Numer. Math. 34, 439–455 (1980)

Numerische Mathematik

Strong Uniqueness and Second Order Convergence in Nonlinear Discrete Approximation

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Summary: Strong uniqueness has proved to be an important condition in demonstrating the second order convergence of the generalised Gauss-Newton method for discrete nonlinear approximation problems [4]. Here we compare strong uniqueness with the multiplier condition which has also been used for this purpose. We describe strong uniqueness in terms of the local geometry of the unit ball and properties of the problem functions at the minimum point. When the norm is polyhedral we are able to give necessary and sufficient conditions for the second order convergence of the generalised Gauss-Newton algorithm.

Subject Classifications: AMS(MOS): 65D15; CR: 5.13.

1. Introduction

Consider the discrete approximation problem

$$\min_{\mathbf{x}\in R^p} \|\mathbf{f}(\mathbf{x})\|_1 \tag{1.1}$$

where the function $\|\cdot\|_1: R^n \to R^1$, defines a norm on R^n, the $f_i(\mathbf{x})$, $i=1,2,\ldots,n$, $\varepsilon\, C^2[S]$ where $S\subseteq R^p$ is large enough to contain all values of $\mathbf{x}$ of interest, and $p<n$. Let the minimum value of $\|\mathbf{f}\|_1$ be attained at $\mathbf{x}^*$. We say that the minimum is *strongly unique* or that *strong uniqueness* obtains at $\mathbf{x}^*$ if $\exists\gamma>0$ such that

$$\|\mathbf{f}(\mathbf{x})\|_1 > \|\mathbf{f}(\mathbf{x}^*)\|_1 + \gamma\|\mathbf{x}-\mathbf{x}^*\|_2 \tag{1.2}$$

$\forall\mathbf{x}$ in some ball about $\mathbf{x}^*$ where $\|\cdot\|_2$ is an appropriate norm on R^p. If $\mathbf{f}$ is a linear function, $\nabla\mathbf{f}$ satisfies the Haar condition so that all $p\times p$ minors formed from the rows of $\nabla\mathbf{f}$ are nonzero, and $\|\cdot\|_1$ is the maximum norm, then it is known that the solution to (1.1) is strongly unique. The corresponding continuous problem is the best uniform norm approximation of a function $g(t)$ by

0029-599X/80/0034/0439/$03.40

linear combinations of a finite set of functions on a compact interval I. It is known that strong uniqueness obtains for $\forall g \in C[I]$ iff the set of approximating functions form a Chebyshev set [3]. Recently Cromme [4] has studied the consequences of strong uniqueness for nonlinear discrete approximation in the maximum norm amongst other problems. Consider the generalised Gauss-Newton algorithm in the form:

(i) at $\mathbf{x}_i$ determine $\mathbf{t}_i$ to minimize the linear subproblem (LSP)

$$\min_{\mathbf{t}} \|\mathbf{f}(\mathbf{x}_i) + \nabla \mathbf{f}(\mathbf{x}_i)\mathbf{t}\|_1 \tag{1.3}$$

(ii) set $\mathbf{x}_{i+1} = \mathbf{x}_i + \mathbf{t}_i$.

Cromme shows that strong uniqueness of (1.1) is a sufficient condition for the second order convergence of this algorithm. Second order convergence has also been established [2] in the case that $\|\cdot\|_1$ is polyhedral. That is the norm is defined by a consistent system of linear inequalities with matrix B, such that the set $\{\mathbf{f}; B\mathbf{f} \leqq \mathbf{e}\}$ is bounded and balanced, by

$$\|\mathbf{f}\|_1 = \min h \ni B\mathbf{f} \leqq h\,\mathbf{e} \tag{1.4}$$

where $\mathbf{e}$ is a vector each component of which is 1. This result extended earlier work for the maximum norm [8], and both these studies made use of a condition called the *multiplier condition* to prove second order convergence. Note that (1.3) reduces to a linear programming problem when the norm is polyhedral. The multiplier condition requires that $\exists$ a common optimal reference for this linear programming problem for all $\mathbf{x}$ in a neighbourhood of $\mathbf{x}^*$ such that

(i) $\exists$ an index set $\sigma^*, |\sigma^*| = p+1$, $\rho_i(B)\,\mathbf{f}(\mathbf{x}^*) = \|\mathbf{f}(\mathbf{x}^*)\|_1$, $i \in \sigma^*$,

(ii) $\exists \lambda_i(\mathbf{x}) \geqq \lambda > 0 \ni \sum_{i \in \sigma^*} \lambda_i(\mathbf{x})\,\rho_i(B)\,\nabla \mathbf{f}(\mathbf{x}) = 0$, and

(iii) rank $\{\rho_i(B)\,\nabla \mathbf{f}(\mathbf{x}^*),\, i \in \sigma^*\} = p$

where $\rho_i(B)$ stands for the i'th row of the matrix B.

However, the multiplier condition is used only to show that $\mathbf{t}_i$ minimizing (1.3) satisfies the $(p+1) \times (p+1)$ system of linear equations for $(h_i, \mathbf{t}_i^T)$

$$\rho_j(B)\,(\mathbf{f}(\mathbf{x}_i) + \nabla \mathbf{f}(\mathbf{x}_i)\,\mathbf{t}_i) = h_i,\, j \in \sigma^*, \tag{1.5}$$

provided $\|\mathbf{x}_i - \mathbf{x}^*\|_2$ is small enough where $h_i = \|\mathbf{f}(\mathbf{x}_i) + \nabla \mathbf{f}(\mathbf{x}_i)\,\mathbf{t}_i\|_1$.

This condition has an important geometric interpretation. For consider the definition of the norm by dilation of the associated convex set forming the unit ball (see [7] for example). We know that

$$\|\mathbf{f}\|_1 = \mathbf{v}_\alpha^T \mathbf{f},\ \forall \mathbf{v}_\alpha \in \partial \|\mathbf{f}\|_1 \tag{1.6}$$

where $\partial \|\mathbf{f}\|_1$ denotes the subdifferential of $\|\cdot\|_1$ at $\mathbf{f}$, and where the $\{\mathbf{v}_\alpha\}$, the subgradients, are the normals to the supporting hyperplanes to the dilation of the unit ball at the point $\mathbf{f}$. It is a consequence of the definition of a polyhedral norm that $\rho_i(B) \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1$, $i \in \sigma^*$. Thus the multiplier condition is being used to ensure that both the solution vector $\mathbf{f}(\mathbf{x}^*)$ to (1.1) and also the solution vectors of all LSP's with $\|\mathbf{x}_i - \mathbf{x}^*\|_2$ small enough are aligned with points in the same edge of the unit ball, that is with the set of points on the unit ball having the same subdifferential. Here we show that the rank of $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1$ is at least $p+1$ if the minimum is strongly unique, and that there is a close correspondence between the multiplier condition and strong uniqueness if the rank of $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1$ is exactly $p+1$ in the sense that both imply systems of equations equivalent to (1.5). However, it is easy to see that strong uniqueness need not imply the multiplier condition. For consider

$$\mathbf{f}(\mathbf{x})=\begin{bmatrix} x_1-x_2 \\ x_1 \\ -x_1+x_2 \end{bmatrix}.$$

Then the maximum norm $\|\mathbf{f}(\mathbf{x})\|_1$ is minimised when $x_1=x_2=0$, and clearly (using a maximum norm also for $\mathbf{x}$)

$$\|\mathbf{f}(\mathbf{x})\|_1 \geqq \tfrac{1}{2}\|\mathbf{x}\|_2$$

so that this solution is strongly unique. However,

$$\nabla\mathbf{f}(\mathbf{x})=\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix},$$

and this matrix does not satisfy the Haar condition which in this case is equivalent to the multiplier condition. In fact we show that strong uniqueness is implied by the multiplier condition so that it is strictly a weaker condition.

It is natural to ask if strong uniqueness is the weakest condition that ensures second order convergence of the generalised Gauss-Newton algorithm. It is shown in Sect. 3 that this is not so, and necessary and sufficient conditions are given for second order convergence in the case of polyhedral norms.

Strong uniqueness is unusual in mathematical programming where the usual uniqueness results are a consequence of the well known second order sufficiency conditions. Consider, for example, the problem

$$\min_{\mathbf{x}\in F} g(\mathbf{x});\ F=\{\mathbf{x};\ h_i(\mathbf{x})\geqq \mathbf{0},\ i=1,2,\ldots,m\} \tag{1.7}$$

If the Kuhn-Tucker conditions [5] characterize a stationary point $\mathbf{x}^*\in F$ then $\exists\lambda_i^*\geqq 0$ such that

$$\nabla g(\mathbf{x}^*)=\sum_{i=1}^{m}\lambda_i^*\,\nabla h_i(\mathbf{x}^*) \tag{1.8}$$

and

$$\lambda_i^*\,h_i(\mathbf{x}^*)=0,\ i=1,2,\ldots,m. \tag{1.9}$$

We define the Langrangian function to be

$$L(\mathbf{x},\lambda)=g(\mathbf{x})-\sum_{i=1}^{m}\lambda_i h_i(\mathbf{x}), \tag{1.10}$$

and the second order sufficiency conditions [5] require that $\exists\alpha>0$ such that

$$\mathbf{t}^T \nabla_x^2 L(\mathbf{x}^*, \boldsymbol{\lambda}^*)\mathbf{t} \geqq \alpha\|\mathbf{t}\|^2$$
$$\forall \mathbf{t}\in H(\mathbf{x}^*, \boldsymbol{\lambda}^*)=\{\mathbf{t};\ \nabla h_i(\mathbf{x}^*)\mathbf{t}=0 \quad \text{if} \quad \lambda_i^*>0, i=1,2,\dots,m\}. \tag{1.11}$$

The condition proves to be somewhat stronger than necessary to show that $\mathbf{x}^*$ is a unique local solution of second order by which we mean that $\exists\beta>0$, $\delta>0$ such that

$$g(\mathbf{x})\geqq g(\mathbf{x}^*)+\beta\|\mathbf{x}-\mathbf{x}^*\|^2, \forall \mathbf{x}\in F\cap\{\mathbf{x};\ \|\mathbf{x}-\mathbf{x}^*\|<\delta\}. \tag{1.12}$$

This holds if and only if the inequality (1.11) holds for $\mathbf{t}\in T(\mathbf{x}^*)\cap H(\mathbf{x}^*,\boldsymbol{\lambda}^*)$ [6] where $T(\mathbf{x}^*)$ is the tangent cone to F at $\mathbf{x}^*$ and is defined by

$$T(\mathbf{x}^*)=\{\mathbf{t};\ \exists\{\mu_j\}\geqq 0, \{\mathbf{x}_k\}\to\mathbf{x}^*, \{\mathbf{x}_k\}\subset F\ni \lim_{n\to\infty}\|\mu_n(\mathbf{x}_n-\mathbf{x}^*)-\mathbf{t}\|=\mathbf{0}\}. \tag{1.13}$$

Strong uniqueness, in this context, corresponds to the existence of a unique local solution of first order and so is clearly a more restrictive condition than second order uniqueness. Thus we expect it to imply further information concerning problem structure. Certain results in this direction are summarised in the following theorem.

Theorem 1.1. *If* $\mathbf{x}^*\in F$ *and the Kuhn-Tucker constraint qualification* [5] *holds at* $\mathbf{x}^*$ *then the following statements are equivalent:*

(i) $\exists\alpha>0, \delta>0\ni g(\mathbf{x})-g(\mathbf{x}^*)\geqq\alpha\|\mathbf{x}-\mathbf{x}^*\|, \forall\mathbf{x}\in F\cap\{\mathbf{x};\ \|\mathbf{x}-\mathbf{x}^*\|<\delta\}$, (1.14)

(ii) $\nabla g(\mathbf{x}^*)\mathbf{t}>0 \quad \forall\mathbf{t}\neq 0, \mathbf{t}\in T(\mathbf{x}^*)$, (1.15)

and

(iii) $T(\mathbf{x}^*)\cap H(\mathbf{x}^*, \lambda^*)=\{0\}, \forall\lambda^*$ *satisying* (1.8), (1.9). (1.16)

Proof. (i) $\Rightarrow$ (ii) is an immediate consequence of the constraint qualification which implies the existence of a continuously differentiable arc in F emmanating from $\mathbf{x}^*$ in the direction $\mathbf{t}$, $\forall\mathbf{t}\in T(\mathbf{x}^*)$, (ii) $\Rightarrow$ (i) follows by a straighforward contradiction argument, and (ii) $\Leftrightarrow$ (iii) follows from the Kuhn-Tucker conditions.

Corollary. *For linear programming problems uniqueness implies strong uniqueness because* (1.15) *must hold if the solution of the linear programming problem is unique.*

In the next section we make an application of this theorem to show that the multiplier condition implies strong uniqueness.

2. Properties of the Strong Uniqueness Condition

The interest for us in strong uniqueness lies in its use in proving second order convergence for the generalised Gauss-Newton algorithm, and in the fact that it is a weaker condition than the multiplier condition which has also been used for this purpose. To amplify these statements we develop in this section three kinds of results:

(i) those relating strong uniqueness to properties of the LSP for points close to $\mathbf{x}^*$,

(ii) those relating strong uniqueness to properties of the norm, in particular to properties of $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1$, and

(iii) those relating strong uniqueness and the multiplier condition.

As a preliminary we recall the characterisation of $\mathbf{x}^*$ as a stationary point of (1.1) [9].

Definition. $\mathbf{x}^*$ is a stationary point of $\|\mathbf{f}(\mathbf{x})\|_1$ iff

$$\exists \mathbf{v} \in \partial\|\mathbf{f}(\mathbf{x}^*)\|_1 \ni \mathbf{v}^{*T} \nabla \mathbf{f}(\mathbf{x}^*) = 0. \tag{2.1}$$

This statement is closely related to the Kuhn-Tucker conditions of mathematical programming.

Remark. An alternative statement to (2.1) is that

$$0 \in \{\mathbf{v}_\alpha^T \nabla \mathbf{f}(\mathbf{x}^*);\ \mathbf{v}_\alpha \in \partial\|\mathbf{f}(\mathbf{x}^*)\|_1\}$$

as $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1$ is convex and closed. The fundamental theorem on linear inequalities ([3], p. 19) states that this condition is necessary and sufficient for the system of linear inequalities $\mathbf{v}_\alpha^T \nabla \mathbf{f}(\mathbf{x}^*) < 0$, $\forall \mathbf{v}_\alpha \in \partial\|\mathbf{f}(\mathbf{x}^*)\|_1$ to be inconsistent. We will see that this inconsistency takes a strong form when strong uniqueness holds at $\mathbf{x}^*$ (for example the inequality (2.10)).

To develop Cromme's results we write the LSP in the form

$$\min_{\mathbf{x}} \|\mathbf{r}(\mathbf{x}, \mathbf{y})\|_1 \tag{2.2}$$

where

$$\mathbf{r}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{y}) + \nabla \mathbf{f}(\mathbf{y})\,(\mathbf{x} - \mathbf{y}). \tag{2.3}$$

Lemma 2.1. *Strong uniqueness of* (1.1) *is equivalent to strong uniqueness of* (2.2) *when* $\mathbf{y} = \mathbf{x}^*$.

Remark. When $\mathbf{f}$ is linear (specifically when $\mathbf{f}(\mathbf{x}) = \mathbf{r}(\mathbf{x}, \mathbf{x}^*)$) the inequality (1.2) holds for all $\mathbf{x}$.

Proof. It follows from the assumed smoothness of $\mathbf{f}(\mathbf{x})$ that

$$\mathbf{r}(\mathbf{x}, \mathbf{x}^*) = \mathbf{f}(\mathbf{x}) + \|\mathbf{x} - \mathbf{x}^*\|_2^2\, \mathbf{w}(\mathbf{x}, \mathbf{x}^*) \tag{2.4}$$

where $\mathbf{w}(\mathbf{x}, \mathbf{x}^*)$ is an appropriate vector of mean values. Let problem (1.1) have a strongly unique solution at $\mathbf{x}^*$. Then

$$\begin{aligned}
\|\mathbf{r}(\mathbf{x}, \mathbf{x}^*)\|_1 &= \|\mathbf{f}(\mathbf{x}) + \|\mathbf{x} - \mathbf{x}^*\|_2^2\, \mathbf{w}(\mathbf{x}, \mathbf{x})\|_1 \\
&\geqq \|\mathbf{f}(\mathbf{x})\|_1 - \|\mathbf{x} - \mathbf{x}^*\|_2^2\, \|\mathbf{w}(\mathbf{x}, \mathbf{x}^*)\|_1 \\
&\geqq \|\mathbf{f}(\mathbf{x}^*)\|_1 + \gamma \|\mathbf{x} - \mathbf{x}^*\|_2 + O(\|\mathbf{x} - \mathbf{x}^*\|_2^2) \\
&\geqq \|\mathbf{r}(\mathbf{x}^*, \mathbf{x}^*)\|_1 + \bar{\gamma} \|\mathbf{x} - \mathbf{x}^*\|_2
\end{aligned}$$

for all $\mathbf{x}$ in a small enough neighbourhood of $\mathbf{x}^*$. Clearly the argument can be reversed.

The next result uses Lemma 2.1 show that the successive minimizers of a sequence of LSP's converge to $\mathbf{x}^*$ and that the convergence is second order. This is essentially the main result of [4].

Theorem 2.1. *Let the LSP for* $\mathbf{y}=\mathbf{x}^*$ *be strongly unique and let* $S=\{\mathbf{x};\ \|\mathbf{x}-\mathbf{x}^*\|_2<\xi\}$ *be such that for all* $\mathbf{x}\in S$

$$\text{(i)}\quad \|\mathbf{f}(\mathbf{x}^*)-\mathbf{f}(\mathbf{x})-\nabla\mathbf{f}(\mathbf{x})\,(\mathbf{x}^*-\mathbf{x})\|_1\leqq\kappa\|\mathbf{x}-\mathbf{x}^*\|_2^2, \tag{2.5}$$

$$\text{(ii)}\quad \|(\nabla\mathbf{f}(\mathbf{x})-\nabla\mathbf{f}(\mathbf{x}^*))\,(\mathbf{t}-\mathbf{x}^*)\|_1\leqq\beta\|\mathbf{x}-\mathbf{x}^*\|_2\|\mathbf{t}-\mathbf{x}^*\|_2 \tag{2.6}$$

$$\text{(iii)}\quad \beta\|\mathbf{x}-\mathbf{x}^*\|_2\leqq\gamma/2,\ \textit{and} \tag{2.7}$$

$$\text{(iv)}\quad 4\kappa\|\mathbf{x}-\mathbf{x}^*\|_2/\gamma\leqq\theta<1. \tag{2.8}$$

for positive constants θ, β, κ. *Let* $\{\mathbf{x}_i\}$ *be generated by successively minimizing a corresponding sequence of LSP's. Then* $\mathbf{x}_{i+1}\in S$, *and the rate of convergence of the sequence* $\{\mathbf{x}_i\}$ *is second order.*

Proof. We have using (2.5) and (2.6) that, for all $\mathbf{x}$, and for $\mathbf{x}_i\in S$, $\|\mathbf{r}(\mathbf{x},\mathbf{x}_i)-\mathbf{r}(\mathbf{x},\mathbf{x}^*)\|_1\leqq\kappa\|\mathbf{x}_i-\mathbf{x}^*\|_2^2+\beta\|\mathbf{x}_i-\mathbf{x}^*\|_2\|\mathbf{x}-\mathbf{x}^*\|_2$.

Thus, by strong uniqueness,

$$\begin{aligned}\|\mathbf{r}(\mathbf{x}^*,\mathbf{x}^*)\|_1+\gamma\|\mathbf{x}_{i+1}-\mathbf{x}^*\|_2-\kappa\|\mathbf{x}_i-\mathbf{x}^*\|_2^2-\beta\|\mathbf{x}_i-\mathbf{x}^*\|_2\,\|\mathbf{x}_{i+1}-\mathbf{x}^*\|_2&\\ \leqq\|\mathbf{r}(\mathbf{x}_{i+1},\mathbf{x}^*)\|_1-\|\mathbf{r}(\mathbf{x}_{i+1},\mathbf{x}_i)-\mathbf{r}(\mathbf{x}_{i+1},\mathbf{x}^*)\|_1&\\ \leqq\|\mathbf{r}(\mathbf{x}_{i+1},\mathbf{x}_i)\|_1&\\ \leqq\|\mathbf{r}(\mathbf{x}^*,\mathbf{x}_i)\|_1&\\ \leqq\|\mathbf{r}(\mathbf{x}^*,\mathbf{x}^*)\|_1+\kappa\|\mathbf{x}_i-\mathbf{x}^*\|_2^2&\end{aligned}$$

whence

$$(\gamma-\beta\|\mathbf{x}_i-\mathbf{x}^*\|_2)\|\mathbf{x}_{i+1}-\mathbf{x}^*\|_2\leqq2\kappa\|\mathbf{x}_i-\mathbf{x}^*\|_2^2$$

so that

$$\|\mathbf{x}_{i+1}-\mathbf{x}^*\|_2\leqq\theta\|\mathbf{x}_i-\mathbf{x}^*\|_2. \tag{2.9}$$

Thus $\mathbf{x}_{i+1}\in S$ and the sequence $\{\mathbf{x}_i\}$ converges to $\mathbf{x}^*$. The argument also shows that the ultimate rate of convergence is second order.

Remark. There exists at least one minimizer $\mathbf{x}_{i+1}$ of the LSP in the proof of the above theorem because $\|\mathbf{r}(\mathbf{x},\mathbf{x}_i)\|_1$ is a convex function of $\mathbf{x}$ as $\mathbf{r}(\mathbf{x},\mathbf{x}_i)$ is linear and $\|\cdot\|_1$ is bounded below. It need not be true that $\mathbf{x}_{i+1}$ is unique, but strong uniqueness of (1.1) ensures that all minimizers are close to $\mathbf{x}^*$. Also $\mathbf{x}_{i+1}$ need not minimize the LSP. The argument requires only that $\|\mathbf{r}(\mathbf{x}_{i+1},\mathbf{x}_i)\|_1\leqq\|\mathbf{r}(\mathbf{x}^*,\mathbf{x}_i)\|_1$.

The next example considers the case in which the minimizer of the LSP is not unique.

Example 2.1. Let $\|\cdot\|_1$ be the maximum norm and consider

$$\mathbf{f}(x)=\begin{bmatrix}1+x+\alpha x^2\\ 1-x+\beta x^2\\ 1\end{bmatrix}, \qquad \alpha,\beta>0.$$

The minimum norm problem has a solution at $x=0$, and we have

$$\|\mathbf{f}(x)\|_1-1\geqq|x|+O(x^2)$$

which shows strong uniqueness. In this case we have

$$\mathbf{r}(x,\bar{x})=\begin{bmatrix}1+\bar{x}+\alpha\bar{x}^2+(1+2\alpha\bar{x})(x-\bar{x})\\ 1-\bar{x}+\beta\bar{x}^2+(-1+2\beta\bar{x})(x-\bar{x})\\ 1\end{bmatrix}.$$

If $-\frac{1}{2\alpha}<\bar{x}<\frac{1}{2\beta}$ then the LSP has a minimum of 1 which is attained for

$$\frac{\beta\bar{x}^2}{-1+2\beta\bar{x}}\leqq x\leqq\frac{\alpha\bar{x}^2}{1+2\alpha\bar{x}}.$$

This shows both that the minimizer of the LSP is not unique except at the optimum, and that second order convergence obtains from good enough initial points.

Our next result characterises strong uniqueness in terms of local properties both of the function and of the point of the unit ball which determines its norm.

Theorem 2.2. *Problem* (1.1) *has a minimizer* $\mathbf{x}^*$ *which is strongly unique iff* $\exists\gamma>0$ *such that*

$$\forall\mathbf{t},\ \|\mathbf{t}\|_2=1,\ \exists\mathbf{v}\in\partial\|\mathbf{f}(\mathbf{x}^*)\|_1\ni\mathbf{v}^T\nabla\mathbf{f}(\mathbf{x}^*)\,\mathbf{t}\geqq\gamma. \tag{2.10}$$

Proof. Let $\mathbf{x}^*$ be a strongly unique minimizer of (1.1), and let $\mathbf{v}\in\partial\|\mathbf{f}(\mathbf{x}^*)\|_1$ so that $\|\mathbf{f}(\mathbf{x}^*)\|_1=\mathbf{v}^T\mathbf{f}(\mathbf{x}^*)$. Then the subgradient inequality gives (with $\mathbf{v}_x\in\partial\|\mathbf{f}(\mathbf{x})\|_1$)

$$\begin{aligned}\mathbf{v}^T\mathbf{f}(\mathbf{x}^*)&\geqq\mathbf{v}_x^T\mathbf{f}(\mathbf{x})+\mathbf{v}_x^T(\mathbf{f}(\mathbf{x}^*)-\mathbf{f}(\mathbf{x}))\\ &=\|\mathbf{f}(\mathbf{x})\|_1+\mathbf{v}_x^T(\mathbf{f}(\mathbf{x}^*)-\mathbf{f}(\mathbf{x}))\\ &\geqq\|\mathbf{f}(\mathbf{x}^*)\|_1+\gamma\|\mathbf{x}-\mathbf{x}^*\|_2+\mathbf{v}_x^T(\mathbf{f}(\mathbf{x}^*)-\mathbf{f}(\mathbf{x}))\end{aligned}$$

using strong uniqueness. This gives

$$\mathbf{v}_x^T\nabla\mathbf{f}(\mathbf{x}^*)\,(\mathbf{x}-\mathbf{x}^*)\geqq\gamma\|\mathbf{x}-\mathbf{x}^*\|_2+O(\|\mathbf{x}-\mathbf{x}^*\|_2^2) \tag{2.11}$$

and holds for all $\mathbf{v}_x\in\partial\|\mathbf{f}(\mathbf{x})\|_1$. Now let $\mathbf{x}\to\mathbf{x}^*$ in the direction $\mathbf{t}$. Then $\exists$ a subsequence $\ni\{\mathbf{v}_x\}\to\mathbf{v}\in\partial\|\mathbf{f}(\mathbf{x}^*)\|_1$ [10]. This demonstrates necessity.

To show sufficiency let $\{\mathbf{x}_i\}\to\mathbf{x}^*$ in the direction $\mathbf{t}$. Then the subgradient inequality gives

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}_i)\|_1&\geqq\|\mathbf{f}(\mathbf{x}^*)\|_1+\mathbf{v}^T(\mathbf{f}(\mathbf{x}_i)-\mathbf{f}(\mathbf{x}^*)),\ \forall\mathbf{v}\in\partial\|\mathbf{f}(\mathbf{x}^*)\|_1\\ &\geqq\|\mathbf{f}(\mathbf{x}^*)\|_1+\mathbf{v}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{x}_i-\mathbf{x}^*)+O(\|\mathbf{x}_i-\mathbf{x}^*\|_2^2),\\ &\geqq\|\mathbf{f}(\mathbf{x}^*)\|_1+\gamma\|\mathbf{x}_i-\mathbf{x}^*\|_2+O(\|\mathbf{x}_i-\mathbf{x}^*\|_2^2)\end{aligned}$$

using (2.10) to specialize $\mathbf{v}$. This demonstrates strong uniqueness.

Corollary. *Strong uniqueness implies that rank* $(\nabla \mathbf{f}(\mathbf{x}^*))$ *and rank* $\{\mathbf{v}^T \nabla \mathbf{f}(\mathbf{x}^*), \mathbf{v} \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1\} = p$.

Proof. Assume the contrary. Then $\exists \mathbf{t}, \mathbf{t} \neq 0, \ni \nabla \mathbf{f}(\mathbf{x}^*)\mathbf{t} = 0$ contradicting strong uniqueness of the LSP at $\mathbf{x}$, or $\mathbf{v}^T \nabla \mathbf{f}(\mathbf{x}^*)\mathbf{t} = 0 \ \forall \mathbf{v} \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1$.

These results imply that a necessary condition for strong uniqueness is that the point $\mathbf{f}(\mathbf{x}^*)$ in R^n corresponds to a point at which $\|\cdot\|_1$ is not smooth (this would require a unique supporting hyperplane). It follows from the above corollary that rank $\{\mathbf{v}; \mathbf{v} \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1\}$ is at least p. This result is improved in the next theorem.

Theorem 2.3. *If* $\mathbf{x}^*$ *is a unique local solution of first order of* (1.1) *or* (2.2) *then the rank of* $\{\mathbf{v}; \mathbf{v} \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1\}$ *is at least* $p+1$.

Proof. This result is trivially true if $\mathbf{f}(\mathbf{x}^*) = 0$. Thus we assume $\|\mathbf{f}(\mathbf{x}^*)\|_1 > 0$. As rank $(\nabla \mathbf{f}(\mathbf{x}^*)) = p$ we can find an orthogonal matrix A such that

$$A \nabla \mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} U \\ 0 \end{bmatrix} \tag{2.12}$$

where U is $p \times p$ nonsingular. Let

$$\mathbf{w}_\alpha = \begin{bmatrix} \mathbf{w}_\alpha^{(1)} \\ \mathbf{w}_\alpha^{(2)} \end{bmatrix} = A \mathbf{v}_\alpha, \ \mathbf{v}_\alpha \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1 \tag{2.13}$$

where $\mathbf{w}_\alpha$ is partitioned in conformity with (2.12). Clearly $\{\mathbf{w}_\alpha\}$ is also closed and convex. Now strong uniqueness gives (using (2.10) and noting that vectors of the form $U\mathbf{t}$ span R^p)

$$\forall \mathbf{t} \exists \mathbf{w}_\alpha \ni \mathbf{w}_\alpha^{(1)T} \mathbf{t} > 0. \tag{2.14}$$

Thus inconsistency of the linear inequalities $\mathbf{w}_\alpha^{(1)T} \mathbf{t} \geqq 0$, $\forall \mathbf{w}_\alpha$ is a consequence of the inconsistency of the inequalities $\mathbf{v}_\alpha^T \nabla \mathbf{f}(\mathbf{x}^*)\mathbf{u} < 0$, $\forall \mathbf{u} \in R^p$, $\forall \mathbf{v}_\alpha \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1$. It follows that

$$\begin{aligned} &\text{(i) rank } \{\mathbf{w}_\alpha^{(1)}\} = p, \text{ and} \\ &\text{(ii) } 0 \in \{\mathbf{w}_\alpha^{(1)}\}. \end{aligned} \tag{2.15}$$

Now, by definition,

$$\mathbf{v}_\alpha^T \mathbf{f}(\mathbf{x}^*) = \mathbf{w}_\alpha^T A \mathbf{f}(\mathbf{x}^*) = \|\mathbf{f}(\mathbf{x}^*)\|_1, \ \forall \mathbf{v}_\alpha \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1 \tag{2.16}$$

and $\exists$, by (2.15),

$$\mathbf{w}_{p+1} = \begin{bmatrix} 0 \\ \mathbf{w}_{p+1}^{(2)} \end{bmatrix}. \tag{2.17}$$

As $\|\mathbf{f}(\mathbf{x}^*)\|_1 > 0$ it follows from (2.16) that $\mathbf{w}_{p+1}^{(2)} \neq 0$. Select $\mathbf{w}_i^{(1)}, i = 1, 2, \ldots, p$ $\ni \text{rank}\{\mathbf{w}_i^{(1)}\} = p$ and consider

$$W = \begin{bmatrix} \mathbf{w}_1^{(1)T} & (\mathbf{w}_1^{(2)})_k \\ \vdots & \vdots \\ \mathbf{w}_p^{(1)T} & (\mathbf{w}_p^{(2)})_k \\ 0 & (\mathbf{w}_{p+1}^{(2)})_k \end{bmatrix}. \tag{2.18}$$

where k is chosen such that $(\mathbf{w}_{p+1}^{(2)})_k \neq 0$. This matrix is clearly nonsingular and has rank $(p+1)$ so that rank $\{\mathbf{w}_\alpha\}$ and therefore rank $\{\mathbf{v}_\alpha\}$ is at least $p+1$.

We consider now the case where the solution (1.1) is strongly unique and $\mathbf{f}(\mathbf{x}^*)$ corresponds to a point on the dilated unit ball which lies in a 'flat' subset Q made up from the intersection of the surface of the ball with the family of hyperplanes defined by $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1$ where rank $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1 = p+1$. Q is assumed to have a nontrivial relative interior, and we show that the solution of the LSP (2.2) is strongly unique and corresponds to a point in Q if $\|\mathbf{y}-\mathbf{x}^*\|_2$ is small enough. Thus strong uniqueness ensures exactly the same behaviour as the multiplier condition (see the discussion following (1.5)) for polyhedral norms when rank $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1 = p+1$.

Remark. Examples where rank $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1 = p+1$ include both the maximum and L_1 norms when the characterization theorems are precisely satisfied so that there are exactly $p+1$ extrema in the first case and exactly p zeros in the second.

Theorem 2.4. *Let $\mathbf{x}^*$ be a unique local solution of first order to* (1.1). *Then $\mathbf{g} = \mathbf{f}(\mathbf{x}^*)/\|\mathbf{f}(\mathbf{x}^*)\|$ is the image of $\mathbf{f}(\mathbf{x}^*)$ on the unit ball under dilation so that $\partial\|\mathbf{g}\|_1 = \partial\|\mathbf{f}(\mathbf{x}^*)\|_1 = \{\mathbf{v}_\alpha\}$, and rank $\{\mathbf{v}_\alpha\} = p+1$. We define Q to be the set*

$$Q = \{\mathbf{t} \in R^n, \|\mathbf{t}\|_1 = 1, \mathbf{v}^T(\mathbf{t}-\mathbf{g}) = 0 \ \forall\, \mathbf{v} \in \{\mathbf{v}_\alpha\}\} \tag{2.19}$$

and we require Q to have a non-empty relative interior in the strong sense that $\exists \varepsilon > 0$

$$\mathbf{v}^T(\mathbf{t}-\mathbf{g}) = 0 \ \forall\, \mathbf{v} \in \{\mathbf{v}_\alpha\}, \quad \text{and} \quad \|\mathbf{t}-\mathbf{g}\|_1 < \varepsilon \Rightarrow \mathbf{t} \in Q. \tag{2.20}$$

Then the solution of the LSP (2.2) *is strongly unique and is attained at a point in a dilation of Q provided $\|\mathbf{x}^*-\mathbf{y}\|_2$ is small enough.*

Proof. Let $\mathbf{v}_i$, $i=1,\ldots,p+1$ be a basis for $\{\mathbf{v}_\alpha\}$ corresponding to the set $\mathbf{w}_i$, $i=1,\ldots p+1$ constructed in the proof of the previous Theorem. Then

$$\mathbf{v} \in \{\mathbf{v}_\alpha\} \Rightarrow \mathbf{v} = \sum_{i=1}^{p+1} \mu_i \mathbf{v}_i \tag{2.21}$$

where the μ_1 are unique, bounded, and satisfy $\sum_{i=1}^{p+1} \mu_1 = 1$. The conditions for $\mathbf{r}(\mathbf{x},\mathbf{y})$ to be in a dilation of Q, expressed in terms of the basis for $\{\mathbf{v}_\alpha\}$, are

$$\mathbf{v}_i^T(\theta\mathbf{f}(\mathbf{x}^*) - \mathbf{r}(\mathbf{x},\mathbf{y})) = 0, \; i = 1, 2, \ldots, p+1, \tag{2.22}$$

and

$$\|\theta\mathbf{f}(\mathbf{x}^*) - \mathbf{r}(\mathbf{x},\mathbf{y})\|_1 \quad \textit{small enough.}$$

Let

$$A\nabla\mathbf{f}(\mathbf{y}) = \begin{bmatrix} U \\ 0 \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

where E_1, E_2 are small for $\|\mathbf{y}-\mathbf{x}^*\|_2$ small. The system (2.22) is equivalent to

$$\mathbf{v}_i^T \mathbf{f}(\mathbf{x}^*)\,\theta - \mathbf{w}_i^T\left\{\begin{bmatrix} U \\ 0 \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\right\}(\mathbf{x}-\mathbf{y}) = \mathbf{v}_i^T \mathbf{f}(\mathbf{y}),\ i=1,2,\dots,p+1. \tag{2.23}$$

This system is clearly nonsingular when E_1, E_2 are small. Thus the solution is a continuous function of $\mathbf{y}$, and $\begin{bmatrix} \theta \\ \mathbf{x}-\mathbf{y} \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{y}\to\mathbf{x}^*$. Thus $\exists \mathbf{x}(\mathbf{y}) \ni \mathbf{r}(\mathbf{x},\mathbf{y}) \in \theta \|\mathbf{f}(\mathbf{x}^*)\|_1 Q$ provided $\|\mathbf{y}-\mathbf{x}^*\|_2$ is sufficiently small. Now it follows from strong uniqueness at $\mathbf{x}^*$ and the continuity of $\nabla\mathbf{f}$ that

$$\forall \mathbf{t},\ \|\mathbf{t}\|_2 = 1,\ \exists \gamma(\mathbf{y}),\ \mathbf{v}\in\{\mathbf{v}_\alpha\} \ni \mathbf{v}^T \nabla\mathbf{f}(\mathbf{y})\,\mathbf{t} > \gamma(\mathbf{y}) > \mathbf{0} \tag{2.24}$$

$\forall \mathbf{y} \ni \|\mathbf{x}^* - \mathbf{y}\|_2$ sufficiently small. This permits us to show that $\mathbf{x}(\mathbf{y})$ determined by (2.22) solves (2.2). By the subgradient inequality and the definition of Q (this implies that $\{\mathbf{v}_\alpha\} \subseteq \partial\|\mathbf{r}(\mathbf{x}(\mathbf{y}),\mathbf{y})\|_1$)

$$\begin{aligned} \|\mathbf{r}(\mathbf{x},\mathbf{y})\|_1 &\geqq \|\mathbf{r}(\mathbf{x}(\mathbf{y}),\mathbf{y})\|_1 + \mathbf{v}^T(\mathbf{r}(\mathbf{x},\mathbf{y}) - \mathbf{r}(\mathbf{x}(\mathbf{y}),\mathbf{y})),\ \forall \mathbf{v}\in\{\mathbf{v}_\alpha\}, \\ &\geqq \|\mathbf{r}(\mathbf{x}(\mathbf{y}),\mathbf{y})\|_1 + \mathbf{v}^T \nabla\mathbf{f}(\mathbf{y})(\mathbf{x}-\mathbf{x}(\mathbf{y})) \\ &\geqq \|\mathbf{r}(\mathbf{x}(\mathbf{y}),\mathbf{y})\|_1 + \gamma(\mathbf{y})\|\mathbf{x}-\mathbf{x}(\mathbf{y})\|_2 \end{aligned} \tag{2.25}$$

by choosing $\mathbf{v}$ such that (2.24) is satisfied. This last inequality also shows strong uniqueness.

Remark. (*i*) The condition that rank $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1 = p+1$ is required to ensure that (2.22) has unique solution. Example 2.1 shows one possible situation when this condition is not satisfied. Here the solution of the LSP for $\mathbf{y} \neq \mathbf{x}^*$ is not unique so that strong uniqueness cannot obtain. Note that equation (2.22) corresponds to the system (1.5) which we wrote down as a consequence of the multiplier condition.

(ii) If $n=p+1$ then Q must reduce to a single point. However, equation (2.22) ensures that $\mathbf{r}(\mathbf{x}(\mathbf{y}),\mathbf{y}) \| \mathbf{f}(\mathbf{x}^*)$ in this case. Thus $\partial\|\mathbf{r}(\mathbf{x}(\mathbf{y}),\mathbf{y})\|_1 = \{\mathbf{v}_\alpha\}$ so that the conclusion of Theorem 2.4 remains valid.

To conclude this section we complete the demonstration that the multiplier condition is the stronger condition by showing that it implies strong uniqueness. This result follows by an easy contradiction argument, but it is instructive to prove it using Theorem 1.1 which provides a link between our considerations and standard mathematical programming.

Theorem 2.5. *For polhedral norm problems, the multiplier condition implies that the minimum to* (1.1) *is strongly unique.*

Proof. Let $\mathbf{y}\in R\times R^p$ and consider the problem

$$\min_{\mathbf{y}\in R^{p+1}} \mathbf{e}_1^T\mathbf{y} \tag{2.26}$$

where $\mathbf{e}_1$ is the usual coordinate vector, subject to

$$[1,\ -\mathbf{v}_{\alpha_i}^T \nabla\mathbf{f}(\mathbf{x}^*)]\,\mathbf{y} \geqq \|\mathbf{f}(\mathbf{x}^*)\|_1,\ i=1,2,\dots,p+1, \tag{2.27}$$

where the $\mathbf{v}_{\alpha_i}$ are the vectors aligned with $\mathbf{f}(\mathbf{x}^*)$ in the multiplier condition. This linear programming problem has the unique solution

$$\mathbf{y}^* = \begin{bmatrix} \|\mathbf{f}(\mathbf{x}^*)\|_1 \\ 0 \end{bmatrix},$$

and we have by the rank condition

$$H = \{\mathbf{t};\, [1, \mathbf{v}_{\alpha_i}^T \nabla \mathbf{f}(\mathbf{x}^*)]\,\mathbf{t} = 0, i = 1, 2, \ldots, p+1\} = \{0\}.$$

Thus, by Theorem 1.1,

$$\exists \gamma > 0 \ni h - \|\mathbf{f}(\mathbf{x}^*)\|_1 \geqq \gamma \|\mathbf{y} - \mathbf{y}^*\| \geqq \gamma \|\mathbf{x} - \mathbf{x}^*\|_2 \tag{2.28}$$

for all feasible $\mathbf{y}$ under reasonable assumptions on the norm on R^{p+1}. Now consider points $\mathbf{y}$ such that

$$h - \mathbf{v}_\alpha^T \nabla \mathbf{f}(\mathbf{x}^*)\,(\mathbf{x} - \mathbf{x}^*) \geqq \|\mathbf{f}(\mathbf{x}^*)\|_1, \forall\, \mathbf{v}_\alpha \in \partial \|\mathbf{f}(\mathbf{x}^*)\|_1. \tag{2.29}$$

Such points are feasible for (2.27) and thus satisfy (2.28). For a particular $\mathbf{x}$ now specialize h to be the infimum such that (2.29) holds. As $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1$ is closed $\exists\, \mathbf{v}_\alpha$ such that equality holds in (2.29). Substituting into (2.28) gives

$$\|\mathbf{r}(\mathbf{x}, \mathbf{x}^*)\|_1 \geqq \mathbf{v}_\alpha^T \mathbf{r}(\mathbf{x}, \mathbf{x}^*) \geqq \|\mathbf{f}(\mathbf{x}^*)\|_1 + \gamma \|\mathbf{x} - \mathbf{x}^*\|_2. \tag{2.30}$$

This demonstrates strong uniqueness for the LSP and thus for (1.1) by Lemma 2.2.

Remark. In the above argument the multiplier condition is used to form a restriction of the LSP which is a linear programming problem with a unique solution. However, uniqueness at the optimum implies (2.28) as a consequence of the corollary to Theorem 1.1. Thus, in particular, uniqueness of the LSP at the optimum implies strong uniqueness for polyhedral norms.

3. Weakest Conditions for Second Order Convergence (Polyhedral Norm Case)

We have shown that strong uniqueness is the weakest condition that has been used to demonstrate second order convergence of the generalised Gauss-Newton algorithm. Thus it is natural to ask if this condition is necessary as well as sufficient. Unfortunately this is not so as the following example shows.

Example 3.1. Consider[1]

$$\mathbf{f}(x) = \begin{bmatrix} x + \alpha x^3 \\ 1 + x - x^3 \end{bmatrix}, \qquad \alpha > 0.$$

a simple calculation shows that in the L_1 norm

$$\begin{aligned} \|\mathbf{f}\|_1 &= 1 + 2|x| - (1-\alpha)|x|^3, \qquad x > 0, \\ &= 1 + (1+\alpha)|x|^3, \qquad x < 0, \end{aligned}$$

[1] An even simpler example in the max norm was suggested by the referee. Let $\mathbf{f} = \begin{bmatrix} 1+x \\ 1-x^3 \end{bmatrix}$. Then $\|\mathbf{f}\| = 1 + x$, $x > 0$, $1 + |x|^3$, $x < 0$, and the LSP gives $t = -\dfrac{x + x^3}{1 + 3x^2}$

so that $\|\mathbf{f}\|_1$ is minimized when $x=0$, and clearly this minimum is not strongly unique. The solution of the LSP is found by minimizing $\phi(t)$ where

$$\phi=|\bar{x}+\alpha\bar{x}^3+(1+3\alpha\bar{x}^2)t|+|1+\bar{x}-\bar{x}^3+(1-3\bar{x}^2)t|,$$

and this minimum is attained at

$$t=-\frac{\bar{x}+\alpha\bar{x}^3}{1+3\alpha\bar{x}^2}, \qquad \bar{x}\neq 0.$$

Thus the generalised Gauss-Newton iteration is at least second order convergent for we have

$$x_{i+1}=x_i-\frac{x_i+\alpha x_i^3}{1+3\alpha x_i^2}=\frac{2\alpha x_i^3}{1+3\alpha x_i^2}.$$

However, when $\bar{x}=0$ the LSP is minimized for $-1\leqq t\leqq 0$ and so does not have a unique solution.

For the particular case of polyhedral norms the weakest conditions which ensure second order convergence of the generalised Gauss-Newton algorithm can be given. Here the LSP reduces to the linear programming problem [1]

$$\min_{[h,\mathbf{x}]\in Z} h;\ Z=\left\{\begin{bmatrix} h \\ \mathbf{x}\end{bmatrix};\ h\geqq 0,\ \rho_j(B)\,\mathbf{r}(\mathbf{x},\bar{\mathbf{x}})\leqq h,\ \forall j\right\}. \tag{3.1}$$

We make the natural assumption that $\nabla\mathbf{f}(\mathbf{x}^*)$ has rank p and this ensures that the solution set of the LSP (2.2) is bounded for $\|\bar{\mathbf{x}}-\mathbf{x}^*\|_2$ small enough [3, p. 96]. The simplex algorithm applied solve the LSP returns a solution at a vertex of the feasible region where $p+1$ of the inequalities hold as equations and where the matrix of this system of equations is nonsingular. To describe the solution set of the LSP let the optimal vertices be indexed $1, 2, \ldots, q(\bar{\mathbf{x}})$, the $\rho_j(B)$ determining the corresponding sets of $(p+1)$ equations be pointed to by index sets $\sigma^i(\bar{\mathbf{x}})$, $i=1, 2, \ldots, q(\bar{\mathbf{x}})$, and the matrices of these sets of equations and corresponding solutions vectors be denoted by $A^i(\bar{\mathbf{x}})$, $\mathbf{x}^i(\bar{\mathbf{x}})$, $i=1, 2, \ldots, q(\bar{\mathbf{x}})$ respectively. Also, if $\bar{\mathbf{x}}$ is restricted to be one of the iterates $\mathbf{x}_k$, $k=1, 2\ldots$ produced by the generalised Gauss-Newton algorithm then the dependence on $\bar{\mathbf{x}}$ is abbreviated to the use of a subscript (for example $\mathbf{x}_k^i$). Consider

$$X_i=\{\mathbf{x};\ \mathbf{x} \text{ minimizes } \|\mathbf{r}(\mathbf{x},\mathbf{x}_i)\|_1\}=\operatorname{conv}\{\mathbf{x}_i^j, j=1, 2, \ldots q_i\}. \tag{3.2}$$

The algorithm requires only that

$$\mathbf{x}_{i+1}\in X_i,\ i=1, 2, \ldots. \tag{3.3}$$

Obviously it is undesirable that the properties of the algorithm depend on the choice of $\mathbf{x}_{i+1}$ (consider example 2.1). Thus we say that it is second order convergent only if $\exists k>0$, i_0 large enough such that

$$\|\mathbf{x}-\mathbf{x}^*\|_2 \leqq k\|\mathbf{x}_i-\mathbf{x}^*\|_2^2, \forall \mathbf{x}\in X_i, \forall i \geqq i_0. \tag{3.4}$$

It follows from Theorem 2.1 that strong uniqueness is a sufficient condition for second order convergence in this sense. When $\|\cdot\|_1$ is polyhedral it follows from (3.2) that it is only necessary to verify (3.4) for $\mathbf{x}_{i+1}=\mathbf{x}_i^j, j=1,2\ldots,q_i$.

Lemma 3.1. *If*

$$\exists \mathbf{x}\in X(\bar{\mathbf{x}}) \ni \rho_j(B)\,\mathbf{r}(\mathbf{x},\bar{\mathbf{x}})=h(\bar{\mathbf{x}})$$

then

$$\exists i\in\{1,\ldots,q(\bar{\mathbf{x}})\}\ni \rho_j(B)\in\partial\|\mathbf{r}(\mathbf{x}^i(\bar{\mathbf{x}}),\bar{\mathbf{x}})\|_1. \tag{3.5}$$

Proof. From the definition of $X(\bar{\mathbf{x}})$ it follows that

$$\exists \alpha_i\geqq 0, \sum_{i=1}^{q(\bar{\mathbf{x}})}\alpha_i=1\ni \mathbf{x}=\sum_{i=1}^{q(\bar{\mathbf{x}})}\alpha_i\,\mathbf{x}^i(\bar{\mathbf{x}}),$$

whence, using linearity,

$$\mathbf{r}(\mathbf{x},\bar{\mathbf{x}})=\sum_{i=1}^{q(\bar{\mathbf{x}})}\alpha_i\,\mathbf{r}(\mathbf{x}^i(\bar{\mathbf{x}}),\bar{\mathbf{x}})$$

and the result follows as

$$\rho_j(B)\,\mathbf{r}(\mathbf{x}^i(\bar{\mathbf{x}}),\bar{\mathbf{x}})\leqq\|\mathbf{r}(\mathbf{x}^i(\bar{\mathbf{x}}),\bar{\mathbf{x}})\|_1=h(\bar{\mathbf{x}})$$

with equality only if (3.5) holds.

Theorem 3.1. *Let $\|\cdot\|_1$ be a polyhedral norm and $\{\mathbf{x}_i\}$ be a sequence of points generated by the generalised Gauss-Newton algorithm. If $\exists$ a neighbourhood N of $\mathbf{x}^*$ such that A_i^k has a uniformly bounded inverse for each $k\in\{1,2,\ldots q_i\}$, and all $\mathbf{x}_i\in N$ then the sequence $\{\mathbf{x}_i\}$ is second order convergent if and only if, $\forall\,\mathbf{x}_i$, $\mathbf{x}_i\neq\mathbf{x}^*$, $\|\mathbf{x}_i-\mathbf{x}^*\|_2$ small enough,*

$$\bigcup_{\mathbf{x}\in X_i}\partial\|\mathbf{r}(\mathbf{x},\mathbf{x}_i)\|_1\subseteq\partial\|\mathbf{f}(\mathbf{x}^*)\|_1. \tag{3.6}$$

Proof. To show sufficiency we follow the approach used in [2]. Note that by (3.6)

$$\rho_j(B)\,\mathbf{f}(\mathbf{x}^*)=\|\mathbf{f}(\mathbf{x}^*)\|_1, j\in\sigma_i^k \tag{3.7}$$

for each k. Thus, using (2.4),

$$\rho_j(B)\,\mathbf{r}(\mathbf{x}^*,\mathbf{x}_i)=\|\mathbf{f}(\mathbf{x}^*)\|_1-\|\mathbf{x}^*-\mathbf{x}_i\|_2^2\,\rho_j(B)\,\mathbf{w}(\mathbf{x}^*,\mathbf{x}_i). \tag{3.8}$$

Subtracting this equation from the corresponding equation determining $\mathbf{x}_i^k$ for $\forall j \in \sigma_i^k$ gives

$$A_i^k \begin{bmatrix} h_i - \|\mathbf{f}(\mathbf{x}^*)\|_1 \\ \mathbf{x}_i^k - \mathbf{x}^* \end{bmatrix} = -\|\mathbf{x}_i - \mathbf{x}^*\|_2^2 \begin{bmatrix} \vdots \\ \rho_j(B)\,\mathbf{w}(\mathbf{x}^*, \mathbf{x}_i) \\ \vdots \end{bmatrix} \tag{3.9}$$

for each k. This demonstrates second order convergence as A_i^k has a uniformly bounded inverse for $\|\mathbf{x}_i - \mathbf{x}^*\|_2$ small enough.

To show necessity we assume that the sequence $\{\mathbf{x}_i\}$ converges and that the inclusion (3.6) does not hold. As the set $\rho_j(B)$ is finite it follows, by restriction to a subsequence if necessary, that

$$\exists k, \{\mathbf{x}_i\} \to \mathbf{x}^* \ni \rho_k(B) \in \partial \|\mathbf{r}(\mathbf{x}(\mathbf{x}_i), \mathbf{x}_i)\|_1 \tag{3.10}$$

but $\rho_k(B) \notin \partial \|\mathbf{f}(\mathbf{x}^*)\|_1$ where $\mathbf{x}(\mathbf{x}_i)$ is the Gauss-Newton iterate. From (3.10) it follows that $\rho_k(B) \in \partial \|\mathbf{r}((\mathbf{x}^*), \mathbf{x}^*)\|_1$ by the limiting properties of subdifferentials [10]. As $\mathbf{x}^*$ minimizes $\|\mathbf{f}(\mathbf{x})\|_1$ it is known, [9], that

$$\|\mathbf{f}(\mathbf{x}^*)\|_1 = \|\mathbf{r}(\mathbf{x}(\mathbf{x}^*), \mathbf{x}^*)\|_1 \tag{3.11}$$

By assumption, $\forall i$ large enough,

$$\rho_k(B)\mathbf{f}(\mathbf{x}_i) < \|\mathbf{f}(\mathbf{x}^*)\|_1.$$

Thus

$$\exists \varepsilon > 0, i_0(\varepsilon) \ni \rho_k(B)\,\mathbf{f}(\mathbf{x}_i) \leqq \|\mathbf{r}(\mathbf{x}(\mathbf{x}_i), \mathbf{x}_i)\|_1 - \varepsilon, i > i_0.$$

By Lemma 3.1

$$\exists \alpha_j \geqq 0, \sum_{j=1}^{q_i} \alpha_j = 1 \ni \|\mathbf{r}(\mathbf{x}(\mathbf{x}_i), \mathbf{x}_i)\|_1 = \rho_k(B)\,\mathbf{f}(\mathbf{x}_i) + \rho_k(B)\,\nabla\mathbf{f}(\mathbf{x}_i) \sum_{j=1}^{q_i} \alpha_j(\mathbf{x}_i^j - \mathbf{x}_i)$$

whence, for at least one j, (say $j(i)$),

$$\rho_k(B)\,\nabla\mathbf{f}(\mathbf{x}_i)(\mathbf{x}_i^{j(i)} - \mathbf{x}_i) \geqq \varepsilon.$$

It follows that $\|\mathbf{x}_i^{j(i)} - \mathbf{x}^*\|_2 \not\to 0, i \to \infty$, contradicting (3.4).

Remark. (i) The assumption that $(A_i^k)^{-1}$ be uniformly bounded corresponds to the assumption, usually made in analyzing Newton's method, that the Jacobian matrix be nonsingular. If this condition is not assumed then equation (3.9) shows that second order convergence will not be obtained in general even if $\nabla\mathbf{f}(\mathbf{x}^*)$ has rank p and the inclusion (3.6) holds. Consider

$$\mathbf{f} = \begin{bmatrix} 1 + x_1 & + x_2 \\ 1 + x_1^2 & \\ 1 & + x_2^2 \\ 1 + \alpha x_1 + \beta x & \end{bmatrix} \quad \text{with } \alpha - \beta < 0.$$

In the maximum norm this problem has a minimum of 1 when $x_1 = x_2 = 0$. If we consider the LSP at $x_1 = \delta_1$, $x_2 = \delta_2$, $\delta_1 > \delta_2$ and δ_1, δ_2 small then the first three components of $\mathbf{r}$ define the optimum vertex. The defining equations are

$$\begin{gathered} 1+\delta_1+\delta_2+t_1+t_2=h \\ 1+\delta_1^2+2\delta_1 t_1=h \\ 1+\delta_2^2+2\delta_2 t_2=h \end{gathered}$$

giving

$$\begin{aligned} t_1 &= -\frac{\delta_1}{2}-\frac{\delta_2}{2}-\frac{\delta_1\delta_2^2}{\delta_1+\delta_2-2\delta_1\delta_2}, \\ t_2 &= -\frac{\delta_1}{2}-\frac{\delta_2}{2}-\frac{\delta_1^2\delta_2}{\delta_1+\delta_2-2\delta_1\delta_2}, \\ h &= 1-\delta_1\delta_2-\frac{2(\delta_1\delta_2)^2}{\delta_1+\delta_2-2\delta_1\delta_2}. \end{aligned}$$

Evaluating the last inequality in the LSP gives

$$\begin{aligned} &1+\alpha\delta_1+\beta\delta_2+\alpha t_1+\beta t_2 \\ &\quad = 1+(\alpha-\beta)(\delta_1-\delta_2)/2-\frac{\delta_1\delta_2}{\delta_1+\delta_2-2\delta_1\delta_2}(\alpha\delta_2+\beta\delta_1) \\ &\quad < h \end{aligned}$$

so that $\partial\|\mathbf{r}(\mathbf{t},\mathbf{x})\|_1 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \partial\|\mathbf{f}(0)\|_1 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$.

The successive iterates satisfy $\delta_1 - \delta_2 > 0$ but not (3.4), and

$$A(\mathbf{x}) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -2x_1 & \\ 1 & & -2x_2 \end{bmatrix}$$

is singular at the origin although $\nabla\mathbf{f}$ has rank 2 there. The solution of the LSP at the origin is not unique.

(ii) The simplest way that the inclusion (3.6) can fail to hold is for rank $\partial\|\mathbf{f}(\mathbf{x}^*)\|_1 < p+1$ as it is straight forward to show that rank $\{\rho_j(B), j \in \sigma_i^k\} = p+1$. All we need do is follow the argument of Theorem 2.3 as the properties (2.15) follow in this case because the simplex algorithm returns a solution point corresponding to a vertex of the feasible region for (3.1).

Consider, for example,

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_1+x_2 \\ 1-x_1+x_2^2 \\ 1+x_1^2-x_2 \end{bmatrix}.$$

In the L_1 norm we have

$$\|\mathbf{f}\|_1 \geq 2+x_1^2+x_2^2$$

so that the minimum is attained for $x_1 = x_2 = 0$.

However, only $f_1(0,0)=0$ so that

$$\partial \|\mathbf{f}(0,0)\|_1 = \operatorname{conv}\{[1,1,1],[-1,1,1]\}$$

which has rank $2 < p+1 = 3$.

(iii) It is not sufficient for the inclusion (3.6) to hold that rank $\partial \|\mathbf{f}(\mathbf{x}^*)\|_1 = p+1$ as the following example shows. Let

$$\mathbf{f}(x) = \begin{bmatrix} x + \alpha x^2 \\ 1 + x + x^2 \end{bmatrix}, \quad 0 < \alpha < 1.$$

In the L_1 norm, provided $x > -\frac{1}{\alpha}$,

$$\begin{aligned} \|\mathbf{f}\|_1 &= 1 + (1-\alpha)x^2, \quad x < 0 \\ &\quad 1 + 2x + (1+\alpha)x^2, \quad x > 0, \end{aligned}$$

so that $\|\mathbf{f}\|_1$ is minimized when $x = 0$, and the minimum is not strongly unique. At the minimum $f_1(x) = 0$. If we now consider the correction given by the generalised Gauss-Newton method we find two cases:

(i) $x_i < 0$, $x_{i+1} = x_i - \dfrac{x_i + \alpha x_i^2}{1 + 2\alpha x_i} = \dfrac{\alpha x_i^2}{1 + 2\alpha x_i} > 0$, and $r_1(x_{i+1}, x_i) = 0$.

(ii) $x_i > 0$, $x_{i+1} = x_i - \dfrac{1 + x_i + x_i^2}{1 + 2x_i} = \dfrac{-1 + x_i^2}{1 + 2x_i} < 0$, and $r_2(x_{i+1}, x_i) = 0$.

In the second case $\partial \|\mathbf{r}(x_{i+1}, x_i)\|_1 \nsubseteq \partial \|\mathbf{f}(\mathbf{x}^*)\|_1$, but both sets have rank $p+1 = 2$.

Remark. If the inclusion (3.6) does not hold then the proof of Theorem 3.1 shows that the Gauss-Newton method cannot be expected to converge unless modified to include a line search along the direction given by the LSP. One possible approach is indicated in [9].

4. Conclusion

Strong uniqueness is important in nonlinear discrete approximation problems because it guarantees the second order convergence of the generalised Gauss-Newton method. Here we have been able to show that strong uniqueness is a weaker condition than the multiplier condition which has also been used for the same purpose. An alternative formulation of strong uniqueness has been given which emphasises both the local geometry of the unit ball and the structure of the problem functions, and some progress has been made towards a characterisation of strong uniqueness. For example, if the point of minimum norm in (1.1) is strongly unique then the corresponding point of the unit ball determining $\|\cdot\|_1$ cannot be smooth, and the rank of the set of subgradients at this point is at least $p+1$. Also we have shown that the multiplier condition is sufficient, and that uniqueness of the LSP at the optimum is both necessary and sufficient for strong uniqueness in polyhedral norm problems. It would be of interest to characterise the weakest conditions which ensure second order convergence of the generalised Gauss-Newton method as our results in this direction are limited to the polyhedral norm case. In this connection we note that the results in [9] suggest that it is likely to be necessary that the point of minimum norm correspond to a non smooth point on the unit ball.

Axknowledgement. We are grateful to David Anderson for comments on a first draft of this paper and for example 3.1, to Alistair Watson for suggestions and comments, and to the referee for his carful reading of the paper, for suggesting several points where the presentation should be improved, and for several alternative examples.

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Received November 28, 1978/Revised September 12, 1979