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The Brauer Group of a Rational Surface

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Let k be a finite field of characteristic p and let X be an algebraic surface which is projective and smooth over k and which is geometrically connected. Then, motivated by the relation between Brauer groups and Tate-Šafarevič groups, Tate and Artin have conjectured [6]:

(a) the Brauer group, $Br(X)$, of X is finite;

(b) there is a canonical non-degenerate skew-symmetric form on $Br(X)$;

(c) $$P_2(X, q^{-s}) \sim \frac{[Br(X)]\,|\det(D_i \cdot D_j)|\,(1-q^{1-s})^{\rho(X)}}{q^{\alpha(X)}[NS(X)_{\text{tors}}]^2} \quad \text{as } s \to 1,$$

where $[S]$ denotes the order of a set $S, q=[k]$, $\alpha(X)=\chi(X, O_X)-1+\dim(\operatorname{Pic Var}(X))$, $\rho(X)$ is the rank of the Néron-Severi group $NS(X)$ of X, $(D_i)_{1\leqq i\leqq \rho}$ is a basis for $NS(X)$ modulo torsion, and $P_2(X, T)$ is the characteristic polynomial of the endomorphism of $H_l^2(\bar{X}_{\text{et}})$ induced by the Frobenius endomorphism of X.

It has been proved [6] that (a) implies (b) and (c) for the components of $Br(X)$ prime to p, and when X is a product of curves the conjectures have been proved in their entirety [4]. Nevertheless, it may be of interest that for the simplest surfaces, viz. the rational surfaces, the conjectures are an almost trivial consequence of known facts.

Thus, let X be a rational surface over k of the above type, let $\bar{k}$ be the algebraic closure of k, and let k' be a finite extension of k such that $NS(X')=NS(\bar{X})$ where $X'=X\otimes_k k'$ and $\bar{X}=X\otimes_k \bar{k}$. Write Γ and Γ' for the Galois groups of $\bar{k}$ over k and k' respectively and write $\Gamma''=\Gamma/\Gamma'$.

$NS(\bar{X})$ is torsion-free and the pairing $NS(\bar{X})\times NS(\bar{X})\to \mathbf{Z}$ defined by the intersection product has discriminant ± 1. Indeed, both these statements are true for $\mathbf{P}_{\bar{k}}^2$ and their validity is obviously preserved by dilations.

The Brauer group of X is isomorphic to $H^1(\Gamma, NS(\bar{X}))$. This remark is due to Artin and may be proved as follows. The Hochschild-Serre spectral sequence for $\bar{X}_{\text{et}}/X_{\text{et}}$ applied to the sheaf $\mathbf{G}_m$ gives an exact sequence

$$0\to H^1(\Gamma, NS(\bar{X}))\to H^2(X, \mathbf{G}_m)\to H^2(\bar{X}, \mathbf{G}_m).$$

$H^2(\bar{X}, \mathbf{G}_m) = Br(\bar{X}) = 0$ because $Br(\bar{X})$ is birationally invariant [1] and $\bar{X}$ is birationally equivalent to $\mathbf{P}^1_{\bar{k}} \times \mathbf{P}^1_{\bar{k}}$. If $f: \mathbf{P}^1_{\bar{k}} \times \mathbf{P}^1_{\bar{k}} \to \mathbf{P}^1_{\bar{k}}$ denotes a projection onto one of the factors then $R^0 f_* \mathbf{G}_m = \mathbf{G}_m$, $R^1 f_* \mathbf{G}_m = \mathbf{Z}$, and $R^s f_* \mathbf{G}_m = 0$ for $s > 1$. Since $H^r(\mathbf{P}^1_{\bar{k}}, \mathbf{G}_m) = 0$ for $r > 1$ and $H^1(\mathbf{P}^1_{\bar{k}}, \mathbf{Z}) = 0$, the Leray spectral sequence for f shows that $H^2(\mathbf{P}^1_{\bar{k}} \times \mathbf{P}^1_{\bar{k}}, \mathbf{G}_m) = 0$. Hence $H^1(\Gamma, NS(\bar{X})) \approx Br(X)$.

There is an exact sequence

$$0 \to H^1(\Gamma'', NS(X')) \to H^1(\Gamma, NS(\bar{X})) \to H^1(\Gamma', NS(\bar{X})).$$

Γ' acts trivially on $NS(\bar{X})$, and so

$$H^1(\Gamma', NS(\bar{X})) = \text{Conts Hom}(\Gamma', NS(\bar{X})),$$

which is zero because $NS(\bar{X})$ has no finite subgroups. Hence

$$Br(X) \approx H^1(\Gamma, NS(\bar{X})) \approx H^1(\Gamma'', NS(X')).$$

This last group is finite because Γ'' is finite and $NS(X')$ is finitely generated. This proves (a).

$\mathbf{Z}$, regarded as a Γ'' module with trivial action, is a class module for Γ'' in the sense of [2, p. 94]. Since the intersection product induces a natural isomorphism $NS(\bar{X}) \approx \text{Hom}(NS(\bar{X}), \mathbf{Z})$, [2, IV Thm. 14] shows that the cup-product pairing

$$H^1(\Gamma'', NS(\bar{X})) \times H^1(\Gamma'', NS(\bar{X})) \to H^2(\Gamma'', \mathbf{Z}) \approx \mathbf{Z}/n\mathbf{Z} \qquad (n = [\Gamma''])$$

is non-degenerate. This pairing agrees with the pairing on $Br(X)$ (non p) defined in [6]. The general properties of cup-products show that the pairing is skew-symmetric but (pace [6, p. 19]) it need not be alternating and so the order of $Br(X)$ may be twice a square. For examples where $[Br(X)] = 2$, see [3, 3.28]. This completes the proof of (b).

For (c), consider the commutative diagram:

$$\begin{array}{ccc} NS(X) & \xrightarrow{e} & \text{Hom}(NS(X), \mathbf{Z}) \\ \| & & \uparrow g \\ NS(\bar{X})^{\Gamma} & \xrightarrow{f} & NS(\bar{X})/(\sigma - 1)\, NS(\bar{X}) \end{array}$$

where σ is the canonical topological generator of Γ, f is induced by the identity map of $NS(\bar{X})$, and e and g are both induced by the intersection product. We will say that a homomorphism h of $\mathbf{Z}$-modules is a quasi-isomorphism if both $\ker(h)$ and $\text{coker}(h)$ are finite, and in that case we write $z(h) = \dfrac{[\text{coker}(h)]}{[\ker(h)]}$. Lemmas analogous to those on pp. 19, 20 of [6] hold for this definition of z. In particular, $z(e) = |\det(D_i \cdot D_j)|$

where (D_i) is a basis for $NS(X)$. (Notice that, unlike the corresponding determinant for $NS(\bar{X})$, this need not be 1. For example, if X is a non-degenerate del Pezzo surface whose degree d is square-free, then $|\det(D_i \cdot D_j)| = (\omega_X \cdot \omega_X) = d$.)

Consider the pairing

$$NS(\bar{X}) \times NS(X) \to \mathbf{Z}$$

defined by the intersection product. Suppose $D \in NS(\bar{X})$ is such that $ND = \sum_{i=0}^{n-1} \sigma^i D = 0$ (where $n = [\Gamma'']$, so $\Gamma'' = \{1, \bar{\sigma}, \ldots, \bar{\sigma}^{n-1}\}$). Then, for any $E \in NS(X)$, $n(D \cdot E) = \sum_{i=0}^{n-1} (D \cdot \sigma^{-i} E) = (ND \cdot E) = 0$.

Hence $(D \cdot E) = 0$. Conversely, if $(D \cdot E) = 0$ for all $E \in NS(X)$ then $(ND \cdot E) = n(D \cdot E) = 0$ for all E, and since $ND \in NS(X)$, this implies that $ND = 0$. This shows that the kernel of g is $\ker(N: NS(\bar{X}) \to NS(\bar{X}))/(\sigma - 1)\, NS(\bar{X}) = H^1(\Gamma'', NS(\bar{X}))$. Since g is obviously surjective, we find that $z(g) = [Br(X)]^{-1}$.

The étale cohomology sequence of

$$0 \to \mu_{l^n} \to \mathbf{G}_m \xrightarrow{l^n} \mathbf{G}_m \to 0 \qquad (l \neq p)$$

gives an isomorphism $NS(\bar{X})/l^n NS(\bar{X}) \approx H^2(\bar{X}, \mu_{l^n})$. Hence

$$NS(\bar{X}) \otimes \mathbf{Z}_l \approx \varprojlim H^2(X, \mu_{l^n}),$$

and $NS(\bar{X}) \otimes \mathbf{Q}_l \approx H_l^2(\bar{X})(1)$ in the notation of [5]. Thus [5, p. 101] if σ_2 is the automorphism of $NS(\bar{X}) \otimes \mathbf{Q}_l$ induced by σ then $\det(1 - \sigma_2 T) = P_2(X, q^{-1} T)$ (see also [7]). g and e both being quasi-isomorphisms imply that f is a quasi-isomorphism. Thus, by the analogue of [6, z. 4], if θ is the map $\sigma - 1$: $NS(\bar{X}) \to NS(\bar{X})$, then $\det(T - \theta \otimes 1) = T^\rho R(T)$ where $\rho = \operatorname{rank}(NS(X))$. Also, $z(f) = R(0) = \Pi \left(1 - \frac{\alpha_i}{q}\right)$ where the α_i are the roots of $P_2(X, T)$ which are not equal to q. Now the equality $z(f)\, z(g) = z(e)$ shows that

$$P_2(X, q^{-s}) \sim [Br(X)]\, |\det(D_i \cdot D_j)|\, (1 - q^{1-s})^{\rho(X)} \qquad \text{as } s \to 1.$$

This implies (c) because in this case $\alpha(X) = 1 - 1 + 0 = 0$.

Example. Let k contain the cube roots of 1 and have characteristic $\neq 3$, and let a be an element of k which is not a cube in k. Then

$$X: Z_0^3 + Z_1^3 + Z_2^3 = a Z_3^3$$

is a rational surface which over $k' = k(\sqrt[3]{a})$, becomes isomorphic to $\mathbf{P}_k^2$, with 6 points blown up. Moreover, $NS(X)$ has rank 1. It follows (using that $NS(\bar{X})$ has rank 7 and that $\Gamma'' \approx \mathbf{Z}/3\mathbf{Z}$ has only one non-trivial representation over $\mathbf{Q}$) that

$$P_2(X, T) = (1 - qT)(1 - \rho q T)^3 (1 - \rho^2 q T)^3$$

where ρ is a primitive cube root of 1. Hence

$$P_2(X, q^{-s}) \sim 27(1 - q^{1-s}) \quad \text{as } s \to 1.$$

By Noether's formula $(\omega_X \cdot \omega_X) + \operatorname{rank}(NS(\bar{X})) = 10$, and so $(\omega_X \cdot \omega_X) = 3$. It follows that ω_X generates $NS(X)$ and that $[Br(X)] = 9$. Because of the self-duality of $Br(X)$, this implies that

$$Br(X) \approx \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}.$$

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