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Higher Order Twisted Cohomology Operations*

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This paper gives a definition and axiomatization of higher order twisted cohomology operations and an application to the problem of classifying cross sections of fibrations. The axioms are analogous to, and essentially include, those given by Maunder [22] for ordinary higher order operations. The operations are associated with relations in $A(D)=H^*(D; Z_p)\odot A_p$ where D is a fixed space, A_p the mod p Steenrod algebra, and $\odot$ the semi-tensor product operation of Massey and Peterson [21]. The operations are defined on the underlying vector space of $\hat{x}\, H^*(X, B; Z_p)$ where $(X, B, \hat{x})$ is a pair over D (meaning $B\subset X$ and $\hat{x}: X\to D$) and $\hat{x}\, H^*(X, B; Z_p)$ is simply $H^*(X, B; Z_p)$ viewed as an $A(D)$-module via $\hat{x}$.

There are four parts and an appendix. An outline follows.

Part I. Motivation, Statement of Axioms. A particular twisted secondary operation is examined from several viewpoints. The purpose of this is to motivate the definition that is ultimately given for the general case. (The casual reader may wish to read this section and then skip to Part V.) The axioms and existence and quasi-uniqueness theorems are stated.

Part II. The Category $\mathcal{T}(u)$, $u: C\to D$. The study of the homotopy properties of this category of spaces and maps "under C and over D by u" is motivated by I. It is shown that $\mathcal{T}D=\mathcal{T}(\mathrm{id})$, $\mathrm{id}: D\to D$, has all of the good properties of $\mathcal{T}*=$ the category of base pointed spaces and maps. Homotopy operations are developed in the category and a Peterson-Stein formula is proved relating functional and secondary operations.

Part III. Proofs of the Theorems of I. The theorems are restated in the language of $\mathcal{T}D$. There the techniques of Adams [1] and Maunder [22] could be applied directly. However, a slightly different method is outlined which owes something to Spanier [30, 31] and Alvez [2].

Part IV. Some Exact Sequences. Some special cases of the general results of Part II are considered in this part. The functional cup product of Steenrod [34] and the twisted functional operation of Meyer [24] and Gitler-Stasheff [9] are shown to be special cases of the homotopy operations of Part II.

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Part V. An Application. In this part twisted operations are applied to the problem of classifying cross sections of a fibration. Let $p: Y \to B$ be a fiber bundle over a CW-complex B with an n-connected fiber. Let Sect$[p]$ denote the set of homotopy classes of cross sections of p. Sect$[p]$ has been computed by Steenrod [33], Liao [18], Boltyanski, James-Thomas [16], and James [12], for dim $B \leqq n+i$ and i small. Their results often apply both to the stable range ($i \leqq n$) and the unstable range. The methods of the present paper are applied only to the stable range – however, the results there are a substantial generalization of the earlier results. Theoretically the problem is solved (in various ways) for the entire stable range (dim $B \leqq 2n$). The answer is stated in terms of a decomposition of p (e.g. a modified Postnikov system in the sense of Mahowald [19]) and twisted operations in the cohomology of B. The general result is made concrete in a case involving secondary operations. An application to the problem of classifying immersions is given. The theme of this part can be stated: twisted operations bear the same relation to the classification of liftings as ordinary operations do to the classification of mappings.

The operations of this paper, based on relations in $A(D)$, were conceived and developed by the author by analogy with ordinary operations based on $A(*)=A$. The motivation was certain relations in the calculations of Mahowald [19] (see also Thomas [36]). However, similar operations were considered independently and earlier by others: (1) Meyer gave in [24] both a chain complex and an invariant definition of functional twisted operations and posed the problem of axiomatizing twisted operations; and (2) Gitler and Stasheff, in [9], gave an invariant definition of the functional operation and used it to define a large indeterminacy version of a twisted secondary operation. However, these authors do not prove the additivity property of their twisted secondary operation. This property is immediate from the definition of Part III. Also, it would be difficult to handle the indeterminacy of operations of higher than the second order by their methods (or even to define such operations). The thesis of this paper is that the "natural" category for $A(D)$-operations is $\mathscr{T} D$ and that general properties of the operations are best proved by stating them in the language of $\mathscr{T} D$, proving them, then translating into more common language.

Higher order twisted operations are also mentioned by Lamore [17]. They have been applied by Thomas [37] to the problem of the existence of cross sections of a fibration.

This paper is a revised and extended version of my thesis [20]. I wish to thank Professor E. Thomas for serving as my thesis advisor. I have profited from his lectures on fiber spaces and I am grateful for having had the opportunity of writing up several of them. I also wish to thank Professor W. Massey for expressing his interest in this work.

Part I. Motivation, Statement of Theorems

1. Primary Operations. Let $A(D)=H^*(D;Z_p)\odot A_p$. Recall that as a vector space this is simply $H^*(D;Z_p)\otimes A_p$. The product is given by: $(d\otimes a)(e\otimes b)=\sum_i(-1)^* d\,a_i'(e)\otimes a_i''\,b$ where $*=(\deg a_i'')(\deg e)$ and $\Psi a=\sum_i a_i'\otimes a_i''$, Ψ being the coproduct in A_p. $A(D)$ is an associative algebra containing A and D as sub-algebras (Massey-Peterson [21]).

Let $\hat{x}:X\to D$ be a fixed map and $B\subset X$. Then $H^*(X,B)$ is an $A(D)$-module via $(d\otimes a)\cdot h=f^*(d)\,a(h)$ (use Z_p coefficients everywhere now). If necessary denote this $A(D)$-module by $\hat{x}H^*(X,B)$. The exact sequence of the pair $B\subset X$ (or of a triple $C\subset B\subset X$) is an exact sequence of $A(D)$-modules. If $\hat{y}:Y\to D$ and $f:X\to Y$ is such that $\hat{y}f$ is homotopic to $\hat{x}$ then f^* is an $A(D)$-map.

Consider the following situation

$$\begin{array}{ccc} E_f & \longrightarrow & E \\ \downarrow & \nearrow{\scriptstyle l} & \downarrow{\scriptstyle p} \\ D & \xrightarrow{f} & B \end{array}$$

where p is a fibration with fibre F and $E_f\to D$ is the induced fibration and $p\,l=f$. Then $E_f\to D$ admits a section and we can form the exact sequence of the "triple" (E,E_f,D) (use mapping cylinders to produce inclusions here and elsewhere when necessary). Assume (B,D) is $(a-1)$-connected and F is $(b-1)$-connected. A theorem of Serre says that $H^*(B,D)\to H^*(E,E_f)$ is isomorphic for $i<a+b$. We get the following exact sequence of $A(D)$-modules.

$$\cdots\to H^i(E,D)\to H^i(E_f,D)\to H^{i+1}(B,D)\to\cdots\to H^{a+b-1}(E_f,D).$$

This relative Serre sequence is useful for computing $H^*(E,D)$.

2. An Example of a Secondary Operation. The category will consist of pairs over D, $(X,B,\hat{x})$ where $B\subset X$ and $\hat{x}:X\to D$, and maps $f:(X,B)\to(X',B')$ such that $\hat{y}f\sim\hat{x}$. For simplicity we take B empty and X a CW complex here. Use Z_2 coefficients everywhere and $A=A_2$. The primary operations are the elements of $A(D)$.

Let $d\in H^2(D)$ be arbitrary, but fixed, and define $L=L_d=d\otimes 1+1\otimes Sq^2\in A(D)$. From the multiplication rule above it follows easily that we have a relation in $A(D)$: $LL+(Sq^1L)Sq^1=0$. Here and elsewhere Sq^i means $1\otimes Sq^i$. The key fact is that $Sq^2d=d^2$. Note that if $D=*$ (a point) then this is simply: $Sq^2Sq^2+Sq^3Sq^1=0$.

We want to associate a secondary operation Φ with this relation. By analogy with the case $D=*$ we should get $\Phi:\hat{x}H^n(X)\to\hat{x}H^{n+3}(X)$ such that (at least)

1) it is an additive relation for the abelian group structure. (See MacLane [20] for "additive relation". Here, as there, Def = domain of definition, Ind = indeterminacy, Ker = kernel, and Im = image.) Alternatively, Φ is a homomorphism defined on a subgroup with values in a quotient group.

2) $\operatorname{Def}\Phi = \ker L \cap \ker Sq^1 \subset H^n(X)$, i.e., Φh is defined iff $Lh = 0 = Sq^1 h$.

3) $\operatorname{Ind}\Phi = \operatorname{Im} L + \operatorname{Im} Sq^1 L$. Alternatively, Φ takes values in $H^{n+3}(X)/(\operatorname{Im} L + \operatorname{Im} Sq^1 L)$.

4) Φ is natural (for maps described above).

We will imitate the technique used by Adams [1] for $D = *$ as closely as possible. Let $K_i = K(Z_2, i)$, an Eilenberg-MacLane CW-complex of type (Z_2, i), with fundamental class $\iota_i \in H^i(K_i)$. First note that L can be realized as a map $s:(D \times K_n, D) \to (K_{n+2}, *)$, i.e. s is determined up to homotopy by $s^* \iota_{n+2} = L(1 \otimes \iota_n)$. Write L for s and $\bar{\iota}_n$ for $1 \otimes \iota_n$. (Maps, their homotopy classes, and the corresponding cohomology classes or operations will not usually be distinguished in the future.) Form the diagram:

$$\begin{array}{ccccc} D \times K_{n+1} \times K_n & \xrightarrow{i} & P & \xrightarrow{\varphi} & K_{n+3} \\ \downarrow & & \downarrow & & \\ D & \xrightarrow{j} & D \times K_n & \xrightarrow{(L, Sq^1)} & K_{n+2} \times K_{n+1} \end{array}$$

P is induced from the path-loop fibration over $K_{n+2} \times K_{n+1}$. Since a representative of (L, Sq^1) can be chosen so that $D = D \times * \subset D \times K_n$ is sent to a point by (L, Sq^1), the fibration over D induced by $(L, Sq^1)j$ is a product as shown. We can use the relative Serre sequence of Section 1 to find $\varphi \in H^{n+3}(P, D)$ such that $i^* \varphi = L\bar{\iota}_{n+1} + Sq^1 L\bar{\iota}_n$.

Let $\hat{x}: X \to D$ and $h \in \hat{x} H^*(X)$. Then $(\hat{x}, h): X \to D \times K_n$ and $(L, Sq^1)^*(\hat{x}, h) = (Lh, Sq^1 h)$, so $(\hat{x}, h)$ lifts to P iff $Lh = 0 = Sq^1 h$.

Define: $\Phi(x) = \{\varphi \hat{h} \mid p\hat{h} = (\hat{x}, h)\} \subset [X, K_{n+3}] = H^{n+3}(X)$. Properties (2) and (4) are immediate. However, (1) and (3) are not so clear. If $D = *$ then (L, Sq^1) is a loop map, P is a loop space, and φ is a loop map for n sufficiently large. This plus the exactness of

$$[X, K_{n+1} \times K_n] \to [X, P] \to [X, K_n] \to [X, K_{n+2} \times K_{n+1}]$$

give (1) and (3). If D is not a point none of these things is true. One could, with strain, establish the indeterminacy. However, the strain becomes greater for higher order operations and, furthermore, can be avoided entirely by working in the correct category.

Let $\mathscr{T}_D$ be the category of spaces and maps over D, i.e. of pairs $(X, \hat{x})$ where $\hat{x}: X \to D$ and maps $f: X \to Y$ such that $\hat{y} f = \hat{x}$ (strict equality, different from the first sentence of this section). Let $\mathscr{T}D$ be the category of spaces and maps under and over D, i.e., all $(X, \check{x}, \hat{x})$ where $\check{x}: D \to X$, $\hat{x}: X \to D$ and $\hat{x}\check{x} = 1$, and maps $f: X \to Y$ such that $\hat{y}f = \hat{x}$ and $f\check{x} = \check{y}$. Alternatively, $\mathscr{T}D$ consists of all pairs (R, r) such that $D \subset R$ and $r: R \to D$ is a retraction, and maps $f: R \to S$ which are the identity on D and such that $sf = r$.

Both categories have a natural notion of homotopy – an ordinary homotopy which at each stage is a map in the category. Denote the corresponding sets of homotopy classes by $[X, Y]_D$ (for $\mathscr{T}_D$) and $\langle X, Y \rangle$ (for $\mathscr{T}D$).

Consider the following commutative diagram

$$\begin{array}{ccccc}
H^n(X) & & \xrightarrow{\quad\Phi\quad} & & H^{n+3}(X) \\
\updownarrow & & & & \Big| \\
[X, K_n] & & & & \Big| \\
\updownarrow & & & & \updownarrow \\
[X, D \times K_n] \supset \{[\hat{x}, h]\} & \xleftarrow{p_*} & [X, P] & \xrightarrow{\varphi^*} & [X, K_{n+3}] \\
\updownarrow & & & & \updownarrow \\
[X, D \times K_n]_D & \longleftarrow & [X, P]_D & \longrightarrow & [X, D \times K_{n+3}]_D \\
\updownarrow & & & & \updownarrow \\
\langle XVD, D \times K_n \rangle & \xleftarrow{p_*} & \langle XVD, P \rangle & \xrightarrow{(q, \varphi)} & \langle XVD, D \times K_{n+s} \rangle
\end{array}$$

where $q = \pi p$ and $\pi: D \times K_n \to D$ is the projection.

Our definition of Φ is given by line $2: \Phi(h) = \varphi * p_*^{-1}[\hat{x}, h]$. More convenient definitions are given by lines 3 and 4.

We could work with line 3 exclusively, i.e., in $\mathscr{T}_D$. It can be proved directly that

$$[X, D \times K_{m+1} \times K_n]_D \to [X, P]_D \to [X, K_n]_D$$

is an exact sequence of abelian groups and that $(q, \varphi)_*$ is a homomorphism (see IV.1). However, in the proofs some ideas involving $\mathscr{T}D$ would eventually intrude; so it seems more natural to work with the definition of Φ given by line 4. We will see that all of the good properties of $\mathscr{T}*$ (= the category of pointed spaces and maps) are also valid for $\mathscr{T}D$. The diagram (1.1) is replaced by the following diagram in $\mathscr{T}D$.

$$\text{(1.2)} \qquad \begin{array}{ccccc}
D \times K_{n+1} \times K_n & \xrightarrow{i} & P & \xrightarrow{(q, \varphi)} & D \times K_{n+3} \\
& & \downarrow & & \\
& & (D \times K_n) & \xrightarrow{(\pi, L, Sq^1)} & D \times K_{n+2} \times K_{n+1}
\end{array}$$

P is the same space as in (1.1). Now, however, $P \to D \times K_n$ is induced from the path-loop fibration in $\mathcal{T}D$:

$$D \times K_{n+1} \times K_n \to D \times PK_{n+2} \times PK_{n+1} \to D \times K_{n+2} \times K_{n+1}.$$

(π, L, Sq^1) is a loop map, P is a loop space, (q, φ) is a loop map, and the appropriate sequence is exact. In brief, all of the established techniques for working with untwisted operations in $\mathcal{T}*$ can be applied to twisted operations when they are placed in $\mathcal{T}D$.

3. *Suspension.*

3.1. Definition. a) Let $\hat{x}: X \to D$. Define $C_D X = D \times 0 \cup X \times I/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation generated by $(x, o) \sim (\hat{x}(x), o)$, $x \in X$. Define $\Sigma_D X = D \times 0 \cup X \times I \cup D \times 1/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation generated by $(x, o) \sim (\hat{x}(x), o)$, $(x, 1) \sim (\hat{x}(x), 1)$, $x \in X$.

b) Let $i: B \subset X$ and $\hat{x}: X \to D$.

Define $C_D i = D \times 0 \cup B \times I \cup X \times 1/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation generated by $(b, o) \sim (\hat{x}(b), o)$, $(b, 1) \sim (i(b), 1)$, $b \in B$.

Note that each of the spaces is a space over D via a map obtained from $X \times I \xrightarrow{\text{proj}} X \xrightarrow{\hat{x}} D$. This map will be denoted by $\hat{x}$ also. Assume henceforth that B is closed in X. This ensures that $C_D B \subset C_D i$ and $\Sigma_D B \subset \Sigma_D X$. This assumption could, and probably should, be avoided in what follows by working with the cohomology of maps instead of the cohomology of pairs.

Let $u: (X, B) \to (C_D i, C_D B)$ and $v: (C_D X, C_D i) \to (\Sigma_D X, \Sigma_D B)$ be the natural inclusion and "collapsing to D" maps. They are isomorphisms on cohomology by excision and deformation retraction. Let δ be the coboundary for the exact sequence of the triple $(C_D X, C_D i, C_D B)$. It is an isomorphism since $H^n(C_D X, C_D B) \approx H^n(D, D) = 0$ for all n.

3.2. Definition. $\Sigma_D: \hat{x} H^n(X, B) \to \hat{x} H^{n+1}(\Sigma_D X, \Sigma_D B)$ is defined to be the $A(D)$-isomorphism $v^* \delta i^{*-1}$.

Suppose $B \subset X \xrightarrow{\hat{x}} D \xrightarrow{t} E$ and $w = t\hat{x}$. Then $H^n(X, B)$ is an $A(E)$-module via w and the following diagram is commutative:

$$\begin{array}{ccc} \hat{x} H^n(X, B) & \xrightarrow{\Sigma_D} & \hat{x} H^{n+1}(\Sigma_D X, \Sigma_D B) = w H^{n+1}(\Sigma_D X, \Sigma_D B) \\ \| & & \nearrow T^* \\ w H^n(X, B) & \xrightarrow{\Sigma_E} & w H^{n+1}(\Sigma_E X, \Sigma_E B) \end{array}$$

All maps are isomorphisms. $T: (\Sigma_D X, \Sigma_D B) \to (\Sigma_E X, \Sigma_E B)$ is defined from t in the obvious way.

$\Sigma_D X$ defined in 3.1 is the bundle suspension of James [13]. It is also a special case of the Whitney join over D (Hall [10]). $\Sigma_D^n X \to D$ is the Whitney join of $\hat{x}: X \to D$ and $\text{proj}: D \times S^{n-1} \to D$.

4. Axioms. The terminology used here ("pyramid of operations", "admissable chain complex") is taken from Maunder's paper [22]. When D is a point the axioms and theorems are essentially those of Maunder.

Let D be a fixed space and

$$C: C_N \xrightarrow{d_n} C_{N-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0$$

be a chain complex of free finitely generated $A(D)$-module. Each C_i is to be graded and satisfy $(C_i)_q = 0$ if $q < i$; each d_i is to be an $A(D)$-map of degree 0. A "pair over D" is a triple $(X, B, \hat{x})$ where $B \subset X$ and $\hat{x}: X \to D$. Suppose that for $N \geqq r > s \geqq 0$ and every $(X, B, \hat{x})$ that $\varphi^{r,s}(X, B; \hat{x})$: $\mathrm{Hom}_k(C_r, \hat{x} H^*(X, B)) \rightharpoonup \mathrm{Hom}_m(C_s, \hat{x} H^*(X, B))$ is an additive relation of degree $-r+s+1$ (so $m = k - r + s + 1$). Here and everywhere else $\mathrm{Hom}_k(M, N)$ means $\mathrm{Hom}_{A(D),k}(M, N) =$ the $A(D)$-maps from M to L of degree k. An element of $\mathrm{Hom}(C_r, \hat{x} H^*(X, B))$ can be thought of as a vector of cohomology classes of the abelian group $H^*(X, B)$.

4.1. Definition. The collection $\{\varphi^{r,s}\}$, $N \geqq r > s \geqq 0$ is said to be a pyramid of stable operations associated with the chain complex C if the following axioms are satisfied:

I) If $N = 1$ then $\varphi^{1,0} = \mathrm{Hom}(d_1, 1)$.

II) If $N > 1$ then

(1) (Induction.) If $r - s < N$ then $\{\varphi^{u,v}\}$, $r \geqq u > v \geqq s$ is associated with the chain complex

$$C_r \to \cdots \to C_s.$$

(2) (Whither-Whence.)

$$\varphi^{N,0}(X, B, \hat{x}): \mathrm{Hom}(C_0, \hat{x} H^*(X, B)) \rightharpoonup \mathrm{Hom}(C_N, \hat{x} H^*(X, B))$$

is an additive relation of degree $-N+1$. $\mathrm{Def}\, \varphi^{N,0} = \mathrm{Ker}\, \varphi^{N-1,0}$ and $\mathrm{Ind}\, \varphi^{N,0} = \mathrm{Im}\, \varphi^{N,1}$.

(3) (Naturality.) Suppose $(X, B, \hat{x})$ and $(X^1, B^1, \hat{x}^1)$ are pairs over D, $g: (X, B) \to (X^1, B^1)$, and $\hat{x}^1 g$ is homotopic to $\hat{x}$. If $\varepsilon \in \mathrm{Def}\, \varphi^{N,0}(X^1, B^1, \hat{x}^1)$ then $g^* \varepsilon \in \mathrm{Def}\, \varphi^{N,0}(X, B; \hat{x})$ and $g^* \varphi^{N,0}(\varepsilon) \subset \varphi^{N,0}(g^* \varepsilon)$.

(4) (Suspension.) $\Sigma_D \varphi^{N,0} = (-1)^{N-1} \varphi^{N,0} \Sigma_D$.

(5) (Peterson-Stein relation.) Suppose $B^1 \subset B \subset X$ and $\hat{x}: X \to D$. If $\eta \in \mathrm{Def}\, \varphi^{n-1,0}(X, B^1, \hat{x})$ and $i^* \eta \in \mathrm{Def}\, \varphi^{N,0}(B, B^1, \hat{x})$ then

$$\varphi^{N,N-1} j^{*-1} \varphi^{N-1,0} \eta \subset -\delta \varphi^{N,0} i^* y,$$

where i^*, j^*, and δ are from the exact sequence of the triple (X, B, B^1).

4.2. Definition. The chain complex $C: C_N \to \cdots \to C_1$ is admissible if I) $N=1$ or II) $N>1$ and there is a pyramid of operators $\{\varphi^{r,s}\}$, $N-1 \geqq r > s \geqq 0$, associated with $C_{N-1} \to \cdots \to C_0$ such that $\varphi^{N,N-1}\varphi^{N-1,0}(\eta)=0$ (with 0 indeterminacy) for each $\eta \in \operatorname{Def}\varphi^{N-1,0}$ where $\varphi^{N,N-1}$ is the primary operation associated with $C_N \to C_{N-1}$.

4.3. Theorem. *If a chain complex has an associated pyramid of operations then it is admissible.*

Proof. Take $B=B^1$ in axiom 5.

The next two theorems will be proved in Part III.

4.4. Theorem (Existence). *If a chain complex is admissible, then it has at least one associated pyramid of operations.*

4.5. Theorem (Quasi-Uniqueness). *If $\varphi_0^{N,0}$ and $\varphi_1^{N,0}$ are two possible apexes of the pyramid of operations $\{\varphi^{r,s}\}$, $N-1 \geqq r > s \geqq 0$, associated with the chain complex*

$$C_N \xrightarrow{d_N} C_{N-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0$$

then there exists an operation X *associated with*

$$C_N \xrightarrow{\bar{d}} C_{N-2} \xrightarrow{d_{N-2}} C_{N-3} \to \cdots \to C_1 \xrightarrow{d_1} C_0$$

for some $\bar{d}$ (X is of the $(N-1)$'st order) such that $\mathrm{X}(\varepsilon) \subset \varphi_0^{N,0}(\varepsilon) - \varphi_1^{N,0}(\varepsilon)$ for each $\varepsilon \in \operatorname{Def}\varphi_0^{N,0} = \operatorname{Def}\varphi_1^{N,0}$.

We will consider what happens when D is varied. First, note that $t: D \to E$ gives $t^*: A(E) \to A(D)$. If $(X, B, \hat{x})$ is a pair over D then $(X, B, t\hat{x})$ is a pair over E and $t\hat{x}H^*(X,B)$ gets its $A(E)$ module structure by pull-back along t^*.

Let C^1 be an $A(E)$-module. $C = A(D) \otimes_{A(E)} C^1$ is an $A(D)$-module. There is a natural isomorphism

$$\alpha: \operatorname{Hom}_{A(E)}(C^1, t\hat{x}H^*(X,B)) \to \operatorname{Hom}_{A(D)}(C, \hat{x}H^*(X,B)).$$

Let $C^1: C_N^1 \to \cdots \to C_0^1$ be a free $A(E)$-module. Then C is a free $A(D)$-module. Let $\{\varphi^{1r,s}\}$ be a pyramid of operations associated with C^1. Define

$$\varphi^{r,s}: \operatorname{Hom}_{A(D),k}(C_s, \hat{x}H^*(X,B)) \rightharpoonup \operatorname{Hom}_{A(D),m}(C_r, \hat{x}H^*(X,B)),$$

$m = k - r + s + 1$, by $\varphi^{r,s} = \alpha\varphi^{1r,s}\alpha^{-1}$ where

$$\varphi^{1r,s}: \operatorname{Hom}_{A(E),k}(C_s^1, t\hat{x}H^*(X,B)) \rightharpoonup \operatorname{Hom}_{A(E),m}(C_s^1, t\hat{x}H^*(X,B)).$$

Then it is not difficult to verify that $\{\varphi^{r,s}\}$ is a pyramid of operations for C. The proof of the suspension axiom depends on the following commutative diagram which is a consequence of the remarks after Definition 3.2.

$$\begin{array}{ccc}
\operatorname{Hom}_k(C_i, \hat{x} H^*(X,B)) \xrightarrow{\Sigma_D} & & \\
\uparrow{\scriptstyle \alpha} & & \\
\operatorname{Hom}_k(C_i, t\hat{x} H^*(X,B)) \xrightarrow{\Sigma_E} & & \\
& \operatorname{Hom}_{k+1}(C_i, \hat{x} H^* \Sigma_D X, \Sigma_D B)) \xleftarrow{\alpha} & \operatorname{Hom}_{k+1}(C_i^1, t\hat{x} H^*(\Sigma_D X, \Sigma_D B)) \\
& & \nearrow{\scriptstyle \operatorname{Hom}(1, T^*)} \\
& \operatorname{Hom}_{k+1}(C_i^1, t\hat{x} H^*(\Sigma_E X, \Sigma_E B)) &
\end{array}$$

φ is defined in terms of φ^1. On the other hand, if $(X, B, \hat{x})$ is a pair over E and it happens that $\hat{x}^1 \sim tx$ for some $\hat{x}: X \to D$ then φ^1 is determined on $(X, A, \hat{x})$ by φ.

As an example consider $t: * \to E$. Then $t^*: A(E) \to A(*) = A$. If φ^1 is a given operation for the $A(E)$-chain complex C^1 then $\varphi = \alpha \varphi^1 \alpha^{-1}$ is associated with an A-chain complex so is an Adams-Maunder operation [13]. The above discussion shows that if $(X, B, \hat{x})$ is a pair over E and $\hat{x}$ is null-homotopic then $\varphi^1(X, B, \hat{x})$ coincides with the untwisted Adams-Maunder operation $\varphi(X, B)$.

Part II. The Category $\mathscr{T}(C \to D)$

1. Definition and General Properties of $\mathscr{T}(C \to D)$. Let $\mathscr{T}$ be the category of topological spaces and continuous functions. Define $\mathscr{T}(C \xrightarrow{u} D)$ for $u: C \to D \in \mathscr{T}$ to be the category whose objects are triples $(X, \check{x}, \hat{x})$ where $C \xrightarrow{\check{x}} X \xrightarrow{\hat{x}} D$ and $\hat{x}\check{x} = u$. A map $\bar{f}: (X, \check{x}, \hat{x}) \to (Y, \check{y}, \hat{y})$ is a map $f: X \to Y$ such that $f\check{x} = \check{y}$ and $\hat{y}f = \hat{x}$. The triple $(X, \check{x}, \hat{x})$ will be denoted by $\bar{X}$ and, when confusion seems unlikely, simply by X. The map $\bar{f}$ will frequently be denoted by f.

It should be noted that the definition makes sense for any category $\mathscr{T}$. The work of this part could be carried out, with some extra effort, in this more general setting. The results would take the form: "If $\mathscr{T}$ has such and such a property then $\mathscr{T}(C \to D)$ has such and such a property". These more general results will be taken for granted later when, for example, $\mathscr{T}$ is allowed to be the category of pairs of spaces.

Some examples are:

(1) $\mathscr{T}(\emptyset \to *)$ where $\emptyset$ is the empty set and $*$ a point. The obvious functor $\mathscr{T} \to \mathscr{T}(\emptyset \to *)$ is an equivalence.

(2) $\mathscr{T}(* \to *)$. This category is equivalent to the category of pointed spaces and pointed maps.

(3) $\mathscr{T}(C\to *)$. This is the category of spaces under C.

(4) $\mathscr{T}(\emptyset\to D)$. This is the category of spaces over D (see Part I).

(5) $\mathscr{T}(*\xrightarrow{u} D)$. Where $u(*)=d_0\in D$. This is the category of pointed spaces over D. If $(X,\check x,\hat x)\in\mathscr{T}(*\xrightarrow{u}D)$ then the base point of X is $\check x(*)$.

(6) $\mathscr{T}(D\xrightarrow{1}D)$. This is the category of spaces under and over D (see Part I). Denote it by $\mathscr{T}D$. If (X,x_0) is a pointed space then $(D\times X,\iota,\pi)\in\mathscr{T}D$ where $\iota:D\to D\times X$ is defined by $\iota(d)=(d,x_0)$ and $\pi:D\times X\to D$ is the projection. More generally, let $D\times x_0\subset R\subset D\times X$. Then (R,ι,π) is in $\mathscr{T}D$.

We observe that $\mathscr{T}(C\to D)=\mathscr{T}u$ has the following categorical properties. See Freyd [8], for example, for the terminology used here.

(1) $\mathscr{T}u$ has an initial object $C\xrightarrow{1}C\xrightarrow{u}D$ and a terminal object $C\xrightarrow{u}D\xrightarrow{1}D$.

(2) $\mathscr{T}u$ has pullbacks and pushouts. Let $\bar f:\bar X\to\bar Z$ and $\bar g:\bar Y\to\bar Z$. Define $K=\{(x,y)\in X\times Y: f(x)=g(y)\}$, i.e., K is the pullback of f and g in $\mathscr{T}$. Define $C\xrightarrow{\check k}K\xrightarrow{\hat k}D$ by $\check k(c)=(\check x(c),\check y(c))$ and $\hat k(x,y)=\hat x(x)=\hat y(y)$. The natural projections (from $\mathscr{T}$) give the following commutative diagram:

$$\begin{array}{ccc} \bar K & \xrightarrow{\bar\pi_2} & \bar Y \\ {\scriptstyle\bar\pi_1}\downarrow & & \downarrow{\scriptstyle\bar g} \\ \bar X & \xrightarrow{\bar f} & \bar Z \end{array}$$

and it is easy to check that this is indeed a pullback diagram. Pushouts are constructed by a dual process.

(3) $\mathscr{T}u$ has products and sums. For the product use the pullback of the terminal maps.

(4) Let $C'\xrightarrow{k}C\xrightarrow{u}D\xrightarrow{m}D'$. Define a functor $F:\mathscr{T}(u)\to\mathscr{T}(muk)$ by $(C\xrightarrow{\check x}X\xrightarrow{\hat x}D)\mapsto(C'\xrightarrow{\check x k}X\xrightarrow{m\hat x}D')$. F preserves initial objects iff $C=C'$, terminal objects iff $D=D'$, pullbacks iff m is one-to-one, pushouts iff k is onto. In particular, $\emptyset\to C\xrightarrow{u}D\xrightarrow{1}D$ gives $\mathscr{T}u\to\mathscr{T}(\emptyset\to D)$ preserving terminal objects and pullbacks, so products.

We now consider homotopy in $\mathscr{T}u$.

(1) Define $W:\mathscr{T}u\to\mathscr{T}u$ as follows: given $C\xrightarrow{\check x}X\xrightarrow{\hat x}D$, set $WX=\{k\in X^I:\hat x\,k(t)=\hat x\,k(t')\ \forall\, t,t'\in I\}$, where I is the unit interval and X^I is the space of continuous functions from I to X with the compact-open topology. Define $W\bar X$ to be $(WX,\check w,\hat w)$ where $\check w(c)=\check x(c)^I$ and $\hat w(k)=\hat x\,k(o)$ $(=\hat x\,k(t)\ \forall\, t\in I)$. Then $W\bar X$ is in $\mathscr{T}u$. W is the free path function (W for weg = free path). Define $\bar w_t: W\bar X\to\bar X$ for $t\in I$ by $\bar w_t(k)=k(t)$. If $\bar f,\bar g:\bar X\to\bar Y$, say that $\bar f$ and $\bar g$ are homotopic, $\bar f\sim\bar g$, if there is an $\bar H:\bar X\to W\bar Y$ such that $\bar w_0\bar H=\bar f$ and $\bar w_1\bar H=\bar g$.

(2) Define $Z:\mathscr{T}u\to\mathscr{T}u$ as follows: given $C\xrightarrow{\check{x}}X\xrightarrow{\hat{x}}D$, set $ZX=X\times I/\mathscr{R}$, where $\mathscr{R}$ is the equivalence relation: $(x,t)\mathscr{R}(x',t')$ iff there is $c\in C$ such that $x=\check{x}(c)=x'$. Let $k:X\times I\to ZX$ be the identification map. Define $C\xrightarrow{\check{z}}ZX\xrightarrow{\hat{z}}D$ by $\check{z}(c)=k(x(c),t)$ and $\hat{z}\,k(x,t)=\hat{x}(x)$. Then $Z\bar{X}=(ZX,\check{z},\hat{z})$ is in $\mathscr{T}u$. Z is the cylinder functor (Z for zylinder = cylinder). Check that $\bar{f}\sim\bar{g}$: $\bar{X}\to\bar{Y}$ iff there is an $\bar{H}:Z\bar{X}\to\bar{Y}$ such that $\bar{H}\bar{z}_0=\bar{f}$ and $\bar{H}\bar{z}_1=\bar{g}$.

The result is that Z and W are adjoint functors and provide a notion of homotopy with the usual properties. The sections which follow give an outline of a development of the elementary homotopy theory of the category $\mathscr{T}u$. The motive for doing this is given in I.2. The method used will be that of Puppe [28] and its Eckmann-Hilton dual as worked out by Nomura [26].

2. *Path-Loop Properties of* $\mathscr{T}u$. The homotopy theory of $\mathscr{T}u$ is outlined here. The theorems can be proved by standard methods such as those of Nomura [25]. The purpose of carrying the outline so far is to demonstrate how much homotopy theory is valid here, before specializing to $u=\mathrm{id}:D\to D$. This is particularly helpful in the dual situation since in Part III we will have to work both with

$$\mathscr{T}(\emptyset\to D)\quad\text{and}\quad\mathscr{T}(D\xrightarrow{\mathrm{id}}D).$$

2.1. Definition. Let $\bar{X}\in\mathscr{T}u$.

a) Define the pointed path space of $\bar{X}$, $P\bar{X}$, to be the pullback of $\check{x}:C\to\bar{X}$ and $w_0:W\bar{X}\to\bar{X}$. Define $p_t:P\bar{X}\to\bar{X}$ as the composite $P\bar{X}\to W\bar{X}\xrightarrow{w_t}X$ for $t\in I$.

b) Define the loop space of $\bar{X}$, $\Omega\bar{X}$, as the pullback of $\check{x}:\bar{C}\to\bar{X}$ and $p_1:P\bar{X}\to\bar{X}$.

Let $\bar{f}:\bar{X}\to\bar{Y}$. Then $P\bar{f}$ and $\Omega\bar{f}$ are defined in the obvious ways and P and Ω become functors: $\mathscr{T}u\to\mathscr{T}u$.

Next, the absolute covering homotopy property and the path lifting property for $f:\bar{X}\to\bar{Y}$ can be defined in the natural way. They can be shown to be equivalent and $\bar{f}$ is said to be a fibration if it has either property. The fiber of $\bar{f}:\bar{X}\to\bar{Y}$ is defined to be the pullback of the initial map $\check{y}:\bar{C}\to\bar{Y}$ and $\bar{f}:\bar{X}\to\bar{Y}$. The mapping space $E\bar{f}$ is defined to be the pullback of $\bar{f}$ and $p_1:P\bar{Y}\to\bar{Y}$. $P\bar{Y}\to\bar{Y}$ and $E\bar{f}\to\bar{X}$ can be shown to be fibrations with fiber $\Omega\bar{Y}$.

Suppose that the following diagram is homotopy commutative by means of H.

$$\begin{array}{ccc}\bar{X} & \xrightarrow{f} & \bar{Y}\\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b}\\ \bar{X}' & \xrightarrow{f'} & \bar{Y}'\end{array}\qquad \begin{array}{l}H:\bar{X}\to\bar{Y}\\ \\ H:bf\sim f'a.\end{array}$$

Define $w = w_H : Ef \to Ef'$ by $w(x, c, k) = (a(x), c, (Pb)k + H(x))$. Then the left square of the following diagram is commutative:

$$\begin{array}{ccccc} Ef & \to & X & \to & Y \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ Ef' & \to & X' & \to & Y' \end{array}$$

where βf and $\beta f'$ are the natural projections.

The path-loop sequence for $f: \overline{X} \to \overline{Y}$ is:

$$\ldots \Omega^2 Y \to \Omega Ef \to \Omega \overline{X} \to \Omega \overline{Y} \to Ef \to \overline{X} \to \overline{Y}$$

and the path-resolution sequence is:

$$\ldots E\beta^2 f \to E\beta f \to Ef \to \overline{X} \to \overline{Y}.$$

They are homotopically equivalent. Furthermore, a homotopy commutative square sending f to f' as above yields a homotopy commutative ladder in both cases and this is preserved under the equivalence. This important naturality property is the basis of the homotopy definition of a twisted secondary operation.

3. *Path-Loop Properties of* $\mathscr{T}(D \xrightarrow{1} D) = \mathscr{T}D$. The objects in $\mathscr{T}D$ are triples $(X, \check{x}, \hat{x})$ where $D \xrightarrow{\check{x}} X \xrightarrow{\hat{x}} D$ and $\hat{x}\check{x} = 1$. Note that $\check{x}$ is actually an inclusion and $\hat{x}$ a retraction of X onto the subspace D. $D \xrightarrow{1} D \xrightarrow{1} D$ is a zero object for $\mathscr{T}D$. The definitions of the previous section take a simpler form here. Each $\hat{x}^{-1}(d) \subset X$ has a natural base point-namely, $\check{x}(d)$. The path space in $\mathscr{T}D$ is formed simply by taking the (disjoint) union of the ordinary path spaces of $\hat{x}^{-1}(d)$ for all $d \in D$. Similarly $\Omega X = \bigcup_{d \in D} \tilde{\Omega}(\hat{x}^{-1}(d))$ where $\tilde{\Omega}$ is the ordinary loop functor (in $\mathscr{T}*$). Also, if $f: \overline{X} \to \overline{Y}$ is a fibration in $\mathscr{T}D$ then the fiber is simply $f^{-1}(D)$ ("rope" might be more appropriate than "fiber" here). Two examples are:

(1) Let $x_0 \in X$ where X is a space in $\mathscr{T}$. Form $\overline{D \times X} = (D \times X, \pi, \iota)$ in $\mathscr{T}D$ where $\iota: D \to D \times X$ and $\pi: D \times X \to D$ are defined by $\iota(d) = (d, x_0)$ and $\pi(d, x) = d$. Then

$$P(\overline{D \times X}) = (D \to D \times \tilde{P}X \to D) \quad \text{and} \quad \Omega(\overline{D \times X}) = (D \to D \times \tilde{\Omega}X \to D)$$

where $\tilde{P}$ and $\tilde{\Omega}$ are the path and loop functors of $\mathscr{T}*$.

I found example (1) and the associated notions of derived function and loop suspension in James-Thomas [14, pp. 501/502]. The definitions of this part are the result of a desire to generalize these notions to cover example (2).

(2) Let $f: D \times X \to Y$ be a map in the ordinary category of spaces and maps such that $f(D \times x_0) = y_0$. Form Ef in the category; so $Ef = \{(d, x, k) \in D \times X \times \tilde{P}Y: f(d, x) = k(1)\}$. Then $D \xrightarrow{\iota} Ef \xrightarrow{\pi} D$ is in $\mathscr{T}D$ where $\iota(d) = (d, x_0, y_0^I)$ and $\pi(d, x, k) = d$. Let $g = (\text{proj}, f): D \times X \to D \times Y$. Then $\bar{g}$ is in $\mathscr{T}D$ and $Ef = E\bar{g}$ and the latter is formed in $\mathscr{T}D$ (cf. the last part of I.2).

Let A and X be in $\mathscr{T}D$. Denote by $\langle A, X\rangle$ the set of homotopy classes of maps from A to X. It is a pointed set with the unique map $A \to D \to X$ as point. This map will be denoted by "O" or "$O_{A,X}$".

3.1. Theorem. *Let $f: X \to Y$ be any map of $\mathscr{T}D$. Then the following sequence of pointed sets is exact for any A in $\mathscr{T}D$*

$$\langle A, Ef\rangle \xrightarrow{\beta f_*} \langle A, X\rangle \xrightarrow{f_*} \langle A, Y\rangle .$$

Next, a homotopy group structure is defined on ΩX. First note that, in $\mathscr{T}D$,

$$\Omega X \times \Omega X = \{(k, k') \in X^I \times X^I : k(0) = k(1) = k'(0) = k'(1) \in x(D)$$
$$\text{and } \hat{x}\, k(t) = \hat{x}\, k(t')\ \forall\, t, t' \in I\}.$$

The product is formed in the category (see II.1). Hence $m: \Omega X \times \Omega X \to \Omega X$ defined by $m(k, k') = k + k'$ (ordinary path sum) makes sense and is a map in $\mathscr{T}D$. Now in $\mathscr{T}D$ there is a zero object $\bar{D}$ and hence a zero map between any two maps in the category. This, plus the reversal map $r: \Omega X \to \Omega X$ defined by $r k(t) = k(1-t)$ give:

3.2. Theorem. *$(\Omega X, m)$ is a homotopy group in the category $\mathscr{T}D$.*

For a discussion of "homotopy group in a category" see Eckman-Hilton [5] and [6]. Just as in the case $D = *$ we get that $\langle A, \Omega X\rangle$ is a group, abelian if ΩX is replaced by $\Omega^2 X$, and $\Omega f_*: \langle A, \Omega X\rangle \to \langle A, \Omega Y\rangle$ is a homomorphism, etc.

There is also a homotopy operation $n: \Omega Y \times Ef \to Ef$.

$$\Omega Y \times Ef = \{(k', x, k) \in Y^I \times X \times Y^I : k'(o) = k'(1) = k(o) \in \hat{x}(D)$$

and $\hat{y}\, k(t) = \hat{y}\, k'(t) = \hat{x}(x) \forall t \in I\}$. n is defined by $n(k', x, k) = (x, k + k')$. It is easy to check that it has the expected properties.

The above is summed up in the following theorem.

3.3. Theorem. *Let $f: X \to Y$ be a map of $\mathscr{T}D$ and A a space of $\mathscr{T}D$. Then there is an exact sequence of pointed sets:*

$$\cdots \langle A, \Omega^2 Y\rangle \xrightarrow{\Omega\gamma f_*} \langle A, \Omega Ef\rangle \xrightarrow{\Omega\beta f_*} \langle A, \Omega X\rangle \xrightarrow{\Omega f_*} \langle A, \Omega Y\rangle$$
$$\longrightarrow \langle A, Ef\rangle \longrightarrow \langle A, X\rangle \longrightarrow \langle A, Y\rangle$$

and an action:

$$n_*: \langle A, \Omega Y\rangle \times \langle A, Ef\rangle \to \langle A, Ef\rangle$$

such that

(1) *the sequence consists of groups and homomorphisms after the third term and abelian groups after the sixth term.*

(2) $o \cdot v = v$, $(u+u') \cdot v = u \cdot (u' \cdot v)$ *and* $(\gamma f)_*(u+u') = u \cdot (\gamma f)_* u'$ *where* $u, u' \in \langle A, \Omega Y\rangle$ *and* $v \in \langle A, Ef\rangle$.

(3) $\beta f_* v = \beta f_* v'$ *iff there is a* u *in* $\langle A, \Omega Y\rangle$ *such that* $v' = u \cdot v$.

(4) $\gamma f_* u = \gamma f_* u'$ *iff* $-u+u'$ *is in the image of* Ωf_*.

There are two kinds of naturality for the sequence and action of the theorem. First, it is quite easy to see that a map $A \to A'$ gives a transformation of the corresponding sequences and action when f is fixed. Second, a homotopy commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ X' & \xrightarrow{f'} & Y' \end{array} \qquad H: bf \sim f'a$$

gives a transformation of both sequence and action which depends on the homotopy H. These facts about naturality will be used later to define homotopy operations in the category.

4. *Cone-Suspension Properties of* $\mathscr{T}(C \to D)$ *and* $\mathscr{T}D$. The definitions and theorems of this section are dual to those of the previous two sections. So they will not be stated.

Notice that here, in contrast to Sections 2 and 3, the case $C = \emptyset$ is interesting. The theorems for $\mathscr{T}(\emptyset \to *)$ and $\mathscr{T}(* \to *)$ give a simultaneous development of unreduced and reduced homotopy. Also, the cone, suspension, and mapping cone constructions for $\mathscr{T}(\emptyset \to D)$ are those used in I.3. For example, if $(X, \hat{x}) \in \mathscr{T}(\emptyset \to D)$, $\hat{x}: X \to D$, then its suspension is $\Sigma_D X \to D$ and $\Sigma_D X$ is the disjoint union of the ordinary unreduced suspension of each $\hat{x}^{-1}(d)$ (of each "fiber"). In $\mathscr{T}D$ the suspension is formed by taking the reduced suspension of each $x^{-1}(d)$.

5. *Secondary Operations in* $\mathscr{T}D$. The method of this section was inspired by Spanier's article [30]. The first operation considered (the bracket operation) is the one treated by Spanier. However, the actual definitions used here are more in the style of Steenrod [34] and Peterson-Stein [27]. The other operations (the box operations) seem not to have been observed before in their general form (even for $D = *$). All the operations depend on naturality properties of the path-loop and cone-suspension sequences of Sections 3 and 4.

5.1A. *The Bracket Operation-Loop Version.* Suppose given

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

in $\mathscr{T}D$ and suppose gf and hg are null homotopic. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
\langle X,\Omega Y\rangle & \longrightarrow & \langle X,\Omega Z\rangle & \longrightarrow & \langle X,Eh\rangle & \xrightarrow{p_*} & \langle X,Y\rangle & \longrightarrow & \langle X,Z\rangle \\
\downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow & & \downarrow \\
\langle W,\Omega Y\rangle & \xrightarrow{\Omega h_*} & \langle W,\Omega Z\rangle & \xrightarrow{i_*} & \langle W,Eh\rangle & \longrightarrow & \langle W,Y\rangle & \longrightarrow & \langle W,Z\rangle .
\end{array}$$

Define $\langle f,g,h\rangle' = i_*^{-1} f^* p_*^{-1}\langle g\rangle$. It is a double coset of $f^*\langle X,\Omega Z\rangle$ and $\Omega h_*\langle W,\Omega Y\rangle$ in $\langle W,\Omega Z\rangle$ and depends only on $\langle f\rangle, \langle g\rangle$, and $\langle h\rangle$. $\langle f,g,h\rangle'$ can also be defined by putting together null-homotopies as in Spanier [30]. The operation defined here differs in sign from that of [30]. If $D=*$, this operation has the Peterson definition of the functional cohomology operation as a special case. More generally, it has a special case the functional twisted operation defined by Meyer in [24] and Gitler-Stasheff in [9]. This will be made more explicit in Part IV.

5.1B. *The Bracket Operation-Suspension Version.* Start with the same data: $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathscr{T}D$; gf and hg both null-homotopic.

Form:

$$\begin{array}{ccccccccc}
\langle \Sigma X,Z\rangle & \xrightarrow{\Sigma f^*} & \langle \Sigma W,Z\rangle & \xrightarrow{k^*} & \langle Cf,Z\rangle & \longrightarrow & \langle X,Z\rangle & \longrightarrow & \langle W,Z\rangle \\
\uparrow & & \uparrow{\scriptstyle h_*} & & \uparrow{\scriptstyle h_*} & & \uparrow & & \uparrow \\
\langle \Sigma X,Y\rangle & \longrightarrow & \langle \Sigma W,Y\rangle & \longrightarrow & \langle Cf,Y\rangle & \xrightarrow{i^*} & \langle X,Y\rangle & \longrightarrow & \langle W,Y\rangle .
\end{array}$$

Define $\langle f,g,h\rangle = k^{*-1} h_* i^{*-1}\langle g\rangle$. It is a double coset of $h^*\langle \Sigma W,Y\rangle$ and $\Sigma f^*\langle \Sigma X,Y\rangle$ in $\langle \Sigma W,Z\rangle$ and depends only on $\langle f\rangle, \langle g\rangle$, and $\langle h\rangle$. It differs only in sign from $\langle f,g,h\rangle'$. If $D=*$ this version has the Steenrod definition [34] of the functional cohomology operation as a special case. More generally, it gives the definition of twisted functional cohomology operation which is actually used by Meyer and Gitler-Stasheff.

5.2. *The Box Operation-Loop Versions.* Suppose that the following data in $\mathscr{T}D$ is given:

$$\begin{array}{ccccc}
X & \xrightarrow{x} & A & \xrightarrow{f} & B \\
 & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} \\
 & & L & \xrightarrow{g} & M
\end{array}
\qquad H: A \to WM \qquad \begin{array}{l} p_0 H = sf \\ p_1 H = gr \end{array}$$

i.e., the box is homotopy commutative by means of a fixed homotopy H. Suppose also that fx and rx are null-homotopic. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\langle X, \Omega A\rangle & \longrightarrow & \langle X, \Omega B\rangle & \longrightarrow & \langle X, Ef\rangle & \xrightarrow{p_*} & \langle X, A\rangle & \longrightarrow & \langle X, B\rangle \\
\downarrow & & \downarrow{\scriptstyle \Omega s_*} & & \downarrow{\scriptstyle w_*} & & \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle s_*} \\
\langle X, \Omega L\rangle & \xrightarrow{\Omega g_*} & \langle X, \Omega M\rangle & \xrightarrow{i_*} & \langle X, Eg\rangle & \longrightarrow & \langle X, L\rangle & \longrightarrow & \langle X, M\rangle
\end{array}$$

where $w = w_H$ (see 2.15). Define $[x, f, g, r, s, H] = i_*^{-1} w_* p_*^{-1} \langle x\rangle$. It is a double coset of $\Omega s_* \langle X, \Omega B\rangle$ and $\Omega g_* \langle X, \Omega L\rangle$ in $\langle X, \Omega M\rangle$. If one builds upward using Er and Es, instead of to the left with Es and Eg, then the result is an operation which differs from the one defined here only in sign. The operation can be defined directly, by putting together the null-homotopies of sf and gr by means of H.

In the case $D = *$ the operation has the Adams operations [1] as special cases. In the general case this operation gives the secondary twisted operations of I.4.

5.3. *The Box Operation-Suspension Versions.* Suppose that the following data in $\mathcal{T}D$ is given:

$$\begin{array}{ccccc}
A & \xrightarrow{f} & B & & \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \\
L & \xrightarrow{g} & M & \longrightarrow & Y
\end{array}
\qquad H: ZA \to M \qquad \begin{array}{l} H i_0 = sf \\ H i_1 = gr \end{array}$$

Suppose also that ys and yg are null-homotopic. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
\langle \Sigma B, Y\rangle & \longrightarrow & \langle \Sigma A, Y\rangle & \longrightarrow & \langle Cf, Y\rangle & \longrightarrow & \langle B, Y\rangle & \longrightarrow & \langle A, Y\rangle \\
\uparrow & & \uparrow{\scriptstyle \Sigma r^*} & & \uparrow{\scriptstyle w^*} & & \uparrow{\scriptstyle s^*} & & \uparrow{\scriptstyle r^*} \\
\langle \Sigma M, Y\rangle & \longrightarrow & \langle \Sigma L, Y\rangle & \longrightarrow & \langle Cg, Y\rangle & \xrightarrow{i^*} & \langle M, Y\rangle & \longrightarrow & \langle L, Y\rangle
\end{array}$$

where $w = w_H$ (see 3.15). Define $[y, f, g, r, s, H] = k^{*-1} w^* i^{*-1} \langle y\rangle$. It is a double coset of $\Sigma r^* \langle \Sigma L, Y\rangle$ and $\Sigma f^* \langle \Sigma B, Y\rangle$ in $\langle \Sigma A, Y\rangle$. It can be defined vertically (with a different sign) via Cr and Cs or directly by putting together null-homotopies.

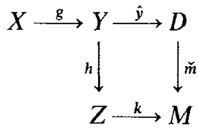
If $D = *$ and Y is an Eilenberg-MacLane space this operation has fiber space suspension (in cohomology) as a special case. For any D it gives a suspension for a fiber space in $\mathcal{T}D$ and this will be used in Part III.

The operations 5.1 – 5.3 are related by several formulas of the Peterson-Stein type. One such formula is given here.

Suppose the following diagram in $\mathcal{T}D$ is given:

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \xrightarrow{k} M.$$

Suppose also that gf, hg, and kh are null-homotopic and that H is a null-homotopy of kh. Define $\Theta_H(g)=[g, h, \check{m}, \check{y}, k]\subset\langle X, \Omega M\rangle$ based on the following diagram in $\mathscr{T}D$.

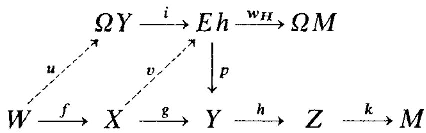


the given null-homotopy H, and a vertical diagram using Eh and $E\check{m}=\Omega M$.

5.4. Theorem. (Cf. Theorem 4.5 of Spanier [30].) $f^*\,\Theta_H(g)=\Omega k_*\langle f, g, h\rangle'$ *for any choice of null-homotopy* H.

Proof. It follows easily from the definitions that the indeterminacies of the left and right hand sides of the equation are the same. Hence it suffices to find a common element.

Consider the following diagram in $\mathscr{T}D$.



$w=w_H$ is the map, depending on H, which is used in the definition of $\Theta_H(g)$. It follows (as in II.2.15) that $w_H\,i\sim\Omega k$. Let v be any lifting of g (using that hg is null-homotopic) and u any lifting of v (using that gf is null-homotopic). Then the diagram as shown is homotopy commutative. By the definitions of the operations involved, wv is a representative of $\Theta_H(g)$ and $\langle u\rangle$ is a representative of $\langle f, g, h\rangle'$. Hence $f^*\langle wv\rangle=\langle wiu\rangle=\langle(\Omega k)\,u\rangle=\Omega k_*\langle u\rangle$ is a common element of the cosets of the theorem. Q.E.D.

Part III. Proofs of the Theorems of I

1. Transference. The axioms in I.4 are stated for the category $\mathscr{T}_D^2$ of pairs over D. The object of this section is to show that for the proofs of the existence and quasi-uniqueness theorems it will suffice to consider a certain full subcategory $\mathscr{P}\mathscr{T}D$ of $\mathscr{T}D$. $\mathscr{P}\mathscr{T}D$ consists of spaces which contain D as a strongly non-degenerate retract.

1.1. Definition. Let $(X, A, \hat{x})$ be a pair over D. Define $(X/A)_D$ as $X\cup D/\mathscr{R}$ where $\mathscr{R}$ is the equivalence relation generated by: $a\sim\hat{x}(a)$ for each $a\in A$. A map $(X/A)_D\to D$ is defined by $\hat{x}\cup 1$. Denote this map by $\hat{x}$ also.

The result is that $\hat{x}$ is a retraction of $(X/A)_D$ onto D. Also, if $A\subset X$ has the $AHEP$ over D, it is easy to check that $D\subset(X/A)_D$ has it as well.

1.2. Definition. $\kappa(X, A, \hat{x}):(X, A, \hat{x}) \to ((X/A)_D, D, \hat{x})$ is defined to be the composite $(X, A) \xrightarrow{m} (C_D i, C_D A) \xrightarrow{k} ((X/A)_D, D)$.

Note that both m and k are maps over D and m is an isomorphism on cohomology by excision and deformation retraction. It follows from II.4.12 that if $A \subset X$ has the $AHEP$ over D then k is a homotopy equivalence of pairs over D.

Denote by $\mathscr{P}\mathscr{T}D$ the full subcategory of $\mathscr{T}D$ consisting of all $(Y, \check{x}, \hat{y})$ such that $\check{y}: D \subset Y$ has the $AHEP$ over D. Such a Y can be said to contain D as a strongly non-degenerate retract.

The transfer from $\mathscr{T}_D^2$ to $\mathscr{P}\mathscr{T}D$ is in two stages. First, replace $(X, A, \hat{x}) \in \mathscr{T}_D^2$ by $(M, A, \hat{x})$ where M is the ordinary mapping cylinder of $A \subset X$ (M formed in $\mathscr{T}_D$ is the same as M formed in $\mathscr{T}$). The map $M \to D$ defined from $\hat{x}$ is still denoted by $\hat{x}$. Then $(X, A, \hat{x}) \to (M, A, \hat{x})$ is a homotopy equivalence of pairs over D and $A \subset M$ has the $AHEP$ over D. It is easy to check that all axioms can be transferred to the intermediate category. Next, apply κ. It is easy to see that all the axioms except possibly the suspension axiom are then transferred to $\mathscr{P}\mathscr{T}D$. For the suspension axiom we must relate two different notions of suspension.

Let $(Y, \check{y}, \hat{y}) \in \mathscr{T}D$. Then by II.4.1 we have $(\Sigma Y, \check{y}, \hat{y}) \in \mathscr{T}D$. Now consider $(Y, D, \hat{y}) \in \mathscr{T}_D^2$. Then by I.3.1 we have $(\Sigma_D Y, D, \hat{y}) \in \mathscr{T}_D^2$. Note that ΣY is obtained from $\Sigma_D Y$ by collapsing $\Sigma_D D$ to D. $\Sigma_D Y$ is an unreduced suspension and ΣY is a reduced suspension. There is a map $k:(\Sigma_D Y, \Sigma_D D) \to (\Sigma Y, D)$. It can be checked that if $D \subset Y$ has the $AHEP$ over D then k is a homotopy equivalence of pairs under and over D.

1.3. Definition. Let $(Y, \check{y}, \hat{y}) \in \mathscr{P}\mathscr{T}D$. Define an $A(D)$-isomorphism $\Sigma: H^n(Y, D) \to H^{n+1}(\Sigma Y, D)$ by $\Sigma = k^{*-1}\Sigma_D$.

Now let $(X, A, \hat{x})$ be a pair over D and A be closed in X. The following diagram in $\mathscr{T}_D^2$ is commutative:

$$\begin{array}{ccc}
(\Sigma_D X, \Sigma_D A) & \xrightarrow{\Sigma_D \kappa(X,A)} & (\Sigma_D (X/A)_D, \Sigma_D D) \\
 & \searrow^{\kappa(\Sigma_D X, \Sigma_D A)} & \downarrow k \\
 & & (\Sigma_D (X/A)_D, D) \\
 & & \| \\
 & & ((\Sigma_D X/\Sigma_D A)_D, D).
\end{array}$$

This implies that the following diagram of $A(D)$-modules is commutative:

$$\begin{array}{ccc}
H^n(X, A) & \xleftarrow{\kappa(X,A)^*} & H^n((X/A)_D, D) \\
\downarrow \Sigma_D & & \downarrow \Sigma \\
H^{n+1}(\Sigma_D X, \Sigma_D A) & \xleftarrow{\kappa(\Sigma_D X, \Sigma_D A)^*} & H^{n+1}(\Sigma (X/A)_D, D)
\end{array}$$

where all maps are isomorphisms and $(\Sigma_D X/\Sigma_D A)_D$ has been identified with $\Sigma(X/A)_D$.

The result is that in order to prove the existence Theorem I.4.4 it suffices to define operations on $\mathscr{P}\mathscr{T}D$ and check the reformulated axioms – where "reformulation" in axiom 4 means replacing suspension by reduced suspension.

2. *Suspension Operations.* In this section two suspension operations are defined, one of which is dual to the reduced suspension of the previous section. Also, sufficient conditions are given for the operations to be surjective. All spaces and maps are assumed to be in $\mathscr{T}D$. So the functors P, Ω, C, Σ send $\mathscr{T}D$ to $\mathscr{T}D$ and, in particular, Σ coincides with the Σ of the previous section.

Consider now the following special case of the operation of II.5.3. Let $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2$ be a diagram in $\mathscr{T}D$ and assume $f_2 f_1$ is null-homotopic by a fixed null-homotopy H. Then we have the following homotopy commutative diagram

$$\begin{array}{ccccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & Cf_1 & \xrightarrow{k} & \Sigma A_0 & \xrightarrow{\Sigma f_1} & \Sigma A_1 \\ \downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle w_H} & & \downarrow & & \downarrow \\ D & \longrightarrow & A_2 & \longrightarrow & A_2 & \longrightarrow & D & \longrightarrow & \Sigma A_2 . \end{array}$$

If we apply $\langle\ ,Z\rangle$ to this diagram the resulting rows are exact in the sense of II.4.20.

2.1. Definition. $\rho=\rho_H : \langle A_2, Z\rangle \rightharpoonup \langle A_0, \Omega Z\rangle$ is defined to be $k^{*-1} w^*$.

ρ is defined on $\operatorname{Ker}(f_2^*)$ and its indeterminacy is $f_1^* \langle A_1, \Omega Z\rangle$. Let $\langle g\rangle \in \langle A_2, Z\rangle$ and let G be a null-homotopy of $g f_1$. Then a representative for $\rho\langle g\rangle$ is $(Pg)H - Gf_1$. If $A_0 \to A_1 \to A_2$ along with H is thought of as a fibration then ρ can be thought of as fiber space suspension.

Consider $\Omega A_0 \xrightarrow{\Omega f_1} \Omega A_1 \xrightarrow{\Omega f_2} \Omega A_2$ and a null-homotopy H' of $\Omega f_2 \Omega f_1$ determined from H in the natural way. Let

$$\rho' = \rho_{H'} : \langle \Omega A_2, Z\rangle \rightharpoonup \langle \Omega A_0, \Omega Z\rangle$$

be the corresponding operation. Then it is not difficult to prove the following theorem.

2.2. Theorem. $\rho'\langle \Omega g\rangle = -\Omega(\rho\langle g\rangle)$.

An important special case of the operation is obtained by letting $A_0 \to A_1 \to A_2$ be the path-loop fibration and H the constant null-homotopy. In case Z is $D \times K(Z_p, n)$ there is a definition in terms of cohomology.

2.3. Definition. Let $(Y, \check{y}, \hat{y})$ be in $\mathscr{T}D$. Define an $A(D)$ homomorphism $\sigma:\hat{y}H^n(Y,D)\to\hat{y}H^{n-1}(\Omega Y,D)$ as δp^* where $\bar{p}:(PY,\Omega Y)\to(Y,D)$ and δ is the coboundary from the exact sequence of the triple $(PY,\Omega Y,D)$, (an isomorphism).

In order to see that this is dual to the reduced suspension of the previous section, consider the following alternate definition of the latter.

2.4. Definition. Let $(Y,\check{y},\hat{y})$ be in $\mathscr{T}D$. Define $\Sigma:\hat{y}H^n(Y,D)\rightharpoonup\hat{y}H^{n+1}(\Sigma Y,D)$ as $\bar{k}^{*-1}\delta$ where $\bar{k}:(CX,X)\to(X,D)$ and δ is the coboundary from the exact sequence of the triple (CX,X,D) (an isomorphism).

This definition coincides with III.1.3 under the (more general) conditions here. Under the more restrictive conditions of II.1.3 ($D\subset Y$ has the $AHEP$ over D) the operation is an isomorphism – although here it may merely be an additive relation. Note, however, that Σ^{-1} is always a homomorphism.

Let $f:X\to\Omega Y$ be in $\mathscr{T}D$ and $f':\Sigma X\to Y$ be its adjoint. So

$$f:(X,D)\to(\Omega Y,D) \quad\text{and}\quad f':(\Sigma X,D)\to(Y,D).$$

2.5. Lemma. $f^*\sigma=\Sigma^{-1}f'^*$.

Proof. Exactly as in Adams [1, p. 62].

In order to apply the relative Serre theorem (Serre [29], or Spanier [32, p. 506]) to the $\bar{p}$ of 2.3 we need to know that $\bar{p}$ is a fibration in $\mathscr{T}=$ category of spaces and maps. This is easily proved. The Serre theorem implies that $\bar{p}^*:H^k(Y,D)\to H^k(PY,\Omega Y)$ is isomorphic for $k\leqq 2$ (connectivity of (Y,D))-1. Hence σ of 2.3 is isomorphic in the same range. If $A_1\to A_2$ in 2.1 is actually a fibration in $\mathscr{T}$ and $Z=D\times K(Z_p,n)$ then it follows similarly that $\rho:H^n(A_2,D)\rightharpoonup H^{n-1}(A_0,D)$ is epimorphic if $n\leqq \operatorname{conn}(A_2,D)+\operatorname{conn}(A_0,D)$.

3. Proofs of the Existence and Quasi-Uniqueness Theorems. Let $d:C\to C'$ be a map of free finitely generated graded $A(D)$-modules. A map $f:B\to B'$ in $\mathscr{T}D$ will be called a *realization of d of degree k* provided:

(1) $B=D\times X_i K(Z_p,\deg(c_i)+k)$ and $B'=D\times X_j K(Z_p,\deg(c_j')+k)$ where the c_i's generate C and the (c_j')'s generate C';

(2) $f=(\text{proj},h)$ and

$$h:(D\times X_i K(Z_p,\deg(c_i)+k),D)\to X_j K(Z_p,\deg(c_j')+k)$$

is determined by $h^*\iota(i,k)=\Sigma s(i,j)(1\otimes\iota(j,k)')$ where $d(c_i)=\Sigma s(i,j)c_j'$, $s(i,j)\in A(D)$, and the ι's are fundamental classes.

Suppose that

$$C: C_N\xrightarrow{d_n}C_{N-1}\longrightarrow\cdots\longrightarrow C_1\xrightarrow{d_1}C_0$$

is a chain complex as in I.4. Call

$$B(o) \xrightarrow{f(1)} B(1) \longrightarrow \cdots \longrightarrow B(N-1) \xrightarrow{f(N)} B(N)$$

in $\mathcal{T}D$ a *horizontal realization of C of degree k* provided that $f(i)$ is a realization of degree k of d_i. Consider the following diagram in $\mathcal{T}D$.

Diagram $\mathcal{U}_k(N)$

$$\begin{array}{llllll}
\Omega^{N-1}B(N-1)\to\cdots E(2,N-1)\to & E(1,N-1) & \xrightarrow{g(N)} \Omega^{N-1}B(N) & & & \\
\quad\downarrow p(N-1,N-2) & \downarrow & & & & \\
\Omega^{N-2}B(N-2)\to\cdots E(2,N-2)\to & E(1,N-2) & \xrightarrow{g(N-1)} \Omega^{N-2}B(N-1) & \xrightarrow{\Omega^{N-2}f(N)} \Omega^{N-2}B(N) & & \\
\quad\downarrow & \downarrow & & & & \\
\quad\vdots & \vdots & & & & \\
\Omega^2B(2)\to \quad E(2,2) \to & E(1,2) & \xrightarrow{g(3)} \Omega^2B(3)\ldots\Omega^2B(N) & & & \\
\quad\downarrow & \downarrow & & & & \\
\quad \Omega B(1) \to & E(1,1) & \xrightarrow{g(2)} \Omega B(2) & \xrightarrow{\Omega f(3)} \Omega B(3) & \to\cdots\Omega B(N) & \\
 & \downarrow p(1,0) & & & & \\
 & B(0) & \xrightarrow{f(1)} B(1) & \xrightarrow{f(2)} B(2) & \xrightarrow{f(3)} B(3) & \to\cdots\to B(N)
\end{array}$$

Denote the diagram by $\mathcal{U}_k(N)$. Each $E(m,n)\to E(m,n-1)$ is a principal fibration induced by the map $E(m,n-1)\to\Omega^{m-1}B(n)$ where $B(0)$ is thought of as $E(1,0)$, $\Omega B(1)$ as $E(2,1)$ etc. All spaces and maps depend on k and this will be indicated by a subscript if necessary. $\mathcal{U}_K(N)$ will be called a *universal example in degree k* provided that $\langle g(i)\rangle\in\rho\,\langle\Omega^{i-2}f(i)\rangle$, $2\leqq i\leqq N$. ρ is defined as in III.2.1 using the natural null-homotopy of $E(1,i-1)\to E(1,i-2)\to\Omega^{i-2}B(i-1)$.

Define $\Omega\mathcal{U}_k(N)$ to be the diagram obtained from $\mathcal{U}_k(N)$ by replacing all spaces by their loops and all maps by their loops except the $g(n)$'s. Replace $g(n)$ by $(-1)^{n-1}\Omega g(n)$.

Consider $\mathcal{U}_k(N)$ and $\mathcal{U}_{k+1}(N)$. We can find a sequence of homotopy equivalences from the bottom row of $\mathcal{U}_k(N)$ to the bottom row of $\Omega\mathcal{U}_{k+1}(N)$ – giving a homotopy commutative ladder. Pick any such sequence. Then homotopy equivalences are induced on all rows to give a map: $\alpha_k(N)\colon\mathcal{U}_k(N)\to\Omega\mathcal{U}_{k+1}(N)$ with homotopy commutativity throughout. The sign introduced with the g's in $\Omega\mathcal{U}_{k+1}(N)$ was to guarantee this homotopy commutativity (see III.2.2).

The collection $\{\mathcal{U}_k(N),\alpha_k(N)\}=\big(\mathcal{U}(N),\alpha(N)\big)$ is called a *universal example for C* or a vertical realization of C.

Fundamental classes for the universal example are chosen as follows. First make a choice for the $K(Z_p,n)$'s of the bottom row of $\mathcal{U}_k(N)$ for a fixed k. Then always assume $\sigma\,\iota_n=\iota_{n-1}$ where (here only)

$$\sigma\colon H^n\big(K(Z_p,n);Z_p\big)\to H^{n-1}\big(\Omega K(Z_p,n-1);Z_p\big)$$

is the ordinary cohomology suspension. This gives fundamental classes for all of $\mathscr{U}_k(N)$. The α's then determine fundamental classes for all $\mathscr{U}_j(N)$'s. These fundamental classes are used to set up a correspondence between homotopy and cohomology in what follows. In order for this correspondence to be valid it is assumed for all spaces Y (in $\mathscr{P}\mathscr{T}D$) under consideration that (Y, D) has the homotopy type of a CW pair. The results can be extendend to the general case by standard techniques (e.g., CW-approximation).

3.1. Lemma. *Let* $C: C_N \to \cdots \to C_0$ *be a chain complex as in* I.4 *and let* $(\mathscr{U}(N), \alpha(N))$ *be a universal example for* C. *If the operations* $\{\varphi^{r,s}\}$ *are defined by*

$$\Omega^s B(s) \leftarrow E(s+1, r-1) \rightarrow \Omega^r B(r) \tag{3.2}$$

then they satisfy the axioms given in I.4.

Proof. Use induction. The full definition of $\varphi = \varphi^{N,0}$ is $\varphi = g_* p_*^{-1}$ where:

$$\begin{array}{l} \mathrm{Hom}_k(C_0, \hat{y} H^*(Y, D)) \\ \quad \| \\ \langle Y, B_k(0) \rangle \xleftarrow{p_*} \langle Y, E_k(1, N-1) \rangle \xrightarrow{g_*} \langle Y, \Omega^{N-1} B_k(N) \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Hom}_{k-N+1}(C_N, \hat{y} H^*(Y, D)) \end{array}$$

and $g = g(N)$, $p = p(1, 0) \ldots p(1, N-2)$. The details are omitted (see [23], cf. (Maunder [22]))

3.2. Lemma. *Let* $C: C_N \to \cdots \to C_0$ *be a chain complex as in* I.4. *Suppose* $\varphi^{N,0}$ *is the peak of a pyramid of operations associated with* C. *Then there exists a universal example for* C *such that* $\varphi^{N,0}(\langle p \rangle) \ni g(N)$ *where* $p = p(1, 0) \ldots p(1, N-2)$.

Proof. This is omitted. In order to retain the homotopic flavor of the development use

$$\rho: \langle \Omega^{N-2} B(N-1), \Omega^{N-2} B(N) \rangle \rightarrow \langle E(1, N-1), \Omega^{N-1} B(N) \rangle$$

to get g from $\Omega^{N-2} f(N)$. ρ is defined as in III.2.1 by the canonical null-homotopy of $g(N-1)\, p(1, N-2)$.

Theorems I.4.4 and I.4.5 follow without difficulty from these lemmas.

Part IV. Some Exact Sequences

Two special cases of the exact sequence of II.3.3 are considered here. The first is stated principally in the language of $\mathscr{T}_D$ (spaces and maps over D). The second is a further specialization and is stated partly in terms of $\mathscr{T}_D$ and partly in terms of $\mathscr{T}*$ (= the category of pointed spaces and maps). The second sequence and its dual are used to show that the definition of the functional twisted cohomology operation given by Meyer [24] and Gitler-Stasheff [9] can be viewed as a special case of the bracket operation of II.5.1. It follows that the functional operation is related to the corresponding secondary operation of part III by the Peterson-Stein formula III.5.4.

Suppose that $(X, A, \hat{x})$ is a pair over D (meaning merely that $A \subset X$ and $\hat{x}: X \to D$) and $(Y, B, \check{y}, \hat{y})$ is a pair under and over D (meaning $\check{y}: D \subset B$, $B \subset Y$, $\hat{y}: Y \to D$, and $\hat{y}\,\check{y} = 1$). Denote the set of homotopy classes of maps, classified by homotopy over D, by $[(X, A); (Y, B)]_D$. This can be thought of as taking place in $\mathscr{T}^2(\emptyset \to D)$ where $\mathscr{T}^2$ is the category of pairs of spaces. (Y, B) is sent into this category by the forgetful function. Then $[(X, A); (Y, B)]_D$ has $[\check{y}\,\hat{x}]_D$ as a distinguished element. Recall from III.1.1 that $((X/A)_D, \check{x}, \hat{x})$ is in $\mathscr{T}D$. The procedure of passing from $(X, A, \hat{x})$ to $((X/A)_D, \check{x}, \hat{x})$ is a variation on "adding a disjoint base point" and could be viewed as simply that – adding a disjoint copy of D – by working in $\mathscr{T}^2 D$ instead of $\mathscr{T}D$. Now, note that $\langle (X/A)_D; Y \rangle = [(X, A); (Y, D)]_D$. This plus the exact sequence of III.3.3 is all that is needed to prove the following theorem.

1. Theorem. *Suppose that $f: Y \to Z$ is a map of $\mathscr{T}D$ and that $(X, A, \hat{x})$ is a pair over D. Then there is an exact sequence of pointed sets:*

$$\cdots \to [(X, A); (\Omega^2 Z, D)]_D$$
$$\to [(X, A); (\Omega Ef, D)]_D \to [(X, A); (\Omega Y, D)]_D \to [(X, A); (\Omega Z, D)]_D$$
$$\to [(X, A); (Ef, D)]_D \to [(X, A); (Y, D)]_D \to [(X, A); (Z, D)]_D$$

and an action

$$[(X, A); (\Omega Z, D)]_D \times [(X, A); (Ef, D)]_D \to [(X, A); (Ef, D)]_D$$

where ΩY, ΩZ, and Ef are formed in $\mathscr{T}D$. The sequence consists of groups and homomorphisms after the third term and abelian groups after the sixth. The action has properties like those in III.3.3.

The following definition is needed for the next theorem.

2. Definition. (James-Thomas [14, p. 501].) Let $g: (D \times K, D \times *) \to (M, *)$ be a map of pairs. Define $\Delta g: (D \times \Omega K, D \times *) \to (\Omega M, *)$ by $\Delta g(d, a)(t) = g(d, a(t))$.

Let $g: (D \times K, D \times *) \to (M, *)$ be a map of pairs of spaces. Let $Eg \to D \times K$ be the principal fibration induced by g in $\mathscr{T}*$. Let ΩK and ΩM be the ordinary loop spaces formed in $\mathscr{T}*$. Let $(X, A, \hat{x})$ be a pair over D.

Define

$$\alpha g:[(X,A);(K,*)]\to[(X,A);(M,*)],\ \beta g:[(X,A);(Eg,D)]_D \to[(X,A);(K,*)],$$

and $\gamma g:[(X,A);(\Omega M,*)]\to[(X,A);(Eg,D)]_D$ as follows: $\alpha g[v]=$ the class of $(X,A)\xrightarrow{(x,v)}(D\times K, D\times *)\xrightarrow{g}(M,*)$, $\beta g[w]_D=$ the class of

$$(X,A)\xrightarrow{w}(Eg,D)\xrightarrow{\text{proj}}(D\times K, D\times *),$$

and $\gamma g[u]=$ the class over D of

$$(X,A)\xrightarrow{(x,u)}(D\times\Omega M, D\times *)\xrightarrow{i}(Eg,D).$$

3. Theorem. *Use the notation above. There is an exact sequence of pointed sets*

$$\cdots\longrightarrow[(X,A);(\Omega^2 M,*)]$$
$$\xrightarrow{\gamma\Delta g}[(X,A);(E\Delta g,D)]_D\xrightarrow{\beta\Delta g}[(X,A);(\Omega K,*)]\xrightarrow{\alpha\Delta g}[(X,A);(\Omega M,*)]$$
$$\xrightarrow{\gamma g}[(X,A);(Eg,D)]_D\xrightarrow{\beta g}[(X,A);(K,*)]\xrightarrow{\alpha g}[(X,A);(M,*)]$$

and an action

$$[(X,A);(\Omega M,*)]\times[(X,A);(Eg,D)]_D\to[(X,A);(Eg,D)].$$

The sequence consists of groups and homomorphisms after the third term and abelian groups after the sixth term. The action has properties like those in III.3.3.

Proof. In Theorem 2 replace Y by $D\times K$, Z by $D\times M$, and f by (proj, g). Note that Ef, formed in $\mathcal{T}D$ is the same as Eg which is defined in $\mathcal{T}*$. Also, Ωf is (proj, Δg). Finally observe that $[(X,A);(D\times K,D)]_D=[(X,A);(K,*)]$. Q.E.D.

4. *Example.* Let (X,A) and $(D,\emptyset)$ be relative CW complexes. Let $K=K(Z,n)$, $M=K(Z,n+1)$, and $d\in H^k(D,Z)$. Define, for each non-negative integer n, $g(n):D\times K(Z,n)\to K(Z,n+1)$ by $g(n)^*\,\iota(n+1)=d\otimes\iota(n)$. Then for a pair over D, $(X,A,\hat{x})$, Theorem 3 gives the following exact sequence of abelian groups and homomorphisms:

$$\ldots H^{n-1}(X,A)\xrightarrow{\varphi}H^{n-1+k}(X,A)\to[(X,A);(Eg(n),D)]_D$$
$$\to H^n(X,A)\xrightarrow{\varphi}H^{n+k}(X,A)\to[(X,A);(Eg(n+1),D)]_D\to\cdots$$

where $\varphi(w)=\hat{x}^*(d)\,w$.

Note that if $D=X$, $\hat{x}=$ identity, and $d=x\in H^k(X)$ then $\varphi=x\cup$: $H^n(X,A)\to H^{n+k}(X,A)$. So φ is the homomorphism defined by cupping with x and we have succeeded in including it in an exact sequence in a natural way.

5. *Example.* Suppose that $f:(X,A)\to(Y,B)$ in $\mathscr{T}^2$ and $u\in H^k(Y;Z)$ are given. Consider (X,A) to be a pair over Y by means of f. Let $w\in H^n(Y,B)$ be such that $f^*w=0$ and $u\cdot w=0$. Define $g=u\otimes\iota:Y\times K(Z,n)\to K(Z,n+k)$ and form

$$(X/A)_Y \xrightarrow{f} (Y/B)_Y \xrightarrow{(1,w)} Y\times K(Z,n) \xrightarrow{(\mathrm{proj},g)} Y\times K(Z,n+k)$$

in $\mathscr{T}Y$. If we form the diagram for $\langle f,(1,w),(\mathrm{proj},g)\rangle$ as in III.5.1 B and make the same sort of identifications as in the proof of Theorem 3, then the result is seen to be exactly the Steenrod definition of the functional cup product $u\cup_f w$. If III.5.1A is used instead then, as in Example 4, we get the following dual definition of the functional cup product.

$$\begin{array}{ccccc}
H^{n-1}(X,A) & \xrightarrow{f^*u\cup} & H^{n-1+k}(X,A) & \xrightarrow{\gamma g} & [(X,A);(E,Y)]_Y \\
\uparrow & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\
H^{n-1}(Y,B) & \longrightarrow & H^{n-1+k}(Y,B) & \longrightarrow & [(Y,B);(E,Y)]_Y
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^{n+k}(X,A) & \xrightarrow{f^*u\cup} & H^{n+k}(X,A) \\
 & \uparrow & & \uparrow \\
\longrightarrow & H^n(Y,B) & \xrightarrow{u\cup} & H^{n+k}(Y,B)
\end{array}$$

where $E=Eg$. That is, $-u\cup_f w=(\gamma g)^{-1}f^*(\beta g)^{-1}w=a$ coset of $f^*H^{n-1+k}(Y,B)+f^*uH^{n-1+k}(X,A)$ in $H^{n-1+k}(X,A)$.

6. *Example.* Let (X,A) and $(D,\emptyset)$ be relative CW complexes. Let $s\in A(D)=H^*(D;Z_p)\odot A_p$. Represent s in each dimension n by a map $s(n)=d\otimes\iota(n)+1\otimes Sq^2\,\iota(n):D\times K(Z_p,n)\to K(Z_p,n+k)$ where k is the degree of s. For example, if $d\in H^2(D;Z_2)$ then $s=d\otimes 1+1\otimes Sq^2$ is represented by $s(n)=d\otimes\iota(n)+1\otimes Sq^2\,\iota(n):D\times K(Z_2,n)\to K(Z_2,n+2)$. For $(X,A,\hat{x})$, a pair over D, Theorem 3 gives the following exact sequence of abelian groups and homomorphisms:

$$\ldots H^{n-1}(X,A;Z_p)\to H^{n-1+k}(X,A;Z_p)\to\big[(X,A);(Es(n),D)\big]_D$$
$$\to H^n(X,A;Z_p)\to\ H^{n+k}(X,A;Z_p)\ \to\big[(X,A);(Es(n+1),D)\big]_D\ldots.$$

7. *Example.* Suppose that $f:(X,\hat{x})\to(Y,\hat{y})$ is a map of spaces over D. Suppose also that $w\in H^n(Y;Z_p)$, $s\in A(D)$, $f^*w=0$, and $s(w)=0$. Consider the following diagram in $\mathscr{T}D$:

$$X^+ \xrightarrow{f^+} Y^+ \xrightarrow{(y,w)} D\times K(Z_p,n) \xrightarrow{(\mathrm{proj},s)} D\times K(Z_p,n+1)$$

where $X^+=(X/\emptyset)_D=X\cup D$ (disjoint union), $y=\hat{y}\cup\mathrm{id}$, and w means $w\cup *$.

First apply Definition III.5.1B. After the appropriate identifications are made we get the following diagram:

$$\begin{array}{ccccccccc} H^{n-1}(Y) & \longrightarrow & H^{n-1}(X) & \longrightarrow & H^n(Y,X) & \xrightarrow{j^*} & H^n(Y) & \longrightarrow & H^n(X) \\ \downarrow & & \downarrow s & & \downarrow s & & \downarrow s & & \downarrow \\ H^{m-1}(Y) & \longrightarrow & H^{m-1}(X) & \xrightarrow{i^*} & H^m(Y,X) & \longrightarrow & H^m(Y) & \longrightarrow & H^m(X) \end{array}$$

where $m=n+k$ and (Y,X) means (Mf,X) where Mf is the ordinary mapping cylinder. Then $s_f w = \langle f^+, (y,w), (\text{proj}, s)\rangle = i^{*-1} s j^{*-1} w$, a coset of $sH^{n-1}(X)+f^* H^{n-1+k}(Y)$ in $H^{n-1+k}(X)$.

This is exactly the definition of the twisted functional operation given by Meyer in [24] and Gitler-Stasheff in [9].

Definition III.5.1A gives, as in Example 6, the following diagram:

$$\begin{array}{ccccccccc} H^{n-1}(X) & \xrightarrow{s(n-1)} & H^{m-1}(X) & \xrightarrow{\gamma s(m)} & [X,E]_D & \longrightarrow & H^n(X) & \longrightarrow & H^m(X) \\ \uparrow & & \uparrow f^* & & \uparrow f^* & & \uparrow & & \uparrow \\ H^{n-1}(Y) & \xrightarrow{s(n-1)} & H^{m-1}(Y) & \longrightarrow & [X,E]_D & \xrightarrow{\beta s(n)} & H^n(Y) & \longrightarrow & H^m(Y) \end{array}$$

where $m=n+k$ and $E=Es(n)$. So $-s_f w = \langle f^+, (y,w), (\text{proj}, s)\rangle' = (\gamma s(n))^{-1} f^* (\beta s(n))^{-1} w$ is an alternate, equivalent, definition of the twisted functional operation.

Part V. An Application

Twisted operations occur naturally in connection with existence and classification of cross-sections of a fibration. The classification problem will be taken up here.

Consider the following commutative diagram of pointed spaces and maps.

$$(1) \qquad \begin{array}{ccc} & & Y \\ & \nearrow & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

where p is a fibration and X is a CW complex. g will be spoken of as a "map over f". A "homotopy over f" is a base point preserving homotopy $H_t: X \to Y$ such that $pH_t = f$ for each $t \in I$. Let $[X,Y]_B$ be the set of pointed homotopy classes of pointed maps over f. Denote it by $[X,Y]_{f,p}$ if necessary.

It will be shown that $[X,Y]_B$ can be calculated in terms of a tower of fibrations which factors p and various twisted operations in $H^*(X)$. The method described here is designed for use with Postnikov systems. The results could be stated in terms of a spectral sequence. A modification of the method yields a spectral sequence of the Adams type – this will be the subject of a separate paper.

1. Generalities.

A_k) assume that diagram (1) can be embedded in the following diagram

$$
\begin{array}{ccccccc}
 & & Y^2 & \longrightarrow & Y & & \\
 & \swarrow & & \searrow & & \searrow & \\
E''(k) & \longrightarrow & E'(k) & \longrightarrow & E(k) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
E''(k-1) & \longrightarrow & E'(k-1) & \longrightarrow & E(k-1) & \xrightarrow{f(k-1)} & C(k-1) \\
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
E''(2) & \longrightarrow & E'(2) & \longrightarrow & E(2) & \xrightarrow{f(2)} & C(2) \\
\downarrow & & \downarrow & & \downarrow & & \\
Y\times\Omega C(0) & \xrightarrow{v(1)} & E'(1) & \xrightarrow{u(1)} & E(1) & \xrightarrow{f(1)} & C(1) \\
 & \searrow & \downarrow & & \downarrow & & \\
X & \xrightarrow{g} & Y & \xrightarrow{p} & B & \xrightarrow{f(0)} & C(0)
\end{array}
\tag{2}
$$

where

(1) $g: Y^2 \to Y$ is the square of p-i.e., the pullback of p by p.

(2) $E(i+1) \to E(i)$, $E'(i+1) \to E'(i)$, and $E''(i+1) \to E''(i)$ are induced from $PC(i) \to C(i)$, the path-loop fibration over $C(i)$.

(3) $v(1)$ is a fiber homotopy equivalence due to $f(0)p \sim 0$.

Note that g has a lifting to Y^2-namely, sg where $s: Y \to Y^2$ is the canonical cross section of $g: Y^2 \to Y$.

B_k) assume $\pi_i(Y) \to \pi_i(E(k))$ is isomorphic for $i \leqq N_k$.

Then the same is true for $Y^2 \to E''(k)$ and by the approximation theorem for liftings (James-Thomas [15]) we have that

$$[X, Y]_B = [X, Y^2]_Y \to [X, E''(k)]_Y$$

is bijective for dimension $X \leqq N_k$.

C_k) assume $Y \subset Y \times \Omega C(0)$ has the HEP (homotopy extension property) with respect to $C(1)$. Then $f(1)u(1)v(1)$ is homotopic to $g(1)$ and $g(1)|Y = *$. Let $P(2)$ be induced by $g(1)$. Then $P(1) \to Y$ is fiber homotopically equivalent to $E''(2) \to Y$ and admits a section. Replace $E''(2)$ by $P(2)$ and repeat the process (with a new assumption).

The result is:

$$
\begin{array}{ccccccccccccc}
 & & & & C(k-1) & & & & C(2) & & C(1) & & \\
 & & & & \uparrow{\scriptstyle g(k-1)} & & & & \uparrow{\scriptstyle g(2)} & & \uparrow{\scriptstyle g(1)} & & \\
Y^2 & \to & P(k) & \to & P(k-1) & \to & \cdots & & P(2) & \to & Y\times\Omega C(0) & \to & Y
\end{array}
\tag{3}
$$

where $P(i) \to Y$ has, say, $s(i)$ as a section and $g(i)|s(i)(Y) = *$. Also $[X, Y]_B = [X, Y^2]_Y \to [X, P(k)]_Y$ is bijective for dimension $X \leqq N_k$.

At this point we can transfer the problem to the category $\mathscr{T}Y$ (see Part II for the properties of $\mathscr{T}Y$). We have the following diagram in $\mathscr{T}Y$.

$$
\begin{array}{ccccc}
Y \times \Omega C(k) & \longrightarrow & P(k) & & \\
 & & \downarrow & & \\
Y \times \Omega C(k-1) & \xrightarrow{i(k-1)} & P(k-1) & \xrightarrow{g'(k)} & Y \times C(k) \\
 & & \vdots & & \\
Y \times \Omega C(2) & \longrightarrow & P(2) & \xrightarrow{g'(3)} & Y \times C(3) \\
 & & \downarrow & & \\
Y \times \Omega C(1) & \xrightarrow{i(1)} & P(1) & \xrightarrow{g'(2)} & Y \times C(2) \\
 & & \downarrow & & \\
 & & Y \times \Omega C(0) & \xrightarrow{g'(1)} & Y \times C(1).
\end{array}
\tag{4}
$$

Here $g'(i) = (p(i), g(i))$, $p(i): P(i) \to Y$ is from (3), and $P(i+1) \to P(i)$ is induced from the path-loop fibration in $\mathscr{T}Y: Y \times \Omega C(i) \to Y \times PC(i) \to Y \times C(i)$. ($P$ and Ω are still the path and loop functors on $\mathscr{T}* =$ the ordinary category of pointed spaces and maps. Let $\bar{P}$ and $\bar{\Omega}$ denote the corresponding functors on $\mathscr{T}Y$. Recall from Part II that $\bar{P}(Y \times C(i)) = Y \times PC(i)$ and $\bar{\Omega}(Y \times C(i)) = Y \times \bar{\Omega} C(i)$.) Also $[X, Y]_B = \langle X \vee Y, P(i) \rangle$ where $\langle\, ,\rangle$ denotes the set of homotopy classes in $\mathscr{T}Y$.

D_k) assume $C(0)$ is a loop space in $\mathscr{T}*$ and $C(1), \ldots, C(k)$ are double loop spaces in $\mathscr{T}*$. Assume that $g'(i)$ is a loop map in $\mathscr{T}Y$.

Theorem 1 (classical). *Assume* A_1, B_1, C_1, D_1. *If* $\dim X \leqq N_1$ *then* $[X, Y]_B \leftrightarrow [X, \Omega C(0)]$.

Theorem 2. *Assume* A_2, B_2, C_2, D_2. *If* $\dim X \leqq N_2$ *then*

a) $[X, Y]_B$ *has a natural group structure.*

b) (James-Thomas [16])

$$[X, Y]_B = L_0 \supset L_1 \supset L_2 = 0$$

and

$$L_0/L_1 = \operatorname{Ker} \alpha \subset [X, \Omega C(0)],$$

$$L_1/L_2 = [X, \Omega C(1)]/\operatorname{Im} \alpha$$

where α *is the twisted primary operation determined by* $g(1)$.

Theorem 3. *Assume* A_3, B_3, C_3, D_3. *If* $\dim X \leqq N_3$ *then*

a) $[X, Y]_B$ *has a natural group structure.*

b) $[X, Y]_B = L_0 \supset L_1 \supset L_2 \supset L_3 = 0$

$$L_0/L_1 = \operatorname{Ker} \varphi \subset [X, \Omega C(0)],$$

$$L_1/L_2 = \operatorname{Ker} \beta / \operatorname{Im} \alpha,$$

$$L_2/L_3 = [X, \Omega C(2)] / \operatorname{Im} \Phi$$

where $\alpha: [X, \Omega^2 C(0)] \to [X, \Omega C(1)]$ *and* $\beta: [X, \Omega C(1)] \to [X, C(2)]$ *are the twisted primary operations determined by* $g(1)$ *and* $g(2)\, i(1)$, *respectively, and* Φ *is the twisted secondary operation due to the relation* $\beta\alpha = 0$.

Theorem k. *Assume* A_k, B_k, C_k, D_k. *If* $\dim X \leqq N_k$ *then*

a) $[X, Y]_B$ *has a natural group structure.*

b) $[X, Y]_B = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ *and the quotients* L_i/L_{i+1} *are determined by twisted operations of order* $\leqq k-1$. *The operations are determined by the diagram* (2).

Proofs. Apply the functor $\langle X \vee Y, - \rangle$ to the diagram (4). By II.3.3 the result is part of an exact couple of abelian groups. The theorems can now be proved by standard techniques.

Comment. Becker proved in (3) that if $\dim X \leqq 2 \operatorname{Conn} F$ (F = fiber of p) then $[X, Y]_B$ has a natural abelian group structure. His theorem can be proved by the methods used here.

2. *A Guiding Principle.* Associated with every modified Postnikov system for $p: Y \to B$ there is a classification theorem. The twisted operations involved can be read off from the data of the Postnikov system. See Mahowald [19] or Thomas [36] for "modified Postnikov system".

3. *A Specific Example.* Consider

$$V_{m, m-k} \to BSO(k) \xrightarrow{p} BSO(m)$$

where $k = 4s+1$, $V_{m,m-k}$ is the Stiefel manifold of $(m-k)$-frames in R^m and $BSO(n)$ is the classifying space for $SO(n)$. The following system was constructed by Mahowald in [19].

$$\begin{array}{ccccc}
 & & B_k \searrow & & \\
 & & E_3 & & \\
 & & \downarrow & & \\
B_k \times K_{4s+2} \times K_{4s+3} \times K_{4s+4} & \xrightarrow{i_2} & E_2 & \xrightarrow{k^3} & K_{4s+4} \\
\downarrow & & \downarrow & & \\
B_k \to B_k \times K_{4s+1} \times K_{4s+3} & \xrightarrow{i_1} & E_1 & \xrightarrow{(k_1^2, k_2^2, k_3^2)} & K_{4s+3} \times K_{4s+4} \times K_{4s+5} \\
 & & \downarrow & & \\
 & B_k \xrightarrow{p} & B_m & \xrightarrow{(W_{4s+2}, W_{4s+4})} & K_{4s+2} \times K_{4s+4}
\end{array}$$

where $K_i = K(Z_2, i)$ is an Eilenberg-MacLane space of type (Z_2, i), $B_n = BSO(n)$,

$$k_1^2 i_1 = L\bar{\iota}_{4s+1} \qquad (L = 1 \otimes Sq^2 + W_2 \otimes 1 \in A(B_k),\ \bar{\iota}_j = 1 \otimes \iota_j \otimes 1),$$
$$k_2^2 i_1 = LSq^1 \bar{\iota}_{4s+1} + Sq^1 \bar{\iota}_{4s+3},$$
$$k_3^2 i_1 = (1 \otimes Sq^4 + W_4 \otimes 1)\bar{\iota}_{4s+1} + Q\bar{\iota}_{4s+3},$$

where $Q = W_2 \otimes 1$ if $K = 8s+1$ and $Q = 1 \otimes Sq^2$ if $K = 8s+5$, and

$$k^3 i_2 = L\bar{\iota}_{4s+2} + Sq^1 \bar{\iota}_{4s+3}.$$

The system has the property that $\pi_i(B_k) \to \pi_i(E_3)$ is isomorphic for $i \leqq 4s+5$.

Assumptions A_3, B_3, and D_3 of V.I are clearly satisfied with $N_3 = 4s+5$. The classifying spaces and the K_i's can be taken to be metric simplicial complexes (Milnor [25]). This and Theorems 11.3 and 2.2 of Hu [11, p. 106/117] imply C_3. Theorem 3 of V.I gives the following theorem.

3.1. Theorem. *Let $f: X \to BSO(m)$, $g: X \to BSO(k)$, and $pg = f$. Assume* dim $X \leqq 4s+5$. *Then $[X, BSO(k)]_{BSO(m)} = L_0 \supset L_1 \supset L_2 \supset L_3 = 0$ and*

$$L_0/L_1 = \text{Ker}\,\varphi$$
$$L_1/L_2 = \text{Ker}\,\beta/\text{Im}\,\alpha$$
$$L_2/L_3 = H^{4s+3}(X)/\text{Im}\,\varphi$$

where

$$\alpha: H^i(X) \oplus H^{i+2}(X) \to H^{i+2}(X) \oplus H^{i+3}(X) \oplus H^{i+4}(X)$$
$$\beta: H^i(X) \oplus H^{i+1}(X) \oplus H^{i+2}(X) \to H^{i+2}(X)$$
$$\varphi: H^i(X) \oplus H^{i+2}(X) \to H^{i+3}(X)$$
$$\alpha(x, y) = (Lx,\ LSq^1 x + Sq^1 y,\ Sq^4 x + W_4 x + Qy),$$
$$\beta(x, y, z) = Lx + Sq^1 y,$$

and $\varphi = \varphi^{2,0}$ is due to the relation $\beta\alpha = 0$.

If $X = RP^{4s+5}$ (real projective space) the operations are not difficult to evaluate. φ is seen to be trivial by first evaluating it in complex projective space and then using the natural map $RP^{2n} \to CP^n$ and the naturality of φ. The following theorem is the result.

3.2. Theorem. *Let ξ be a vector bundle over RP^{k+4}, $k=4s+1$. Assume that ξ has $m-k$ linearly independent cross sections. Let*

$$V_{m,m-k} \to E \xrightarrow{p} RP^{k+4}$$

be the associated bundle. It has a cross section. Denote by Sect$[p]$ *the group of homotopy classes of sections of p. Then the order of* Sect$[p]$ *is given by the following table.*

			$\lvert \mathrm{Sect}[p] \rvert$	
1)	$W_2(\xi)=0$,	$W_4(\xi)=0$	16	*s even*
			8	*s odd*
2)	$W_2(\xi)=0$,	$W_4(\xi)=x^4$	4	*s even*
			4	*s odd*
3)	$W_2(\xi)=x^2$,	$W_4(\xi)=0$	1	
4)	$W_2(\xi)=x^2$,	$W_4(\xi)=x^4$	1	

If $\xi=\nu$ is the stable normal bundle for RP^{4s+5} then $W_2(\nu)=x^2$ and $W_4(\nu)=0$. The following corollary now follows from the Smale-Hirsh theorem as formulated by James and Thomas in [15].

3.3. Corollary. *Any two immersions of RP^{4s+5} in RP^{8s+6} are regularly homotopic.*

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