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The Gliding Hump Property in Vector Sequence Spaces

By

Charles Swartz, Las Cruces

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Abstract. It is shown that vector sequence spaces with a gliding hump property have many of the properties of complete spaces. For example, it is shown that the β-dual of certain vector sequence spaces with a gliding hump property are sequentially complete with respect to the topology of pointwise convergence and also versions of the Banach–Steinhaus Theorem are established for such spaces.

1. Introduction

The gliding hump property for sequence spaces has been used to treat various topics in the theory of sequence spaces [16, 17, 19]. The gliding hump property has at least some formal resemblance to the $\mathscr{K}$-property for normed linear spaces introduced in [1], and the $\mathscr{K}$-property has been shown to be a useful substitute for completeness in treating various topics in functional analysis such as the uniform boundedness principle, the Mazur–Orlicz Theorem for separately continuous bilinear maps and the closed graph theorem [3]. This suggests that the gliding hump property may serve as a substitute for completeness in sequence spaces. In this note we show that this is indeed the case. As in the case with the $\mathscr{K}$-property, we show that the Basic Matrix Theorem of Antosik and Mikusinski ([3] 2.2) can be used to show that sequence spaces with the gliding hump property have many of the properties of complete spaces, and, in fact, the Basic Matrix Theorem can be used to treat the case of vector-valued sequence spaces with operator-valued β-duals as introduced by Maddox [15]. Our vector results give generalizations of scalar results of Noll [16], and the proofs given by the Basic Matrix Theorem give interesting contrasts to the previous scalar proofs.

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1. Preliminaries

Henceforth, X and Y will denote Hausdorff topological vector spaces and $L(X, Y)$ will denote the space of all continuous linear operators from X into Y. We will say that the pair (X, Y) has the *weak Banach–Steinhaus property* if $\{T_k\} \subseteq L(X, Y)$ and $\lim_k T_k x = Tx$ exists for each $x \in X$ implies that $T \in L(X, Y)$; if in addition $\lim_k T_k x = Tx$ uniformly for x in precompact subsets of X, the pair (X, Y) will be said to have the *strong Banach–Steinhaus property*. For example, if X is barrelled and Y is locally convex, (X, Y) has the strong Banach–Steinhaus property [24].

We give a description of the General Uniform Boundedness Principle (General UBP) which will be used below. A sequence $\{x_j\} \subseteq X$ is said to be $\mathscr{K}$ convergent if every subsequence has a further subsequence $\{x_{n_j}\}$ such that the subseries $\sum_{j=1}^{\infty} x_{n_j}$ converges in X; a subset $B \subseteq X$ is $\mathscr{K}$ bounded if $\{x_j\} \subseteq B$ and $t_j \to 0$ implies that $\{t_j x_j\}$ is $\mathscr{K}$ convergent [1, 3]. A $\mathscr{K}$ bounded set is always bounded but not conversely, in general. A space in which a bounded set is $\mathscr{K}$ bounded is called an *$\mathscr{A}$-space* [14]; for example, F-spaces are $\mathscr{A}$-spaces. For $\mathscr{A}$-spaces, we have the following General UBP: if X is an $\mathscr{A}$-space and $\Gamma \subseteq L(X, Y)$ is pointwise bounded on X, then Γ is uniformly bounded on bounded subsets of X ([14], Corollary 4).

Let $s(X)$ be the vector space of all X-valued sequences, where the operations of addition and scalar multiplication are coordinatewise. Let E be a topological vector space which is a subspace of $s(X)$. If $x \in E$, the k^{th} coordinate of x will be denoted by x_k, i.e., $x = \{x_k\}$, and the coordinate function $x \to x_k$ will be denoted by Q_k. We call E a *$K(X)$ space* if each Q_k is continuous [7]; if X is the scalar field and the coordinate functionals are continuous, E is called a K-space. For each n, let P_n be the section map $E \to E$ which sends $x = (x_1, x_2, \ldots) \to (x_1, \ldots, x_n, 0, \ldots)$. We say E has the property AK (respectively, AB, SB, SUB) if each P_n is continuous and $P_n x \to x$ for each $x \in E$ (respectively, $\{P_n x\}$ is bounded for each $x \in E$, each P_n is bounded, $\{P_n\}$ is uniformly bounded on bounded subsets of E).

Following NOLL [16] we say a sequence, $\{z^n\}$, of non-zero vectors from $s(X)$ is a *block sequence* if there exists a strictly increasing sequence of positive integers $\{k_j\}$ such that

$$z^n = (0, \ldots, 0, z^n_{k_{n-1}+1}, \ldots, z^n_{k_n}, 0, \ldots).$$

We say E has the *strong gliding humps property* (*SGHP*) if given any block sequence, $\{z^n\}$, which is bounded in E there is a sequence $\{n_k\}$ such that $z = \sum_{k=1}^{\infty} z^{n_k} \in E$ (the convergence of the series is understood to be coordinatewise). We say E has the *weak gliding humps property* (*WGHP*) if given $x \in E$ and any block sequence $\{x^k\}$ with $x = \sum_{k=1}^{\infty} x^k$ (pointwise sum), then every subsequence has a further subsequence $\{n_k\}$ with $\tilde{x} = \sum_{k=1}^{\infty} x^{n_k} \in E$ [16].

If $x \in X$ and e_j is the scalar sequence with a 1 in the j^{th} coordinate and 0 elsewhere, we write $e_j \otimes x$ for the X-valued sequence with x in the j^{th} coordinate and 0 elsewhere. Let $\Phi(X)$ be the linear span of $\{e_j \otimes x : j \in \mathbb{N},\ x \in X\}$ in $s(X)$, i.e. $\Phi(X)$ is the subspace of all X-valued sequences with only a finite number of non-zero coordinates.

We assume, henceforth, that $E \supseteq \Phi(X)$. Using the notation of [7], $E^{\beta Y}$ will denote all sequences $T = \{T_k\} \subseteq L(X, Y)$ such that the series $\sum_{k=1}^{\infty} T_k x_k$ converges for all $x = \{x_k\} \in E$ (we require that T_k be continuous as contrasted with Maddox in [15]). We write $T \cdot x = \sum_{k=1}^{\infty} T_k x_k$ when $T \in E^{\beta Y}$, $x \in E$. If X and Y are the scalar field, we write, as usual, $E^{\beta Y} = E^{\beta}$.

2. Results

We begin by establishing a sequential continuity result for spaces with SGHP.

Recall that the topology of any topological vector space Y is generated by the family of continuous quasi-norms on Y so a sequence $\{y_j\} \subseteq Y$ converges to 0 in Y if and only if $|y_j| \to 0$ for every continuous quasi-norm, $|\ \ |$, on Y [9].

Theorem 1. *Let $T \in E^{\beta Y}$ and assume that E is a $K(X)$ space having SGHP. If $x^i \to 0$ in E, then $T \cdot x^i \to 0$ in Y, i.e., T is sequentially continuous.*

Proof. If not, there is a continuous quasi-norm $|\ \ |$ on Y, a sequence $x^i \to 0$ in E and $\varepsilon > 0$ such that $|T \cdot x^i| > \varepsilon$ for all i. Put $m_1 = 1$. There exists n_1 such that $|\sum_{k=1}^{n_1} T_k x_k^{m_1}| > \varepsilon$. By the K-space property, $\lim_i x_k^i = 0$ for each k and since T_k is continuous, there exists $m_2 > m_1$ such that $\sum_{k=1}^{n_1} |T_k x_k^{m_2}| < \varepsilon/2$. There exists $n_2 > n_1$ such that $|\sum_{k=1}^{n_2} T_k x_k^{m_2}| > \varepsilon$. Hence, $|\sum_{k=n_1+1}^{n_2} T_k x_k^{m_2}| > \varepsilon/2$. Continuing this construction produces increasing sequences of positive integers $\{m_k\}$,

$\{n_k\}$ such that

$$\left| \sum_{k=n_j+1}^{n_{j+1}} T_k x_k^{m_{j+1}} \right| > \varepsilon/2. \tag{1}$$

Define a block sequence $\{z^j\}$ by $z^j = (0, \ldots, 0, x_{n_j+1}^{m_{j+1}}, \ldots, x_{n_{j+1}}^{m_{j+1}}, 0, \ldots)$. By SGHP, there exists $\{p_k\}$ such that

$$z = \{z_k\} = \sum_{k=1}^{\infty} z^{p_k} \in E.$$

Then $T \cdot z = \sum_{k=1}^{\infty} T_k z_k$, doesn't converge since the partial sums of this series are not Cauchy by (1).

Compare Theorem 1 with 7.2.9 of [25] and Exer. 3.8 of [13].

Corollary 2. *Let E be as in Theorem 1. Each* $T \in E^{\beta Y}$ *is a bounded operator from E to Y; if E is bornological, T is continuous.*

If E satisfies the hypothesis of Theorem 1 and is bornological, then by Corollary 2 we may consider $E^{\beta Y}$ to be a subspace of $L(E, Y)$. Let us say that the sequence space E has *property* (*I*) if for each j the injection $x \to e_j \otimes x$ is continuous from X into E. If E has properties (I) and AK, then $E^{\beta Y} = L(E, Y)$; for if $A \in L(E, Y)$, then $T_k x = A(e_k \otimes x)$ defines a continuous linear operator $T_k \in L(X, Y)$ and $\sum_{k=1}^{n} T_k x_k \to Ax$ for each $x \in E$ so $T = \{T_k\} \in E^{\beta Y}$ and $Ax = T \cdot x$. Compare this statement with 7.2.9 of [25].

If X is the scalar field in Corollary 2, let E^s be the space of sequentially continuous linear functionals on E. From Corollary 2, we get the following result.

Corollary 3. *Assume X is the scalar field and E is a K-space with SGHP. Then* $E^{\beta} \subseteq E^s$; *if E is a bornological AK space, then* $E^{\beta} = E'$.

Proof. If E is an AK space, $E' \subseteq E^{\beta}$ ([13] p. 60) so the result follows from Corollary 2.

This result can be compared with 2.3.9 of [13] which indicates that SGHP can be used as a substitute for barrelledness.

In [2] (see also [3] and [14]), a general version of the uniform boundedness principle was established which contained the classical version of the uniform boundedness principle for F-spaces as a special case. We next show that such a version of the uniform boundedness principle holds for certain spaces with the SGHP. As was the case in

[2], [3] and [14], our principle tool used in the proof is the Basic Matrix Theorem of Antosik and Mikusinski ([3] 2.2).

Theorem 4. *Assume that E has properties SUB and SGHP and that X is an $\mathscr{A}$-space. If $\Gamma \subseteq E^{\beta Y}$ is pointwise bounded on E, then Γ is uniformly bounded on bounded subsets of E.*

Proof. If not, there exist a continuous quasi-norm, $|\ |$, on Y, $\{T^k\} \subseteq \Gamma$, $\{x^k\} \subseteq E$ bounded, $t_k \to 0$ and $\delta > 0$ such that $|t_k T^k \cdot x^k| > \delta$ for all k.

Before beginning our construction, we make a preliminary observation for use in the construction. For each n, the section P_n is bounded (property SB) and has range in the subspace $\prod_{i=1}^{n} X \times \{0, \ldots\} = X_n$ which is an $\mathscr{A}$-space since X is a $\mathscr{A}$-space. The sequence $\{T^k\}$ is pointwise bounded on X_n so $\{T^k\}$ is uniformly bounded on bounded subsets of X_n by the General UBP discussed above ([14] Cor. 4). Hence, for each n,

$$\lim_i t_i \sum_{k=1}^{n} T_k^i x_k^i = 0. \tag{2}$$

Set $m_1 = 1$. Pick n_1 such that $|t_{m_1} \sum_{k=1}^{n_1} T_k^{m_1} x_k^{m_1}| > \delta$. By (2), there is $m_2 > m_1$ such that $|t_{m_2} \sum_{k=1}^{n_1} T_k^{m_2} x_k^{m_2}| < \delta/2$. There exists $n_2 > n_1$ such that $|t_{m_2} \sum_{k=1}^{n_2} T_k^{m_2} x_k^{m_2}| > \delta$. Thus,

$$\left| t_{m_2} \sum_{k=n_1+1}^{n_2} T_k^{m_2} x_k^{m_2} \right| > \delta/2.$$

Continuing this construction produces two increasing sequences $\{m_k\}$, $\{n_k\}$ such that

$$\left| t_{m_{j+1}} \sum_{k=n_j+1}^{n_{j+1}} T_k^{m_{j+1}} x_k^{m_{j+1}} \right| > \delta/2 \qquad \text{for all } j. \tag{3}$$

Define a block sequence $\{z^j\}$ by $z^j = (0 \ldots, x_{n_{j-1}+1}^{m_j}, \ldots, x_{n_j}^{m_j}, 0 \ldots)$ and consider the matrix $M = [t_{m_i} T^{m_i} \cdot z^j]$. We claim that M is a $\mathscr{K}$-matrix (see [3] §2). The columns of M converge to 0 by the pointwise boundedness assumption. By SUB and SGHP, given any subsequence there is a further subsequence $\{p_j\}$ such that $z = \sum_{j=1}^{\infty} z^{p_j} \in E$. Then

$$\sum_{j=1}^{\infty} t_{m_i} T^{m_i} \cdot z^{p_j} = t_{m_i} T^{m_i} \cdot z \to 0$$

by the pointwise boundedness assumption, and M is, indeed, a $\mathscr{K}$-matrix. By the Basic Matrix Theorem of Antosik–Mikusinski ([3] 2.2, [14]), the diagonal of M converges to 0 contradicting (3).

If E satisfies the hypothesis of Theorem 4 and if E is in addition a normed space and $T \in E^{\beta Y}$, then $\{TP_n\}_n$ is pointwise bounded on E so $\{TP_n\}_n$ is bounded on the unit ball of E. Thus, one can define a β-norm for $T, \beta(T) = \sup\{\|\sum_{k=1}^{n} T_k x_k\| : n, \ \|x\| \leqslant 1\}$. Even in the scalar case, the β-norm is usually only defined for B-spaces, so Theorem 4 allows a relaxation of this requirement ([25], 4.3.16).

Corollary 5. *Let E be as in Theorems 1 and 4. If E is bornological and quasi-barrelled, then Γ is equicontinuous.*

Proof. If E is bornological, $E^{\beta Y} \subseteq L(E, Y)$ by Corollary 2 and Γ is equicontinuous by Proposition 11 of [14].

We now consider the case when X is the scalar field.

Corollary 6. *Assume X is the scalar field and E has properties SUB and SGHP.*

(i) *If $E^{\beta} \subseteq E'$, then $\sigma(E', E)$ bounded subsets of E^{β} are $\beta(E', E)$ bounded.*

(ii) *If $E^{\beta} = E'$, then $\sigma(E', E)$ bounded subsets of E' are $\beta(E', E)$ bounded so E' (and also E) is a Banach–Mackey space* ([24] 10.4).

(iii) *If $E^{\beta} = E'$ and E is quasi-barrelled, then E is barrelled.*

Proof. (i) follows immediately from Theorem 4; (ii) follows from (i); (iii) follows from (ii) and 10.1.11 of [24].

Note from Corollary 3, if E is a bornological K-space with SGHP, $E^{\beta} \subseteq E'$ so the hypothesis of (i) is satisfied; if, in addition, E is an AK space, $E' = E^{\beta}$ so the hypothesis in (ii) and (iii) is satisfied.

Corollary 6 can be used to drop the completeness hypothesis from Theorem 2 of [16]. In Theorem 2 of [16], NOLL used the completeness of the space E to invoke a form of the Banach–Steinhaus Theorem. From Corollary 6(i) it follows that if E is a normed (actually even bornological ([24] 10.1.10)) K space with properties SUB and SGHP, then any pointwise bounded family of elements of E^{β} is equicontinuous and the form of the Banach–Steinhaus Theorem used by NOLL is then available.

Corollary 6(iii) is a result of the same nature as Theorem 4.1 of [5], Satz 3.1 of [21] and Theorem 1 of [17]. The assumptions on the space E are quite different than those in the results cited above; in particular, the space E is required to be normed in the results above, whereas this condition is relaxed in (iii).

From (iii) it follows that the Closed Graph Theorem is applicable to any quasi-barrelled K space with properties SUB and SGHP.

We next consider the sequential completeness of $E^{\beta Y}$ with respect to the topology of pointwise convergence on E. In the scalar case our result gives a generalization of NOLL's result on the sequential completeness of $(E^{\beta}, \sigma(E^{\beta}, E))$ when E has WGHP ([16], Theorem 5).

Theorem 7. *Let E have WGHP and assume that Y is sequentially complete and (X, Y) satisfies the weak Banach–Steinhaus property. If $\{T^k\} \subseteq E^{\beta Y}$ is such that $\lim_k T^k \cdot x$ exists in Y for each $x \in E$ and if $T_j x = \lim_k T^k \cdot (e_j \otimes x) = \lim_k T^k_j x$ for $x \in X$, then $T = \{T_j\} \in E^{\beta Y}$ and $\lim_k T^k \cdot x = T \cdot x$. (Note T_j as defined above belongs to $L(X, Y)$ by the weak Banach–Steinhaus property.)*

Proof. First, we claim that $T = \{T_j\} \in E^{\beta Y}$. If not, there exists $x \in E$ such that $\sum T_j x_j$ doesn't converge. By the sequential completeness of Y there exists a continuous quasi-norm, $|\ \ |$, on Y, an $\varepsilon > 0$ and increasing sequences $\{m_k\}$, $\{n_k\}$ with $n_k < m_k < n_{k+1}$ such that

$$\left| \sum_{k=n_j}^{m_j} T_k x_k \right| > \varepsilon. \tag{4}$$

Let $I_j = \{k : n_j \leqslant k \leqslant m_j\}$ and consider the matrix $M = [T^i \cdot C_{I_j} x]$, where C_I is the characteristic function of I. We claim that M is a $\mathscr{K}$-matrix. First, each column of M converges to $T \cdot C_{I_j} x$. Next, if $\{p_j\}$ is an increasing sequence of positive integers, by WGHP $\{p_j\}$ has a subsequence $\{q_j\}$ such that $\tilde{x} = \sum_{j=1}^{\infty} C_{I_{q_j}} x \in E$. Hence, $\sum_{j=1}^{\infty} T^i \cdot C_{I_{q_j}} x = T^i \cdot \tilde{x}$ converges in Y by hypothesis. Thus, M is a $\mathscr{K}$-matrix and by the Basic Matrix Theorem ([3] 2.2, [14]),

$$0 = \lim_i \lim_j T^i \cdot C_{I_j} x = \lim_j \lim_i T^i \cdot C_{I_j} x = \lim_j \sum_{k \in I_j} T_k x_k$$

contradicting (4).

Next, we claim that $\lim_i T^i \cdot x = T \cdot x$ for each $x \in E$. If not, we may assume that there exist a continuous quasi-norm, $|\ \ |$, on Y, an $\varepsilon > 0$ and $x \in E$ such that $|(T^i - T) \cdot x| > \varepsilon$ for all i. Put $m_1 = 1$ and pick n_1

such that $|\sum_{k=1}^{n_1}(T_k^i - T_k)x_k| > \varepsilon$. There exists $m_2 > m_1$ such that

$$\left|\sum_{k=1}^{n_1}(T_k^{m_2} - T_k)x_k\right| < \varepsilon/2$$

and there exists $n_2 > n_1$ such that

$$\left|\sum_{k=1}^{n_2}(T_k^{m_2} - T_k)x_k\right| > \varepsilon$$

so

$$\left|\sum_{k=n_1+1}^{n_2}(T_k^{m_2} - T_k)x_k\right| > \varepsilon/2.$$

Continuing produces increasing sequences $\{m_j\}$, $\{n_j\}$ such that

$$\left|\sum_{k=n_{j-1}+1}^{n_j}(T_k^{m_j} - T_k)x_k\right| > \varepsilon/2. \tag{5}$$

As above the matrix $M = [\sum_{k=n_{j-1}+1}^{n_j}(T_k^{m_i} - T_k)x_k]$ is a $\mathscr{K}$-matrix so its diagonal converges to 0 contradicting (5).

From Theorem 7 it follows that $E^{\beta Y}$ is sequentially complete with respect to the topology of pointwise convergence on E. In particular, when X is the scalar field and E has WGHP, E^β is $\sigma(E^\beta, E)$ sequentially complete. This is a result of NOLL ([16], Theorem 6); the proof above, even for the vector case, should be contrasted with NOLL's proof. In this case $(E^\beta, \sigma(E^\beta, E))$ is a Banach–Mackey space and $\sigma(E^\beta, E)$ bounded sets are $\beta(E^\beta, E)$ bounded ([24] 10.4.8).

A sequence space E is said to be monotone if $m_0E = E$, where m_0 is the space of scalar sequences with finite range and m_0E is the pointwise product [6]. Bennett has shown that if E is monotone, then $\sigma(E^\beta, E)$ is sequentially complete [6]. A monotone space has WGHP so NOLL's result ([16], Theorem 6) gives a generalization of Bennett's result.

Finally, we consider a form of the Banach–Steinhaus Theorem for sequence spaces with SGHP. Recall that if X is an F-space and $\{T_k\} \subseteq L(X, Y)$ is such that $\lim_k T_kx = Tx$ exists for each $x \in X$, then the Banach–Steinhaus Theorem asserts that T is continuous and, moreover, the limit, $\lim_k T_kx = Tx$, is uniform for x in compact subsets of X ([3] 5.2). For $E^{\beta Y}$, the first conclusion of the Banach–Steinhaus Theorem was addressed in Theorems 1 and 7. We now address the second conclusion for sequence spaces with SGHP.

Theorem 8. *Let E have properties SGHP and SUB and assume that Y is sequentially complete and (X, Y) has the strong Banach–Steinhaus property. Let $\{T^i\} \subseteq E^{\beta Y}$ be such that* $\lim_i T^i \cdot x = T \cdot x$ *exists for each $x \in E$ (Theorem 7 is applicable so $T \in E^{\beta Y}$). If K is bounded in E and such that $Q_j K$ is precompact in X for each j, then* $\lim_i T^i \cdot x = T \cdot x$ *uniformly for $x \in K$. (In particular, if K is precompact and the coordinate functions Q_j are continuous, the hypothesis is satisfied.)*

Proof. By Theorem 7, $T \in E^{\beta Y}$ so we may assume that $T = 0$. It suffices to show that $\lim T^i \cdot x^i = 0$ for $\{x^i\} \subseteq K$. If this fails, we may assume that there exist a continuous quasi-norm, $|\ |$, on Y, an $\varepsilon > 0$ and $\{x^i\} \subseteq K$ with $|T^i \cdot x^i| > \varepsilon$ for all i. Put $m_1 = 1$ and choose n_1 such that $|\sum_{k=1}^{n_1} T^{m_1} \cdot x^{m_1}| > \varepsilon$.

For each k, $\lim_i T_k^i x = 0$ for $x \in X$ so the convergence is uniform on precompact sets by the strong Banach–Steinhaus property. Hence,

$$\lim_i T_k^i x_k = 0 \qquad \text{uniformly for } x \in K \tag{6}$$

since $Q_k K$ is precompact.

By (6) there exists $m_2 > m_1$ such that $|\sum_{k=1}^{n_1} T_k^{m_2} x_k^{m_2}| < \varepsilon/2$. There exists $n_2 > n_1$ such that $|\sum_{k=1}^{n_2} T_k^{m_2} x_k^{m_2}| > \varepsilon$ so $|\sum_{k=n_1+1}^{n_2} T_k^{m_2} x_k^{m_2}| > \varepsilon/2$. Continuing produces increasing sequences $\{m_k\}$, $\{n_k\}$ such that

$$\left| \sum_{k=n_{j-1}+1}^{n_j} T_k^{m_j} x_k^{m_j} \right| > \varepsilon/2. \tag{7}$$

Define a block sequence $z^j = (0, \ldots, x_{n_{j-1}+1}^{m_j}, \ldots, x_{n_j}^{m_j}, 0 \ldots)$. Then $\{z^j\}$ is bounded in E since K is bounded and E has SUB. Consider the matrix $M = [T^{m_i} \cdot z^j]$. We claim that M is a $\mathscr{K}$-matrix. First, the columns of M converge to 0 since $z^j \in E$. Next, if $\{p_j\}$ is an increasing sequence, by SGHP $\{p_j\}$ has a subsequence $\{q_j\}$ such that $z = \sum_{j=1}^{\infty} z^{q_j} \in E$ so $\sum_{j=1}^{\infty} T^{m_i} \cdot z^{q_j} = T^{m_i} \cdot z \to 0$ by hypothesis. Hence, M is a $\mathscr{K}$-matrix and by the Basic Matrix Theorem the diagonal of M converges to 0 contradicting (7).

Corollary 9. (Sequential Equicontinuity). *Let E satisfy the hypothesis of Theorem 8 and let E be a $K(X)$ space. If $x^j \to 0$, then* $\lim_j T^i \cdot x^j = 0$ *uniformly for $i \in \mathbb{N}$. In particular, if E is a metric linear space, then $\{T^i\}$ is equicontinuous.*

Proof. The result follows from Theorems 1 and 8.

We now consider some consequences of Theorem 8 for the scalar case.

Corollary 10. *Let X be the scalar field. Let E be a K-space with $SGHP$ and SUB. Then $x^i \to 0$ in $\sigma(E^\beta, E)$ if and only if $x^i \to 0$ uniformly on bounded subsets of E, i.e., if and only if $x^i \to 0$ in the strong topology $\beta(E^\beta, E)$ of E^β. In particular, a subset $B \subseteq E^\beta$ is $\sigma(E^\beta, E)$ bounded if and only if B is strongly bounded so E is a Banach–Mackey space* ([24] 10.4.3).

We can also obtain a version of the Schur Theorem from summability on the equivalence of weak and norm convergence of sequences in ℓ^1. Denote by c_{00} the space of all scalar sequences which are eventually zero equipped with the sup-norm.

Corollary 11. *Let $c_{00} \subseteq E \subset \ell^\infty$ and suppose that $E^\beta = \ell^1$ with the inclusion map from c_{00} into E continuous. If E is a K-space with $SGHP$ and SUB, then $y^i \to 0$ in $\sigma(\ell^1, E)$ if and only if $\| y^i \|_1 \to 0$.*

Proof. The set $\{C_\sigma : \sigma \subseteq \mathbb{N} \text{ finite}\} = \mathscr{F}$ is bounded in c_{00} and, therefore, in E. By Corollary 10, $y^i \cdot C_\sigma \to 0$ uniformly for $C_\sigma \in \mathscr{F}$. Thus, given $\varepsilon > 0$, there exists n such that $i \geqslant n$ and σ finite implies $|\sum_{j \in \sigma} y^i_j| < \varepsilon$. Hence, $\sum_{j=1}^\infty |y^i_j| = \| y^i \|_1 \leqslant 2\varepsilon$ for $i \geqslant n$.

If $E = \ell^\infty$, this is a result due to Schur, sometimes referred to as Schur's Lemma [18]. The result was extended to $E = m_0$ by Hahn [10]; although m_0 does not have SGHP, we can obtain Hahn's result from Corollary 11 by employing the fact that m_0 is a barrelled space ([24] 15.1.3). Indeed, if $E = m_0$ in Corollary 11, then $\{y^i\}$ is pointwise bounded on E and, therefore, norm bounded since m_0 is barrelled. Then the sequence $\{y^i\}$ is equicontinuous and converges pointwise on m_0 which is a dense subspace of ℓ^∞ so $\{y^i\}$ converges pointwise on ℓ^∞ and Corollary 11 implies $\| y^i \|_1 \to 0$.

The gliding hump property bears a strong resemblance to the quasi-σ-family property of SAMARATUNGA and SEMBER and Corollary 11 has much the same flavor as Theorem 2.5 of [20].

3. Hellinger–Toeplitz Result

The classical Hellinger–Toeplitz Theorem asserts that any infinite (scalar) matrix $A = [a_{ij}]$ which maps ℓ^2 into ℓ^2, i.e., $\forall x \in \ell^2$, $\{\sum_{j=1}^\infty a_{ij} x_j\} = Ax \in \ell^2$, is (norm) continuous [11]. In this section we establish analogues of this theorem for operator-valued matrices which map vector sequence spaces to vector sequence spaces. Let $A_{ij} \in L(X, Y)$ for $i,j \in \mathbb{N}$ and let A be the matrix $[A_{ij}]$. Let F be a sequence space of Y-valued sequences equipped with a Hausdorff vector topology. We say that $A \in (E, F)$ if and only if $Ax = \{\sum_{j=1}^{\infty} A_{ij} x_j\} \in F$ for every $x \in E$, i.e., if and only if A maps E into F. We seek conditions which guarantee the continuity of A with respect to the original topologies of E and F as in the classical Hellinger–Toeplitz Theorem. Such a result for vector FK spaces has been established by BARIC ([4] 2.7); see also [23] and [12] 34.7, for scalar results. We can now give our vector form of the Hellinger–Toeplitz Theorem.

Theorem 12. *Assume that each $T \in E^{\beta Y}$ induces an operator in $L(E, Y)$ (e.g., Corollary 2) and that (E, F) has the weak Banach–Steinhaus property. Assume that F is an AK space with property (I). Then any matrix A which maps E into F, i.e. $A \in (E, F)$ is continuous.*

Proof. Let A^i be the i^{th} row of the matrix A. Since $A \in (E, F)$, each $A^i \in E^{\beta Y}$ and by hypothesis the composition map $x \to A^i \cdot x \to e_j \otimes (A^i \cdot x)$ is continuous from E into F for each j. Therefore, $T_n : x \to (A^1 \cdot x, \ldots, A^n \cdot x, 0, \ldots)$ is continuous from E into F for each n. By hypothesis, $T_n x \to Ax$ for each $x \in E$, and since (E, F) has the weak Banach–Steinhaus property, A is continuous.

If E is a bornological, $K(X)$ space with SGHP, each $T \in E^{\beta Y}$ induces an element of $L(E, Y)$ by Corollary 2 so if (E, F) has the weak Banach–Steinhaus property and F satisfies the hypothesis of Theorem 12, Theorem 12 is applicable in this situation. For example, if X is a Banach space and $\ell^p(X)$ is space of all X-valued sequences $x = \{x_k\}$ with $\{\|x_k\|\} \in l^p$, $1 \leqslant p \leqslant \infty$, then $\ell^p(X)$ has SGHP, I and is a $K(X)$ space if $\ell^p(X)$ is equipped with the (complete) norm $\|x\|_p = \|\{\|x_k\|\}\|_p$. If $1 \leqslant p < \infty$, $\ell^p(X)$ also has AK. Similarly, if $c_0(X)$ is the space of all X-valued sequences with $\lim \|x_k\| = 0$ and $c_0(X)$ is equipped with the sup-norm, then $c_0(X)$ is a $K(X)$ space with I and AK but not SGHP. If $\Phi(X)$ is equipped with the sup-norm, then $\Phi(X)$ is a $K(X)$-space with I and AK. Thus, if Y is also a B-space, Theorem 12 is applicable with E equal to $\ell^p(X)$ for $1 \leqslant p \leqslant \infty$ and F equal to $\ell^p(Y)$ for $1 \leqslant p < \infty$ or $c_0(Y)$ or $\phi(Y)$. BARIC ([4] 2.7]) has given a similar result for complete metrizable $K(X)$-spaces; the case $F = \Phi(Y)$ is not covered by BARIC's result. In particular, the scalar case of this result clearly implies the classical Hellinger–Toeplitz Theorem.

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CHARLES SWARTZ
Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88005
U.S.A.