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A Unified Approach to Methods for the Simultaneous Computation of All Zeros of Generalized Polynomials

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Summary. Applying Newton's method to a particular system of nonlinear equations we derive methods for the simultaneous computation of all zeros of generalized polynomials. These generalized polynomials are from a function space satisfying a condition similar to Haar's condition. By this approach we bring together recent methods for trigonometric and exponential polynomials and a well-known method for ordinary polynomials. The quadratic convergence of these methods is an immediate consequence of our approach and needs not to be proved explicitly. Moreover, our approach yields new interesting methods for ordinary, trigonometric and exponential polynomials and methods for other functions occuring in approximation theory.

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1. Introduction

Suppose that we are given a polynomial p of degree n which has exactly n distinct zeros $w_1, \ldots, w_n$ in the complex plane. If we normalize p to have leading coefficient one the other coefficients of p are related to $w_1, \ldots, w_n$ in a well-known manner via the elementary symmetric functions. So we can build up a system of n nonlinear equations expressing the dependence of the coefficients of p on the values of the zeros of p. Usually we know the coefficients of p whereas the zeros are unknown. To approximate the zeros of p numerically we can now apply the n-dimensional Newton's method to our system of equations. This yields a surprisingly simple iterative method for the simultaneous computation of all zeros of p. The Q-rate of convergence of this method is at least 2 (cf. [10, Definition 9.1.5] for a definition of the Q-rate), since all the conditions of the standard theorem on quadratic convergence of Newton's method are fulfilled (e.g. [10, Theorem 10.2.2]).

The method sketched above occurs frequently in literature and seems to have been rediscovered several times using different approaches. The resulting formulae can already be found in Weierstraß [11]. The approach we outlined

here was used in Durand [5] and Kerner [6], whereas Dočev [4] obtained the method using a continuation principle. Note that Dočev had to prove the quadratic convergence explicitly.

Recently, methods for the simultaneous computation of all zeros of a given trigonometric or exponential polynomial were developed by Angelova and Semerdzhiev [2] and Makrelov and Semerdzhiev [7], respectively. These methods were obtained using a continuation principle similar to that of Dočev [4]. The Q-rate of convergence of these methods is again at least 2 as is shown by rather laborious calculations in [2] and [7].

Our unified approach is to generalize the approach of Durand and Kerner to other functions than polynomials. Suppose that we are given an $(n+1)$-dimensional space V of functions and an element f of V having exactly n distinct simple zeros. We assume that the coefficients of f with respect to some base of V depend in a unique manner on the zeros of f provided f satisfies one additional linear constraint. This constraint may be viewed as a generalization of the normalizing condition for polynomials. We then apply Newton's method to the nonlinear system which expresses the coefficients of f as a function of its zeros. The resulting computational scheme will be seen to be very simple. If the elements of V are sufficiently smooth functions, the theorem on local convergence of Newton's method establishes that our method converges with a Q-order of at least 2. For special choices of the space V we obtain the convergence results for the methods for trigonometric and exponential polynomials of [2] and [7] without any further calculations. Moreover, we will show that our approach not only yields new quadratically convergent methods for ordinary, trigonometric and exponential polynomials but also applies to other important spaces V such as exponential sums or sums of sines which are not trigonometric polynomials.

Finally, let us just mention that our approach is related to a particular method which can be used in approximation theory to perform the simultaneous exchange step in the iterative method of Remez (cf. [8, p. 115], [9]).

2. Notation

Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Since we will have to distinguish between vectors and scalars we reserve the bold characters $\mathbf{w}, \mathbf{x}$ and $\mathbf{y}$ for vectors in $\mathbb{K}^n$ with $\mathbf{x}=(x_1, \ldots, x_n)^T$ etc. Alternatively, we sometimes will find it useful to denote the i-th component of a vector $\mathbf{x}$ by $(\mathbf{x})_i$.

Given a function $f: U \subseteq \mathbb{K}^n \to \mathbb{K}$ we write $\partial_i f(\mathbf{x})$ to denote the partial derivative of f with respect to the i-th variable evaluated at $\mathbf{x}$. Analogously, we write $\partial_{ij} f(\mathbf{x})$ for the corresponding second partial derivative.

If $f: U \subseteq \mathbb{K} \to \mathbb{K}$ is a function of only one variable t we write $\dot{f}(t)$ to denote the derivative of f evaluated at t.

We will often be concerned with functions g of the form $g=g(\mathbf{x}, t)$ where $\mathbf{x} \in \mathbb{K}^n$, $t \in \mathbb{K}$. In this case, the partial derivative with respect to the variable t is again denoted by $\dot{g}(\mathbf{x}, t)$, whereas the partial derivative with respect to the i-th component of $\mathbf{x}$ is written in the form $\partial_i g(\mathbf{x}, t)$.

3. The Unified Approach

Let I be a subset of $\mathbb{K}$ which, for the sake of simplicity, we assume to be open. Let V denote an $(n+1)$-dimensional subspace of $C^2(I)$, spanned by the twice continuously differentiable functions $g_i: I \to \mathbb{K}$, $i=1, \ldots, n+1$.

Suppose that we are given a function $f \in V$ having exactly n distinct simple zeros $w_1, \ldots, w_n$ in I, i.e. we have $f(w_i)=0$ and $\dot{f}(w_i) \neq 0$ for $i=1, \ldots, n$. Usually, f will be expressed in terms of the base functions g_i:

$$f=\sum_{i=1}^{n+1} a_i g_i, \quad \text{where } a_i \in \mathbb{K} \quad \text{for } i=1, \ldots, n+1.$$

Our problem is to approximate the zeros of f numerically. To this purpose suppose that we are given a linear functional

$$\ell: V \to \mathbb{K}$$

such that for x_i sufficiently close to w_i there is exactly one function $g \in V$ satisfying

$$g(x_i)=0 \quad \text{for } i=1, \ldots, n \tag{3.1}$$

and

$$\ell(g)=\ell(f) \quad \text{with} \quad \ell(f) \neq 0. \tag{3.2}$$

Since all scalar multiples of g satisfy (3.1) too, equation (3.2) may be viewed as a normalizing constraint to obtain uniqueness of g. The above conditions are fulfilled in two typical situations:

a) The functional ℓ is given by $\ell: g \to g(t_0)$ with $t_0 \neq w_i$ for $i=1, \ldots, n$ and V satisfies the $(n+1)$-dimensional Haar-condition:

Given $(n+1)$ distinct points $t_0, \ldots, t_n \in I$ and $(n+1)$ numbers $a_0, \ldots, a_n \in \mathbb{K}$ there is exactly one function $g \in V$ with $g(t_i)=a_i$, $i=0, \ldots, n$.

b) The functional ℓ is given by

$$\ell: g=\sum_{i=1}^{n+1} b_i g_i \to b_j.$$

where $j \in \{1, \ldots, n+1\}$ is fixed, and the space spanned by the functions g_i, $i=1, \ldots, n+1$, $i \neq j$, satisfies the n-dimensional Haar-condition.

For the moment let us choose the base functions g_i in such a manner that we have $g_{n+1}=f$ and $\ell(g_i)=0$ for $i=1, \ldots, n$. This is always possible. A rapid calculation now shows that g satisfies (3.1) and (3.2) if and only if

$$g=\sum_{i=1}^{n} b_i g_i + f$$

with

$$\begin{bmatrix} b_1(\mathbf{x}) \\ \vdots \\ b_n(\mathbf{x}) \end{bmatrix} = - \begin{bmatrix} g_1(x_1) \dots g_n(x_1) \\ \vdots \qquad \vdots \\ g_1(x_n) \dots g_n(x_n) \end{bmatrix}^{-1} \cdot \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix} \tag{3.3}$$

This is a system of n nonlinear equations expressing the coefficients b_i of g as a function of the vector $\mathbf{x}=(x_1, \dots, x_n)^T$, the vector of zeros of g. Since g itself depends on $\mathbf{x}$ it will be useful to write

$$g=g(t)=g(\mathbf{x}, t)=\sum_{i=1}^{n} b_i(\mathbf{x})\, g_i(t)+f(t). \tag{3.4}$$

By our assumptions the function

$$\mathbf{B}(\mathbf{x})=(b_1(\mathbf{x}), \dots, b_n(\mathbf{x}))^T \tag{3.5}$$

is defined and twice continuously differentiable in a neighbourhood $U \subseteq \mathbb{K}^n$ of $\mathbf{w}=(w_1, \dots, w_n)^T$, the vector of zeros of f. Moreover, $\mathbf{B}(\mathbf{w})=\mathbf{0}$. To approximate $\mathbf{w}$ we may hence apply Newton's method to $\mathbf{B}$. For this we have to calculate the inverse of the Jacobian $\mathbf{B}'(\mathbf{x})$ what will be done in the following lemma.

Lemma 3.1. *If* $\mathbf{B}'(\mathbf{x})$ *is non-singular its inverse is given by the matrix* $(\beta_{ij}(\mathbf{x}))_{i,j=1,\dots,n}$, *where*

$$\beta_{ij}(\mathbf{x})=-g_j(x_i)/\dot{g}(\mathbf{x}, x_i). \tag{3.6}$$

Proof. We calculate

$$\begin{aligned} \sum_{j=1}^{n} \beta_{ij}(\mathbf{x})\, \partial_k\, b_j(\mathbf{x}) &= \frac{-1}{\dot{g}(\mathbf{x}, x_i)}\, \partial_k \left(\sum_{j=1}^{n} b_j(\mathbf{x})\, g_j(x_i) \right) \\ &= \frac{-1}{\dot{g}(\mathbf{x}, x_i)}\, \partial_k (g(\mathbf{x}, x_i)-f(x_i)). \end{aligned} \tag{3.7}$$

The last equality holds by (3.4). Now $g(\mathbf{x}, x_i)$, as a function of the n variables $x_1, \dots, x_n$, vanishes identically. Thus we have

$$\partial_k\, g(\mathbf{x}, x_i)=0 \quad \text{for } k=1, \dots, n, \quad k \neq i$$

and

$$\partial_i\, g(\mathbf{x}, x_i)+\dot{g}(\mathbf{x}, x_i)=0.$$

Since $\partial_k\, f(x_i)=0$ for $i=1, \dots, n$ we see that the right side of (3.7) is 0 for $i \neq k$ and 1 for $i=k$. This finishes the proof. □

Note that the proof of Lemma 3.1 also shows that $\mathbf{B}'(\mathbf{w})$ is non-singular, since $g(\mathbf{w}, t)=f(t)$ and $\dot{f}(w_i) \neq 0$ for $i=1, \dots, n$.

By Lemma 3.1 we see the i-th component of the Newton-correction $\mathbf{B}'(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x})$ to be

$$\sum_{j=1}^{n} \beta_{ij}(\mathbf{x})\, b_j(\mathbf{x}) = \frac{-1}{\dot{g}(\mathbf{x}, x_i)} \left(\sum_{j=1}^{n} b_j(\mathbf{x})\, g_j(x_i) \right).$$

By (3.4) the last sum is equal to $g(\mathbf{x}, x_i) - f(x_i)$ or just $-f(x_i)$ since $g(\mathbf{x}, x_i) = 0$. Hence we have

$$(\mathbf{B}'(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x}))_i = f(x_i)/\dot{g}(\mathbf{x}, x_i). \tag{3.8}$$

Putting all our investigations together we obtain the following theorem.

Theorem 3.2. *Let* (M) *denote the iterative method which is calculating* $\mathbf{x}^{k+1}$ *from* $\mathbf{x}^k$ *by the formula*

$$x_i^{k+1} = x_i^k - f(x_i^k)/\dot{g}(\mathbf{x}^k, x_i^k),$$
$$i = 1, \ldots, n, \quad k = 0, 1 \ldots . \tag{M}$$

Then there is a neighbourhood $U \subseteq \mathbb{K}^n$ of $\mathbf{w} = (w_1, \ldots, w_n)^T$, the vector of zeros of f, such that the iterates of method (M) converge to $\mathbf{w}$ whenever $\mathbf{x}^0 \in U$. The Q-order of method (M) is at least 2.

Proof. By the preceding discussion, method (M) is just Newton's method for the function $\mathbf{B}$. $\mathbf{B}$ was already seen to be twice continuously differentiable in a neighbourhood of $\mathbf{w}$ with $\mathbf{B}(\mathbf{w}) = \mathbf{0}$ and non-singular $\mathbf{B}'(\mathbf{w})$. So the theorem follows directly from the standard theorem on quadratic convergence of Newton's method (e.g. [10, Theorem 10.2.2]). □

Note that method (M) is independent of the special choice of the base $g_1, \ldots, g_{n+1}$ of V.

To perform method (M) we need to construct the functions $g(\mathbf{x}^k, t)$. This can always be done by solving the linear system for the coefficients of $g(\mathbf{x}^k, t)$ arising from (3.1) and (3.2). In the case of ordinary, trigonometric and exponential polynomials, however, the functions $g(\mathbf{x}^k, t)$ can be obtained in a more simple manner as we will see in the next section.

4. Applications

Here we discuss method (M) for particular choices of the space V and the functional ℓ. The methods from [2] and [7] will turn out to be special cases of method (M). Moreover, we will obtain other interesting quadratically convergent methods which seem to have not been discussed before.

a) Polynomials

Let V be the space

$$V = \left\{ g(t) = \sum_{i=0}^{n} b_i\, t^i,\ b_i \in \mathbb{C} \right\}$$

with $t \in \mathbb{C}$. Take

$$f(t) = \sum_{i=0}^{n} a_i t^i$$

and let the functional ℓ be given by

$$\ell: \sum_{i=0}^{n} b_i t^i \to b_n.$$

Then $g(\mathbf{x}^k, t) = a_n \cdot \prod_{i=1}^{n} (t - x_i^k)$ and method (M) gives

$$x_i^{k+1} = x_i^k - f(x_i^k) \Big/ \left[a_n \cdot \prod_{j=1, j \neq i}^{n} (x_i^k - x_j^k) \right],$$
$$i = 1, \ldots, n, \quad k = 0, 1, \ldots . \tag{4.1}$$

This is precisely the method from Durand [5], Dočev [4] and Kerner [6]. Usually f is normalized to have $a_n = 1$.

Another possible choice for the functional ℓ is

$$\ell: g \to g(t_0),$$

provided t_0 is not a zero of f. Method (M) then reads

$$x_i^{k+1} = x_i^k - f(x_i^k) \Big/ \left[A^k \cdot \prod_{j=1, j \neq i}^{n} (x_i^k - x_j^k) \right], \tag{4.2}$$

where

$$A^k = f(t_0) \Big/ \prod_{i=1}^{n} (t_0 - x_i^k), \quad i = 1, \ldots, n, \quad k = 0, 1, \ldots .$$

Of course, method (4.2) requires slightly more computational work than (4.1). However, it might be useful in the case where we can evaluate f but ignore the coefficient a_n. Another interesting method can be derived from (4.2). Assume that f is a polynomial with $a_0 \neq 0$ and $a_n = 1$. Consider the polynomial $h(t) = \sum_{i=0}^{n} a_i t^{n-i}$. We have $h(0) = 1$ and $h(t) = t^n \cdot f(1/t)$ for $t \neq 0$. Applying method (4.2) with $t_0 = 0$ to h we get iterates x_i^k which approximate the inverses of the zeros of f. Reformulating (4.2) by substituting $y_i^k = 1/x_i^k$ we finally obtain

$$y_i^{k+1} = y_i^k \Big/ \left[1 + f(y_i^k) \Big/ \left(y_i^k \cdot \prod_{j=1, j \neq i}^{n} (y_i^k - y_j^k) \right) \right],$$
$$i = 1, \ldots, n, \quad k = 0, 1, \ldots . \tag{4.3}$$

Herein the iterates y_i^k converge again quadratically to the zeros of f since their inverses converge quadratically.

b) Trigonometric Polynomials

Consider the $(2m+1)$-dimensional space of trigonometric polynomials

$$V=\left\{g(t)=a_0+\sum_{k=1}^{m}(a_k\cos kt+b_k\sin kt),\ a_0,a_k,b_k\in\mathbb{R}\right\}$$

with $t\in(-\pi,\pi)$ and take

$$f(t)=c_0+\sum_{k=1}^{m}(c_k\cos kt+d_k\sin kt). \tag{4.4}$$

A trigonometric polynomial $g\in V$ having $2m$ distinct zeros $x_1,\ldots,x_{2m}$ can be represented in the form (cf. Angelova and Semerdzhiev [2]):

$$g(t)=c\cdot\prod_{i=1}^{2m}\sin[(t-x_i)/2]\quad\text{with } c\in\mathbb{R}. \tag{4.5}$$

Thus, if we take the functional

$$\ell\colon\ g\to g(t_0),$$

where t_0 is not a zero of f we have

$$g(\mathbf{x}^k,t)=B^k\prod_{i=1}^{2m}\sin[(t-x_i^k)/2]$$

with

$$B^k=f(t_0)\Big/\left(\prod_{i=1}^{2m}\sin[(t_0-x_i^k)/2]\right). \tag{4.6}$$

Method (M) now reads

$$x_i^{k+1}=x_i^k-f(x_i^k)\Big/\left(\frac{B^k}{2}\cdot\prod_{j=1,j\neq i}^{2m}\sin[(x_i^k-x_j^k)/2]\right),$$
$$i=1,\ldots,2m,\qquad k=0,1,\ldots. \tag{4.7}$$

This method was derived by Angelova and Semerdzhiev in [2] using a continuation principle. In [2] the authors had to perform some rather tedious calculations to show that method (4.7) has a Q-order of at least 2. Such additional labour is not required in our approach. Furthermore, we believe that our approach produces the numbers B^k in a more natural manner than the approach of [2].

Let us discuss another choice for the functional ℓ. Since f is supposed to have exactly $2m$ zeros the numbers c_m and d_m of (4.4) cannot vanish both at

a time. So we may assume that $c_m \neq 0$, if necessary passing from $f(t)$ to $f\left(\frac{\pi}{2}-t\right)$. Therefore, we can take the functional

$$\ell\colon\ g(t)=a_0+\sum_{k=1}^{m}(a_k\cos kt+b_k\sin kt)\to a_m.$$

We then have

$$g(\mathbf{x}^k,t)=\bar{B}^k\prod_{i=1}^{2m}\sin[(t-x_i^k)/2],$$

where

$$\bar{B}^k=c_m\cdot\frac{(-4)^m}{2}\Big/\left[\cos\left(\frac{1}{2}\sum_{i=1}^{2m}x_i^k\right)\right],$$
$$i=1,\ldots,2m,\quad k=0,1,\ldots. \tag{4.8}$$

You may easily verify that the function $g(\mathbf{x}^k,t)$ from (4.8) actually satisfies $\ell(g(\mathbf{x}^k,t))=c_m$ by considering the corresponding complex polynomial in e^{it}. Method (M) now reads

$$x_i^{k+1}=x_i^k-f(x_i^k)\Big/\left(\frac{\bar{B}^k}{2}\cdot\prod_{j=1,j\neq i}^{2m}\sin[(x_i^k-x_j^k)/2]\right),$$
$$i=1,\ldots,2m,\quad k=0,1,\ldots. \tag{4.9}$$

This method has not been given in literature before. Similar to method (4.7), it has a Q-order of at least 2. Numerical experiments show that method (4.9) has a relatively wide domain of convergence whereas the domain of convergence of method (4.7) depends on the choice of the number t_0 which should be far from the (usually unknown) zeros of f. In addition, the evaluation of $\bar{B}^k$ in method (4.9) requires less computational work than the evaluation of B^k in method (4.7). So we conclude that method (4.9) should usually be preferred to method (4.7).

c) *Exponential Polynomials*

In a similar manner as in b) we can derive quadratically convergent methods for the simultaneous computation of all $2m$ zeros of a given exponential polynomial

$$f(t)=\sum_{k=-m}^{m}a_k e^{kt},\quad a_k\in\mathbb{R}\quad\text{for } k=-m,\ldots,m,$$

with $t \in \mathbb{R}$. The resulting formulae are just the same as (4.6)–(4.9) with all trigonometric functions replaced by the corresponding hyperbolic functions.

The analog of method (4.7) for exponential polynomials was derived by Makrelov and Semerdzhiev [7]. Again, the quadratic convergence of the method had to be proven explicitly by laborious calculations in [7]. The analog of method (4.9) for exponential polynomials is new. It appears superior to the analog of method (4.7) for the same reasons as in b).

d) Exponential Sums

Let $\{\lambda_1, \ldots, \lambda_{n+1}\}$ be a set of $(n+1)$ distinct real numbers, take

$$V = \left\{ g(t) = \sum_{i=1}^{n+1} b_i \exp(\lambda_i t), b_i \in \mathbb{R} \right\}.$$

with $t \in \mathbb{R}$ and let $f \in V$ be an exponential sum with n distinct zeros. The space V satisfies the $(n+1)$-dimensional Haar-condition (e.g. [3, p. 189]). We hence can choose the linear functional ℓ to be

$$\ell\colon\ g \to g(t_0),$$

where t_0 is not a zero of f and apply method (M). This time there is no product-representation for the functions $g(\mathbf{x}^k, t)$. Therefore, these functions must be constructed by solving the linear system for the coefficients of $g(\mathbf{x}^k, t)$ arising from Eq. (3.1) and (3.2).

Alternatively, we can also choose ℓ to be the functional which maps $g \in V$ on its coefficient with respect to $\exp(\lambda_r t)$ (r fixed), provided the coefficient of f with respect to $\exp(\lambda_r t)$ is non-zero. This choice for ℓ is possible since the space spanned by the remaining functions $\exp(\lambda_i t)$, $i = 1, \ldots, n+1$, $i \neq r$ satisfies the n-dimensional Haar-condition. As in b) and c) this choice for ℓ is often favourable.

In a similar manner as in d) we can treat various other spaces of functions arising in approximation theory. Some of these spaces are listed in [3, pp. 189–190], including the spaces

$$\left\{ g(t) = \sum_{i=1}^{n+1} b_i/(1 - \lambda_i t), b_i \in \mathbb{R} \right\},$$

where

$$\{\lambda_1, \ldots, \lambda_{n+1}\} \subseteq (-1, 1), \qquad t \in (-1, 1)$$

and

$$\left\{ g(t) = \sum_{i=1}^{n+1} b_i \sin(\lambda_i t), b_i \in \mathbb{R} \right\},$$

where

$$\{\lambda_1, \ldots, \lambda_{n+1}\} \subseteq (0, 1), \qquad t \in (0, \pi).$$

Note that in the latter applications each iterative step of method (M) requires additional arithmetic work due to the solution of the linear system for the coefficients of $g(\mathbf{x}^k, t)$. Nevertheless, method (M) remains an attractive method since numerical experiments show that it often has a wider domain of convergence than the one-dimensional Newton's method for the individual searching for zeros of f. This fact is explained by Makrelov and Semerdzhiev [7] who consider a continuous analog of method (Q) in the case of real ordinary, trigonometric and exponential polynomials. Their explanation carries over to all the above applications of method (M).

5. Numerical Examples

Here we report two numerical examples which illustrate the typical features of method (M).

a) Let V be the 5-dimensional space of exponential sums given by

$$V = \left\{ g(t) = \sum_{i=1}^{5} b_i \exp(\lambda_i t), b_i \in \mathbb{R} \right\} \quad \text{with} \quad t \in \mathbb{R},$$

where $\lambda_1 = -2$, $\lambda_2 = -0.5$, $\lambda_3 = 0$, $\lambda_4 = 0.7$ and $\lambda_5 = 2$. We take the function

$$f(t) = \sum_{i=1}^{5} a_i \exp(\lambda_i t),$$

where the coefficients a_i were determined such that $a_5 = 1$ and that f has the zeros $w_1 = -4$, $w_2 = -2$, $w_3 = 0$, $w_4 = 2$. To apply method (M) we choose the linear functional

$$\ell: \sum_{i=1}^{5} b_i \exp(\lambda_i t) \to b_5.$$

The numerical results are written down in Table 1, where we report the values of $|x_i^k - w_i|$ rounded to the second non-zero digit. The initial guesses x_i^0 were

Table 1

i	1	2	3	4
x_i^0	-5.0	-1.0	1.0	3.0
$\lvert x_i^1 - w_i \rvert$	$8.7 \cdot 10^{-1}$	$9.4 \cdot 10^{-1}$	$8.3 \cdot 10^{-1}$	$3.5 \cdot 10^{-1}$
$\lvert x_i^2 - w_i \rvert$	$6.0 \cdot 10^{-1}$	$7.7 \cdot 10^{-1}$	$4.3 \cdot 10^{-1}$	$6.3 \cdot 10^{-3}$
$\lvert x_i^3 - w_i \rvert$	$2.7 \cdot 10^{-1}$	$4.3 \cdot 10^{-1}$	$1.4 \cdot 10^{-2}$	$7.9 \cdot 10^{-4}$
$\lvert x_i^4 - w_i \rvert$	$5.2 \cdot 10^{-2}$	$7.2 \cdot 10^{-2}$	$1.6 \cdot 10^{-3}$	$1.5 \cdot 10^{-5}$
$\lvert x_i^5 - w_i \rvert$	$1.8 \cdot 10^{-3}$	$2.2 \cdot 10^{-3}$	$2.4 \cdot 10^{-5}$	$2.9 \cdot 10^{-8}$
$\lvert x_i^6 - w_i \rvert$	$1.9 \cdot 10^{-6}$	$2.3 \cdot 10^{-6}$	$1.0 \cdot 10^{-8}$	0
$\lvert x_i^7 - w_i \rvert$	0	0	$8.6 \cdot 10^{-13}$	0

deliberately chosen far from the zeros of f to illustrate the wide domain of convergence of the method. Note that the initial guess -1 is very unfavourable for the one-dimensional Newton's method for individual searching for zeros of f since the first iteration step then yields -19.08. The subsequent iterates change very slowly. For example, the 15th iterate is still less than -12.

All calculations were done on a kws personal computer using the M 68020 micro-processor with 13 decimal digits.

b) Let V be the 5-dimensional space of trigonometric polynomials given by

$$V=\left\{g(t)=a_0+\sum_{k=1}^{2}(a_k\cos kt+b_k\sin kt),\ a_k, b_k\in\mathbb{R}\right\}$$

with $t\in(-\pi, \pi)$ and let $f\in V$ be the function

$$f(t)=\prod_{i=1}^{4}\sin[(t-w_i)/2],$$

where $w_1=-1.7$, $w_2=0.3$, $w_3=0.5$, $w_4=1.7$. (This example is taken from Angelova and Semerdzhiev [2].) Table 2 reports the numerical results when applying method (4.9) of application 4 b) to f. We took the same initial guesses as Angelova and Semerdzhiev [2] who applied method (4.7) to f. Unfortunately, the authors do not communicate their choice for the value of t_0 which is required in method (4.7), but numerical calculations indicate that it was $t_0=1$.

Comparing Table 2 with the numerical results of [2] we see that our method (4.9) requires only 5 iterations to achieve the accuracy which is attained by method (4.7) after 7 iterations.

Table 2

i	1	2	3	4
x_i^0	-1.5	0.0	0.7	1.4
$\lvert x_i^1-w_i\rvert$	$2.7\cdot10^{-2}$	$1.3\cdot10^{-2}$	$2.5\cdot10^{-2}$	$2.2\cdot10^{-2}$
$\lvert x_i^2-w_i\rvert$	$5.2\cdot10^{-5}$	$1.6\cdot10^{-3}$	$1.9\cdot10^{-3}$	$2.4\cdot10^{-4}$
$\lvert x_i^3-w_i\rvert$	$5.3\cdot10^{-10}$	$1.4\cdot10^{-5}$	$1.4\cdot10^{-5}$	$1.1\cdot10^{-7}$
$\lvert x_i^4-w_i\rvert$	0	$9.3\cdot10^{-10}$	$9.3\cdot10^{-10}$	0
$\lvert x_i^5-w_i\rvert$	0	0	0	0

6. Conclusion

We explained the quadratic convergence of several existing methods for the simultaneous computation of all zeros of generalized polynomials by showing that they actually result from Newton's method. Besides a better understanding

of existing methods, our approach also yields new methods applicable to a larger class of generalized polynomials.

Numerical examples show that simultaneous searching for zeros is often preferable to the individual searching for one zero by Newton's method. This is due to a rather large domain of convergence for the simultaneous methods and can heuristically be explained by considering continuous analogs (see [7]).

There exist methods for the simultaneous computation of zeros which converge faster than quadratically. For example, in the case of an ordinary polynomial p, we have the well-known Ehrlich's method

$$x_i^{k+1} = x_i^k - p(x_i^k) \Big/ \left(p'(x_i^k) - p(x_i^k) \cdot \sum_{j=1, j \neq i}^{n} (x_i^k - x_j^k)^{-1} \right),$$

$$i = 1, \ldots, n,$$

which is known to converge cubically. Note that this method requires the evaluation of the first derivative of p.

The simultaneous methods for ordinary, trigonometric and exponential polynomials exhibit a natural parallelism which seems to make them particularly attractive for use on a vector or parallel computer. For example, method (4.1) has proved to be efficient on a CRAY-1 (see Alt [1]).

We did not address the important question of how to find suitable initial approximations required for the simultaneous methods.

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