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Approximating Derivations on Ideals of C^*-Algebras

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Abstract. For each *-derivation δ of a separable C^*-algebra A and each $\varepsilon>0$ there is an essential ideal I of A and a self-adjoint multiplier x of I such that $\|(\delta-\operatorname{ad}(ix))|I\|<\varepsilon$ and $\|x\|\leqq\|\delta\|$.

Introduction

The best understood derivations on a C^*-algebra A have the form $\operatorname{ad}(h)$, where $h\in A$ or, slightly more general, where $h\in M(A)$—the multiplier algebra of A. Even though not all derivations have this form, the multiplier derivations are so attractive that one may naturally ask whether some weakening of the multiplier condition could lead to a general result. For example, if δ is a derivation of A, does there exist a closed ideal I of A such that $\delta|I$ is given by a multiplier (of I) or such that δ/I is given by a multiplier (of A/I)? Unfortunately, the answer to both these questions is negative, see [1, Example 6.5] and [13].

We show in this paper that if δ is a derivation of a separable C^*-algebra A, there is for each $\varepsilon>0$ an essential ideal I of A (i.e. I intersects every non-zero ideal of A) and an element x in $M(I)$ such that $\|(\delta-\operatorname{ad}(x))|I\|\leqq\varepsilon$. Moreover, if $\delta^*=-\delta$ (or $\delta^*=\delta$) we can choose $x=x^*$ (or $x=-x^*$) such that $\|x\|\leqq\|\delta\|$ (whence $\|\operatorname{ad}(x)\|\leqq 2\|\delta\|$). A similar result was obtained by Elliott in [5, Theorem] for the case of a separable approximately finite-dimensional C^*-algebra A. The ideal there is not shown to be essential and the bound on $\|x\|$ is $248\|\delta\|$, but, more important, Elliott finds an x in $M(I)$ that derives A. This can probably also be obtained in the general case, but one will have to generalize [4, Theorem 3.3] to arbitrary (separable) C^*-algebras.

Recall that an element b in A_+ is strictly positive if $\varphi(b)>0$ for every state φ of A. The existence of a strictly positive element in A is equivalent with the existence of a countable approximate unit for A (contained in $C^*(b)$). Thus every separable C^*-algebra has strictly positive elements. If A'' denotes the enveloping von Neumann algebra of A we denote by $M(A)$ the elements x in A'' such that $xA\subset A$ and $Ax\subset A$. If b is strictly positive in A then $x\in M(A)$ if only $xb\in A$ and $bx\in A$ by [3, Proposition 2.6].

Main Results

1. Lemma. Let B be a hereditary C^*-subalgebra of a C^*-algebra A and assume that no non-trivial closed ideal of A contains B. For each x in A_+ there is a sequence (y_n) in A such that $x=\sum y_n^* y_n$, $\sum \|y_n\| < \infty$ and $y_n y_n^* \in B$ for every n.

Proof. Consider the set

$$J=\{x\in A_+ \mid \exists\, y_1, \ldots, y_n \in A,\ x=\sum y_k^* y_k,\ y_k y_k^* \in B\}.$$

Since B_+ is a hereditary cone in A_+, it follows from [9, Corollary 1.2] that J is a strongly invariant hereditary cone in A_+. Consequently, the norm closure of J is the positive part of a closed ideal in A, and since $B_+ \subset J$ it follows from our assumption that J is dense in A_+.

Let $\tilde{A}$ denote the C^*-algebra obtained by adjoining a unit to A. If $x\in A_+$ and $\varepsilon>0$ there is by the first part of the proof a finite set (y_n) in A, with $y_n y_n^* \in B$ for all n, such that $\|x-\sum y_n^* y_n\| \leqq \varepsilon$. But then

$$x \leqq \sum y_n^* y_n + \varepsilon 1,$$

whence, again by [9, Corollary 1.2], we have a decomposition

$$x=\sum z_n^* z_n + z_0^* z_0, \qquad z_n z_n^* \leqq y_n y_n^*,\ z_0 z_0^* \leqq \varepsilon 1.$$

It follows that $z_n z_n^* \in B$ and that $\|x-\sum z_n^* z_n\| \leqq \varepsilon$. Repeating the argument with $x-\sum z_n^* z_n$ in place of x, and continuing by induction (say with $\varepsilon=2^{-n}$) we obtain the lemma.

2. Lemma. Let δ be a derivation of a C^*-algebra A having a strictly positive element b. There is then an approximate unit (u_n) for A contained in $C^*(b)$ such that for every n we have

$$u_{n+1} u_n = u_n, \qquad \|(1-u_n)b\| < 2^{-n}, \qquad \|\delta(u_n)\| < 2^{-n}.$$

Proof. Let K denote the convex set of monotone increasing continuous functions f on R_+ such that $f(t)=0$ for all $t \leqq t_0$ for some $t_0>0$, and $f(t)=1$ for all $t \geqq t_1$ for some $t_1>t_0$. Note that K is also a net (the partial ordering being the ordering of functions) and converges to the characteristic function for the set $\mathsf{R}_+ \setminus \{0\}$. Since b is strictly positive it follows that the net $\{f(b) \mid f\in K\}$ is an approximate unit for A and converges to 1 σ-weakly in A''. Extending δ to a σ-weakly continuous derivation of A'', this implies that for each f_0 in K the set

$$E=\{\delta(f(b)) \mid f\in K, f \geqq f_0\}$$

contains 0 as a σ-weak limit point. Since E is convex (δ is linear and K is convex), and the σ-weak topology on A is the $\sigma(A, A^*)$-topology, it follows from the Hahn-Banach theorem that E contains 0 as a limit point in norm. Thus with $u_n=f_n(b)$, for a suitable sequential subnet (f_n) in K, we find by induction an approximate unit (u_n) satisfying the three conditions in the lemma.

1. Proposition. Let A be a C^*-algebra containing a strictly positive element, and let B be a hereditary C^*-subalgebra of A not contained in any non-trivial closed ideal of A. Assume that δ is a skew-adjoint derivation of A such that $\delta(B) \subset B$. There is then a derivation δ' of A of the form $\delta' = \mathrm{ad}(x)$, where $x \in M(A)_{sa}$, $\|x\| \leqq \|\delta\|$ and $\|\delta - \delta'\| \leqq \|\delta | B\|$.

Proof. Let b be a strictly positive element in A, and choose an approximate unit (u_n) in $C^*(b)$ satisfying the conditions in Lemma 2. Let $e_n = u_n - u_{n-1}$ (with $u_0 = 0$), so that $\sum e_n = 1$. Moreover, $e_n e_m = 0$ if $|n-m| > 1$, and

$$\|e_n b\| \leqq 2^{-n+1}, \qquad \|\delta(e_n)\| \leqq 2^{-n+1}$$

for all n.

For each n, choose by Lemma 1 a sequence (y_{nk}) in A with $\sum y_{nk}^* y_{nk} = e_n$, $\sum \|y_{nk}\| < \infty$ and $y_{nk} y_{nk}^* \in B$ for every k. Define

$$x_n = \sum \delta(y_{nk}^*) y_{nk}, \qquad x_n' = \sum y_{nk}^* \delta(y_{nk}),$$

and note that $x_n, x_n' \in A$. Moreover, $x_n + x_n' = \delta(e_n)$, whence $\|x_n + x_n'\| \leqq 2^{-n+1}$.

Choose by [7, Corollary 3.5.8] an element h in A''_+ such that $\delta = \mathrm{ad}(h)$ and $\|\delta\| = \|h\|$. Then

$$x_n = \sum (h y_{nk}^* - y_{nk}^* h) y_{nk} = h e_n - \sum y_{nk}^* h y_{nk}. \qquad (*)$$

The last term is positive and dominated by $\|h\| \sum y_{nk}^* y_{nk} = \|h\| e_n$, from which we conclude that $\|x_n\| \leqq 2\|\delta\|$. Similarly, $\|x_n'\| \leqq 2\|\delta\|$. But $(*)$ also shows that $x = \sum x_n \in A''$, with

$$x = \sum_n h e_n - \sum_{n,k} y_{nk}^* h y_{nk} = h - \sum y_{nk}^* h y_{nk}.$$

We see that $x = x^*$, and that x is the difference between two positive elements, both dominated by $\|h\|$, and thus $\|x\| \leqq \|h\| \, (= \|\delta\|)$.

To prove that $x \in M(A)$ if suffices to show that $xb \in A$. To this end consider the partial sum

$$s_{ml} = \sum_{n=m}^{l} x_{2n} b$$

(for $m < l$). We have

$$s_{ml} s_{ml}^* = \sum_{n=m}^{l} \sum_{k=m}^{l} x_{2n} b^2 x_{2k}^* = \sum_{n=m}^{l} x_{2n} b^2 x_{2n}^*,$$

because if $n \neq k$ then

$$x_{2n} b^2 x_{2k}^* = \sum_{i,j} \delta(y_{2n,i}^*) y_{2n,i} b^2 y_{2k,j}^* \delta(y_{2k,j}^*)^* = 0,$$

since $y_{2n,i}^* y_{2n,i} \leqq e_{2n}$, $y_{2k,j}^* y_{2k,j} \leqq e_{2k}$, and $e_{2n} b^2 e_{2k} = 0$. Using that $\|x_n - x_n'\| < 2^{-n+1}$, this gives

$$\|s_{ml}\|^2 \leqq \left\| \sum_{n=m}^{l} x'_{2n} b^2 x'^*_{2n} \right\| + 4\|\delta\| \sum_{n=m}^{l} 2^{-2n+1}$$
$$\leqq \operatorname{Max}_{m \leqq n \leqq l} \|x'_{2n} b^2 x'^*_{2n}\| + 8\|\delta\| 4^{-m+1},$$

because the elements $x'_{2n} b^2 x'^*_{2n}$ are pairwise orthogonal (each is dominated by a multiple of e_{2n}). Finally, $x_{2n} = x_{2n}(1 - u_{2n-2})$, whence

$$\|x'_{2n} b\| \leqq 2^{-2n+1} + \|x_{2n} b\| \leqq 2^{-2n+1} + 2\|\delta\| 2^{-2n+2}.$$

Combining these inequalities we obtain

$$\|s_{ml}\|^2 \leqq (2^{-2m+1} + 2\|\delta\| 2^{-2m+2})^2 + 8\|\delta\| 4^{-m+1},$$

which shows that $\sum x_{2n} b \in A$. In the exact same manner we show that $\sum x_{2n-1} b \in A$, whence by summation $xb \in A$, i.e. $x \in M(A)$.

Take, again by [7, Corollary 3.5.8], an element k in B''_+ ($\subset A''$) such that $\delta|B = \operatorname{ad} k$ and $\|k\| = \|\delta|B\|$. Set $y = \sum_{n,k} y^*_{nk} k y_{nk}$, and note that $y \in A''_+$ with $\|y\| \leqq \|k\|$. For each a in A we then use the fact that $\sum_{n,k} y^*_{nk} y_{nk} = 1$ to compute

$$\delta(a) = \delta\Big(\sum_{n,k,m,l} y^*_{nk} y_{nk} a y^*_{ml} y_{ml}\Big)$$
$$= \sum_{n,k,m,l} \delta(y^*_{nk}) y_{nk} a y^*_{ml} y_{ml} + y^*_{nk} \delta(y_{nk} a y^*_{ml}) y_{ml} + y^*_{nk} y_{nk} a y^*_{ml} \delta(y_{ml})$$
$$= xa + \sum_{n,k,m,l} y^*_{nk}(k y_{nk} a y^*_{ml} - y_{nk} a y^*_{ml} k) y_{ml} - ax$$
$$= xa + ya - ay - ax = \operatorname{ad}(x+y)(a).$$

Consequently, with $\delta' = \operatorname{ad}(x)$ we have

$$\|\delta - \delta'\| = \|\operatorname{ad}(y)\| \leqq \|y\| \leqq \|\delta|B\|,$$

as desired.

1. Theorem. Let δ be a derivation of a separable C^*-algebra A. For each $\varepsilon > 0$ there is an essential ideal I of A and an element x in $M(I)$ such that $\|(\delta - \operatorname{ad}(x))|I\| \leqq \varepsilon$. If δ is skew-adjoint we can choose $x = x^*$ with $\|x\| \leqq \|\delta\|$.

Proof. We may assume that δ is skew-adjoint and that $\|\delta\| = 1$. Let h be the minimal positive generator for δ in A'' (cf. [7, Corollary 3.5.8]). We know from [8, Theorem 2.1] that h is a lower semi-continuous element in A''_+. This result is obtained by showing that each spectral projection of h, corresponding to an interval $[0, t]$, is a closed projection (namely the annihilator of $M_\alpha(t, \infty)$). Its complement, denoted by $q(t)$, is therefore open, i.e. there exists a hereditary C^*-subalgebra $B(t)$ of A such that $q(t)$ is the strong limit of an approximate unite for $B(t)$. Moreover, $B(t) = q(t) A'' q(t) \cap A$. (In the case at hand $B(t) = R(t) \cap R(t)^*$, where the right ideal $R(t)$ is the closure of $M_\alpha(t, \infty) A$.)

Let $q_1 = q(1-\varepsilon)$ and $B_1 = B(1-\varepsilon)$; and let I_1 denote the closed ideal generated by B_1. Clearly $\delta(B_1) \subset B_1$ since q_1 is a spectral projection for h. Moreover,

$$\delta|B_1 = \mathrm{ad}(q_1 h), \quad (1-\varepsilon)q_1 \leqq q_1 h \leqq q_1;$$

so that $\|\delta|B_1\| \leqq \varepsilon$. Since $\|\delta\| = 1$ we have $q_1 \neq 0$, whence $B_1 \neq 0$ and $I_1 \neq 0$. Choose by Zorn's lemma a closed ideal A_1 of A which is maximal with respect to the property $A_1 \cap I_1 = \{0\}$. Then $A_1 + I_1$ is an essential ideal in A and

$$\delta|A_1 = \mathrm{ad}((1-q_1)h)|A_1,$$

whence $\|\delta|A_1\| \leqq 1-\varepsilon$. Repeating the argument above with A_1 in place of A we find a non-zero hereditary C^*-subalgebra B_2 of A_1 with $\delta(B_2) \subset B_2$ and $\|\delta|B_2\| \leqq \varepsilon$. Moreover, if I_2 is the closed ideal generated by B_2, and if A_2 is a closed ideal in A_1 orthogonal to I_2 such that $I_2 + A_2$ is essential in A_1, then

$$\|\delta|A_2\| \leqq \|\delta|A_1\| - \varepsilon.$$

Since $\varepsilon > 0$ this process must eventually terminate, and we are left with δ-invariant hereditary C^*-subalgebras $B_1, B_2, \ldots, B_n$, generating pairwise orthogonal ideals $I_1, I_2, \ldots, I_n$ such that $\|\delta|B_k\| \leqq \varepsilon$ for each k and $I_1 + I_2 + \cdots + I_n$ is essential in A.

Let $B = B_1 + B_2 + \cdots + B_n$ and $I = I_1 + I_2 + \cdots + I_n$. Then $\delta(B) \subset B$ and $\|\delta|B\| \leqq \varepsilon$. Applying Proposition 1 there is then an element x in $M(I)_{sa}$ with $\|x\| \leqq \|\delta\|$ such that $\|(\delta - \mathrm{ad}(x))|I\| \leqq \varepsilon$. Since I is essential in A the theorem follows.

Essential Multipliers

Given a C^*-algebra A let $\{I_\lambda | \lambda \in \Lambda\}$ be the net, partially ordered under inclusion ($\lambda \prec \mu$ if $I_\lambda \supset I_\mu$), of essential ideals in A. (Note that if I and J are essential then so is $I \cap J$.) If $\lambda \prec \mu$ we have the inclusions

$$I_\mu \subset I_\lambda \subset A \subset M(A) \subset M(I_\lambda) \subset M(I_\mu)$$

by [10, Proposition 2.6]. We define $M^\infty(A)$ as the inductive limit of the system $\{M(I_\lambda) | \lambda \in \Lambda\}$, and shall refer to it as the C^*-algebra of essential multipliers.

We invite the reader to verify the following statements:

(1) $M^\infty(A)$ is non-separable unless A is separable, simple and has a unit (whence $M^\infty(A) = A$).

(2) If I is a minimal ideal in A then $M^\infty(A) = M(I)$.

(3) If I is an essential ideal in A then $M^\infty(A) = M^\infty(I)$.

(4) If I is not essential in A then $M^\infty(I)$ is a direct summand of $M^\infty(A)$.

(5) Each non-zero ideal in $M^\infty(A)$ has a non-zero intersection with A.

(6) If A is prime then $M^\infty(A)$ is prime.

(7) If A has a faithful factorial representation π then it extends to a faithful representation of $M^\infty(A)$ such that $\pi(M^\infty(A)) \subset \pi(A)''$.

(8) If $A = C([0,1])$ then $M^\infty(A)$ can be identified with the algebra of bounded functions on $[0,1]$ that are continuous except on a set of first category, modulo the ideal of functions that are zero except on a set of first category.

2. Proposition. (cf. [6, 2.4]). Each derivation δ of a separable C^*-algebra A extends to an inner derivation $\bar{\delta}$ of $M^\infty(A)$. Moreover, if δ is skew-adjoint we can for each $\varepsilon>0$ choose x in $M^\infty(A)_{sa}$ with $\bar{\delta}=\mathrm{ad}(x)$ such that

$$\|x\| \leqq \|\delta\| + \varepsilon.$$

Proof. Each derivation of a C^*-algebra B has a unique (norm-preserving) extension to $M(B)$ by [1, Lemma 6.1]. It follows that each derivation of an essential ideal of A has a unique (norm-preserving) extension to $M^\infty(A)$.

We may assume that δ is skew-adjoint and that $\|\delta\|=1$. Applying Theorem 1 and proceeding by induction we obtain a decreasing sequence (I_n) of essential ideals (I_{n+1} is essential in I_n and therefore essential in A), and a sequence of self-adjoint multipliers (x_n) (i.e. $x_n \in M(I_n)$ for each n), such that (with $I_0=A$ and $x_0=0$)

$$\left\| \left(\delta - \sum_{k=1}^{n} \mathrm{ad}(x_k)\right) \middle| I_n \right\| < 2^{-n}\varepsilon; \tag{$*$}$$

$$\|x_n\| \leqq \left\| \left(\delta - \sum_{k=0}^{n-1} \mathrm{ad}(x_k)\right) \middle| I_{n-1} \right\|. \tag{$**$}$$

If follows from $(*)$ and $(**)$ that $\|x_1\| \leqq 1$ and $\|x_{n+1}\| \leqq 2^{-n}\varepsilon$, so that we can define $x=\sum x_n$ in $M^\infty(A)$ with $\|x\| \leqq 1+\varepsilon$. By $(*)$ we have

$$\|(\delta-\mathrm{ad}(x))|I_n\| \leqq 2^{-n}\varepsilon + 2\sum_{k>n} \|x_k\| \leqq 6\cdot 2^{-n}\varepsilon.$$

With $\bar{\delta}$ as the unique extension of δ to $M^\infty(A)$ we must therefore have $\|\bar{\delta}-\mathrm{ad}(x)\| \leqq 6\cdot 2^{-n}\varepsilon$, whence $\bar{\delta}=\mathrm{ad}(x)$, as desired.

Let A be a primitive C^*-algebra which is anti-liminary (otherwise A has a minimal ideal), and assume that A acts irreducibly on some Hilbert space H. From (7) we know that $M^\infty(A) \subset B(H)$. Note that $M^\infty(A) \neq B(H)$; in fact $M^\infty(A) \cap C(H)=\{0\}$, since otherwise $A \cap C(H) \neq \{0\}$ by (5). It is possible, but probably not true in general, that $M^\infty(A)$ is simple; but if not we define a well-ordered chain of C^*-algebras $M^\infty_\alpha(A)$, increasing with α, such that

$$M^\infty_\alpha(A) = M^\infty(M^\infty_{\alpha-1}(A))$$

if α is not a limit ordinal, and

$$M^\infty_\alpha(A) = \Big(\bigcup_{\beta<\alpha} M^\infty_\beta(A)\Big)^-$$

if α is a limit ordinal. Let ω be the first ordinal for which $M^\infty_\omega(A)=M^\infty_{\omega+1}(A)$ and put $M^\infty_\omega(A)=M^\omega(A)$. Note that $M^\omega(A) \cap C(H)=\{0\}$ since the assumption that there is a first α with $M^\infty_\alpha(A) \cap C(H) \neq \{0\}$ is not compatible with (5).

The C^*-algebra $M^\omega(A)$ has the property that it contains every multiplier of any non-zero ideal in $M^\omega(A)$. If therefore Theorem 1 could be extended to cover some reasonable non-separable cases (it holds for any C^*-algebra for which every (essential) ideal contains a strictly positive element), we could use Proposition 2 to conclude that every derivation of $M^{\omega}(A)$ is inner. As it stands there is no obvious advantage in using $M^{\omega}(A)$ instead of $M^{\infty}(A)$.

Note that by Proposition 2 both $M^{\infty}(A)$ and (in the case of a primitive algebra) $M^{\omega}(A)$ is a deriving algebra in the sense of Sakai (see [11] and [12]).

References

1. Akemann, C.A., Elliott, G.A., Pedersen, G.K., Tomiyama, J.: Derivations and multipliers of C^*-algebras. Amer. J. Math. **98**, 679–708 (1976)
2. Akemann, C.A., Pedersen, G.K.: Complications of semi-continuity in C^*-algebra theory. Duke Math. J. **40**, 785–795 (1973)
3. Akemann, C.A., Pedersen, G.K., Tomiyama, J.: Multipliers of C^*-algebras. J. Functional Analysis **13**, 277–301 (1973)
4. Elliott, G.A.: On lifting and extending derivations of approximately finite-dimensional C^*-algebras. J. Functional Analysis **17**, 395–408 (1974)
5. Elliott, G.A.: Derivations determined by multipliers on ideals of a C^*-algebra. Publ. R.I.M.S. Kyoto Univ. **10**, 721–728 (1975)
6. Elliott, G.A.: Automorphisms determined by multipliers on ideals of a C^*-algebra. J. Functional Analysis **23**, 1–10 (1976)
7. Olesen, D.: On spectral subspaces and their applications to automorphism groups. Symposia Math. **20**, 253–296. Bologna 1976
8. Olesen, D., Pedersen, G.K.: Derivations of C^*-algebras have semi-continuous generators. Pacific J. Math. **53**, 563–572 (1974)
9. Pedersen, G.K.: Measure theory for C^*-algebras III. Math. Scand. **25**, 71–93 (1969)
10. Pedersen, G.K.: Applications of weak* semicontinuity in C^*-algebra theory. Duke Math. J. **39**, 431–450 (1972)
11. Sakai, S.: Derived C^*-algebras of primitive C^*-algebras. Tôhoku Math. J. **25**, 307–316 (1973)
12. Tomiyama, J.: Derived algebras of C^*-algebras. C^*-algebras and their applications to statistical mechanics and quantum field theory, 147–153. Edited by D. Kastler. Amsterdam: North-Holland Publ. Comp. 1976
13. Tomiyama, J.: Derivations of C^*-algebras which are not determined by multipliers in any quotient algebra. Proc. Amer. Math. Soc. **47**, 265–267 (1975)

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