = (d-1)^m$. Thus all polynomials in this open dense set are tame. The case $d = 1$ is trivial.

To prove a) we use induction on $\dim T$; our main tools for the induction step are the semi-continuity of Milnor numbers, Proposition 2.3, and "Bertini-Sard"-like theorems of Verdier [V]. To state the latter let $p: M \to T$ be a map of varieties with $p(M)$ dense in T and let $\mathcal{S}$ be a Whitney stratification of M. Then Corollaire (5.1) of [V] states that there is an open subset $U \subseteq T$ such that $p: p^{-1}(U) \to U$ is locally trivial with respect to the stratification, i.e., for any two nearby points $t_1, t_2 \in U$, there is a homeomorphism $h: p^{-1}(t_1) \to p^{-1}(t_2)$ such that h maps $S \cap p^{-1}(t_1)$ homeomorphically into $S \cap p^{-1}(t_2)$ for every $S \in \mathcal{S}$. If T is irreducible, U is irreducible, hence connected and the above holds for all $t_1, t_2 \in U$.

Let T^τ denote the subset of tame polynomials in the set T. Our induction hypothesis is:

(H_n) If T is a constructible subset of $V_{m,d}$ with $\dim T \leqq n$, then T^τ is constructible.

For $\dim T = 0$, the result is trivial. Now let T be a constructible set of dimension $n+1$ and assume (H_n) holds. Since T is a union of locally closed irreducible varieties of dimension not exeeding $n+1$, it suffices to prove (H_{n+1}) for irreducible, locally closed varieties of dimension $n+1$. Further, we need only find an open set $U \subseteq T$ such that $U^\tau = U$ or $U^\tau = \emptyset$ since T^τ would equal $U \cup (T-U)^\tau$ or $(T-U)^\tau$ both of which are constructible.

We may further assume that $\mu(F_t) < \infty$ for all $t \in T$. Let

$$G = \{(t, w) \in T \times \mathbb{C}^m: \partial F_t(w) = 0\},$$

$p: G \to T$ the projection $(t, w) \to t$ and

$$G_1 = \{(t, w) \in G \,|\, \dim_{(t,w)} p^{-1}(p(t, w) \geqq 1\}.$$

The set G_1 is closed and $p(G_1)$ is the set $\{t \in T: \mu(F_t) = \infty\}$. If $p(G_1)$ is dense in T then there is an open dense $U \subseteq p(G_1)$, and $U^\tau = 0$ by Proposition 3.1. We may then replace T by an open subset, which we still call T, on which $\mu(F_t) < \infty$.

Next we show that we may assume that for each $t \in T$, $\mu(F_t^w) < \infty$ for $|w| < \varepsilon$, $0 < \varepsilon$, ε depending on t. Let $G_2 = \{(t, w): \mu(F_t^w) = \infty\}$, by the above argument G_2 is a constructible subset of $T \times \mathbb{C}^m$ disjoint from $T \times \{0\}$. Give $T \times \mathbb{C}^m$ a Whitney stratification $\mathcal{S}$ such that G_2 and $T \times \{0\}$ are unions of strata. In the above description of Verdier's result let $M = T \times \mathbb{C}^m$ and $p: T \times \mathbb{C}^m \to T$ the map $(t, w) \to t$. Let U, t_1, t_2, h be as above. Since $U \times \{0\}$ lies in $T \times \{0\}$, a union of strata, $U \times \{0\} \cap p^{-1}(t) = \{(t, 0)\}$, and since G_2 is a union of strata, then the homeomorphism h maps $(t_1, 0)$ to $(t_2, 0)$ and maps $\{t_1\} \times \mathbb{C}^m \cap G_2$ to $\{t_2\} \times \mathbb{C}^m \cap G_2$. Thus $(t, 0)$ is an accumulation point of $\{t\} \times \mathbb{C}^m \cap G_2$, either for all $t \in U$ or for no t in U. In the former case $U^\tau = \emptyset$. Thus, by replacing T by U, we may assume that for each $t \in T$, $\mu(F_t^w) < \infty$ for $|w| < \varepsilon$, $0 < \varepsilon$, ε depending on t.

Next consider $\bar{\mu}(F_t^w)$ on $T \times \mathbb{C}^m$. Since the sets $(T \times \mathbb{C}^m)_n = \{(t, w) \colon \bar{\mu}(F_t^w) \leq n\}$ are Zariski-closed subsets and $T \times \mathbb{C}^m = \bigcup_n (T \times \mathbb{C}^m)_n$, then $T \times \mathbb{C}^m = (T \times \mathbb{C}^m)_N$ for all large N, by the Baire Category Theorem. Let this N be chosen as small as possible so that $G_3 = (T \times \mathbb{C}^m)_{N-1}$ is a proper closed subset of $T \times \mathbb{C}^m$. Again, using a stratification argument, there is an open subset $U \subseteq T$ such that for $t_1, t_2 \in U$ there is a homeomorphism $h \colon \{t_1\} \times \mathbb{C}^m \to \{t_2\} \times \mathbb{C}^m$, taking $(t_1, 0)$ to $(t_2, 0)$ and $\{t_1\} \times \mathbb{C}^m \cap G_3$ onto $\{t_2\} \times \mathbb{C}^m \cap G_3$. Since G_3 is a closed proper subset of $T \times \mathbb{C}^m$ we have $(t, 0) \in G_3$ either for all $t \in U$ or for no $t \in U$. In the former case for each $t \in U$ $\mu(F_t) = \bar{\mu}(F_t) < \bar{\mu}(F_t^w) = \mu(F_t^w)$ for infinitely many w arbitrarily close to 0. From Proposition 3.1 it follows that $U^\tau = \emptyset$. In the latter case we argue similarily that $\mu(F_t) = \mu(F_t^w)$ for $|w| < \varepsilon$, $\varepsilon > 0$ depending on t and hence $U^\tau = U$. All is now proven.

Kouchnirenko's results. Next we compare the notion of tameness to the non-degeneracy conditions of Kouchnirenko, first recalling some of his definitions. For a polynomial F of m variables, write $F = \sum_\alpha a_\alpha X^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$, $X^\alpha = X_1^{\alpha_1} \ldots X_m^{\alpha_m}$. Let $\operatorname{supp}(F) = \{\alpha \colon a_\alpha \neq 0\}$; the *Newton polyhedron* of F (at infinity), $\tilde{\Gamma}_-(F)$, is the convex hull in R^m of $\{0\} \cup \operatorname{supp}(F)$. The *Newton boundary* of F (at infinity), $\tilde{\Gamma}(F)$, is the union of the closed faces of $\tilde{\Gamma}_-(F)$ which do not contain 0. A polynomial F is called *convenient* if a monomial of the form $X_i^{\alpha_i}$, $\alpha_i > 0$ occurs in F with a non-zero coefficient for every i. For a subset Δ of $\tilde{\Gamma}_-(F)$ and a polynomial $G = \sum b_\alpha X^\alpha$, set $G_\Delta = \sum_{\alpha \in \Delta} b_\alpha X^\alpha$; $F_{\tilde{\Gamma}(F)}$ is called the *Newtonian principal part* of F (at infinity). A polynomial is *non-degenerate* with respect to its Newton boundary if $\left(X_1 \dfrac{\partial F}{\partial X_1}\right)_\Delta, \ldots, \left(X_m \dfrac{\partial F}{\partial X_m}\right)_\Delta$ all have no common zero in $(\mathbb{C} - 0)^m$ for every closed face Δ of the Newton boundary $\tilde{\Gamma}(F)$. A polynomial is non-degenerate if and only if the Newtonian principal part is non-degenerate. For $0 < q \leq m$ let V_q denote the sum of the (standard) q-dimensional volumes of the q-dimensional faces of $\tilde{\Gamma}(F)$ that contain 0, ($V_m = \operatorname{vol} \tilde{\Gamma}(F)$). The *Newton number* of F, $\tilde{\nu}(F)$, is defined by

$$\tilde{\nu}(F) = m!\, V_m - (m-1)!\, V_{m-1} + \ldots + (-1)^m.$$

The following proposition is Theorem 1.5, [K], rephrased slightly.

Proposition 3.3. *Let F be a convenient polynomial. If $\mu(F) < \infty$ then*

i) $\mu(F) \leq \tilde{\nu}(F)$.

If F is non-degenerate with respect to its Newton boundary then $\mu(F) < \infty$ and

ii) $\mu(F) = \tilde{\nu}(F)$.

iii) *In the variety of polynomials with a given Newton boundary the non-degenerate polynomials form an open, dense subvariety.*

From this proposition we can prove:

Proposition 3.4. *If F is a convenient polynomial, non-degenerate with respect to its Newton boundary, then it is tame.*

Proof. Suppose that F is convenient and non-degenerate, then $\mu(F)<\infty$. Let $\{F^w: w\in\mathbb{C}^m\}$ be the family of linear perturbations of F. Since F is convenient the set $U=\{w\in\mathbb{C}^m: \tilde{\Gamma}(F)=\tilde{\Gamma}(F^w)\}$ is open and dense in $\mathbb{C}^m$ and for a possibly smaller subset $V\subseteq U$, F^w will be convenient and non-degenerate, by iii) above. Since F^w has the same Newton polyhedron for all $w\in V$, and $0\in V$ then by ii)

$$\mu(F)=\tilde{\nu}(F)=\tilde{\nu}F^w)=\mu(F^w).$$

Thus, $\mu(F^w)=\mu(F)$ for all small w and F is tame. All is now proven.

Example 3.3. Let $F(X,Y)=X^2+2XY^2+Y^4+Y^2=(X+Y^2)+Y^2$. If $w=(a,b)$ and $a\neq -1$, then by direct calculation F^w has one non-degenerate critical point and $\mu(F^w)=1$, so F is tame. However $\tilde{\nu}(F)=3>\mu(F)$, so F must be degenerate, as may be directly verified.

The above proposition and example show that Theorem 2 generalizes the result of Kouchnirenko and, that for the purposes of Theorem 1.2, tameness is a more natural condition than the non-degeneracy conditions.

§4. Proof of Theorem 1.2

First we recall a result of Durfee on semialgebraic neighbourhoods (cf. [Du]).

Proposition 4.1. *Let $M\subseteq\mathbb{R}^n$ be a semi-algebraic set and $\alpha: M\to\mathbb{R}$ a proper semi-algebraic map, such that $\alpha(x)\geqq 0$ for $x\in M$. Then for all sufficiently small δ, $\alpha^{-1}(0)$ is a deformation retract of $\alpha^{-1}([0,\delta])$.*

An α as above is called a *semi-algebraic rug function*, and $\alpha^{-1}([0,\delta])$, for small δ as above, a *semi-algebraic neighbourhood*. In [Du] Durfee only claims that the inclusion $\alpha^{-1}[0]\subseteq\alpha^{-1}[0,\delta]$ is a homotopy equivalence though it is clear that we actually get a deformation retract. We will use Proposition 4.1 in the form of the following Corollary.

Corollary 4.2. *Suppose $f: \mathbb{C}^n\to\mathbb{C}^r$ is a polynomial map, B_ε and B_δ balls centred at x_0 and $y_0=f(x_0)$ respectively. Then:*

i) *Given B_ε, then for all sufficiently small δ, $B_\varepsilon\cap f^{-1}(y_0)$ is a deformation retract of $B_\varepsilon\cap f^{-1}(B_\delta)$.*

ii) *Given ε sufficiently small there is a δ' such that for all $\delta<\delta'$, $B_\varepsilon\cap f^{-1}(B_\delta)$ is contractible.*

Proof. i) Apply Proposition 4.1 to the function $\alpha(x)=\|f(x)-y_0\|^2$ defined on B_ε.

ii) By the Conic Structure Lemma $f^{-1}(y_0)\cap B_\varepsilon$ is a cone for all small ε. Now apply i). See [Mil], [B-V] for the Conic Structure Lemma.

Next we prove a lemma. For $U\subseteq\mathbb{C}$ (or $c\in\mathbb{C}$) we denote $F^{-1}(U)$, $F^{-1}(c)$ by F_U, F_c respectively.

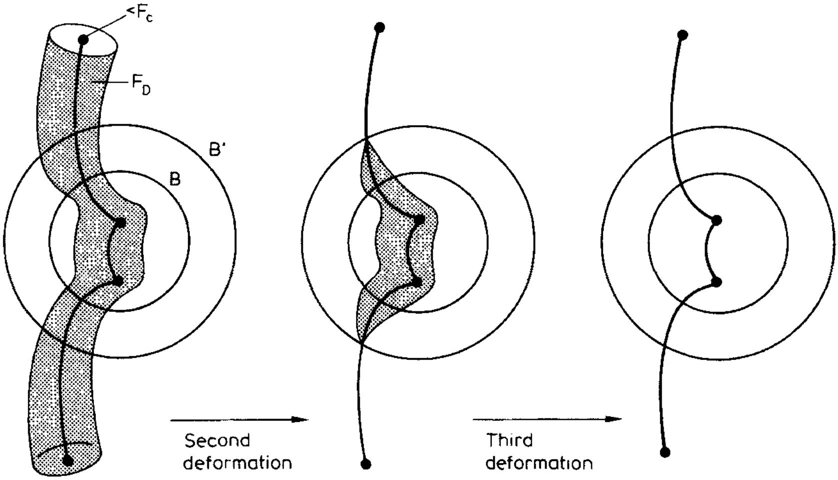

Fig. 1

Lemma 4.3. *Let F be a tame polynomial, $c \in \mathbb{C}$ and Δ a closed disc about c such that F has no critical points in $F_\Delta - F_c$. Then F_c is a deformation retract of F_Δ.*

Proof. The retracting deformation of F_Δ onto F_c may be constructed as a succession of three deformations. The first of these retracts F_Δ into F_D where D is a sufficiently smaller disc. The second and third deformations are pictured in Fig. 1. Now the details.

Let $(x, y) = x_1 y_1 + \ldots + x_{n+1} y_{n+1}$, $x, y \in \mathbb{C}^{n+1}$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$ (bar denotes conjugation), $r(x) = (x, \bar{x}) = \|x\|^2$. The level set $r^{-1}(\varepsilon^2)$ is just the sphere of radius ε. For $x \in F_c \cap S_\varepsilon$ the formulae:

$$dF_x(v) = (\partial F(x), v), \qquad dr_x(v) = 2\,\mathrm{Re}(\bar{x}, v)$$

imply that the map $F\colon S_\varepsilon \to \mathbb{C}$ is full rank at x if and only if $\partial F(x)$ and $\bar{x}$ are linearly independent over $\mathbb{C}$, and similarly $r|F_c \to \mathbb{R}$ is full rank at x if and only if $\partial F(x)$ and $\bar{x}$ are linearly independent over $\mathbb{C}$. Since F_c is an affine algebraic variety $r|F_c$ has only finitely many critical values [Mil] Corollary 2.8. (for a more general function Sard's Theorem could be used). Therefore, we may choose a large ball B containing all the critical points of F and such that $F\colon \partial B \to \mathbb{C}$ is full rank at every point of $F_c \cap \partial B$ (see Fig. 1). Thus $\partial F(x)$ and $\bar{x}$ are linearly independent over $\mathbb{C}$ in a neighborhood of $S_\varepsilon \cap F_c$. It follows that sufficiently small open discs D centered at c the map f: $\partial B \cap F_D \to D$ will be full rank on its domain and F will have no critical points in $\overline{F_D} - B$.

The set $F_D - B^0$ (B^0 = interior of B) is a manifold with boundary $\partial B \cap F_D$. We construct a vector field on $F_D - B^0$ satisfying:

(4.1) i) $dF_x(\xi(x)) = (\partial F(x), \xi(x)) = 1$.

ii) $z\xi(x)$ is tangent to $\partial B \cap F_D$ for $x \in \partial B \cap F_D$, $z \in \mathbb{C}$, i.e., $\mathrm{Re}(\bar{x}, z\xi(x)) = 0$,

iii) $\|\xi(x)\|$ is bounded.

The vector field $\xi_1(x)=\overline{\partial F(x)}/\|\partial F(x)\|^2$ is defined on F_D-B and trivially satisfies i) there. Since F is tame, $\|\partial F(x)\|$ is bounded away from zero on $\overline{F_D-B}$ and so iii) holds for ξ_1 as well. In a compact neighbourhood U of $\partial B\cap F_D$, $\partial F(x)$ and $\bar{x}$ are linearly independent over $\mathbb{C}$ so we can find a smooth $\xi_2(x)$, defined on U, satisfying $(\partial F(x), \xi_2(x))=1$, $(\bar{x}, \xi_2(x))=0$. Thus ξ_2 satisfies i)–iii) on U. If ϕ is a smooth function equal to 1 on $F_D\cap\partial B$ and with support in U then

$$\xi(x)=\phi(x)\,\xi_2(x)+(1-\phi(x))\,\xi_1(x), \qquad x\in F_D-B^0$$

satisfies i)–iii) above. For $z\in\mathbb{C}$ let $\phi_t(z,x)$ be the integral curve of $z\xi$, passing through x at $t=0$, and let $h\colon (F_c-B^0)\times D\to F_D-B^0$ be the map $h(x,z)=\phi_1(z-c,x)$. By i) we have $F(\phi_t(z,x))=F(x)+tz$, so $F(h(x,z))=z$, if defined. By ii) the trajectories cannot pass through ∂B and so h maps $(\partial B\cap F_c)\times D$ onto $\partial B\cap F_D$, again if h is defined. By iii) the integral curves $\phi_t(z-c,x)$ $0\leqq t\leqq 1$ are defined as long as they remain within F_D-B^0. But, as above, this is guaranteed by i) and ii); thus h is well-defined and continuous. It follows that the map $F\colon F_D-B^0\to D$ is a trivial fibration, via the trivializing homeomorphism h.

If we take ξ to be the vector field ξ_1 defined above and forget about the ball B and 4.1.ii) then we may prove, in similar fashion, the following: For $D_1\subseteq D_2$, closed discs, $F\colon F_{D_2^0}-F_{D_1}\to D_2^0-D_1$ is locally trivial if F has no critical points in $F_{D^0}-F_{D_1}$.

Pick D as above and let $\Delta'\supseteq\Delta$ be a slightly larger open disc and $D'\subseteq D$ a slightly smaller closed disc, so that $F\colon F_{\Delta'}=F_{D'}\to\Delta'-D'$ is locally trivial. By the covering homotopy theorem we may construct a retracting deformation of F_Δ into F_D to arrive at the first picture in Fig. 1. Now let B' be another closed ball centred at the origin larger than B. Using the trivialization homeomorphism h we may construct a homotopy $g\colon F_D\times I\to F_D$ such that

$$g(t,x)=x, \qquad x\in F_D\cap B$$

$$g\colon F_D\cap(B'-B^0)\times I\subseteq F_D\cap(B'-B^0)$$

and

$$g(x,1)\in F_c-(B')^0, \qquad x\in F_D-(B')^0.$$

This brings us to the second picture in Fig. 1.

The subset $B'\cap F_c$ is a deformation retract of $B'\cap F_D$ for all small D. Once this is proven, we may extend to a retracting deformation $((F_D\cap B')\cup F_c)\times[0,1]\to F_c$ by letting the homotopy be stationary on F_c-B'. This achieves the third stage of the homotopy in Fig. 1. To show that there is a retracting deformation of $B'\cap F_D$ to $B'\cap F_c$ we may use results on Prill's *good neighbourhoods* [Pr] or appeal to Durfee's results above.

Proof of Theorem 1.2. We shall use a Morse theory argument to show that $\mathbb{C}^{n+1}$ can be built up from F_c, up to homotopy, by adjoining $\mu-\mu^c$ cells of dimension $n+1$. From the long exact homotopy sequence of the pair $(\mathbb{C}^{n+1}, F_c)$, we obtain as a consequence:

$$\pi_q(F_c)=0, \qquad q<n,$$

$$\pi_n(F_c)=\mathbb{Z}^{\mu-\mu^c}.$$

If $n>1$, then by Whitehead's Theorem F_c has the homotopy type of a bouquet of $\mu-\mu^c$ n-spheres (cf. [Mi1], p. 58). If $n=1$ is is well-known that F_c is homotopy equivalent to a bouquet of circles. From the long exact homology sequence of $(\mathbb{C}^{n+1}, F_c)$ we obtain $H_1(F_c, \mathbb{Z})=\mathbb{Z}^{\mu-\mu^c}$, hence the number of circles in the bouquet is $\mu-\mu^c$.

To construct the desired Morse function we first modify F in neighborhood of its critical points. Let $P_1, \ldots, P_s$ be the critical points of F which do not lie on F_c. Let $B_i \subseteq B_i'$ be balls centered at P_i, such that B_i' is larger than B_i, all the B_i' are pairwise disjoint and no B_i' meets F_c. Let u_i be a smooth function with support in B_i' and identically equal to 1 on B_i. Let $a^1, \ldots, a^s$ be vectors in $\mathbb{C}^{n+1}$, to be chosen later, and let $L_i(X)$ be the function $L_i(X)=a_1^i X_1+\ldots+a_{n+1}^i X_{n+1}$. We define G by:

$$G(x)=F(x)-(u_1(x)\,L_1(x)+\ldots+u_s(x)\,L_s(x)),$$

and set $g(x)=|G(x)-c|^2$. Observe that $G(x)=F(x)$ away from $\bigcup_i B_i'$; by picking the a_i sufficiently small the difference $|G(x)-F(x)|$ can be made uniformly small on $\mathbb{C}^{n+1}$. With the a^i so chosen $F^{-1}(\Delta)=G^{-1}(\Delta)$, for all sufficiently small discs Δ, centered at the origin.

Let $\mathbb{C}^{n+1}$ be given its standard metric and let grad g denote the gradient vector field of g with respect to this metric.

$$\operatorname{grad} g=(F(x)-c)\,\overline{\partial F}(x), \qquad x\in\mathbb{C}^{n+1}-\bigcup_i B_i',$$

$$\begin{aligned}\operatorname{grad} g&=(G(x)-c)(\overline{\partial G}(x))\\&=(G(x)-c)(\overline{\partial F}(x)-\bar{a}^i), \qquad \text{on } B_i,\end{aligned}$$

the latter since G is holomorphic on B_i. If $\|\operatorname{grad} g(x)-\operatorname{grad} f(x)\|$ is sufficiently small on $\mathbb{C}^{n+1}$, then the only critical points of g other than those on F_c must lie in $\bigcup_i B_i$ and coincide with critical points of G. According to Proposition 2.2 the a^i may be chosen so that G has $\mu-\mu^c=\mu_{P_1}+\ldots+\mu_{P_s}$ non-degenerate critical points in $\bigcup_i B_i$. The Hessian quadratic form, $H(v,v)$, associated to G at such a critical point, P, is easily determined:

$$H(v,v)=2\operatorname{Re}(v^t J(P)v)$$

where v^t is the transpose of the $1\times(n+1)$ column vector $v\in\mathbb{C}^{n+1}$ and $J(P)$ is the matrix $\left(\overline{(G(P)-c)}\,\frac{\partial^2 G}{\partial X_i\,\partial X_j}(P)\right)$. Since $G(P)\neq c$ and P is a non-degenerate critical point of G, $J(P)$ is non-singular, and thus P is a non-degenerate critical point of g. Since $H(iv, iv)=-H(v,v)$ the Morse index of g at P is $n+1$, one-half the real dimension of $\mathbb{C}^{n+1}$.

Now consider g as a Morse function on $\mathbb{C}^{n+1}$. By Lemma 4.3 $g^{-1}(0)=F_c$ is a deformation retract of $M_a=\{x\in C^{n+1}: g(x)\leq a\}$ for all sufficiently small a. If g were a proper map, then we could use standard Morse theory arguments to show $\mathbb{C}^{n+1}$ can be built up from M_a by adding $\mu-\mu^c$ $(n+1)$-cells. Since

F_c is a deformation retract of M_a, then $\mathbb{C}^{n+1}$ may be obtained, up to homotopy, by adding $\mu-\mu^c$ $(n+1)$-cells to F_c itself.

Even though g is not a proper map, we may show that the standard Morse theory arguments still apply because of the following considerations:

(4.2) i) For $0<a<b$, $M_{a,b}=\{x\in\mathbb{C}^{n+1}: a\leqq g(x)\leqq b\}$ is a complete subset of $\mathbb{C}^{n+1}$ with respect to the standard Hermitian metric on $\mathbb{C}^{n+1}$.

ii) grad $g(x)$ is bounded away from 0 off a compact neighborhood of the critical points of g in $M_{a,b}$.

In §3 of Milnor's book on Morse theory [Mi2], (4.2) i), ii) may be used to establish the completeness of the vector field grad $g(x)/\|\text{grad } g(x)\|^2$, and vector fields obtained from it by modification, in the set $M_{a,b}$. Therefore, all the theorems of standard Morse theory occurring in §3 of [Mi2], remain valid under the weaker hypotheses (4.2), and the Morse theory arguments above are valid.

§5. Topology of general polynomial hypersurfaces

In this section we prove Theorem 1.1 of the Introduction. Also for a class of polynomials more general than tame polynomials we completely determine the homology of the generic level set. For this class of polynomials the abelian group A of Theorem 1.1 will be non-zero in general.

Proof of Theorem 1.1. Let $\alpha_F=\{b_1, \dots, b_s\}$ denote the atypical values of F. Let Δ_i be a small, closed disc centered at b_i such that the discs are pairwise disjoint and let $a_i\in\Delta_i-\{b_i\}$. For the vanishing homology groups described in the introduction we have the isomorphism [Br1]:

$$V_q^i=H_{q+1}(F_{\Delta_i}, F_{a_i}).$$

The latter group does not depend, up to isomorphism, on Δ_i or a_i if Δ_i is sufficiently small.

The isomorphism (1.2) becomes:

$$\tilde{H}_q(F_c)\simeq\bigoplus_{i=1}^{s} H_{q+1}(F_{\Delta_i}, F_{a_i}). \tag{5.1}$$

If $\bar{\mu}^{b_i}$ is the sum of the local Milnor numbers at isolated critical points on F_{b_i}, then $\bar{\mu}=\bar{\mu}^{b_1}+\dots+\bar{\mu}^{b_s}$. We can prove the desired conclusion, $\tilde{H}_n(F_c)\simeq\mathbb{Z}^{\bar{\mu}}\oplus A$, by proving:

$$H_{n+1}(F_{\Delta_i}, F_{a_i})\simeq\mathbb{Z}^{\bar{\mu}^{b_i}}\oplus A_i, \tag{5.2}$$

where A_i is some abelian group. We prove (5.2) using a Mayer-Vietoris argument. Set $\Delta=\Delta_i$, $b=b_i$, $a=a_i$ and let $P_1, \dots, P_l$ be the isolated critical points of F, lying on $F^{-1}(b)$. We may choose small, pairwise disjoint balls B_i, centered at P_i, and shrink Δ if necessary, so that over a slightly larger open disc $\Delta'\supseteq\Delta$, $F: \partial B_i\cap F^{-1}(\Delta')\to\Delta'$ is a proper submersion. Let $B=\bigcup_i B_i$, $R_\Delta=F_\Delta\cap B$, T_Δ

$=\overline{F_\Delta - R_\Delta}$, $S_\Delta = \partial B \cap F_\Delta = R_\Delta \cap T_\Delta$ and let $R_a = R_\Delta \cap F_a$, $S_a = S_\Delta \cap F_a$, $T_a = T_\Delta \cap F_a$. By the hypothesis on $F: S_\Delta \to \Delta$ and Ehresmann's Fibration Theorem, $F: S_\Delta \to \Delta$ is locally trivial, so S_a is a deformation retract of S_Δ. The pairs (R_Δ, S_a), (S_Δ, S_a), (T_Δ, T_a) satisfy $(R_\Delta, R_a) \cup (T_\Delta, T_a) = (F_\Delta, F_a)$ and $(R_\Delta, R_a) \cap (T_\Delta, T_a) = (S_\Delta, S_a)$. Thus we get a Mayer-Vietoris sequence:

$$\ldots \to H_q(S_\Delta, S_a) \to H_q(R_\Delta, R_a) \oplus H_q(T_\Delta, T_a) \to H_q(F_\Delta, F_a) \to H_{q-1}(S_\Delta, S_q) \to \ldots .$$

Since S_a is a deformation retract of S_Δ then the end groups above are trivial and the middle arrow is an isomorphism. Since the B_i are pairwise disjoint $H_q(R_\Delta, R_a) \simeq \bigoplus_{i=1}^{l} H_q(F_\Delta \cap B_i, F_a \cap B_i)$. Now, for Δ, B_i small enough $F_\Delta \cap B_i$ is contractible by ii) of Corollary 4.2. Thus:

$$H_q(R_\Delta, R_a) \simeq \bigoplus_{i=1}^{l} \tilde{H}_{q-1}(F_a \cap B_i).$$

Since $F_a \cap B_i$ is a Milnor fibre with the homotopy type of a bouquet of μ_{P_i} n-spheres:

$$\tilde{H}_n(F_a \cap B_i) = \mathbb{Z}^{\mu_{P_i}}.$$

Taking $A_i = H_{n+1}(T_\Delta, T_a)$ the proof of the theorem is complete.

Remarks. 1) Recall the homomorphism, (1.2), $h_*: \tilde{H}_n(F_a \cap B) \to H_n(F_c)$ obtained by trivializing $F: \mathbb{C}^{n+1} - F^{-1}(\alpha_F) \to \mathbb{C} - \alpha_F$ along a path in $\mathbb{C} - \alpha_F$ from a to c. If we choose a set of non-intersecting paths from $a_1, \ldots, a_s$ to c and if $B_{i,j}$ are pairwise disjoint balls about the critical points $P_1, \ldots, P_j, \ldots, P_{l_i}$ lying on F_b, then the sum of subgroups:

$$\sum_{i,j} h_*(\tilde{H}_n(F_{a_i} \cap B_{i,j}))$$

is an internal direct summand of $\tilde{H}_n(F_c)$ isomorphic to Z^μ. This follows from the above proof and the discussion in [Br 1].

2) If F has only isolated critical points then $H_q(F_c)$ can be split into two parts, one localized at the critical points of F and the other localized at infinity. Let $b \in \alpha_F$, let $B_1, \ldots, B_l$, $B = \bigcup_i B_i$ be as above and U be the complement in $\mathbb{C}^{n+1}$ of a large ball centered at the origin. If ∂B and ∂U are transverse to F_b then for small Δ, $F: F_\Delta - (B \cup U) \to \Delta$ is locally trivial and applying the Mayer-Vietoris argument above (twice) we get:

$$H_{q+1}(F_\Delta, F_a) \simeq H_{q+1}(F_\Delta \cap B, F_a \cap B) \oplus H_{q+1}(F_\Delta \cap U, F_a \cap U).$$

By picking the B_i arbitrarily small and Δ small as well we can "localize" the first summand into arbitrarily small neighbourhoods of the critical points of F lying on F_b. By picking U to be the complement of arbitrarily large balls we can "localize" the second summand at infinity. Do this for each atypical value and use (5.1) to split $\tilde{H}_q(F_c)$ as claimed.

Compactification. In order to calculate the subgroup of $\tilde{H}_n(F_c)$ "localized at infinity" it is useful to introduce the following compactification of $F: \mathbb{C}^{n+1} \to \mathbb{C}$. For convenience, temporarily set $m=n+1$. Let $e: \mathbb{C}^m \to \mathbb{P}^m$, be the embedding

$$x=(x_1, \ldots, x_m) \to (x_1: \ldots: x_n: 1)=(x: 1),$$

$E: \mathbb{C}^m \to \mathbb{P}^m \times \mathbb{C}$ the map $x \to (e(x), F(x))$, M the Zariski closure of $E(\mathbb{C}^m)$ in $\mathbb{P}^m \times \mathbb{C}$ and $p: M \to \mathbb{C}$ the map $(y, z) \to z$. We have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^m & \xrightarrow{E} & M \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow{\text{id}} & \mathbb{C} \end{array}$$

The map $p: M \to \mathbb{C}$ is proper; we will call $P: M \to \mathbb{C}$ a *compactification* of $F: \mathbb{C}^m \to \mathbb{C}$. Write $F(X)=F^d(X)+\ldots+F^0(X)$ where F^j is homogeneous of degree j, $X=(X_1, \ldots, X_m)$, and define:

$$G(X, Z, t)=F^d(X)+ZF^{d-1}(X)+\ldots+Z^d F^0(X)-tZ^d.$$

The set M is given by: $M=\{((x: z), t) \in \mathbb{P}^m \times \mathbb{C}: G(x, z, t)=0\}$. Thus $M_c=p^{-1}(c)$ equals the set $\{(x: z): G(x, z, c)=0\}$, i.e., $M_c=E(F_c) \cup (A \times \{c\})$, where $A \subseteq \mathbb{P}^m$ is the *axis* of M, defined by:

$$A=\{(x: z): z=G(x, z, t)=0\}=\{(x: z): F^d(x)=z=0\}.$$

According to Verdier [V], Cor. 5.1, the map $p: M \to \mathbb{C}$ is a locally trivial fibration except over finitely many points; we extend the terminology of near-fiberings and typical and atypical values to this situation, denoting the set of atypical values by α_p. Denote $p^{-1}(U)$ by M_U, $U \subseteq \mathbb{C}$. The near-fiberings $p: M \to \mathbb{C}$ and $F: \mathbb{C}^m \to \mathbb{C}$ are conveniently related by:

Proposition 5.1. *Let $F: \mathbb{C}^m \to \mathbb{C}$ be a polynomial map and $p: M \to \mathbb{C}$ the associated compactification and let Δ be a disc centered at $b \in \mathbb{C}$ such that $\Delta-\{b\}$ does not meet α_F or α_p, and let $a \in \Delta-\{b\}$. Then:*

(5.3) $$H_q(F_\Delta, F_a) \simeq H^{2m-q}(M_\Delta, M_a).$$

We defer the proof of this until Sect. 6.

The compactification M need not be a smooth variety; by the Jacobian criterion its singular locus is given by

$$\frac{\partial G}{\partial X_1}=\ldots=\frac{\partial G}{\partial X_m}=\frac{\partial G}{\partial Z}=\frac{\partial G}{\partial t}=0,$$

since G is a square free polynomial of the $m+2$ variables $X_1, \ldots, X_m, Z, t$. We may equivalently write these equations as:

(5.4) $$\frac{\partial F^d}{\partial X_1}=\ldots=\frac{\partial F^d}{\partial X_m}=F^{d-1}=Z=0.$$

Thus the singular locus has the form $\Sigma\times\mathbb{C}$ where $\Sigma\subseteq A$ is defined by (5.4). Furthermore, the critical points of $p\colon M-(\Sigma\times\mathbb{C})\to\mathbb{C}$ are of the form $(e(P), F(P))$ where P is a critical point of F. We now restrict our attention to those polynomials having only isolated critical points and such that Σ is a finite set. The set Σ will be finite if and only if the subvariety S of $\mathbb{C}^m$ defined by

$$\frac{\partial F^d}{\partial X_1}=\ldots=\frac{\partial F^d}{\partial X_m}=F^{d-1}=0$$

satisfies $\dim S\leqq 1$. If F is a non-constant polynomial of two variables $\dim S\leqq 1$ is automatic. If F is a polynomial in three variables $\dim S\geqq 2$ iff there is an irreducible non-zero factor of F^{d-1} whose square divides F^d. In particular, if F^d is square-free, then Σ is finite.

Let $\tilde{w}\in\Sigma$. Pick a j so that $\tilde{w}$ belongs to the open set in $\mathbb{P}^m$ defined by $X_j\neq 0$. Define the polynomial of m variables, H_t, by:

$$H_t(Y_1,\ldots,Y_{m-1},Z)=G(Y_1,\ldots,Y_{j-1},1,Y_{j+1},\ldots,Y_m,Z,t)$$

(H_t will not denote $H^{-1}(t)$ though F_c will still denote $F^{-1}(c)$). The point $\tilde{w}$ determines a point $(x_1,\ldots,x_{j-1},1,x_{j+1},\ldots,x_m,0)\in\mathbb{C}^{m+1}$; let

$$w=(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_m,0).$$

By construction $H_t(w)=0$ for all $t\in\mathbb{C}$. Moreover for each $t\in\mathbb{C}$:

(5.5) i) w is an isolated, critical point of H_t,

ii) there is a ball B centered at w such that H_s has no critical points on $H_s^{-1}(0)\cup(B-\{w\})$ for $s\in\Delta$, Δ some disc centered at t.

To see this we argue as follows. Let $t\in\mathbb{C}$ be fixed, and note that $F(X_1,\ldots,X_m)-t=G(X_1,\ldots,X_m,1,t)$ and

$$H_t(Y_1,\ldots,Y_m,Z)=G(Y_1,\ldots,Y_{j-1},1,Y_j,\ldots,Y_{m-1},Z,t).$$

Thus the hypersurfaces in $\mathbb{C}^m$, defined by $F(X)=t$, and $H_t(Y,Z)=0$ may be considered as subsets of $\mathbb{P}^m$ both of whose closures is the projective hypersurface $G(X,Z,t)=0$. The polynomial $F(X)-t$ is a square-free polynomial in the variables $X_1,\ldots,X_m$. Otherwise the intersection of the critical point set of F with $F^{-1}(t)$ would have dimension $m-1$, and F has only isolated critical points. Thus $G(X,Z,t)$ is a square-free polynomial in the variables $X_1,\ldots,X_m,Z$, and hence $H_t(Y,Z)$ is a square-free polynomial in the variables $Y_1,\ldots,Y_{m-1},Z$. This implies that the critical points of H_t lying on $H_t^{-1}(0)$ are precisely the singular points of $H_t^{-1}(0)$. These singular points form a subset of the singular points of $G(X,Z,t)=0$, and these latter consist of all $\tilde{w}\in\Sigma$, in addition to the critical points of F lying on $F^{-1}(t)$. Therefore, the critical points of H_t which lie on $H_t^{-1}(0)$ consist of those $w\in\mathbb{C}^m$ for which $\tilde{w}\in\Sigma$ lies in the subset of $\mathbb{P}^m$ defined by $X_j\neq 0$, in addition to the points corresponding to those critical points of F lying on $F^{-1}(t)$ and in the set defined by $X_j\neq 0$. This clearly implies i) and ii) above.

By Proposition 2.3 for each $\tilde{w}\in\Sigma$ there is an $N_w\geqq 1$ such that $N_w\leqq\mu_w(H_t)<\infty$ with equality for all but finitely many $t\in\mathbb{C}$. Set

$$\lambda_w^t = \mu_w(H_t) - N_w,$$
$$\lambda^t = \sum_{\tilde{w}\in\Sigma} \lambda_w^t,$$
$$\lambda = \sum_{t\in\mathbb{C}} \lambda^t.$$

We may now state the main theorem of this section.

Theorem 5.2. *Let $F: \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial with only isolated critical points and such that the Eq. (5.4) with $m=n+1$ have only finitely many solutions in $\mathbb{P}^{n+1}$. Let λ, λ^c be as defined above and μ the total Milnor number of F. Then if the level set F_c is smooth and $\lambda^c=0$, we have:*

$$\tilde{H}_n(F_c) = \mathbb{Z}^{\mu+\lambda}$$
$$\tilde{H}_q(F_c) = 0 \quad q \neq n.$$

Proof. Let $b\in\mathbb{C}$, Δ a disc centered at b such that $(\Delta - \{b\}) \cap \alpha_F = 0$ and let $a\in\Delta - \{0\}$. We shall show below:

$$(5.6)\qquad \begin{aligned} H_{n+1}(F_\Delta, F_a) &= \mathbb{Z}^{\mu^b+\lambda^b},\\ H_q(F_\Delta, F_a) &= 0, \quad q \neq n+1. \end{aligned}$$

Assuming (5.6), if $b=c$ is a typical value then, $F: F_\Delta \to \Delta$ is locally trivial and we conclude that $\mu^c = \lambda^c = 0$. Since F has only isolated critical points, F_c is smooth if and only if ∂F has no zeros on F_c, if and only if $\mu^c = 0$. Therefore, a typical level set F_c is smooth and satisfies $\lambda^c=0$. On the other hand, if F_c is smooth and $\lambda^c=0$, it does not follow (by a simple argument at least) that c is a typical value. Thus, the typical values could conceivably form a proper subset of the values for which we want to prove the theorem. Therefore, we shall prove the theorem in two steps:

Step 1: Prove the theorem for c a typical value.

Step 2: Use Step 1 to prove the theorem for any smooth F_c with $\lambda^c=0$.

Step 1. Let c be a typical value. Let b, Δ, a be as above and let $p: M \to \mathbb{C}$ be the compactification defined earlier. By the formulae (5.1) it suffices to prove (5.6). In turn, by (5.3) of Proposition 5.1, the formulae (5.6) follow from (shrinking Δ if necessary and setting $m=n+1$):

$$(5.7)\qquad \begin{aligned} H^{n+1}(M_\Delta, M_a) &\simeq \mathbb{Z}^{\mu^b+\lambda^b}\\ H^q(M_\Delta, M_a) &\simeq 0, \quad q \neq n+1. \end{aligned}$$

Let $\tilde{w}\in\Sigma$ and X_j, H_t be as previously defined. The subset W of $\mathbb{P}^{n+1}$ defined by $X_j \neq 0$ is an open neighbourhood of $\tilde{w}$ which may be identified with $\mathbb{C}^{n+1}$; H_t is a polynomial defined on W, let $w\in\mathbb{C}^{n+1}$ be the point associated to $\tilde{w}$. Let $B_w \subseteq W$ be a closed ball centered at w; under the identifications above, $M_a \cap (B_w \times \mathbb{C})$ is identified with $(H_a^{-1}(0) \cap B_w) \times \{a\}$. Let $U = M \cap (\bigcup_{\tilde{w}\in\Sigma} B_w \times \mathbb{C})$, it is an closed neighborhood of $\Sigma \times \mathbb{C}$ in M. If we choose B_w sufficiently small then $H_b^{-1}(0)$ will meet ∂B_w transversely; doing this for all $\tilde{w} \in \Sigma$ we ensure that M_a meets ∂U transversely for all $a \in \mathbb{C}$ near b. Next, surround the critical points of F that lie on F_b by small pairwise disjoint closed balls; call the union B. Choose B so that $F: \partial B \cap F_\Delta \to \Delta$ is full rank for small Δ. The set B may be identified with a subset of M via the embedding $E: \mathbb{C}^{n+1} \to M$. For small open Δ, $T_\Delta = M_\Delta - (B^0 - U^0)$ is a manifold with boundary $(M_\Delta \cap \partial B) \cup (M_\Delta \cap \partial U)$ and $p: T_\Delta \to \Delta$ is a proper submersion, hence locally trivial. By a Mayer-Vietoris argument, we may excise $(T_\Delta, T_\Delta \cap M_a)$ from (M_Δ, M_d) to obtain:

$$H^q(M_\Delta, M_a) \simeq H^q(M_\Delta \cap B, M_a \cap B) \oplus H^q(M_\Delta \cap U, M_a \cap U).$$

Now $(M_\Delta \cap B, M_a \cap B) \simeq (F_\Delta \cap B, F_a \cap B)$, so by previous calculation and the Universal Coefficient Theorem $H^{n+1}(M_\Delta \cap B, M_a \cap B) = \mathbb{Z}^{\mu^b}$, $H^q(M_\Delta \cap B, M_a \cap B) \simeq 0$, $q \neq n+1$. Thus to prove (5.7), it suffices to show

$$(5.8) \qquad \begin{aligned} &H^{n+1}(M_\Delta \cap U, M_a \cap U) \simeq \mathbb{Z}^\lambda, \\ &H^q(M_\Delta \cap U, M_a \cap U) \simeq 0, \qquad q \neq 0. \end{aligned}$$

This calculation is deferred to Sect. 6.

Step 2. F_c is smooth, $\lambda^c = 0$. Let Δ be a small open disc surrounding c, as constructed in Step 1. It suffices to prove that $H_q(F_c) \simeq H_q(F_\gamma)$ for some $\gamma \in \Delta - \{c\}$, since these γ are all typical values. Pick such a γ. Let $A \subseteq \mathbb{P}^m$ be the axis of M defined above, set $A_c = A \times \{c\}$ so that $F_c = M_c - A_c$. By the triangulation theorem [Lo] for compact algebraic varieties (M_c, A_c) is a compact polyhedral pair and $M_c - A_c = F_c$ is an orientable $2n$-dimensional manifold. Thus, we have a duality isomorphism

$$H_q(F_c) \simeq H^{2n-q}(M_c, A_c).$$

Since F_γ is also smooth then $H_q(F_\gamma) \simeq H^{2n-q}(M_\gamma, A_\gamma)$, so we need only show $H^q(M_c, A_c) \simeq H^q(M_\gamma, A_\gamma)$ $q \geqq 0$. Consider the homeomorphism of exact sequences:

$$\begin{array}{ccccccccc} H^{q-1}(M_c) & \to & H^{q-1}(A_c) & \to & H^q(M_c, A_c) & \to & H^q(M_c) & \to & H^q(A_c) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^{q-1}(M_\Delta) & \to & H^{q-1}(A_\Delta) & \to & H^q(M_\Delta, A_\Delta) & \to & H^q(M_\Delta) & \to & H^q(A_\Delta). \end{array}$$

According to Proposition 4.1, above, M_c is a deformation retract of M_Δ and clearly $A_c = A \times \{c\}$ is a deformation retract of $A_\Delta = A \times \Delta$. Therefore, in the above diagram the four outside vertical arrows are isomorphisms and, hence, so is the middle arrow by the Five Lemma. Now again consider the same diagram except that c is replaced by γ. The second and fifth vertical arrows are still isomorphisms for the same reasons. Since $\lambda^c = \mu^c = 0$ then, by (5.7) of Step 1, $H^q(M_\Delta, M_\gamma) = 0$ for all q and the first and fourth arrows are also isomorphisms. Thus $H_q(F_c) \simeq H^{2n-q}(M_\Delta, A_\Delta) \simeq H_q(F_\gamma)$. All is now proven.

Remark. If F_c is smooth but $\lambda^c \neq 0$ we still get a map $\beta: H_q(F_c) \to H_q(F_\gamma)$ via:

$$\beta: H_q(F_c) \xrightarrow{\sim} H^{2n-q}(M_c, A_c) \xleftarrow{\sim} H^{2n-q}(M_\Delta, A_\Delta) \to H^{2n-q}(M_\gamma, A_\gamma) \xrightarrow{\sim} H_q(F_\gamma).$$

Using the Five Lemma we see that β is an isomorphism for $q \neq n-1, n$ and injective for $q=n$. A bit of diagram chasing yields:

$$\lambda^c = rk(H_{n-1}(F_c)) + rk(H_n(F_y)/H_n(F_c)).$$

Example 5.1. Let $F(X, Y, Z) = XYZ - X - Y$. There are three solutions to the Eq. (5.4) only one of which yields a jump in the Milnor number; we have: $\lambda^0 = 2$, $\lambda^c = 0$, $c \neq 0$, $\lambda = 2$. Since F has no critical points we have for $c \neq 0$:

$$H_2(F_c) = \mathbb{Z}^2$$
$$\tilde{H}_q(F_c) = 0 \qquad q \neq 2.$$

By considering the Morse function $r(X, Y, Z) = |X|^2 + |Y|^2 + |Z|^2$ on $XYZ = X + Y$ we determine:

$$H_2(F_0) = \mathbb{Z}$$
$$H_1(F_0) = \mathbb{Z}$$
$$\tilde{H}_q(F_0) = 0 \qquad q \neq 1, 2.$$

Here, at an atypical value we have a jump in the Betti numbers for other than the middle dimension; this phenomenon did not occur for tame polynomials.

§ 6. Proofs of remaining propositions

Proof of Proposition 5.1. We shall use a Lefschetz-type duality. If $K \subseteq L \subseteq T$ are compact polyhedra such that $T - K$ is an orientable n-manifold then:

$$(6.1) \qquad H_q(T-K, T-L) \simeq H^{n-q}(L, K),$$

Dold ([D], Proposition 7.14). In particular, the above will hold if K, L, are compact semi-algebraic sets, according to the triangulation theorem on semi-algebraic sets.

Let V be the union of the shaded disc and blackened line shown in Fig. 2. Let $U = \Delta^0 - V$ be the singly-hatched open region in Fig. 2. Let $a \in \Delta$ be the point given in the hypotheses and let $\beta \in U$. Then

$$H_q(F_\Delta, F_a) \simeq H_q(F_\Delta, F_\beta) \simeq H_q(F_\Delta, F_U) \simeq H_q(F_{\Delta^0}, F_U).$$

The first two isomorphisms follow from the local triviality of $\{F: F_\Delta - F_b \to \Delta - b\}$, the Covering Homotopy Theorem and the fact that U may be contracted to β within $\Delta - \{b\}$. The third isomorphism follows from the fact that the inclusion $F_{\Delta^0} \to F_\Delta$ is a homotopy equivalence, since for any slightly smaller closed disc Δ', concentric with Δ, $F_{\Delta'}$ is deformation retract of both F_{Δ^0} and F_Δ, by the Covering Homotopy Theorem.

Set $T = M_\Delta$, $L = M_V \cup M_{\partial\Delta} \cup A_\Delta$, $K = M_{\partial\Delta} \cup A_\Delta$ then $T - K = F_{\Delta^0}$, $T - L = F_U$. Then T, L, K are as in (6.1) and $T - L$ is a $2m$-dimensional, orientable (real)

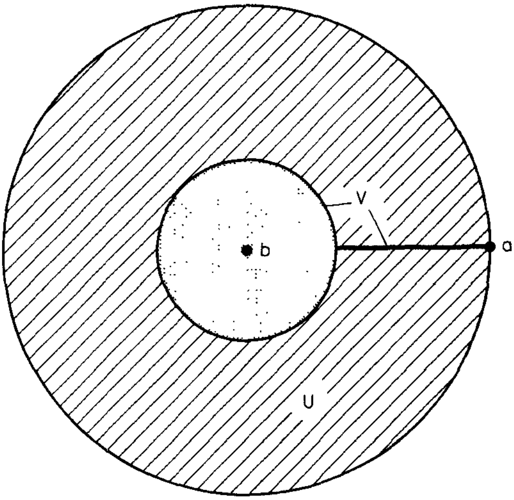

Fig. 2

manifold, so:

$$H_q(F_{\Delta^0}, F_U) \simeq H^{2m-q}(M_V \cup M_{\partial\Delta} \cup A_\Delta, M_{\partial\Delta} \cup A_\Delta).$$

Let $\{\gamma\} = V \cap \partial\Delta$, excise first $F_{\partial\Delta} - F_\gamma$ and then A_U to obtain

$$H^{2m-q}(M_V \cup M_{\partial\Delta} \cup A_\Delta, M_{\partial\Delta} \cup A_\Delta) \simeq H^{2m-q}(M_V \cup A_\Delta, M_\gamma \cup A_\Delta)$$
$$\simeq H^{2m-q}(M_V, M_\gamma \cup A_V).$$

By the triangulation theorem, everything in sight is a polyhedral pair, so the excisions are valid. Clearly $A_\gamma = A \times \{\gamma\}$ is a deformation retract of $A_V = A \times V$ so we further have:

$$H^{2m-q}(M_V, M_\gamma \cup A_V) \simeq H^{2m-q}(M_V, M_\gamma).$$

By an argument similar to the above M_V is a deformation retract of M_Δ so $H_q(M_V, M_\gamma) \simeq H_q(M_\Delta, M_\gamma)$ and $H^{2m-q}(M_\Delta, M_\gamma) \simeq H^{2m-q}(M_\Delta, M_a)$, also as above. Stringing the isomorphisms together we obtain our result.

Homology calculation. In our final proposition we consider the following situation: Let $H_t: \mathbb{C}^{n+1} \to \mathbb{C}$, $t \in \mathbb{C}$ be a family of polynomials depending polynomially on $t \in \mathbb{C}$. Suppose that for some fixed $P \in \mathbb{C}^{n+1}$, $H_t(P) = 0$ for all $t \in \mathbb{C}$ and that the conditions (5.5) hold, where w is replaced by P. Let B, Δ be as in (5.5) and such that ∂B and $H_t^{-1}(0)$ meet transversely for $t \in \Delta$; let ε and δ be the radius of B and Δ respectively. Let $M = M_{\varepsilon,\delta}$ be the set $\{(x, t) \in B \times \Delta: H_t(x) = 0\}$ and $M_t = M \cap B \times \{t\}$.

Proposition 6.1. *Let $H_t: \mathbb{C}^{n+1} \to \mathbb{C}$, $t \in \mathbb{C}$, $P \in \mathbb{C}^{n+1}$, $b \in \mathbb{C}$ and $M = M_{\varepsilon,\delta}$ be as above. Given an ε, δ may be chosen small enough so that $\mu_P(H_t)$ is independent of $t \in \Delta - \{b\}$*

and such that for all $t \in \Delta - \{b\}$:

$$H^{n+1}(M, M_t) \simeq \mathbb{Z}^\lambda, \qquad \lambda = \mu_P(H_b) - \mu_P(H_t),$$

$$H^q(M, M_t) \simeq 0, \qquad q \neq n+1.$$

Proof. Let $p: M \to \Delta$ denote the map $(x, t) \to t$. We may pick a ball B' about P of radius ε', and shrink δ so that the conditions on $M_{\varepsilon,\delta}$ hold for $M' = M_{\varepsilon',\delta}$ and such that $M'_b = B' \cap H_b^{-1}(0)$ is homeomorphic to a cone. By Ehresman's Fibration Theorem $p: \overline{M - M'} \to \Delta$ is locally trivial since each M_t meets $\partial B \cup \partial B'$ transversely and H_t has no critical points in $H_t^{-1}(0) \cap (B - B')$. By a Mayer Vietoris argument

$$H^q(M, M_t) \simeq H^q(M', M'_t), \qquad M'_t = M_t \cap M'.$$

Dropping, primes, we may assume M_b is a cone.

By ii) of Corollary 4.2, M is contractible for small enough δ and by the long exact sequence of the pair (M, M_t) we get

$$H^q(M, M_t) \simeq \tilde{H}^{q-1}(M_t).$$

Thus it suffices to show that M_t has the homotopy type of a bouquet of λ n-spheres. According to Timourian, [T], Lemma 2, there is a disc D, centered at 0, such that for t sufficiently close to b, $H_t^{-1}(D) \cap B$ is homeomorphic to $H_b^{-1}(D) \cap B$. Moreover, given D, we may shrink Δ so that for $t \in D$

$$\sum_i \mu_{P_i}(H_t) = \mu_P(H_b) - \mu_P(H_t) = \lambda$$

where $P_1, \ldots, P_s$ are the critical points of H_t contained in $H_t^{-1}(D) \cap B - \{P\}$. By hypothesis no P_i lies on M_t. For each $t \in \Delta$, $H_t^{-1}(D) \cap B$ is homeomorphic to $H_b^{-1}(D) \cap B$ which will be contractible for small D. Now, using the fact that H_t: $\partial B \cap H_t^{-1}(D) \to D$ is a submersion, the Morse Theory argument of §4 may be easily modified to prove that M_t has the homotopy type of a bouquet of λ n-spheres. All is now proven.

References

[Br 1] Broughton, S.A.: On the Topology of Polynomial Hypersurfaces. Proceedings of Symposia in Pure Mathematics, Vol. 40 (Arcata Singularities Conference) Am. Math. Soc., pp. 167–178, 1983

[Br 2] Broughton, S.A.: On the Topology of Polynomial Hypersurfaces. Ph.D. Thesis, Queen's University, Kingston, Ontario, 1982

[B-V] Burghelea, D., Verona, A.: Local Homological Properties of Analytic Sets. Manuscr. Math. **7**, 55–56 (1972)

[D] Dold, A.: Lectures on Algebraic Topology. Berlin Heidelberg New York: Springer 1972

[Du] Durfee, A.: Neighbourhoods of Algebraic Sets. Transactions of Am. Math. Soc. **276**, 517–530 (1983)

[G] Grothendieck, A.: Éléments de Géométrie Algebrique. IV, Étude Local de Schémas et de Morphismes de Schémas, Publications Mathématiques I.H.E.S., No. 28, 1966

[H] Hartshorne, R.: Algebraic Geometry. Graduate Texts in Math., Berlin Heidelberg New York: Springer 1977

[K] Kouchnirenko, A.G.: Polyèdres de Newton et Nombres de Milnor. Invent. Math. **32**, 1–31 (1976)

[Lo] Lojasiewicz, S.: Triangulations of Semi-Analytic Sets. Ann. Scuola Norm. Super. Pisa, Sci. Fis. Mat. Ser. 3, **18**. 339–373 (1964)

[L-T] Lejeune-Jalabert, M., Teissier, B.: Contribution à l'étude des singularités de point de vue du poly gone de Newton. Cycle évanoissants, plis évanouissants et conditions de whitney. Thesis, Université de Paris VII, Paris 1973

[Ma] Malgrange, B.: Méthodes de la Phase Stationaire et Sommation Borel, in Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory. (Lect. Notes Phys., vol. 126) Berlin Heidelberg New York: Springer 1980, pp. 170–177

[Mi1] Milnor, J.: Singular Points of Complex Hypersurfaces, Ann. Math. Stud. vol. 61, Princeton University Press, 1968

[Mi2] Milnor, J.: Morse Theory. Ann. Math. Stud. vol. 51, Princeton University Press 1963

[Or] Orlik, P.: The Multiplicity of a Holomorphic Map at an Isolated Critical Point. Nordic Summer School/NAVF, Symposium in Pure Mathematics (1976), pp. 405–474

[Pa] Palamadov, V.P.: Multiplicity of Holomorphic Mapping, Functional Analysis and its Applications, 1967, pp. 218–266

[Ph] Pham, F.: Vanishing Homologies and the *n*-Variable Saddlepoint Method. Proceedings of the Symposia in Pure and Applied Math., Vol. 40, Part 2, Am. Math. Soc. 1983, pp. 319–334

[Pr] Prill, D.: Local Classification of Quotients of Complex Manifolds by Discontinuous Groups. Duke Math. J. **34**, 375–386 (1967)

[S] Shafarevich, I.R.: Basic Algebraic Geometry. Berlin Heidelberg New York: Springer 1977

[T] Timourian, J.: The Invariance of Milnor's Number Implies Topological Triviality. Am. J. Math. **99**, 437–446 (1977)

[V] Verdier, J.L.: Stratification de Whitney et Théorème de Bertini-Sard. Invent. Math. **99**, 295–312 (1977)

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