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Almost Every Curve in $\mathbf{R}^3$ Bounds a Unique Area Minimizing Surface

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1. Introduction

Every Lipschitz simple closed curve in $\mathbf{R}^n$ bounds an *area minimizing surface*, that is, a surface S such that any other surface S' with the same boundary satisfies

$$\operatorname{area} S' \geqq \operatorname{area} S.$$

Curiously enough, such a surface need not be unique.

Examples of this nonuniqueness abound. Nitsche [17] thoroughly develops a family of examples by taking intersections of Enneper's minimal surface (Fig. 1) with ellipsoids $x^2+y^2+\frac{2}{3}z^2=a^2$. (See Fig. 2.)

The intersection with small ellipsoids is nearly planar (Fig. 2b), and the enclosed portion of Enneper's surface gives the *unique* area minimizing surface. As the ellipsoids become larger, eventually (Fig. 2f) the enclosed portion of Enneper's surface is no longer area minimizing and there are at least two different area minimizing surfaces, which presumably look like Figure 3.

This example provides a one parameter family of nonsimilar curves bounding more than one area minimizing surface. By symmetrically adding small, smooth bumps to these examples, one can obtain a large space of curves bounding more than one area minimizing surface, a space in some sense of the same dimension as the entire space of curves we shall consider in this paper. Nevertheless, we shall prove that the probability of picking such a curve at random is zero.

Nitsche [16, pp. 396–398] refers to many other examples. The author [13] gives an example of an analytic curve in $\mathbf{R}^4$ that bounds a whole continuum of distinct area minimizing surfaces. (See also Fleming [9], Lévy [12, p. 29], Courant [5, pp. 119–122].)

1.1. Previous Results on Uniqueness

Uniqueness for particular boundary curves in $\mathbf{R}^3$ is known only in a couple of special cases:

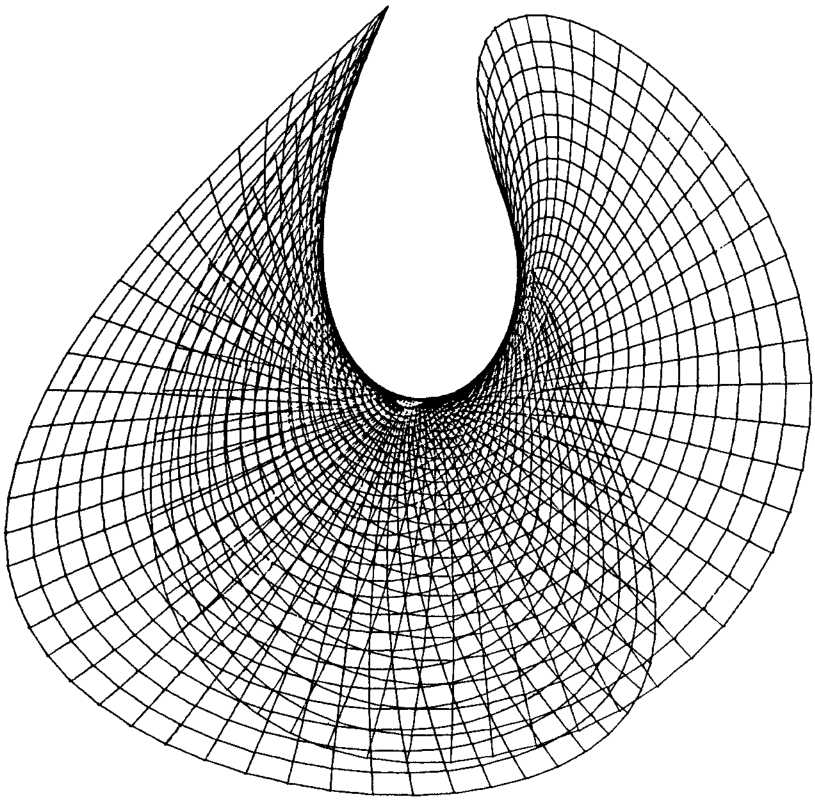

Fig. 1. Enneper's minimal surface

(1) *A Jordan curve whose orthogonal projection on some plane is a simply covered convex curve bounds a unique area minimizing surface* (for various notions of surface in both geometric measure theoretic and mapping contexts). (Rado, 1932, [19, p. 36]; see also [7, 5.4.18].)

(2) *An analytic simple closed curve with total curvature at most* 4π *bounds a geometrically unique immersed disc of least mapping area.* (Nitsche, 1973, [18].)

It seems to be true (as has been recognized for several years, although apparently no one has written a proof) that there is an open dense set of C^2 simple closed curves (under the C^2 norm), such that each curve bounds a unique area minimizing surface. Our own proof incidentally shows that the set of $C^{2,\alpha}$ curves bounding more than one area minimizing surface is a set of the first category (Remark 7.12). But even on the real line there are open dense sets of arbitrarily small Lebesgue measure. We want to say more: we will prove *with respect to a genuine geometrically natural measure* that almost every curve bounds a unique area minimizing surface.

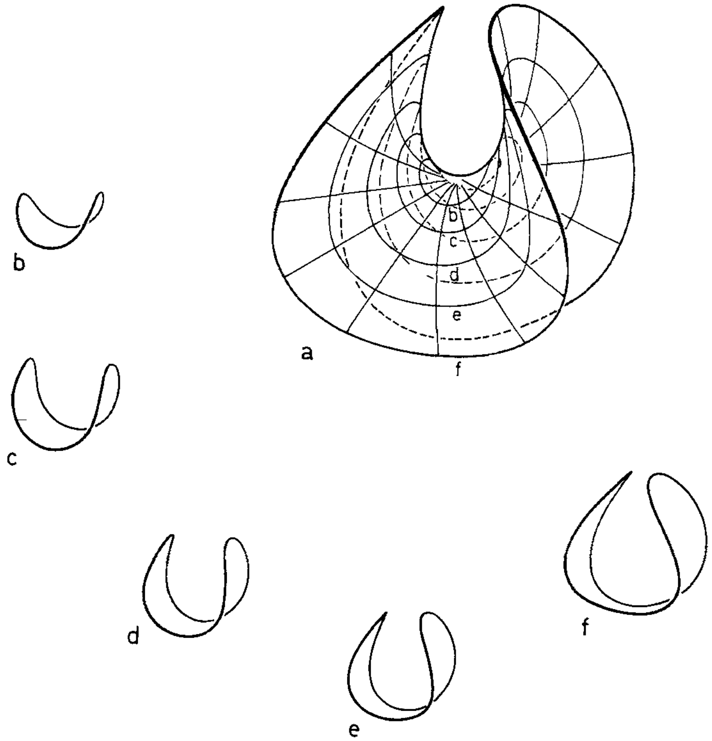

Fig. 2. Shapes of curves given by intersections of Enneper's Surface with ellipsoids

1.2. The Theorem. We study this problem of uniqueness for area minimizing surfaces in the context of geometric measure theory. Our analysis holds equally well for oriented surfaces in the sense of *integral currents* or unoriented surfaces in the sense of *flat chains modulo two.*

To study curves in $\mathbf{R}^3$, it is convenient and equivalent to study the space of parameterizations

$$\mathscr{C} = \{C^{2,\alpha} \text{ maps: } \mathbf{S}^1 \to \mathbf{R}^3\}$$

where $\mathbf{S}^1$ is the circle $\mathbf{R}/2\pi\mathbf{Z}$ and α is a fixed positive number less than one half. We endow $\mathscr{C}$ with the C^2 norm:

$$\|B\| = \max\{\|B\|_\infty, \|B'\|_\infty, \|B''\|_\infty\}.$$

To define a measure on $\mathscr{C}$, we put a measure on the set of formal series

$$B_0 - \sum_{n=1}^{\infty} B_n n^{-3} \cos nt + B_{-n} n^{-3} \sin nt \qquad (B_n \in \mathbf{R}^3)$$

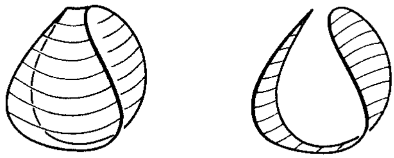

Fig. 3. Two area minimizing surfaces with the same boundary

by giving the coefficients B_n independent Gaussian distributions with mean zero and variance one (except for B_0, which is given a uniform distribution). It follows from some work of G. Hunt [10] that almost every such series converges uniformly to an element of $\mathscr{C}$. The resulting measure μ on $\mathscr{C}$ is very closely related to Brownian motion (see Theorem 4.10) and has many other nice properties. It is invariant under Euclidean motions, and open balls in the C^2 norm on $\mathscr{C}$ are measurable and have positive measure.

We can now state the theorem (7.1, 7.9, 7.10).

Theorem. *Almost every $B \in \mathscr{C}$ bounds a unique area minimizing surface.*

1.3. The Proof. There are three main steps to the proof, the third of which employs a rather novel generalization of a standard density argument.

(1) P.D.E. Lemma (Theorem 5.1). *Two area minimizing surfaces with the same boundary which are tangent along an interval of boundary are equal.*

(2) Geometric Lemma (Theorem 6.7). Here we confine ourselves to a compact set of curves and surfaces and a fixed boundary interval $J \subset \mathbf{S}^1$. *If the tangents to two area minimizing surfaces with the same boundary are close together on J, then the surfaces are close together.*

(3) Density Argument (cf. Remark 7.2). The density of a certain bad set of curves is less than one at all points of the set, and therefore the set has measure zero.

The conclusion of step (3) resembles the standard measure theoretic result for $Z \subset \mathbf{R}^k$ that

(4) *if for almost all $a \in Z$,*

$$\varliminf_{r \to 0} \frac{\mathscr{L}^k(Z \cap \mathbf{B}(a,r))}{\mathscr{L}^k(\mathbf{B}(a,r))} < 1,$$

then $\mathscr{L}^k(Z) = 0$.

But instead of Lebesgue measure $\mathscr{L}^k$ on k dimensional Euclidean space and ordinary balls, we are dealing with a measure μ on the infinite dimensional space of curves $\mathscr{C}$ and balls in the C^2 norm:

$$\mathbf{B}(B_0, r) = \{B \in \mathscr{C} \colon \|B - B_0\| \leqq r\}.$$

One could still apply (4) if, for example, the measure space $(\mathscr{C}, \mu)$ were essentially finite dimensional, in the sense that for some positive integer n, given $B \in \mathscr{C}$, there were positive constants k_0, k_1, such that for small positive r,

(5) $k_0 r^n \leqq \mu(\mathbf{B}(B, r)) \leqq k_1 r^n$.

Although (5) fails for μ, one can decompose $\mathscr{C}$ as a product

$$(\mathscr{C}, \mu) = (\mathscr{C}_N, \mu_N) \times (\mathscr{C}^N, \mu^N)$$

where $\mathscr{C}_N$ is the finite dimensional space of trigonometric polynomials of degree at most N and $\mathscr{C}^N$ is the infinite dimensional subspace of $\mathscr{C}$ consisting of curves whose first $2N+1$ Fourier coefficients vanish. Since μ_N satisfies (5), we can apply (4) to prove that bad subsets of $\mathscr{C}_N$ have μ_N measure zero and then apply Fubini's Theorem to extend the result to $(\mathscr{C}, \mu)$.

Thus the proof boils down to verifying the first part of (3). The basic idea, which depends on steps (1) and (2) and exposes the heart of the theorem, is described in Remark 7.2.

The proof of (1) employs in a new way the old Legendre transformation, which linearizes the minimal surface equation. Subtleties arise from the dependence of the transformation itself on the particular minimal surface.

The proof of (2) involves many uniform estimates on the compact set of curves and surfaces and depends heavily on Allard's boundary regularity results [2].

This theorem seems to be the first application of probability theory to geometric measure theory.

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2. Preliminaries

In general we follow the notation of Federer's treatise [7] and Allard's paper [2].

2.1. Linear Spaces. Let X be a linear space with norm $\| \ \|$. For $a \in X$, $r \in \mathbf{R}$, put

$$\mathbf{B}(a,r) = \{x \in X \colon \|x-a\| \leqq r\}$$

$$\mathbf{U}(a,r) = \{x \in X \colon \|x-a\| < r\}.$$

If $X = \mathbf{R}^n$, we sometimes write $\mathbf{B}^n(a,r)$, $\mathbf{U}^n(a,r)$.

If $x \in X$ and $A \subset X$, we put

$$\operatorname{dist}(A,x) = \inf_{a \in A} \|x-a\|.$$

We define the *Hausdorff metric* on the set of compact subsets of X by

$$\operatorname{HM}(C,D) = \max\{\operatorname{dist}(C,d), \operatorname{dist}(D,c) \colon c \in C, d \in D\}.$$

For $r \in \mathbf{R}$, we define the *homothety*

$$\boldsymbol{\mu}_r \colon X \to X$$

$$\boldsymbol{\mu}_r \colon x \mapsto rx.$$

For $a \in X$, we define the *translation*

$$\tau_a: X \to X$$
$$\tau_a: x \mapsto x + a.$$

Linear subspaces. We will identify a linear subspace of $\mathbf{R}^n$ with the element of $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ which orthogonally projects $\mathbf{R}^n$ onto that subspace.

Tangent cones. If $A \subset X$, $x \in X$, we have the tangent cone of A at x, denoted

$$\mathrm{Tan}(A, x)$$

consisting of all $v \in X$ such that for every $\varepsilon > 0$ there exist

$$a \in A, r > 0 \quad \text{with} \quad \|a - x\| < \varepsilon, \|r(a - x) - v\| < \varepsilon.$$

Smoothness. By C^k we mean of class k, i.e., k times continuously differentiable. For $0 < \alpha < 1$, by $C^{k,\alpha}$ we mean having a k^{th} derivative which is α Hölder continuous.

2.2. Measures [7, 2.1.2, 2.2.3, 2.2.5]. A *measure* over a set X is a function φ: $\mathbf{2}^X \to \mathbf{R}^+ \cup \{0, \infty\}$ such that if F is a countable collection of subsets of X and $A \subset \cup F$, then

$$\varphi(A) \leqq \sum_{B \in F} \varphi(B).$$

A measure φ over a topological space X is called *Borel regular* if open sets are measurable and each subset of X is contained in a Borel set of the same measure. If φ measures a topological space X and open sets are φ measurable, we have its *Borel regularization* $\tilde{\varphi}$ given by

$$\tilde{\varphi}(A) = \min\{\varphi(B): A \subset B, B \text{ Borel}\}.$$

Examples of Borel regular measures on $\mathbf{R}^k$ are given by *Lebesgue measure* $\mathscr{L}^k$ and m dimensional *Hausdorff measure* $\mathscr{H}^m (0 \leqq m \leqq k)$. Put

$$\alpha(k) = \mathscr{L}^k(\mathbf{B}^k(0, 1)).$$

2.3. Densities and mass ratios. If μ measures $\mathbf{R}^n$, $x \in \mathbf{R}^n$, $r > 0$, $k \in \mathbf{Z}^+$, we define the *k dimensional mass ratio of μ at x* by

$$\Theta^k(\mu, x, r) = \frac{\mu(\mathbf{B}^n(x, r))}{\alpha(k) r^k}$$

and we define the *k dimensional density of μ at x* by

$$\Theta^k(\mu, x) = \lim_{r \to 0} \Theta^k(\mu, x, r).$$

We will need to consider more general sorts of densities. We have the following lemma of measure theory.

2.4. Density Lemma. *Suppose φ is a Borel regular measure on a normed linear space X such that for all $x \in X$, there exists $\lambda > 0$, such that for all small $r > 0$,*

$$\varphi(\mathbf{B}(x, 5r)) \leqq \lambda \varphi(\mathbf{B}(x, r)).$$

Let A be any subset of X such that for φ almost all $x \in A$,

$$\liminf_{r \to 0} \frac{\varphi[\mathbf{B}(x,r) \cap A]}{\varphi[\mathbf{B}(x,r)]} < 1.$$

Then $\varphi(A) = 0$.

Proof. [7, 2.9.11, 2.8.17].

2.5. *Jacobians.* Suppose A is a k dimensional C^1 submanifold of $\mathbf{R}^m$ and $f\colon A \to \mathbf{R}^n$ is C^1. Let $a \in A$.

$$Df(a) \in \operatorname{Hom}(\operatorname{Tan}(A, a), \mathbf{R}^n),$$
$$\textstyle\bigwedge_k Df(a) \in \operatorname{Hom}(\bigwedge_k \operatorname{Tan}(A, a), \bigwedge_k \mathbf{R}^n).$$

Put

$$J_k f(a) = \| \textstyle\bigwedge_k Df(a) \|.$$

Then the area formula [7, 3.2.22] holds:

(1) $\displaystyle\int_A J_k f(x)\, d\mathscr{H}^k(x) = \int_{\mathbf{R}^n} \operatorname{card}(f^{-1}(y))\, d\mathscr{H}^k(y).$

Suppose $g\colon A \to \mathbf{R}^n$ is C^1. Then

(2) $|J_k f(a) - J_k g(a)|$
$\leqq k \|Df(a) - Dg(a)\| \max\{\|Df(a)\|^{k-1}, \|Dg(a)\|^{k-1}\}.$

Proof.

$$\begin{aligned}
&|J_k f(a) - J_k g(a)| \leqq \| \textstyle\bigwedge_k Df(a) - \bigwedge_k Dg(a) \| \\
&= |(\textstyle\bigwedge_k Df(a) - \bigwedge_k Dg(a))(\xi_1 \wedge \dots \wedge \xi_k)| \\
&\quad \text{(where } \{\xi_1, \dots, \xi_k\} \text{ is an orthonormal basis for } \operatorname{Tan}(A, a)) \\
&= |Df(a)(\xi_1) \wedge \dots \wedge Df(a)(\xi_k) - Dg(a)(\xi_1) \wedge \dots \wedge Dg(a)(\xi_k)| \\
&\leqq \sum_{i=1}^{k} |Df(a)(\xi_1) \wedge \dots \wedge Df(a)(\xi_{i-1}) \wedge (Df(a) - Dg(a))(\xi_i) \\
&\quad \wedge Dg(a)(\xi_{i+1}) \wedge \dots \wedge Dg(a)(\xi_k)| \\
&\leqq k \|Df(a) - Dg(a)\| \max\{\|Df(a)\|^{k-1}, \|Dg(a)\|^{k-1}\}.
\end{aligned}$$

Finally we note that in case $m = k$,

(3) $\displaystyle J_k f(a) \leqq \prod_{i=1}^{k} |D_i f(a)|$

where $D_i f = \dfrac{\partial f}{\partial x_i}$.

2.6. Integral Currents [7, 4.1.24, 4.2.26]. In $\mathbf{R}^n$ we have the space $\mathbf{I}_k(\mathbf{R}^n)$ of *k dimensional integral currents* (oriented surfaces) with flat norm $\mathscr{F}$ and the corresponding space $\mathbf{I}_k^2(\mathbf{R}^n)$ of unoriented surfaces with flat norm $\mathscr{F}^2$. In general, we will make statements for $\mathbf{I}_k(\mathbf{R}^n)$; analogous (and sometimes easier or stronger) statements hold for $\mathbf{I}_k^2(\mathbf{R}^n)$.

We have a continuous boundary operator

$$\partial\colon \mathbf{I}_k(\mathbf{R}^n)\to\mathbf{I}_{k-1}(\mathbf{R}^n)$$

whose kernel we will denote by $\mathscr{Z}_k(\mathbf{R}^n)$, and a lower semicontinuous mass function

$$\mathbf{M}\colon \mathbf{I}_k(\mathbf{R}^n)\to[0,\infty).$$

[7, 4.1.7].

Associated to $S\in\mathbf{I}_k(\mathbf{R}^n)$ we have the Radon measure $\|S\|$, the set $\operatorname{spt} S=\operatorname{spt}\|S\|$, and the integral varifold

$$|S|=[\mathrm{id}\times\mathrm{Tan}^k(\|S\|,\cdot)]_{\#}\,\|S\|.$$

(Cf. [7, 4.1.5, 4.1.1, 2.2.1, 3.2.16, 4.1.28], [1, 3.5].)

By way of this correspondence, most results on integral varifolds apply as well to integral currents.

The following two fundamental theorems constitute the geometric measure theoretic foundation for all our work.

Compactness Theorem [7, 4.2.17]. *If $c\geqq 0$, then*

$$\{S\in\mathbf{I}_k(\mathbf{R}^n)\colon \mathbf{M}(S)\leqq c, \mathbf{M}(\partial S)\leqq c, \text{ and } \operatorname{spt} S\subset\mathbf{B}^n(0,c)\}$$

is compact.

Isoperimetric Inequality [7, 4.2.10]. *If $T\in\mathscr{Z}_{k-1}(\mathbf{R}^n)$, then there exists $S\in\mathbf{I}_k(\mathbf{R}^n)$* with *$\partial S=T$ and*

$$\mathbf{M}(S)^{(k-1)/k}\leqq 2n^{2k}\mathbf{M}(T).$$

2.7. Mass Minimizing Currents

Definition. We say that an integral current S is *mass minimizing* if whenever S' is an integral current and $\partial S'=\partial S$,

$$\mathbf{M}(S')\geqq\mathbf{M}(S).$$

Existence. It follows from the Compactness Theorem and the isoperimetric inequality that *if B is an integral current with $\partial B=0$, then there is a mass minimizing integral current S with $\partial S=B$.*

Regularity. If $S \in \mathbf{I}_{n-1}(\mathbf{R}^n)$ is mass minimizing, then except for a set of Hausdorff dimension at most n-8, $\operatorname{spt} S - \operatorname{spt} \partial S$ is an analytic manifold ([8, Theorem 1], [14, Theorem 5.8.6]). The question of boundary regularity will be discussed in Chapter 6.

Monotonicity. Suppose $S \in \mathbf{I}_k(\mathbf{R}^n)$ is mass minimizing. If $a \in \operatorname{spt} S - \operatorname{spt} \partial S$, then the function

$$\Theta^k(\|S\|, a, r)$$

is monotonically increasing for $0 < r < \operatorname{dist}(a, \operatorname{spt} \partial S)$ [7, 5.4.4]. If $b \in \operatorname{spt} \partial S$ and for some $R > 0$, $\operatorname{spt} \partial S \cap \mathbf{U}(b, R/3)$ is a smooth manifold B whose curvature is bounded by R^{-1} in the sense that for $x, y \in B$,

$$|(y-x) - \operatorname{Tan}(B, x)(y-x)| \leqq R^{-1}|y-x|^2/2,$$

then

$$\Theta^k(\|S\|, b, r)\, e^{\frac{27}{8} k \frac{r}{R}}$$

is monotonically increasing for $0 < r < R/3$ [2, 3.4(2)].

3. Curves and Surfaces

We give the definitions and basic properties related to the spaces of curves and surfaces we will consider in this paper. We will be particularly interested in compact sets of curves and surfaces, for which we obtain some uniform estimates.

Let $0 < \alpha < \frac{1}{2}$. Put $\mathbf{S}^1 = \mathbf{R}/2\pi\mathbf{Z}$. We also denote by $\mathbf{S}^1$ the associated integral current $\mathbf{E}^1 \llcorner [0, 2\pi]$. Let

$$\mathscr{C} = \{C^{2,\alpha} \text{ maps: } \mathbf{S}^1 \to \mathbf{R}^3\}$$

with the C^2 norm

$$\|B\| = \max\{\|B\|_\infty, \|B'\|_\infty, \|B''\|_\infty\}.$$

Another metric will be handy:

$$\|B\|' = \max\{|(2\pi)^{-1} \int_{\mathbf{S}^1} B(t)\,dt|, \|B''\|_\infty\}.$$

3.1. Lemma.

$$\|B\|' \leqq \|B\| \leqq (1+\pi^2)\|B\|'.$$

Proof. The first inequality is immediate.

To prove the second, let $y \in \mathbf{S}^1$, $\Pi \in \mathbf{O}^*(3, 1)$. Since $\int_{\mathbf{S}^1} B'(t)dt = 0$, $\int_{\mathbf{S}^1} \Pi \circ B'(t)dt = 0$ and hence there is a $y_0 \in \mathbf{S}^1$ such that $\Pi \circ B'(y_0) = 0$. But then, since $|y - y_0| \leqq \pi$,

$$|\Pi \circ B'(y)| \leqq \pi \|(\Pi \circ B')'\|_\infty \leqq \pi \|B''\|_\infty \leqq \pi \|B\|'.$$

Therefore $\|B'\|_\infty \leqq \pi \|B\|'$.

Likewise if $y \in \mathbf{S}^1$, $\Pi \in \mathbf{O}^*$ (3, 1), there is a $y_0 \in \mathbf{S}^1$ such that $\Pi \circ B(y_0) = \Pi((2\pi)^{-1} \int_{\mathbf{S}^1} B(t)dt)$ and hence $|\Pi \circ B(y_0)| \leqq \|B\|'$.

$$\begin{aligned}|\Pi \circ B(y)| &\leqq |\Pi \circ B(y_0)| + |\Pi \circ B(y) - \Pi \circ B(y_0)| \\ &\leqq \|B\|' + \pi \|B'\|_\infty \leqq \|B\|'(1 + \pi^2).\end{aligned}$$

Therefore $\|B\|_\infty \leqq (1 + \pi^2) \|B\|'$.

We conclude that $\|B\| \leqq (1 + \pi^2) \|B\|'$. Note that both norms give the same topology on $\mathscr{C}$.

For $M \in \mathbf{R}$, define

$$\mathscr{C}(M) = \{B \in \mathscr{C}: \text{Hölder constant for } B'' \leqq M\}.$$

By Ascoli's Theorem, if $B_0 \in \mathscr{C}$ and $r_0 > 0$, $\mathbf{B}(B_0, r_0) \cap \mathscr{C}(M)$ is compact. $\mathscr{C}(M)$ is closed and σ-compact. $\mathscr{C} = \bigcup_{M=1}^{\infty} \mathscr{C}(M)$ is σ-compact and hence has a countable basis for its topology.

Let $\mathscr{E} \subset \mathscr{C}$ be the subspace of embeddings of $\mathbf{S}^1$ in $\mathbf{R}^3$:

$$\mathscr{E} = \{B \in \mathscr{C}: |B'| > 0 \text{ and } B(s) = B(t) \Rightarrow s = t\}.$$

Then $\mathscr{E}$ is open.

For $B \in \mathscr{E}$ we define smooth maps

$$\begin{aligned}&\tau, \nu: \operatorname{im} B \to \operatorname{Hom}(\mathbf{R}^3, \mathbf{R}^3)\\ &\tau(b) = \operatorname{Tan}(B, b),\\ &\nu(b) = \tau(b)^\perp,\end{aligned}$$

so that if $x \in \mathbf{R}^3$, $b \in \operatorname{im} B$, $\tau(b)(x) + \nu(b)(x) = x$.

We define a map $\Psi: \mathscr{C} \to \mathscr{Z}_1(\mathbf{R}^3) \subset \mathbf{I}_1(\mathbf{R}^3)$ by

$$\Psi(B) = B_\#(\mathbf{S}^1)$$

(which we will sometimes simply denote B).

Put $\mathscr{D} = \Psi(\mathscr{E})$. We have two topologies on $\mathscr{D}$:

$\mathscr{T}_1$ induced by the flat norm $\mathscr{F}$ on integral currents, and

$\mathscr{T}_2$ induced by $\Psi|\mathscr{E}$ from $\| \ \|$ on $\mathscr{C}$.

3.2. Proposition.

(1) $\mathscr{T}_1 \subset \mathscr{T}_2$.

In fact, for $B, C \in \mathscr{C}$,

$$\mathscr{F}(\Psi B - \Psi C) \leqq \|B - C\|(\text{length } B) + 2\pi \|B - C\|^2.$$

(2) Ψ *is continuous for both topologies.*

(3) $\Psi|\mathscr{E}$ *is open for* $\mathscr{T}_2$.

Proof. To establish (1), let $B, C \in \mathscr{C}$. Consider the map

$$F\colon [0, 2\pi] \times [0, 1] \to \mathbf{R}^3$$
$$F(t, \lambda) = (1 - \lambda) B(t) + \lambda C(t).$$

Put

$$P = F_\#(\mathbf{E}^2 \llcorner [0, 2\pi] \times [0, 1]).$$

Then $\partial P = \Psi B - \Psi C$, so that $\mathscr{F}(\Psi B - \Psi C) \leqq \mathbf{M}(P)$.

$$J_2 F \leqq \left|\frac{\partial F}{\partial t}\right| \left|\frac{\partial F}{\partial \lambda}\right| \quad \text{by } 2.5(3)$$
$$= |(1-\lambda) B'(t) + \lambda C'(t)| \, |C(t) - B(t)| \leqq (|B'(t)| + \|B - C\|) \, \|B - C\|.$$

Hence by the area formula 2.5(1)

$$\mathbf{M}(P) \leqq \int_{[0, 2\pi] \times [0, 1]} (|B'(t)| + \|B - C\|) \, \|B - C\| \, d\mathscr{L}^2$$
$$= \|B - C\| \, (\text{length } B) + 2\pi \|B - C\|^2.$$

(2) follows immediately from (1).

To prove (3), let U be open in $\mathscr{E}$. We must show ΨU is open. Let $\Psi C \in \Psi U$; then C is just a reparametrization of some $B \in U$; i.e., there is a C^2 homeomorphism

$$s\colon \mathbf{S}^1 \to \mathbf{S}^1 \quad \text{such that } B = C \circ s.$$

Put $k = \max\{1, \|s'\|_\infty, \|s'\|_\infty^2, \|s''\|_\infty\}$. Now choose a positive r so small that $\mathbf{B}(B, 2rk) \subset U$. We claim $\Psi \mathbf{B}(C, r) \subset \Psi U$, which will complete the proof.

Let $D \in \mathbf{B}(C, r)$. To show $\Psi D \in \Psi U$, it suffices to show $D \circ s \in U$, for which it suffices to show $\|B - D \circ s\| < 2rk$. But $\|B - D \circ s\| = \|(C - D) \circ s\| < 2rk$, as one easily checks.

Remark. That the map Ψ is open means that every choice of parametrization (from $\mathscr{E}$) induces the same topology at a geometric simple closed curve.

Now fix $B_0 \in \mathscr{E}$, $M \in \mathbf{Z}^+$ and choose $r_1 > 0$ such that

$$\mathbf{B}(B_0, r_1) \subset \mathscr{E}.$$

3.3. Lemma. *There are positive constants c_1, C_1, C_2 such that if*

$$r \leqq r_1, \quad B, C \in \mathbf{B}(B_0, r) \cap \mathscr{C}(M), \quad s, t \in \mathbf{S}^1,$$

then

(1) $\|B\| \leqq C_1$,

(2) $c_1 |t-s| \leqq |B(t)-B(s)| \leqq C_1 |t-s|$

and hence $c_1 \leqq |B'(s)| \leqq C_1$,

(3) $\|\tau B(t)) - \tau(C(t))\| \leqq 2 c_1^{-1} \|B'-C'\|_\infty$,

(4) $|v(B(s))\,(B(t)-B(s))| \leqq C_2 |B(t)-B(s)|^2/2$.

Proof. Just take $C_1 = \|B_0\| + r_1$.

To obtain c_1, consider the function

$$f\colon \mathbf{B}(B_0, r_1) \cap \mathscr{C}(M) \times \mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{R}^+$$

$$f(B,s,t) = \begin{cases} \left| \dfrac{B(t)-B(s)}{t-s} \right| & s \neq t, \\ |B'(u)| & u = s = t. \end{cases}$$

Clearly f is continuous except possibly at points of the form (C, u, u).

But

$$|f(B,s,t) - f(C,u,u)| \leqq |f(B,s,t) - f(B,u,u)| + |f(B,u,u) - f(C,u,u)|.$$

In estimating the first term, we can assume $s, t \approx u$, $s < t$:

$$\begin{aligned} |f(B,s,t) - f(B,u,u)| &\leqq |(t-s)^{-1}(B(t)-B(s)) - B'(u)| \\ &= \left| (t-s)^{-1} \int_s^t (B'(v) - B'(u))\, dv \right| \leqq \left| (t-s)^{-1} \int_s^t C_1 |v-u|\, dv \right| \\ &\leqq \left| (t-s)^{-1} \int_s^t C_1 \max\{|s-u|, |t-u|\}\, dv \right| \leqq C_1 \max\{|s-u|, |t-u|\}. \end{aligned}$$

On the other hand, $|f(B,u,u) - f(C,u,u)| = |B'(u) - C'(u)| \leqq \|B-C\|$.

Therefore f is continuous. As a positive continuous function on a compact set, f has a positive minimum c_1, and (2) holds. (Or if domain $f = \emptyset$, the proposition holds trivially.)

$$\|\tau(B(t)) - \tau(C(t))\| = \left| \frac{B'(t)}{|B'(t)|} - \frac{C'(t)}{|C'(t)|} \right| \leqq 2 c_1^{-1} \|B' - C'\|_\infty;$$

this proves (3).

Finally put $C_2 = 2 C_1 / c_1^2$. Then

$$|v(B(s))\,(B(t)-B(s))| = \left| v(B(s)) \left(B'(s)(t-s) + \int_s^t (t-u)\, B''(u)\, du \right) \right|$$

by Taylor's Theorem with Remainder

$\leqq |s-t|^2 C_1$ because $v(B(s))(B'(s))=0$

$\leqq (C_1/c_1^2)|B(t)-B(s)|^2$ by (2)

$\leqq C_2|B(t)-B(s)|^2/2.$

3.4. Definitions. Put $r_2=\min\{r_1, 1/3\,C_2\}$,

$$A=\{x\in\mathbf{R}^3\colon \operatorname{dist}(x, \operatorname{im} B_0)<r_2\}.$$

Now if $B\in\mathbf{B}(B_0, r_2)$, then

$$A\subset\{x\in\mathbf{R}^3\colon \operatorname{dist}(x, \operatorname{im} B)<2/3\,C_2\}.$$

By [2, 2.1], for any $B\in\mathbf{B}(B_0, r_2)$,

$$\xi=\{(a,b)\in A\times\operatorname{im} B\colon a-b\in v(b)\}$$

$$\rho=\{(a,|a-b|)\colon a\in A,\ b\in\operatorname{im} b,\ a-b\in v(b)\}$$

are continuously differentiable functions on A, $\rho(a)=\operatorname{dist}(a, \operatorname{im} B)$, and if $b=\xi(a)$, then

$$\|D\,\xi(a)-\tau(b)\|\leqq\rho(a)/(C_2^{-1}-\rho(a))\leqq 3\,C_2\,\rho(a)$$

and hence

$$\|D\,\xi(a)\|\leqq 1+3\,C_2\,\rho(a).$$

Next define $\mathscr{S}=\mathscr{S}(r_2)$ as

$\{(S,B)\in\mathbf{I}_2(\mathbf{R}^3)\times\mathbf{B}(B_0, r_2)\cap\mathscr{C}(M)\colon$

S is a mass minimizing current with boundary $B\}$.

We give $\mathscr{S}$ the metric

$$\operatorname{dist}((S_1,B_1),(S_2,B_2))=\max\{\mathscr{F}(S_2-S_1), \|B_2-B_1\|\}.$$

3.5. Lemma. *$\mathscr{S}$ is compact, and if $(S_i,B_i)\to(S,B)$ in $\mathscr{S}$, then* $\operatorname{HM}(\operatorname{spt} S_i, \operatorname{spt} S)\to 0$.

Proof. $\mathbf{B}(B_0, r_2)\cap\mathscr{C}(M)$ is compact. The associated currents, since they minimize mass, are of bounded mass and support. Therefore we can apply the Compactness Theorem 2.6 to conclude that $\mathscr{S}$ is compact.

Assume $(S_i,B_i)\to(S,B)$ in $\mathscr{S}$, but for some $\varepsilon>0$, $\operatorname{HM}(\operatorname{spt} S_i, \operatorname{spt} S)>\varepsilon$. Then we can assume there is a sequence of points $x_1, x_2, x_3, \ldots\to x$ such that one of the following holds:

(a) $x_i\in\operatorname{spt} S$, $\operatorname{dist}(x_i, \operatorname{spt} S_i)>\varepsilon$,

(b) $x_i\in\operatorname{spt} S_i$, $\operatorname{dist}(x_i, \operatorname{spt} S)>\varepsilon$.

In case (a), $x\in\operatorname{spt} S$, $\operatorname{dist}(x, \operatorname{spt} S_i)>\varepsilon/2$ for large i. This contradicts $S_i\to S$.

Hence we can assume (b). Choose $f, g \in C^\infty$ such that $f, g \geqq 0$, $f+g=1$, $\operatorname{spt} f \subset \mathbf{B}^3(x, \varepsilon/2)$, $\operatorname{spt} g \subset \mathbf{R}^3 - \mathbf{B}(x, \varepsilon/3)$. Then

$$S = S \llcorner f + S \llcorner g,$$

$$\mathbf{M}(S) = \mathbf{M}(S \llcorner f) + \mathbf{M}(S \llcorner g);$$

$$S_i = S_i \llcorner f + S_i \llcorner g,$$

$$\mathbf{M}(S_i) = \mathbf{M}(S_i \llcorner f) + \mathbf{M}(S_i \llcorner g).$$

Since $\mathbf{M}$ is lower semicontinuous,

$$\mathbf{M}(S \llcorner f) \leqq \underline{\lim}\, \mathbf{M}(S_i \llcorner f)$$

$$\mathbf{M}(S \llcorner g) \leqq \underline{\lim}\, \mathbf{M}(S_i \llcorner g).$$

But since S_i, S are mass minimizing,

$$\mathbf{M}(S_i) \to \mathbf{M}(S).$$

Combining this fact with the above inequalities yields

$$\mathbf{M}(S_i \llcorner f) \to \mathbf{M}(S \llcorner f).$$

Since by (b) $\operatorname{dist}(x, \operatorname{spt} S) \geqq \varepsilon$,

$$\mathbf{M}(S \llcorner f) = 0.$$

On the other hand, for large i, $\mathbf{B}^3(x_i, \varepsilon/4) \subset \mathbf{B}^3(x, \varepsilon/3)$ and $\operatorname{dist}(x_i, \operatorname{im} B_i) \geqq \varepsilon/4$. Hence by monotonicity (2.7)

$$\mathbf{M}(S_i \llcorner f) \geqq \|S_i\|\,(\mathbf{B}^3(x_i, \varepsilon/4)) \geqq \pi(\varepsilon/4)^2.$$

This contradicts the fact that $\mathbf{M}(S_i \llcorner f) \to \mathbf{M}(S \llcorner f)$ and completes the proof.

For $(S, B) \in \mathscr{S}$, $b \in \operatorname{im} B$, we now define

$$\mathbf{n}_S(b).$$

If $\operatorname{spt} S$ is a C^1 manifold with boundary at b, let $\mathbf{n}_S(b)$ be the unique unit vector in $\ker \tau(b)$ such that

$$\operatorname{Tan}(\operatorname{spt} S, b) = \{\tau(b) + \lambda\, \mathbf{n}_S(b) \colon \lambda \geqq 0\}.$$

Otherwise, put $\mathbf{n}_S(b) = 0$.

4. Measures on the Space of Curves

In this chapter we define a measure μ on $\mathscr{C}$ and exhibit the measure space as a product

$$(\mathscr{C}, \mu) = (\mathscr{C}_N, \mu_N) \times (\mathscr{C}^N, \mu^N).$$

After establishing the basic properties of these measures, we prove an important Lemma (4.6) on uniformly approximating certain subsets of $\mathscr{C}$ by curves in the finite dimensional space $\mathscr{C}_N$. Finally, we prove the close relationship between μ and Brownian motion (4.10).

4.1. Measures on $\mathbf{R}^k$. Let $\mathscr{L}^k$ denote Lebesgue measure on $\mathbf{R}^k$. Define Gaussian measure $\mathscr{G}^k$ as the Borel regular measure such that if E is a Borel subset of $\mathbf{R}^k$,

$$\mathscr{G}^k(E)=(2\pi)^{-\frac{1}{2}k}\int_E e^{-\frac{1}{2}|x|^2}\,d\mathscr{L}^k x.$$

Note that $\mathscr{G}^k=\prod_{n=1}^{k}\mathscr{G}^1$, $\mathscr{G}^k(\mathbf{R}^k)=1$, $\int|x|^2\,d\mathscr{G}^k=k$, and $\mathscr{G}^k$ is $\mathbf{O}(k)$ invariant. We also note that for $r\geqq 0$, $x\in\mathbf{R}^3$,

$$(1)\quad 4^{-1}\exp(-\tfrac{1}{2}(|x|+r)^2)\,r^3\leqq\mathscr{G}^3(\mathbf{B}^3(x,r))\leqq 3^{-1}r^3.$$

4.2. Measure on $\mathscr{C}$. Define a probability space

$$\Omega=\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\mathbf{R}^3$$

with product measure $\Pi\mathscr{G}^3$, which is a Radon measure. (Cf. [7, 2.6.6, 2.5.14]. For the construction it is technically convenient to define $\Pi\mathscr{G}^3$ on $\Pi\bar{\mathbf{R}}^3$, which is compact, and then restrict to Ω, whose complement has measure 0.)

It follows from work by G. Hunt [10, Introduction or Theorems 2, 4] that for almost all $(B_1,B_{-1},B_2,\dots)\in\Omega$,

$$\sum_{n=1}^{\infty}B_n n^{-3}\cos nt+B_{-n}n^{-3}\sin nt$$

converges in $\mathscr{C}$ (in the C^2 norm). Let Ω_0 be the linear subspace of Ω of full measure 1 on which we merely have uniform convergence (in the C^0 norm) to a function in $\mathscr{C}$.

We can define a bijective linear map

$$\Phi\colon\ \mathbf{R}^3\times\Omega_0\to\mathscr{C}$$

$$\Phi\colon\ (B_0,B_1,B_{-1},\dots)\mapsto B_0-\sum_{n=1}^{\infty}B_n n^{-3}\cos nt+B_{-n}n^{-3}\sin nt.$$

Φ is surjective because even a C^1 function has a uniformly convergent Fourier series. Since the maps

$$(B_0,B_1,B_{-1},\dots)\mapsto B_n n^{-3}\cos nt+B_{-n}n^{-3}\sin nt$$

are continuous, their sum Φ is Borel measurable. Φ has a continuous linear inverse

$$\Phi^{-1}\colon\ \mathscr{C}\to\mathbf{R}^3\times\Omega$$

$$\Phi^{-1}\colon\ B\mapsto(B_0,B_1,B_{-1},\dots)$$

which maps B to multiples of its Fourier coefficients:

$$B_0 = (2\pi)^{-1} \int_{\mathbf{S}^1} B(t)\,dt,$$

$$B_n = -n^3 \pi^{-1} \int_{\mathbf{S}^1} B(t) \cos nt\,dt,$$

$$B_{-n} = -n^3 \pi^{-1} \int_{\mathbf{S}^1} B(t) \sin nt\,dt, \qquad (n \in \mathbf{Z}^+).$$

Define a measure μ on $\mathscr{C}$ as the Borel regularization of $\Phi_\#(\mathscr{L}^3 \times \prod \mathscr{G}^3)$ (cf. 2.2), so that if $D \subset \mathscr{C}$ is a Borel set,

$$\mu(D) = \mathscr{L}^3 \times \prod \mathscr{G}^3(\Phi^{-1} D).$$

Of course μ induces the measure $\Psi_\# \mu$ on $\mathscr{Z}_1(\mathbf{R}^3)$.

4.3. Proposition.

(1) *μ is invariant under Euclidean motions.*

(2) *Let $f \in \mathscr{C}$. Suppose f''' is square summable. Then*

$$(\tau_f)_\# \mu \ll \mu.$$

(3) *Open sets have positive μ measure.*

Proof. (1) follows from the facts that any Euclidean motion is a composition of a rotation with a translation, Lebesgue measure is translation invariant, and both Lebesgue and Gaussian measures are rotation invariant.

To prove (2), we will use a theorem of Kadota and Shepp [11, Theorem 1], which says that if $\sum |c_n|^2 < \infty$ then

$$\prod_{n \neq 0} \tau_{c_n \#} \mathscr{G}^3 \ll \prod_{n \neq 0} \mathscr{G}^3.$$

Let $(c_0, c_1, c_{-1}, \ldots) = \Phi^{-1} f$. Integration by parts shows that the vector of Fourier coefficients of f''' is just $(0, c_{-1}, -c_1, c_{-2}, -c_2, \ldots)$. Hence by Plancherel's Formula,

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |c_n|^2 = \int_{\mathbf{S}^1} |f'''(t)|^2\,dt < \infty.$$

Consequently, by the theorem of Kadota and Shepp, $\prod_{n \neq 0} \tau_{c_n \#} \mathscr{G}^3 \ll \prod_{n \neq 0} \mathscr{G}^3$. It follows by Fubini's Theorem and Borel regularity that

$$\mathscr{L}^3 \times \prod \tau_{c_n \#} \mathscr{G}^3 \ll \mathscr{L}^3 \times \prod \mathscr{G}^3.$$

To prove (2), suppose $E \subset \mathscr{C}$, $\mu(E) = 0$, and let D be a Borel set, $E \subset D \subset \mathscr{C}$, such that $\mu(D) = 0$. Since $\mathscr{L}^3 \times \prod \mathscr{G}^3(\Phi^{-1} D) = 0$, $\mathscr{L}^3 \times \prod \tau_{c_n \#} \mathscr{G}^3(\Phi^{-1} D) = 0$. Now,

$$\begin{aligned}\tau_{f\#}\,\mu(E) &\leqq \tau_{f\#}\,\mu(D)\\ &= \mu(D+f) = (\mathscr{L}^3 \times \prod \mathscr{G}^3)(\Phi^{-1}D + (c_0, c_1, c_{-1}, \ldots))\\ &= (\mathscr{L}^3 \times \prod \tau_{c_n\#}\, \mathscr{G}^3)(\Phi^{-1}D) = 0.\end{aligned}$$

Therefore $\tau_{f\#}\,\mu \ll \mu$.

To prove (3), let $U \subset \mathscr{C}$ be open. Since C^∞ functions are dense in $\mathscr{C}$, the translates of U by $C^4 \cap \mathscr{C}$ cover $\mathscr{C}$. Since $\mathscr{C}$ is σ-compact, there is a countable subcover, $\{(U+f): f \in F\}$.

Now suppose $\mu(U)=0$. Then $\mu(U+f)=(\tau_f)_\#\,\mu(U)=0$ by (2). Therefore $\mu(\mathscr{C}) \leqq \sum_{f\in F} \mu(U+f)=0$, a contradiction.

4.4. For $N \in \mathbf{Z}^+$, we can view $\mathbf{R}^3 \times \Omega$ as a product

$$(\mathbf{R}^3 \times \Omega, \mathscr{L}^3 \times \prod_{n\neq 0} \mathscr{G}^3) = (\prod_{|n|\leqq N} \mathbf{R}^3, \mathscr{L}^3 \times \prod_{0<|n|\leqq N} \mathscr{G}^3) \times (\prod_{|n|>N} \mathbf{R}^3, \prod_{|n|>N} \mathscr{G}^3)$$

and denote projection onto the first and second factors by π_N and π^N, so that

$$\pi_N(B_0, B_1, B_{-1}, \ldots) = (B_0, \ldots, B_{-N}),$$
$$\pi^N(B_0, B_1, B_{-1}, \ldots) = (B_{N+1}, \ldots).$$

Furthermore we can view $\mathscr{C}$ as a topological product

$$\mathscr{C} = \mathscr{C}_N \times \mathscr{C}^N$$

where

$$\begin{aligned}\mathscr{C}_N &= \{B \in \mathscr{C}: \pi^N \Phi^{-1} B = 0\}\\ &= \left\{a_0 + \sum_{n=1}^{N} a_n \cos nt + a_{-n} \sin nt: a_n \in \mathbf{R}^3\right\},\\ \mathscr{C}^N &= \{B \in \mathscr{C}: \pi_N \Phi^{-1} B = 0\}.\end{aligned}$$

We also denote projection onto these factors by π_N, π^N.

We observe that $\mathbf{R}^3 \times \Omega_0$, the domain of Φ, is just $\prod_{|n|\leqq N} \mathbf{R}^3 \times \Omega_0^N$, where $\Omega_0^N = \pi^N(\mathbf{R}^3 \times \Omega_0)$, and $(\prod_{|n|>N} \mathscr{G}^3)(\Omega_0^N)=1$. Also, Φ restricts to bijective linear maps

$$\Phi: \prod_{|n|\leqq N} \mathbf{R}^3 \to \mathscr{C}_N,$$

$$\Phi: \Omega_0^N \to \mathscr{C}^N.$$

Imitating the definition of μ, we can define measures μ_N, μ^N on $\mathscr{C}_N$, $\mathscr{C}^N$ as the Borel regularizations of $\Phi_\#(\mathscr{L}^3 \times \prod_{0<|n|\leqq N} \mathscr{G}^3)$, $\Phi_\#(\prod_{|n|>N} \mathscr{G}^3)$.

We claim that $\mu = \mu_N \times \mu^N$ on $\mathscr{C} = \mathscr{C}_N \times \mathscr{C}^N$. Since both are σ-finite Borel regular measures, it suffices to check this for products $D \times E$ of Borel sets in $\mathscr{C}_N$ and $\mathscr{C}^N$, because the algebra of disjoint unions of such products generates the σ-algebra of Borel subsets of $\mathscr{C}$.

$$\begin{aligned}\mu(D\times E) &= (\mathscr{L}^3\times\prod_{n\neq 0}\mathscr{G}^3)(\Phi^{-1}(D\times E))\\ &= (\mathscr{L}^3\times\prod_{n\neq 0}\mathscr{G}^3)(\Phi^{-1}D\times\Phi^{-1}E)\\ &= (\mathscr{L}^3\times\prod_{0<|n|\leq N}\mathscr{G}^3)(\Phi^{-1}D)\cdot(\prod_{|n|>N}\mathscr{G}^3)(\Phi^{-1}E)\\ &= \mu_N(D)\cdot\mu^N(E)\\ &= (\mu_N\times\mu^N)(D\times E).\end{aligned}$$

Therefore we can view $\mathscr{C}$ as a product

$$(\mathscr{C},\mu)=(\mathscr{C}_N,\mu_N)\times(\mathscr{C}^N,\mu^N).$$

The following lemma shows that μ_N is enough like Lebesgue measure to do derivation theory. (Cf. Lemma 2.4.)

4.5. Lemma. *Given $C_1>0$, $N\in\mathbf{Z}^+$, there are positive numbers k_0, k_1 such that if $B\in\mathscr{C}$,*

$$\|B\|_\infty\leqq C_1, \quad \textit{and} \quad r\leqq C_1,$$

then

$$k_0 r^{3(2N+1)}\leqq\mu_N\mathbf{B}(\pi_N B,r)\leqq k_1 r^{3(2N+1)}.$$

Proof. Put

$$k_0=[4\exp(\tfrac{1}{2}C_1^2(2N^2+1)^2)(2N+1)^3]^{-(2N+1)},$$
$$k_1=4\pi(8N^9/3)^{2N+1}.$$

For any $B\in\mathscr{C}$, let $\Phi^{-1}B=(B_0,B_1,B_{-1},\ldots)$. It follows from the characterization of Φ^{-1} in 4.2 that

(1) $|B_n|\leqq 2N^3\|B\|_\infty$ for $|n|\leqq N$.

Now fix $B\in\mathscr{C}$ and suppose $C\in\mathscr{C}_N$.
If $C\in\mathbf{B}(\pi_N B,r)$, then $|C_n-B_n|\leqq 2N^3 r$ for $|n|\leqq N$ by (1), so that

$$\Phi^{-1}\mathbf{B}(\pi_N B,r)\subset\prod_{n=-N}^{N}\mathbf{B}^3(B_n,2N^3 r).$$

Hence,

$$\begin{aligned}\mu_N\mathbf{B}(\pi_N B,r) &= (\mathscr{L}^3\times\prod_{0<|n|\leq N}\mathscr{G}^3)(\Phi^{-1}\mathbf{B}(\pi_N B,r))\\ &\leqq(\mathscr{L}^3\times\prod_{0<|n|\leq N}\mathscr{G}^3)\left(\prod_{n=-N}^{N}\mathbf{B}^3(B_n,2N^3 r)\right)\\ &\leqq\tfrac{4}{3}\pi(2N^3 r)^3\prod_{\substack{n=-N\\ n\neq 0}}^{N}(3^{-1}(2N^3 r)^3) \qquad \text{by 4.1(1)}\\ &\leqq 4\pi(8N^9/3)^{2N+1}r^{3(2N+1)}=k_1 r^{3(2N+1)}.\end{aligned}$$

On the other hand, $C \in \mathbf{B}(\pi_N B, r)$

if $|C_0 - B_0| + |(C_1 - B_1)/1| + \cdots + |(C_{-N} - B_{-N})/N| < r$, which holds
if $|C_n - B_n| < r/(2N+1)$ for $|n| \leqq N$,

so that

$$\Phi^{-1}\mathbf{B}(\pi_N B, r) \supset \prod_{n=-N}^{N} \mathbf{B}^3(B_n, r/(2N+1)),$$

and as above, one obtains the estimate that

$$\mu_N \mathbf{B}(\pi_N B, r) \geqq k_0 r^{3(2N+1)}.$$

The following lemma will enable us to go to $\mathscr{C}_N$ to carry out somewhat intricate geometric density arguments in Chapter 7.

4.6. Approximation Lemma. *Let H be a set of functions: $\mathbf{S}^1 \to \mathbf{R}^3$ with equicontinuous second derivatives which are bounded by a positive constant K_2. Then given $\eta > 0$, there is a positive integer N such that given $h \in H$, there exists $f \in \mathscr{C}_N$, such that*

$$\|f - h\| < \eta.$$

Proof. First choose $\delta > 0$ such that $h \in H$, $|s-t| < \delta \Rightarrow |h''(s) - h''(t)| < \eta/2(1+\pi^2)$. Next choose $N \in \mathbf{Z}^+$ so that the function

$$q \colon \mathbf{S}^1 \to \mathbf{R}$$
$$q(t) = c((1+\cos t)/2)^N$$

(where c is chosen so that $\int_{\mathbf{S}^1} q(t)\,dt = 1$) satisfies

$$\sup\{|q(t)| \colon \delta \leqq t \leqq 2\pi - \delta\} < \eta/4K_2(1+\pi^2).$$

Now let $h \in H$. Define

$$g \colon \mathbf{S}^1 \to \mathbf{R}^3$$
$$g(t) = \int_{\mathbf{S}^1} h''(t-s)\, q(s)\, ds$$
$$= \int_{\mathbf{S}^1} h''(s)\, q(t-s)\, ds.$$

The second expression for $g(t)$ shows that $g \in \mathscr{C}_N$. Using the first we estimate that

$$|g(t) - h''(t)| = \Big|\int_{\mathbf{S}^1} [h''(t-s) - h''(t)]\, q(s)\, ds\Big|$$
$$\leqq \int_{t=-\delta}^{\delta} q(s)\, |h''(t-s) - h''(t)|\, ds + \int_{t=\delta}^{2\pi-\delta} 2K_2\, q(s)\, ds$$
$$\leqq \eta/2(1+\pi^2) + \eta/2(1+\pi^2) = \eta/(1+\pi^2).$$

Furthermore, $\int_{\mathbf{S}^1} g(t)dt = \int_{s\in\mathbf{S}^1} q(s) \int_{t\in\mathbf{S}^1} h''(t-s)\,dt\,ds = 0$ because $\int_{t\in\mathbf{S}^1} h''(t-s)\,dt = \int_{\partial\mathbf{S}^1} h'(t-s) = 0$. Hence we can integrate $g(t)$ term by term and stay inside $\mathscr{C}_N$. To get $f(t)$, do this integration twice and add on a constant of integration so that

$$\int_{\mathbf{S}^1} f(t)\,dt = \int_{\mathbf{S}^1} h(t)\,dt.$$

It follows that $\|f-h\|' \leqq \eta/(1+\pi^2)$, so that by Lemma 3.1,

$$\|f-h\| < \eta.$$

4.7. Brownian Motion. We will now prove the intimate relationship between μ and Brownian motion, which could have been used to define Φ and μ. Theorem 4.10 will show that Φ'' is just Brownian motion with two minor adjustments.

One dimensional Brownian motion is a Gaussian process X: probability space $\times$ real interval, $I \to \mathbf{R}$ characterized by

$$\text{mean } \mathbf{E}X_t \equiv \int X_t = 0 \qquad t\in I$$

$$\text{covariance } \mathbf{E}X_s X_t = \min\{s, t\} \qquad s, t \in I$$

and continuous sample paths (continuity in t).

If $\{X^{(i)}\ 1 \leqq i \leqq n\}$ are independent one dimensional Brownian motions, then $(X^{(1)}, X^{(2)}, \ldots, X^{(n)})$ is *n dimensional Brownian motion.*

4.8. Proposition. *Define*

$$X\colon \prod_{n=-\infty}^{\infty} (\mathbf{R}, \mathscr{G}^1) \times [0, 2\pi) \to \mathbf{R}$$

$$X_t = X((Y_0, Y_1, Y_{-1}, \ldots), t)$$

$$= (2\pi)^{-\frac{1}{2}} Y_0 t + \pi^{-\frac{1}{2}} \sum_{n=1}^{\infty} Y_n n^{-1}(\cos nt - 1) + Y_{-n} n^{-1} \sin nt.$$

Then X_t is Brownian motion on $[0, 2\pi)$.

Proof. Once again it follows from Hunt's work [10, Theorem 2] that almost surely this series converges uniformly to a continuous function of t.

Note that $\{Y_n\}$ is an orthonormal set in $L^2(\prod \mathbf{R})$ and $\mathbf{E}Y_n = 0$. Since

$$\left\| \sum_{n=M}^{N} Y_n n^{-1}(\cos nt - 1) + Y_{-n} n^{-1} \sin nt \right\|_2^2$$

$$= \sum_{n=M}^{N} n^{-2}[(\cos nt - 1)^2 + \sin^2 nt] \leqq 5 \sum_{n=M}^{N} n^{-2},$$

the series for X_t is L^2 Cauchy and hence converges to X_t in L^2 and in L^1. As a sum of Gaussian processes, X_t is Gaussian.

$$\mathbf{E}X_t = 0$$

$$\mathbf{E}X_s X_t = X_s \cdot X_t \quad \text{(inner product in } L^2\text{)}$$

$$= (2\pi)^{-1} s t + \pi^{-1} \sum_{n=1}^{\infty} n^{-2} [(\cos n s - 1)(\cos n t - 1) + \sin n s \sin n t]$$

$$= \min\{s, t\} \quad \text{by the following formula [4, p. 167]:}$$

$$\sum_{n=1}^{\infty} n^{-2} \cos n t = \pi^2/6 - (\pi/2)\, t + (1/4)\, t^2 \qquad (0 \leqq t < 2\pi).$$

Therefore X_t is Brownian motion.

4.9. Corollary.

$$X\colon (\mathbf{R}^3, \mathscr{G}^3) \times (\Omega, \prod_{n \neq 0} \mathscr{G}^3) \times [0, 2\pi) \to \mathbf{R}^3$$

$$X_t = (2\pi)^{-\frac{1}{2}}\, Y t + \pi^{-\frac{1}{2}} \sum_{n=1}^{\infty} B_n n^{-1} (\cos n t - 1) + B_{-n} n^{-1} \sin n t$$

is three dimensional Brownian motion.

Comparing this expression for three dimensional Brownian motion with our definition of Φ, we get the following theorem.

Put $\Omega^* = (\mathbf{R}^3, \mathscr{G}^3) \times (\mathbf{R}^3, \mathscr{L}^3) \times \Omega$ with typical element $(Y, B_0, B_1, B_{-1}, \ldots)$, so that we can view both Φ and Brownian motion X as functions: $\Omega^* \times [0, 2\pi) \to \mathbf{R}^3$, defined almost everywhere, with their defining series convergent in $C^2[0, 2\pi)$ and $C[0, 2\pi)$ (with C^2 norm and C^0 norm) respectively.

4.10. Theorem.

$$\Phi''(t) = \pi^{\frac{1}{2}} \left[X(t) + \left(\frac{1}{2} - \frac{t}{2\pi}\right) X(2\pi) - \frac{1}{2\pi} \int_0^{2\pi} X(t)\, dt \right]$$

and

$$X(t) = \pi^{-\frac{1}{2}} [\Phi''(t) - \Phi''(0) + 2^{-\frac{1}{2}}\, Y t].$$

Remark. Φ'' is Brownian motion with two adjustments to make Φ 2π periodic. First, $\left(\frac{1}{2} - \frac{t}{2\pi}\right) X(2\pi)$ is added to make Φ'' 2π periodic. Second, the average value of $X(t)$ is subtracted to make $\Phi' = \int \Phi''$ periodic. (The constant of integration is chosen to make $\int \Phi'(t)$ zero, i.e., to make Φ periodic. In integrating Φ from Φ', the constant of integration is the random variable B_0.)

5. Uniqueness and P.D.E.

As the first step towards the Uniqueness Theorem 7.1, we prove that two area minimizing surfaces which are tangent along an interval of boundary are equal. Because such surfaces are analytic manifolds on the interior, it suffices to prove the theorem locally, at a boundary point where the two surfaces can be viewed as graphs of functions $u, u'\colon \mathbf{R}^2 \to \mathbf{R}$ satisfying the minimal surface equation

$$(1+u_y^2)\,u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)\,u_{yy} = 0.$$

Thus the problem becomes one of partial differential equations. We know that u and u' have the same values and derivatives along the boundary, and we want to conclude that $u=u'$. The difficulty stems from the nonlinearity of the minimal surface equation. Nevertheless, general results from the theory of partial differential equations due to Aronszajn [3] give the result if u, u' are C^3.

We use a different technique, which only assumes u, u' to be C^2. The Legendre transformation (cf. [6, Volume II, pp. 32–34, 38]) introduces u_x and u_y as new coordinates on the surface. Under this transformation, the minimal surface equation becomes a *linear* elliptic second order partial differential equation for some new function ω corresponding to u and ω' corresponding to u'. On first consideration, this proves nothing, because the Legendre transformation itself depends on the particular function u. However, it turns out rather surprisingly that because the values and first derivatives of u and u' agree along a boundary segment of their common domain, the domains of ω and ω' have a common boundary segment on which the values and first derivatives of ω and ω' agree. It follows that $\omega=\omega'$ and consequently $u=u'$.

5.1. Theorem. *Let S, S' be mass minimizing integral currents with $\partial S = \partial S' = B$, $B \in \mathscr{E}$. Suppose that for all t in some subinterval of* $\mathbf{S}^1$, $\operatorname{spt} S$ *and* $\operatorname{spt} S'$ *are C^2 manifolds with boundary at $B(t)$ and*

$$\mathbf{n}_S(B(t)) = \mathbf{n}_{S'}(B(t)).$$

Then

$$S = S'.$$

5.2. Lemma. *Let S be a mass minimizing integral current with boundary B, $B \in \mathscr{E}$, such that for some $t \in \mathbf{S}^1$,* $\operatorname{spt} S$ *is a C^1 manifold with boundary in a neighborhood of $B(t)$. Then* $\operatorname{spt} S - \operatorname{spt} \partial S$ *is connected.*

Proof. Let D be the component of $\operatorname{spt} S - \operatorname{spt} \partial S$ which intersects the neighborhood of $B(t)$. Since $\operatorname{spt} S - \operatorname{spt} \partial S$ is an analytic manifold (2.7), so is D, and $\partial D \subset \operatorname{spt} \partial S$.

Put $T = S \llcorner D$. Then $\operatorname{spt} \partial T \subset \operatorname{spt} \partial S$. It follows from [7, 4.1.31] that $\partial T = r\,\partial S$ for some $r \in \mathbf{R}$. Considering our neighborhood of $B(t)$, we see we must have $\partial T = \partial S$. Since S minimizes mass, $T = S$. Therefore $\operatorname{spt} S - \operatorname{spt} \partial S$ is connected.

Proof of Theorem. Let b_0 be a boundary point where $\operatorname{spt} S$ and $\operatorname{spt} S'$ are C^2 manifolds with boundary. For convenience we will assume $b_0 = 0$ and

$$\operatorname{Tan}(\operatorname{spt} S, 0) = \operatorname{Tan}(\operatorname{spt} S', 0) \subset \mathbf{R}^2 \times \{0\}.$$

Denote by Π orthogonal projection onto $\mathbf{R}^2 \times \{0\}$.

Choose $\rho_0 > 0$ such that if

$$W = \mathbf{U}^2(0, \rho_0) \times (-\rho_0, \rho_0),$$

then $\operatorname{spt} S \cap W$ and $\operatorname{spt} S' \cap W$ are C^2 manifolds with boundary given nonparametrically as graphs of C^2 functions

$$u, u' \colon G \to \mathbf{R}$$

where

$$G = \Pi(\operatorname{spt} S \cap W) = \Pi(\operatorname{spt} S' \cap W)$$

is a domain in $\mathbf{U}^2(0, \rho_0)$ including the boundary segment

$$B = \Pi(\operatorname{spt} \partial S \cap W) = \Pi(\operatorname{spt} \partial S' \cap W).$$

For $(x, y) \in G$, u must satisfy the minimal surface equation

$$(1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy} = 0,$$

and similarly for u'.

Note that since by hypothesis at points of B, u and u' have the same values and first derivatives, their second partial derivatives involving differentiation along B also agree. Applying the minimal surface equation, we conclude that all their second derivatives agree:

(1) $u|B = u'|B$, $u_x|B = u'_x|B$, $u_y|B = u'_y|B$, $u_{xx}|B = u'_{xx}|B$,
$u_{xy}|B = u'_{xy}|B$, $u_{yy}|B = u'_{yy}|B$.

In particular, if

$$K \colon \operatorname{spt} S \cap W \to \mathbf{R}$$
$$K' \colon \operatorname{spt} S' \cap W \to \mathbf{R}$$

denote Gaussian curvature, then $K|B = K'|B$. Unless $K|B = 0$, we can assume (by moving b_0 and shrinking ρ_0 if necessary) that K, K' are nonvanishing on $\operatorname{spt} S \cap W$, $\operatorname{spt} S' \cap W$. Therefore it suffices to consider two cases:

Case 1. $K|B = 0$.

Case 2. K, K' nonvanishing on $\operatorname{spt} S \cap W$, $\operatorname{spt} S' \cap W$.

Case 1. Since $S \llcorner W$ must be mass minimizing, $\operatorname{spt} S \cap W$ must have mean curvature $H = 0$. But H, K both 0 on $\operatorname{spt} \partial S \cap W$ implies that $\operatorname{Tan}(\operatorname{spt} S, b)$ is constant for $b \in \operatorname{spt} \partial S \cap W$, and hence that $\operatorname{spt} \partial S \cap W = B \subset \mathbf{R}^2 \times \{0\}$. Hence at points of B, the partial derivatives of u, and the second partial derivatives involving differentiation along B are 0. By the minimal surface equation, all the second derivatives are 0. Therefore u can be continued as a C^2 function across B by 0. Since on the interior u is analytic, $u = 0$. Likewise, $u' = 0$. Now by analytic continuation and Lemma 5.2, $S = S'$.

Case 2. Consider C^1 maps

$$f, f' \colon G \to \mathbf{R}^2 = \{(\xi, \eta) \colon \xi, \eta \in \mathbf{R}\}$$
$$f(x, y) = (u_x(x, y), u_y(x, y)),$$
$$f'(x, y) = (u'_x(x, y), u'_y(x, y)),$$

and the continuous map

$$\rho\colon G \to \mathbf{R}$$
$$\rho(x,y) = \begin{vmatrix} u_{xx}(x,y) & u_{xy}(x,y) \\ u_{xy}(x,y) & u_{yy}(x,y) \end{vmatrix}.$$

At every $(x,y) \in G$, $\rho(x,y)$ is a positive multiple of $K(x,y,u(x,y))$ and hence nonzero (negative in fact).

By (1), $Df(0) = Df'(0)$. Since $J_2 f(0) = J_2 f'(0) = |\rho(0)| > 0$, by shrinking ρ_0 if necessary, we can assume that f and f' are C^1 homeomorphisms onto H, $H' \subset \mathbf{R}^2$. Since $f|B = f'|B$, if $H = f(G)$ and $H' = f'(G)$, then $H'' = H \cap H'$ is a domain with a boundary interval $C = f(B) = f'(B)$.

Now via f^{-1} we can view x, y, u as C^1 functions on H. Define

$$\omega\colon H \to \mathbf{R}$$
$$\omega = x\xi + y\eta - u.$$

Differentiating with respect to x and y yields the matrix equation

$$\begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \begin{bmatrix} \omega_\xi \\ \omega_\eta \end{bmatrix} = \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since ρ is nonvanishing, we conclude that

(2) $\omega_\xi = x, \quad \omega_\eta = y.$

This shows us that ω is in fact C^2.

Differentiating these last two equations with respect to x and y we get

$$\begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \begin{bmatrix} \omega_{\xi\xi} & \omega_{\xi\eta} \\ \omega_{\xi\eta} & \omega_{\eta\eta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} \omega_{\xi\xi} & \omega_{\xi\eta} \\ \omega_{\xi\eta} & \omega_{\eta\eta} \end{bmatrix} = \frac{1}{\rho} \begin{bmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{bmatrix}.$$

Now since the minimal surface equation holds for u for all $(x,y) \in G$, we have for all $(\xi,\eta) \in H$,

$$(1+\xi^2)\omega_{\xi\xi} + 2\xi\eta\,\omega_{\xi\eta} + (1+\eta^2)\omega_{\eta\eta} = 0.$$

Of course, with a similar definition for ω' in terms of u', we conclude that for all $(\xi,\eta) \in H'$,

$$(1+\xi^2)\omega'_{\xi\xi} + 2\xi\eta\,\omega'_{\xi\eta} + (1+\eta^2)\omega'_{\eta\eta} = 0.$$

These two equations hold simultaneously in H''. Furthermore, for $(\xi,\eta) \in C \subset H''$,

$$\omega(\xi,\eta) = \omega'(\xi,\eta),$$
$$\omega_\xi(\xi,\eta) = \omega'_\xi(\xi,\eta), \omega_\eta(\xi,\eta) = \omega'_\eta(\xi,\eta)$$

by (1) and (2) and the analogous statement for ω'.

Define

$$\omega'': H'' \to \mathbf{R}$$
$$\omega'' = \omega - \omega'.$$

Then ω'' satisfies the differential equation

$$(1+\xi^2)\omega''_{\xi\xi} + 2\xi\eta\,\omega''_{\xi\eta} + (1+\eta^2)\omega''_{\eta\eta} = 0,$$

and for $(\xi, \eta) \in C$,

$$\omega''(\xi,\eta) = \omega''_{\xi}(\xi,\eta) = \omega''_{\eta}(\xi,\eta) = 0.$$

Hence the second partial derivatives involving differentiation along C are 0. Finally, by the differential equation, we conclude that all the second derivatives are 0:

$$\omega''_{\xi\xi}(\xi,\eta) = \omega''_{\xi\eta}(\xi,\eta) = \omega''_{\eta\eta}(\xi,\eta) = 0.$$

Therefore ω'' can be continued as a C^2 function across C by 0. But on the interior, as the solution to the elliptic partial differential equation, ω'' is analytic. Therefore, $\omega'' = 0$.

But now for $(\xi,\eta) \in H''$,

$$f^{-1}(\xi,\eta) = (\omega_\xi, \omega_\eta) = (\omega'_\xi, \omega'_\eta) = f'^{-1}(\xi,\eta).$$

Hence if $G'' = f^{-1}(H'') = f'^{-1}(H'')$, $f|G'' = f'|G''$. Therefore if $(x,y) \in G''$,

$$\begin{aligned} u(x,y) &= -\omega(f(x,y)) + (x,y)\cdot f(x,y) \\ &= -\omega'(f'(x,y)) + (x,y)\cdot f'(x,y) \\ &= u'(x,y). \end{aligned}$$

By analytic continuation and Lemma 5.2, $S = S'$.

6. Boundary Regularity

Boundary regularity will play an essential role in our arguments. Although it is true that a mass minimizing unoriented surface in $\mathbf{I}_2^2(\mathbf{R}^3)$ with a $C^{2,\alpha}$ boundary is a $C^{2,\alpha}$ manifold with boundary, the corresponding statement of boundary regularity for integral currents is not known. However, Allard [2, p. 429] has proved that any integral varifold with a C^2 boundary whose first variation is summable to a power greater than its dimension is a C^1 manifold with boundary at any boundary point where its density is $\frac{1}{2}$. Proposition 6.1 summarizes his results as they apply to mass minimizing integral currents.

Our Uniform Boundary Regularity Theorem 6.4 produces a small positive number r_5 such that Proposition 6.1 can be applied uniformly to

$$\mathscr{S} = \{(S,B) \in \mathbf{I}_2(\mathbf{R}^3) \times \mathbf{B}(B_0, r_5) \cap \mathscr{C}(M):$$
$$S \text{ is a mass minimizing current with boundary } B\}.$$

It follows that $\mathbf{n}_S(b)$, the inward unit vector normal to the boundary at b and tangent to spt S, is a continuous function on the space of curves and surfaces (Lemma 6.6), and hence that if the tangents to two area minimizing surfaces with close boundaries are close together on J, the surfaces are close together (Theorem 6.7). The proof of the Uniqueness Theorem 7.1 will depend fundamentally on this result. For convenience, we conclude this chapter by formulating from Theorem 6.7 the lemmas we will need in Chapter 7.

6.1. Proposition (Allard [2, p. 429]). *Given ε, $0<\varepsilon<\frac{1}{2}$, there is a positive number δ, such that if $B\in\mathscr{E}$, S is a mass minimizing current with boundary B, $b\in\operatorname{im} B$, and the following two hypotheses are satisfied:*

(1) $|\nu(z)(y-z)|\leqq\delta\,|y-z|^2/2$

for $y, z\in\operatorname{im} B\cap\mathbf{U}^3(b,1)$,

(2) $\Theta^2(\|S\|, b, 1)\leqq\frac{1}{2}+\delta$;

then

(3) spt $S\cap\mathbf{U}^3(b,1-\varepsilon)$ *is a C^1 manifold M with boundary,*

(4) *if $y, z\in M$-im B and T is the plane containing* $\operatorname{Tan}(M,b)$, *we have the estimates*

$$\|\operatorname{Tan}(M,y)-T\|\leqq\varepsilon,$$

$$\|\operatorname{Tan}(M,y)-\operatorname{Tan}(M,z)\|\leqq\varepsilon\,|y-z|^{1/3},$$

(5) *there are an isometry θ of $\mathbf{R}^3\cong\mathbf{R}^2\times\mathbf{R}$ and a continuously differentiable function f: $\mathbf{R}^2\to\mathbf{R}$ such that*

$$\theta M\subset\operatorname{graph} f,$$

$$\theta b=(0,f(0)),\qquad \theta T=\mathbf{R}^2\times\{0\},$$

$$Df(0)=0,\qquad |D_i f(y)-D_i f(z)|\leqq\varepsilon\,|y-z|^{1/3}$$

for $i=1,2$ and y, $z\in\mathbf{R}^2$.

The difficulty in applying this proposition uniformly to $\mathscr{S}(r)$ for some $r>0$ lies in satisfying hypothesis (2) uniformly, since (1) follows from 3.3(4). The next two lemmas overcome this difficulty.

6.2. Lemma. *Given $B_0\in\mathscr{E}$, there are a closed interval $J\subset\mathbf{S}^1$ and an r_5, $0<r_5\leqq r_2$, such that if $(S,B)\in\mathscr{S}(r_5)$, $t\in J$, then*

$$\Theta^2(S, B(t))=\tfrac{1}{2}.$$

Proof. By a lemma of Allard's [2, p. 445], it suffices to show there are independent real linear functions β_1, β_2 such that

$$\operatorname{im} B\subset\{x\in\mathbf{R}^3\colon \beta_i(x-B(t))\geqq 0,\ i=1,2\}.$$

To formulate this condition geometrically, for $a\in\mathbf{R}^3$, v, w unit vectors in $\mathbf{R}^3$, $\eta>0$, we define a wedge

$$W(a,v,w,\eta)=\{x\in\mathbf{R}^3\colon v\cdot(x-a)\geqq\eta\,|w\cdot(x-a)|\}.$$

Now it suffices to produce independent unit vectors v, $w \in \mathbf{R}^3$, $r_4 > 0$, such that

(1) $\operatorname{im} B \subset W(B(t), v, w, r_4)$,

because we can then put

$$\beta_1(z) = (v - r_4 w) \cdot z,$$
$$\beta_2(z) = (v + r_4 w) \cdot z, \qquad (z \in \mathbf{R}^3).$$

It is easy to fit such a wedge about an extreme point of $\operatorname{im} B_0$; we prove it can be done as well for points and curves nearby.

Let $R = \|B_0\|_\infty$ and choose $t_0 \in \mathbf{S}^1$ such that $|B_0(t_0)| = R$. Put

$$r_3 = \min\left\{r_2, \tfrac{1}{2}, \frac{c_1^2}{8R(C_1+1)}, \frac{c_1}{2}\left(\frac{c_1^2}{8MR}\right)^{1/\alpha}\right\},$$

$$r_4 = \min\left\{r_3, \frac{r_3^2}{32(R+1)^2}, \frac{c_1^2}{4RC_1}\right\},$$

$$r_5 = \min\{r_4, c_1 r_4/4, R/2\},$$

$$\delta = \left(\frac{r_5}{11M}\right)^{1/\alpha},$$

$$J = [t_0 - \delta, t_0 + \delta].$$

(2) First we will prove that *if*

$$B \in \mathbf{B}(B_0, 2r_5) \cap \mathscr{C}(M),$$

then there are independent unit vectors v, $w \in \mathbf{R}^3$ *such that*

$$\operatorname{im} B \subset W(B(t_0), v, w, r_4).$$

By choice of R and t_0, at t_0 the derivative of the function $B_0(t) \cdot B_0(t)$ must be 0 and its second derivative must be nonpositive:

$$B_0(t_0) \cdot B_0'(t_0) = 0$$

(3) $B_0''(t_0) \cdot B_0(t_0) + B_0'(t_0) \cdot B_0'(t_0) \leqq 0.$

Secondly, if $B \in \mathbf{B}(B_0, 2r_5) \cap \mathscr{C}(M)$, then

$$\left|\frac{B'(t_0)}{|B'(t_0)|} - \frac{B_0'(t_0)}{|B_0'(t_0)|}\right| \leqq \frac{2\|B - B_0\|}{|B_0'(t_0)|} \leqq 4r_5/c_1 \leqq r_4.$$

Hence there is a unit vector $v \in \mathbf{R}^3$ such that

$$v \cdot B'(t_0) = 0 \quad \text{and} \quad |v + R^{-1} B_0(t_0)| \leqq r_4.$$

Choose a unit vector w independent of v such that $w \cdot B'(t_0) = 0$. Because $\operatorname{im} B \subset \mathbf{B}^3(0, R + 2r_5)$ and $\mathbf{B}^3(B_0(t_0), r_3) \subset \mathbf{B}^3(B(t_0), 2r_3)$, (2) is an immediate consequence of the following two claims:

Claim 1.

$$\operatorname{im} B \cap \mathbf{B}^3(B(t_0), 2r_3) \subset W(B(t_0), v, w, r_4).$$

Claim 2.

$$\mathbf{B}^3(0, R+2r_5) - \mathbf{B}^3(B_0(t_0), r_3) \subset W(B(t_0), v, w, r_4).$$

Proof of Claim 1. If $B(t) \in \mathbf{B}^3(B(t_0), 2r_3)$, then by 3.3(2),

$$|t-t_0| \leqq 2r_3/c_1 \leqq (c_1^2/8MR)^{1/\alpha}$$

by choice of r_3. Hence

$$\begin{aligned} v\cdot B''(t) &= v\cdot(B''(t)-B_0''(t)) + v\cdot(B_0''(t)-B_0''(t_0)) \\ &\quad + (v+R^{-1}B_0(t_0))\cdot B_0''(t_0) - R^{-1}B_0(t_0)\cdot B_0''(t_0) \\ &\geqq -r_3 - M|t-t_0|^\alpha - r_4 C_1 - R^{-1}B_0(t_0)\cdot B_0''(t_0) \\ &\geqq -c_1^2/8R - c_1^2/8R - c_1^2/4R - R^{-1}B_0(t_0)\cdot B_0''(t_0) \\ &\geqq -c_1^2/2R + |B_0'(t_0)|^2/R \qquad \text{by (3)} \\ &\geqq c_1^2/2R. \end{aligned}$$

Applying Taylor's Theorem with Remainder yields

$$v\cdot(B(t)-B(t_0)) \geqq (c_1^2/2R)(\tfrac{1}{2}(t-t_0)^2) = (c_1^2/4R)(t-t_0)^2.$$

On the other hand,

$$\begin{aligned} |w\cdot B''(t)| &\leqq |B''(t)| \\ &\leqq |B_0''(t_0)| + r_3 + M|t-t_0|^\alpha \\ &\leqq C_1 + c_1^2/4R, \end{aligned}$$

so that again by Taylor's Theorem,

$$\begin{aligned} r_4|w\cdot(B(t)-B(t_0))| &\leqq r_3(C_1 + c_1^2/4R)(\tfrac{1}{2}(t-t_0)^2) \\ &\leqq (c_1^2/8R + c_1^2/8R)(\tfrac{1}{2}(t-t_0)^2) \\ &\leqq (c_1^2/8R)(t-t_0)^2. \end{aligned}$$

Therefore $B(t) \in W(B(t_0), v, w, r_4)$ and the first claim is proved.

Proof of Claim 2. Let $y \in \mathbf{B}^3(0, R+2r_5) - \mathbf{U}^3(B_0(t_0), r_3)$. Then

$$r_3 \leqq |y - B_0(t_0)| \leqq 2R + 2r_5.$$

Also

$$\begin{aligned} (R+2r_5)^2 \geqq |y|^2 &= |y - B_0(t_0) + B_0(t_0)|^2 \\ &= |y-B_0(t_0)|^2 + 2B_0(t_0)\cdot(y-B_0(t_0)) + R^2. \end{aligned}$$

Hence

$$-R^{-1}B_0(t_0)\cdot(y-B_0(t_0))\geqq|y-B_0(t_0)|^2/2R-2r_5-2r_5^2/R$$
$$\geqq r_3^2/2R-2r_4-2r_4^2/R$$
$$\geqq r_3^2/2R-r_3^2/16R-r_3^2/16R=3r_3^2/8R.$$

$$v\cdot(y-B(t_0))=-R^{-1}B_0(t_0)\cdot(y-B_0(t_0))+(v+R^{-1}B_0(t_0))\cdot(y-B_0(t_0))$$
$$+v\cdot(B_0(t_0)-B(t_0))$$
$$\geqq 3r_3^2/8R-r_4(2R+2r_5)-2r_5$$
$$\geqq 3r_3^2/8R-(r_3^2/32R^2)(4R)-2r_4$$
$$\geqq 3r_3^2/8R-r_3^2/8R-r_3^2/16R$$
$$\geqq r_3^2/8R.$$

$$r_4|w\cdot(y-B(t_0))|\leqq r_4(2R+4r_5)\leqq r_4(4R)\leqq r_3^2/8R.$$

Therefore $y\in W(B(t_0), v, w, r_4)$ and the second claim is proved.

Now we complete the proof of the lemma by deriving (1) from (2). Let $B\in\mathbf{B}(B_0,r_5)\cap\mathscr{C}(M)$, $t\in J$. Put $C=B\circ\tau_{t-t_0}$. Now

$$\|B-C\|\leqq 11\,\|B-C\|' \quad \text{(Lemma 3.1)}$$
$$=11\max\left\{\left|\frac{1}{2\pi}\int_0^{2\pi}B(t)\,dt-\frac{1}{2\pi}\int_0^{2\pi}B\circ\tau_{t-t_0}(t)\,dt\right|,\ \|B''-(B\circ\tau_{t-t_0})''\|_\infty\right\}$$
$$=11\,\|B''-B''\circ\tau_{t-t_0}\|_\infty$$
$$\leqq 11M\,|t-t_0|^\alpha\leqq 11M\,\delta^\alpha=r_5.$$

Therefore $C\in\mathbf{B}(B_0,2r_5)$, and we can apply (2) to get independent unit vectors v, $w\in\mathbf{R}^3$ such that

$$\operatorname{im}B=\operatorname{im}C\subset W(C(t_0),v,w,r_4)=W(B(t),v,w,r_4).$$

The lemma is proved.

6.3. Lemma. *Given $\delta>0$, there is a positive number s_0, such that if $r\leqq s_0$, $(S,B)\in\mathscr{S}(r_5)$, and $t\in J$, then*

$$\Theta^2(\|S\|,B(t),r)\leqq\tfrac{1}{2}+\delta.$$

Proof. Otherwise, since $\mathscr{S}\times J$ is compact, there are convergent sequences

$$\begin{array}{cccc}(S_1,B_1), & (S_2,B_2), & \cdots\to & (S,B)\in\mathscr{S},\\ t_1, & t_2, & \cdots\to & t\in J,\\ s_1, & s_2, & \cdots\to & 0,\end{array}$$

with

$$\Theta^2(\|S_i\|,B_i(t_i),s_i)>\tfrac{1}{2}+\delta.$$

Since by Lemma 6.2 $\Theta^2(\|S\|, B(t)) = \frac{1}{2}$, we can choose r,

$$0 < r \leqq \min\{r_2, (7C_2)^{-1} \ln(2+4\delta)(2+3\delta)^{-1}\}$$

such that

$$\Theta^2(\|S\|, B(t), r) < (2+\delta)/4.$$

Put $\eta = \frac{1}{2} r(1 - (2+2\delta)^{\frac{1}{2}}(2+3\delta)^{-\frac{1}{2}})$. Choose i large enough that

(1) $s_i \leqq r - 2\eta$,

(2) $\|S_i\| (\mathbf{B}^3(B(t), r-\eta))$
$\leqq \|S\| (\mathbf{B}^3(B(t), r)) + \pi r^2 \delta/4$,

and

(3) $|B(t) - B_i(t_i)| < \eta$.

Now

$$\begin{aligned}\Theta^2(\|S\|, B(t), r) &= \|S\| (\mathbf{B}(B(t), r))/\pi r^2 \\ &\geqq (\|S_i\| (\mathbf{B}(B(t), r-\eta)) - \pi r^2 \delta/4)/\pi r^2 \\ &\geqq (\|S_i\| (\mathbf{B}(B_i(t_i), r-2\eta)))/\pi r^2 - \delta/4 \\ &= \Theta^2(\|S_i\|, B_i(t_i), r-2\eta)[(r-2\eta)/r]^2 - \delta/4 \\ &\geqq \Theta^2(\|S_i\|, B_i(t_i), s_i)\, e^{7C_2 s_i - 7C_2(r-2\eta)}[(r-2\eta)/r]^2 - \delta/4\end{aligned}$$

because by boundary montonicity 2.7, whenever $(S, B) \in \mathscr{S}$, $\Theta^2(S, b, r)\, e^{7C_2 r}$ is nondecreasing in r for $0 < r < r_0$ (since $r_0 \leqq 1/3\,C_2$)

$$\begin{aligned}&\geqq (\tfrac{1}{2} + \delta)\, e^{-7C_2 r}(2+2\delta)(2+3\delta)^{-1} - \delta/4 \\ &\geqq 4^{-1}(2+4\delta)(2+3\delta)(2+4\delta)^{-1}(2+2\delta)(2+3\delta)^{-1} - \delta/4 \\ &= (2+\delta)/4.\end{aligned}$$

But this contradicts the choice of r.

6.4. Uniform Boundary Regularity Theorem. *Given ε_1, $0 < \varepsilon_1 < \frac{1}{2}$, there is a positive number s_1, such that if*

$$(S, B) \in \mathscr{S}(r_5) \quad \text{and} \quad t \in J,$$

then

(1) $\operatorname{spt} S \cap \mathbf{U}^3(B(t), s_1)$ *is a* $C^{2,\alpha}$ *manifold* M *with boundary,*

(2) *if* y, $z \in M$-$\operatorname{im} B$ *and* T *is the plane containing* $\operatorname{Tan}(M, B(t))$, *we have the estimates*

$$\|\operatorname{Tan}(M, y) - T\| \leqq \varepsilon_1,$$

$$\|\operatorname{Tan}(M, y) - \operatorname{Tan}(M, z)\| \leqq \varepsilon_1 |(y-z)/s_1|^{1/3},$$

(3) *there are an isometry* θ *of* $\mathbf{R}^3 \cong \mathbf{R}^2 \times \mathbf{R}$ *and a continuously differentiable function* $f\colon \mathbf{R}^2 \to \mathbf{R}$ *such that*

$$\theta M \subset \text{graph } f,$$

$$\theta(B(t)) = (0, f(0)), \qquad \theta T = \mathbf{R}^2 \times \{0\},$$

$$Df(0) = 0, \qquad |D_i f(y) - D_i f(z)| \leqq \varepsilon_1 \, |(y-z)/s_1|^{1/3}$$

for $i = 1, 2$ *and* $y, z \in \mathbf{R}^2$.
Furthermore,

(4) *if* $y \in M$, $\operatorname{dist}(y, \boldsymbol{\tau}_{B(t)} T) \leqq \sqrt{2}\, \varepsilon_1 \, |y - B(t)|$.

Proof. Given $\varepsilon_1 > 0$, choose $\delta > 0$ so that Proposition 6.1 holds. Then choose $s_0 > 0$ as in Lemma 6.3. Put $s_1 = \frac{1}{2} \min\{s_0, \delta/C_2\}$. For $r \leqq s_1$, we will apply Proposition 6.1 to $\boldsymbol{\mu}_{(2s_1)^{-1}\#} S$ with boundary $(2s_1)^{-1} B$, with $b = (2s_1)^{-1} B(t)$. By Lemmas 3.3(4) and 6.3, the hypotheses of Proposition 6.1 are satisfied. The first three conclusions of our theorem follow immediately from those of the proposition. That the manifold M is $C^{2,\alpha}$ as well as C^1 follows from the higher differentiability theory of Morrey [15, Theorem 7.5].

To verify (4), let $y \in M$. By (3), $\theta y = (x_0, f(x_0))$ for some $x_0 \in \mathbf{B}^2(0, s_1)$, and for all $x \in \mathbf{B}^2(0, s_1)$, $|Df(x)| \leqq \sqrt{2}\, \varepsilon_1$. Consequently,

$$\begin{aligned} \operatorname{dist}(y, T) &= \operatorname{dist}(\theta y, \mathbf{R}^2 \times \{0\}) = |f(x_0)| \\ &\leqq \sqrt{2}\, \varepsilon_1 \, |x_0| \leqq \sqrt{2}\, \varepsilon_1 \, |\theta y - 0| = \sqrt{2}\, \varepsilon_1 \, |y - B(t)|. \end{aligned}$$

6.5. Definitions. For fixed $B_0 \in \mathscr{E}$, let J and r_5 be as in Lemma 6.2. We also denote by J the associated integral current $\mathbf{E}^1 \llcorner J$. Let J_1 be a closed interval in the interior of J. Put $l_1 = \text{length } J_1$. Let $\varphi\colon \mathbf{S}^1 \to [0, 1]$ be a C^∞ function such that $\varphi | J_1 = 1$ and $\operatorname{spt} \varphi$ is contained in the interior of J. We will sometimes denote $\mathscr{S}(r_5/2)$ simply by $\mathscr{S}$.

6.6. Lemma. $\mathbf{n}_S(B(t))$ *is a continuous function on* $\mathscr{S}(r_5/2) \times J$.

Proof. Otherwise there are ζ, $0 < \zeta < 1$, and convergent sequences

$$\begin{array}{lll} (S_1, B_1), (S_2, B_2), \cdots \to (S, B) & \text{in } \mathscr{S} \\ \quad t_1, \qquad t_2, \qquad \cdots \to \quad t & \text{in } J \end{array}$$

such that

$$|\mathbf{n}_S(B(t)) - \mathbf{n}_{S_i}(B_i(t_i))| > \zeta.$$

By replacing B_i by $B_i \circ \tau_{t_i - t}$, we can assume $t_1 = t_2 = \cdots = t$. Similarly by applying small rotations and translations if necessary we can assume

$$\begin{array}{ll} B_i(t) = B(t) & \text{(for simplicity of notation, say } B(t) = 0), \\ B_i'(t), B'(t) & \text{are linearly dependent.} \end{array}$$

Then taking just the tail of the sequence, we can assume $(S_i, B_i) \in \mathscr{S}(r_5)$.

Taking $\varepsilon_1=(\zeta/8)^2$, choose $s_1>0$ so that the conclusions of the Boundary Regularity Theorem 6.4 hold. Choose i large enough so that

$$\mathrm{HM}(\operatorname{spt} S, \operatorname{spt} S_i) \leqq \varepsilon_1 s_1 \sqrt{2}/4.$$

Put $\tau=\tau(B(t))=\tau(B_i(t))$; $v=\tau^\perp$. Put

$$u=\mathbf{n}_S(B(t)), \qquad u_i=\mathbf{n}_{S_i}(B_i(t));$$

$$W=\{x\in\mathbf{R}^3: |vx-vx\cdot u| \leqq \sqrt{2}\,\varepsilon_1 s_1\},$$

$$W_i=\{x\in\mathbf{R}^3: |vx-vx\cdot u_i| \leqq \sqrt{2}\,\varepsilon_1 s_1\}.$$

By 6.4(4),

$$\operatorname{spt} S\cap \mathbf{U}^3(0,s_1)\subset W,$$

$$\operatorname{spt} S_i\cap \mathbf{U}^3(0,s_1)\subset W_i.$$

There is some $y\in\operatorname{spt} S\cap\mathbf{U}^3(0,s_1)\cap v$ with $|y|=\frac{1}{2}s_1$ because $\operatorname{spt} S\cap\mathbf{U}^3(0,s_1)\cap v$ is a 1 manifold in $v\cap\mathbf{U}^3(0,s_1)$ with a single boundary point 0. Since $y\in W$,

$$y\cdot u=|y\cdot u|\geqq|y|-\sqrt{2}\,\varepsilon_1 s_1=s_1/2-\sqrt{2}\,\varepsilon_1 s_1.$$

$$\left|\frac{y}{|y|}-u\right|^2=2-2\frac{y}{|y|}\cdot u$$

$$=2-(4/s_1)(y\cdot u)\leqq 2-2+4\sqrt{2}\,\varepsilon_1=4\sqrt{2}\,\varepsilon_1.$$

Hence $\left|\frac{y}{|y|}-u\right|\leqq 3\sqrt{\varepsilon_1}$.

Since $\mathrm{HM}(\operatorname{spt} S, \operatorname{spt} S_i)\leqq\varepsilon_1 s_1\sqrt{2}/4<s_1/6$, there is a $z\in\operatorname{spt} S_i$ with

$$|y-z|\leqq\varepsilon_1 s_1\sqrt{2}/4<s_1/6; \qquad s_1/3\leqq|vz|\leqq 2s_1/3.$$

Since $z\in W_i$, $vz\cdot u_i\geqq|vz|-\sqrt{2}\,\varepsilon_1 s_1$.

$$\left(\frac{vz}{|vz|}-u_i\right)^2=2-2\frac{vz}{|vz|}\cdot u_i\leqq 2-2+6\sqrt{2}\,\varepsilon_1=6\sqrt{2}\,\varepsilon_1.$$

Hence $\left|\frac{vz}{|vz|}-u_i\right|\leqq 3\sqrt{\varepsilon_1}$. Finally we estimate that

$$\left|\frac{y}{|y|}-\frac{vz}{|vz|}\right|\leqq\frac{2}{|y|}|y-vz|\leqq\frac{4}{s_1}|y-z|\leqq\sqrt{2}\,\varepsilon_1\leqq 2\sqrt{\varepsilon_1}.$$

Therefore $|u-u_i|\leqq 8\sqrt{\varepsilon_1}=\zeta$. Consequently $|\mathbf{n}_S(B(t))-\mathbf{n}_{S_i}(B_i(t))|\leqq\zeta$. This contradiction proves the lemma.

6.7. Theorem. *Given $\eta>0$, there is a positive number δ such that if $(S,B), (R,C)\in\mathscr{S}=\mathscr{S}(r_5/2)$,*

$$\|B-C\|<\delta, \quad \text{and}$$

$$\sup_{t\in J_1} |\mathbf{n}_S(B(t))-\mathbf{n}_R(C(t))|<\delta,$$

then

$$\mathrm{HM}(\operatorname{spt} S, \operatorname{spt} R)<\eta.$$

Proof. The function

$$\tfrac{1}{2}(\|B-C\|+\sup_{t\in J_1}|\mathbf{n}_S(B(t))-\mathbf{n}_R(C(t))|)$$

is by Lemma 6.6 lower semicontinuous on the compact set

$$\{(S,B,R,C)\in\mathscr{S}\times\mathscr{S}: \mathrm{HM}(\operatorname{spt} S, \operatorname{spt} R)\geqq\eta\}.$$

If that set is empty, we are done trivially. Otherwise the function attains its minimum δ. By Theorem 5.1, $\delta>0$.

We conclude this section with two lemmas we will need later.

6.8. Lemma. *Given $\beta>0$, there is a nonnegative C^∞ function ψ: $\mathbf{S}^1\to\mathbf{R}$ such that for all $(S,B)\in\mathscr{S}$,*

$$\sup_{t\in\operatorname{spt}\varphi} |(\mathbf{n}_S\circ B*\psi-\mathbf{n}_S\circ B)(t)|\leqq\beta.$$

Proof. By Lemma 6.6, $\mathbf{n}_S(B(t))$ is a continuous function on $\mathscr{S}\times J$. Since $\mathscr{S}\times J$ is compact, $\mathbf{n}$ is uniformly continuous and we can choose ζ, $0<\zeta<\operatorname{dist}(\operatorname{spt}\varphi, J^c)$, such that for all $(S,B)\in\mathscr{S}$

$$|s-t|<\zeta \Rightarrow |\mathbf{n}_S(B(t))-\mathbf{n}_S(B(s))|<\beta.$$

Now let ψ be a nonnegative C^∞ function: $\mathbf{S}^1\to\mathbf{R}$ such that $\int_{\mathbf{S}^1}\psi=1$ and $\operatorname{spt}\psi\subset(-\zeta,\zeta)$. Then for any $(S,B)\in\mathscr{S}$, $t\in\operatorname{spt}\varphi$,

$$\begin{aligned}
&|\mathbf{n}_S\circ B*\psi(t)-(\mathbf{n}_S\circ B)(t)|\\
&=\Big|\int_{\mathbf{S}^1}[(\mathbf{n}_S\circ B)(t-s)-(\mathbf{n}_S\circ B)(t)]\,\psi(s)\,ds\Big|\\
&\leqq\int_{(-\zeta,\zeta)}|\mathbf{n}_S(B(t-s))-\mathbf{n}_S(B(t))|\,\psi(s)\,ds\\
&\leqq\int_{(-\zeta,\zeta)}\beta\,\psi(s)\,ds=\beta.
\end{aligned}$$

6.9. Lemma. *Given $\eta>0$, there is a positive number ζ such that if*

$$(S,B), \quad (R,C)\in\mathscr{S}$$

and

$$\int_{J_1}|\mathbf{n}_S(B(t))-\mathbf{n}_R(C(t))|^2\,|B'(t)|\,dt\leqq\zeta,$$

then

$$a_0 = \sup_{t \in J_1} |\mathbf{n}_S(B(t)) - \mathbf{n}_R(C(t))|^2 \leqq \eta.$$

Proof. Since by Lemma 6.6, $\mathbf{n}$ is uniformly continuous on $\mathscr{S} \times J$, there is a δ, $0 < \delta < l_1$, such that if (S, B), $(R, C) \in \mathscr{S}$ and $|s - t| < \delta$, then

$$\big| |\mathbf{n}_S(B(t)) - \mathbf{n}_R(C(t))|^2 - |\mathbf{n}_S(B(s)) - \mathbf{n}_R(C(s))|^2 \big| < \tfrac{1}{2}\eta.$$

Put $\zeta = \frac{1}{2}\eta\,\delta\,c_1$. Now by choice of δ, there is a subinterval of J_1 of length δ on which

$$|\mathbf{n}_S(B(t)) - \mathbf{n}_R(C(t))|^2 > a_0 - \tfrac{1}{2}\eta.$$

Consequently,

$$\int_{J_1} |\mathbf{n}_S(B(t)) - \mathbf{n}_R(C(t))|^2 \, |B'(t)| \, dt \geqq (a_0 - \tfrac{1}{2}\eta)\,\delta c_1.$$

Hence $(a_0 - \frac{1}{2}\eta)\,\delta c_1 \leqq \zeta$.

$$a_0 \leqq \tfrac{1}{2}\eta + \zeta/\delta c_1 = \eta.$$

7. The Uniqueness Theorem

We now give the main result of this paper.

7.1. Uniqueness Theorem. *Almost every curve in $\mathscr{D}$ bounds a unique mass minimizing integral current.*

Proof. Let ε be a positive number. Put

$Z = Z_\varepsilon = \{B \in \mathscr{E}$: there are mass minimizing integral currents S, T with boundary B such that $HM(\operatorname{spt} S, \operatorname{spt} T) \geqq \varepsilon\}$.

Let $M \in \mathbf{Z}^+$, $B_0 \in \mathscr{E}$, $r_5 > 0$. Put

$$Z(B_0) = Z \cap \mathbf{B}(B_0, r_5/2) \cap \mathscr{C}(M).$$

$Z(B_0)$ is compact by Lemma 3.5.

It suffices to prove that $\mu(Z(B_0)) = 0$. Indeed, if $\mu(Z(B_0)) = 0$, then $\mu(Z) = 0$ because countably many such sets cover $Z = Z_\varepsilon$. Finally taking a countable sequence of $\varepsilon' s \to 0$, we conclude that

$\mu\{B \in \mathscr{E}$: there are mass minimizing currents S, T with boundary B with $S \neq T\}$
$= \mu(\bigcup_\varepsilon Z_\varepsilon) = 0.$

Hence almost every curve in $\mathscr{E}$ and hence in $\mathscr{D} = \Psi(\mathscr{E})$ bounds a unique mass minimizing integral current.

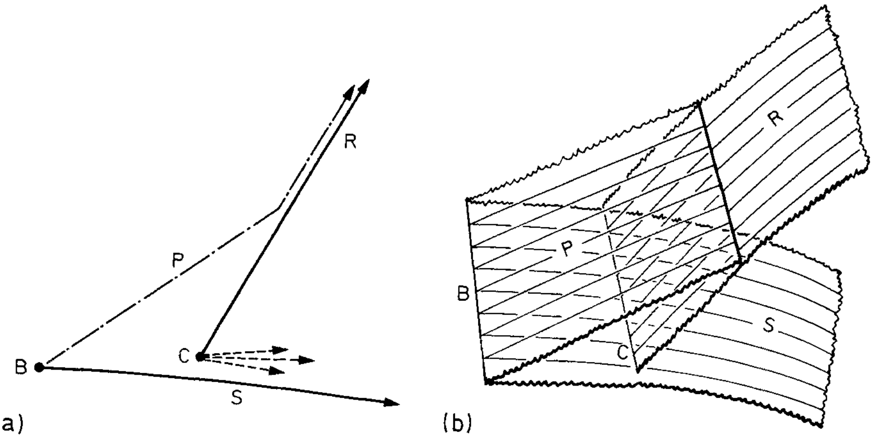

Fig. 4. In **a**, the curves are going into the paper. **b** gives a three-dimensional perspective

Now fix $\varepsilon>0$, $M\in\mathbf{Z}^+$, $B_0\in\mathscr{E}$. The previous chapters associate with B_0 positive constants $r_1\geqq r_2\geqq r_5$, c_1, C_1, C_2, a tubular neighborhood A, intervals $J_1\subset J\subset\mathbf{S}^1$, a C^∞ function φ, and a set $\mathscr{S}=\mathscr{S}(r_5/2)$. We also have the set $Z(B_0)$ as defined above.

7.2. Remark. To prove that $\mu(Z(B_0))=0$, we will show that near any $B\in Z(B_0)$, there are lots of curves that lie outside $Z(B_0)$, and then use a density argument. Indeed, we will show that if C almost lies inside an area minimizing surface S with boundary B, then $C\notin Z(B_0)$.

The heuristic reason $C\notin Z(B_0)$ is illustrated by Figure 4. If all area minimizing surfaces with boundary C go off almost horizontally along the dotted arrows (Fig. 4a), by Theorem 6.7, the surfaces stay within ε and $C\notin Z(B_0)$.

The only possible trouble arises if some area minimizing surface R goes off at some substantial angle like the solid arrow. But that cannot happen: P would have less area than S, a contradiction.

7.3. Definitions. Put $l=2\pi C_1$, so that if $B\in\mathbf{B}(B_0,r_5)\cap\mathscr{C}(M)$, then length $B\leqq l$. Choose $\delta>0$ so that Theorem 6.7 holds with $\eta=\varepsilon/2$. Next by Lemma 6.9, choose β, $0<\beta<\min\{1,\delta/2c_1\}$, so that if (S,B), $(R,C)\in\mathscr{S}$ and

$$\int_{J_1}|\mathbf{n}_S(B(t))-\mathbf{n}_R(C(t))|^2|B'(t)|\,dt\leqq 80\beta\, l,$$

then

$$\sup_{t\in J_1}|\mathbf{n}_S(B(t))-\mathbf{n}_R(C(t))|<\delta.$$

Choose ψ so that Lemma 6.8 holds. Put

$$C_3=6(1+\|\varphi\|+\|\varphi'''\|_\infty)(1+\|\psi\|+\|\psi'''\|_\infty).$$

If $(S,B)\in\mathscr{S}$, then $\varphi(\psi*\mathbf{n}_S\circ B)$ is C^∞, and one can compute that

$$\|\varphi(\psi*\mathbf{n}_S\circ B)\|\leqq 2\pi C_3,$$
$$\|(\varphi(\psi*\mathbf{n}_S\circ B))'''\|_\infty\leqq 2\pi C_3.$$

Therefore by the Approximation Lemma 4.6 we can choose $N\in\mathbf{Z}^+$ such that if $(S,B)\in\mathscr{S}$, there exists $f\in C_N$ such that

$$\|f-\varphi(\psi*\mathbf{n}_S\circ B)\|<\beta/2.$$

Put $\varepsilon_1=\min\{\beta/96,\ \beta c_1/96\,C_1\}$, and choose s_1,

$$0<s_1\leqq\min\{r_5/2,\beta c_1/8\,C_3,\varepsilon_1/C_2\}$$

so that the Regularity Theorem 6.4 holds. Henceforth assume $0<r<s_1$.

Now for $(S,B)\in\mathscr{S}$, put

$$\tilde{S}=S\llcorner\{x\in A:\rho(x)\leqq r\quad\text{and}\quad\xi(x)\in B(J)\}.$$

By Theorem 6.4,

$$F\colon \operatorname{spt}\tilde{S}\to B(J)\times[0,r]$$
$$F(x)=(\xi(x),\rho(x))$$

is a C^1 bijection. Put $G=G_S=F^{-1}$.

7.4. Lemma. *G is continuously differentiable,*

$$\left\|DG(B(t),r)-\left(\frac{B'(t)}{|B'(t)|},\mathbf{n}_S(B(t))\right)\right\|<24\varepsilon_1,$$

and

$$|J_2G-1|<84\varepsilon_1.$$

Proof. From 3.4, for $x\in\operatorname{spt}\tilde{S}$, $b=\xi(x)$,

$$\|D\xi(x)-\tau(b)\|<3\,C_2r\leqq 3\varepsilon_1.$$

Since $\rho(x)=|x-b|$, for $y\in\mathbf{R}^3$,

$$\rho(x)\,D\rho(x)(y)=(x-b)\cdot(y-D\xi(x)(y)).$$

But it follows from 6.4(4) that $|(x-b)-\rho(x)\,\mathbf{n}_S(b)|\leqq 2\varepsilon_1\rho(x)$.

$$\begin{aligned}
\rho(x)&|D\rho(x)(y)-\mathbf{n}_S(b)\cdot y|\\
&\leqq|(x-b)\cdot(y-D\xi(x)(y))-\rho(x)\,\mathbf{n}_S(b)\cdot(y-D\xi(x)(y))|\\
&\quad+|\rho(x)\,\mathbf{n}_S(b)\cdot(y-D\xi(x)(y))-\rho(x)\,\mathbf{n}_S(b)\cdot(y-\tau(b)(y))|\\
&\leqq 2\varepsilon_1\,\rho(x)(3|y|)+\rho(x)(3\varepsilon_1)|y|\\
&\leqq 9\varepsilon_1\,\rho(x)|y|.
\end{aligned}$$

Therefore if $\rho(x)>0$,

$$\|D\rho(x)-\mathbf{n}_S(b)\cdot\|\leqq 9\varepsilon_1.$$

We conclude that

$$\|DF(x)-(\tau(b),\mathbf{n}_S(b)\ \cdot)\|\leqq 12\varepsilon_1.$$

Now let T and T' denote the planes containing $\mathrm{Tan}(\operatorname{spt} S, x)$ and $\mathrm{Tan}(\operatorname{spt} S, b)$ respectively.

$$\begin{aligned}&\|DF(x)\circ T-(\tau(b),\mathbf{n}_S(b)\ \cdot)\circ T'\|\\ &\quad\leqq\|DF(x)\circ T-(\tau(b),\mathbf{n}_S(b)\ \cdot)\circ T\|\\ &\qquad+\|(\tau(b),\mathbf{n}_S(b)\ \cdot)\circ T-(\tau(b),\mathbf{n}_S(b)\ \cdot)\circ T'\|\\ &\quad\leqq 12\varepsilon_1+2\varepsilon_1=14\varepsilon_1 \qquad \text{by 6.4(2)}\\ &|J_2F\circ T(x)-1|\\ &\quad=|J_2F\circ T(x)-J_2(\tau(b),\mathbf{n}_S(\mathbf{b})\ \cdot)\circ T'|\\ &\quad\leqq 2(14\varepsilon_1)(1+14\varepsilon_1) \qquad \text{by 2.5(2)}\\ &\quad<42\varepsilon_1<1.\end{aligned}$$

Hence J_2F is nonvanishing, and G is C^1. Furthermore, since $J_2G=(J_2F)^{-1}$,

$$|J_2G-1|\leqq\frac{42\varepsilon_1}{1-42\varepsilon_1}<84\varepsilon_1.$$

Finally define a linear map

$$L\colon\ T_{B(t)}B(J)\times\mathbf{R}\cong\mathbf{R}^2\to T'\subset\mathbf{R}^3$$

$$L(u,v)=\frac{B'(t)}{|B'(t)|}u+\mathbf{n}_S(B(t))v,$$

so that $(\tau(B(t)),\ \mathbf{n}_S(B(t))\cdot)\circ L=\mathrm{id}$. Then

$$\begin{aligned}\|DG-L\|&\leqq\|T\circ L-DG\|+\|T\circ L-T'\circ L\|\\ &\leqq\|DG\|\,\|DF\circ T-(\tau,\mathbf{n}_S\cdot)\circ T'\|\,\|L\|+\|T-T'\|\,\|L\|\\ &\leqq(1-12\varepsilon_1)^{-1}(14\varepsilon_1)(1)+(\varepsilon_1)(1)\\ &<24\varepsilon_1.\end{aligned}$$

7.5. *Definitions* (see Fig. 5)

Fix $(S,B)\in\mathscr{S}$. Put $\mathbf{b}=\mathbf{n}_S\circ B$. Define a C^1 map

$$\tilde{B}_r\colon\mathbf{S}^1\to\tilde{S}$$

$$\tilde{B}_r(t)=\begin{cases}G_S(B(t),r\varphi(t)) & t\in J,\\ B(t) & t\notin\operatorname{spt}\varphi,\end{cases}$$

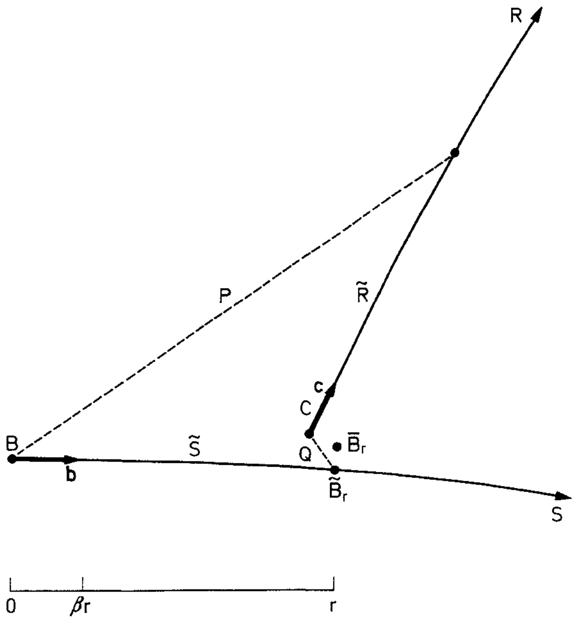

Fig. 5. The curves and surfaces defined in 7.5. The curves are going into the paper

and a C^2 map

$$\bar{B}_r\colon \mathbf{S}^1 \to \mathbf{R}^3$$

$$\bar{B}_r(t) = B(t) + r\varphi(t)(\psi * \mathbf{b}(t)).$$

Suppose $(R, C) \in \mathscr{S}$ and

$$\|C - \bar{B}_r\| \leqq r\beta.$$

Put $\mathbf{c} = \mathbf{n}_R \circ C$. Define a C^1 map

$$\tilde{C}\colon \mathbf{S}^1 \to \tilde{R}$$

$$\tilde{C}(t) = \begin{cases} G_R(C(t), r\varphi(t)) & t \in J, \\ C(t) & t \notin \operatorname{spt} \varphi. \end{cases}$$

7.6. Lemma. *We have following estimates.*

(1) $\|B - \bar{B}_r\| < r C_3$.

(2) $\|B - C\| < \frac{1}{2}\beta c_1$.

(3) $\|B - C\|_\infty < 2r$.

(4) $\|B - \tilde{C}\|_\infty < 3r$.

(5) $|(\tilde{B}_r(t) - B(t)) - r\varphi(t)\,\mathbf{b}(t)| \leqq \beta r \varphi(t)$

and

$$|(\tilde{C}(t)-C(t))-r\varphi(t)\,\mathbf{c}(t)|\leqq \beta r\varphi(t).$$

(6) $\|C-\tilde{B}_r\|_\infty<3\beta r.$

(7) $\|(\tilde{C}-B)-r\varphi(\mathbf{b}+\mathbf{c})\|_\infty<3\beta r.$

(8) $\|\tilde{B}_r'-B'\|_\infty\leqq\frac{1}{2}\beta c_1.$

(9) $\|\tilde{C}'-B'\|_\infty<\beta c_1.$

Proof.

(1) follows from the definitions of $\bar{B}_r$ and C_3.

(2) $\|B-C\|\leqq\|B-\bar{B}_r\|+\|\bar{B}_r-C\|<rC_3+r\beta<\frac{1}{2}\beta c_1.$

(3) $\|B-C\|_\infty\leqq\|B-\bar{B}_r\|_\infty+\|\bar{B}_r-C\|<r+r\beta<2r.$

(4) $\|B-\tilde{C}\|_\infty\leqq\|B-C\|_\infty+\|C-\tilde{C}\|_\infty<2r+r=3r.$

(5) Let $v=\tilde{B}_r(t)-B(t)$. $|v|=r\varphi(t)$. By choice of r and 6.4(4),

$$\operatorname{dist}(\tilde{B}_r(t),\tau_{B(t)}T)\leqq\sqrt{2}\,\varepsilon_1 r\varphi(t)<\tfrac{1}{2}\beta r\varphi(t).$$
$$|v-(v\cdot\mathbf{b}(t))\,\mathbf{b}(t)|\leqq\tfrac{1}{2}\beta r\varphi(t).$$

Squaring both sides yields

$$r^2\varphi(t)^2-(v\cdot\mathbf{b}(t))^2\leqq\tfrac{1}{4}\beta^2r^2\varphi(t)^2.$$
$$(r\varphi(t)-v\cdot\mathbf{b}(t))\,(r\varphi(t)+v\cdot\mathbf{b}(t))\leqq\tfrac{1}{4}\beta^2r^2\varphi(t)^2.$$

Since $v\cdot\mathbf{b}(t)>0$,

$$r\varphi(t)-v\cdot\mathbf{b}(t)\leqq\tfrac{1}{4}\beta^2 r\varphi(t).$$
$$|v-r\varphi(t)\,\mathbf{b}(t)|^2=2r^2\varphi(t)^2-2r\varphi(t)\,v\cdot\mathbf{b}(t)\leqq\beta^2r^2\varphi(t)^2.$$

Therefore, $|(\tilde{B}_r(t)-B(t))-r\varphi(t)\,\mathbf{b}(t)|=|v-r\varphi(t)\,\mathbf{b}(t)|\leqq\beta r\varphi(t)$. The analogous result holds for C.

(6) $$\|C-\tilde{B}_r\|_\infty\leqq\|(\tilde{B}_r-B)-r\varphi\,\mathbf{b}\|_\infty+\|\bar{B}_r-(B+r\varphi\,\mathbf{b})\|_\infty+\|\bar{B}_r-C\|$$
$$<\beta r+\beta r+\beta r=3\beta r$$

by (5); definition of $\bar{B}_r$, choice of ψ; and choice of C.

(7) $$\|(\tilde{C}-B)-r\varphi(\mathbf{b}+\mathbf{c})\|_\infty$$
$$\leqq\|(\bar{B}_r-B)-r\varphi\,\mathbf{b}\|_\infty+\|C-\bar{B}_r\|+\|(\tilde{C}-C)-r\varphi\,\mathbf{c}\|_\infty$$
$$<\beta r+\beta r+\beta r=3\beta r$$

by definition of $\bar{B}_r$, choice of ψ; choice of C; and (5).

(8) and (9) From the definition of $\tilde{B}_r$, we have

$$\tilde{B}_r'(t)=\begin{cases}DG_S(B(t),\,r\varphi(t))\,(|B'(t)|,\,r\varphi'(t)) & t\in J,\\ B'(t) & t\notin\operatorname{spt}\varphi.\end{cases}$$

$$\begin{aligned}|\tilde{B}_r'(t)-B'(t)\leqq &|DG_S(B(t),r\varphi(t))(|B'(t)|,r\varphi'(t))\\&-(B'(t)-r\varphi'(t)\,\mathbf{b}(t))|+|B'(t)+r\varphi'(t)\,\mathbf{b}(t)-B'(t)|\\&<24\varepsilon_1(C_1+rC_3)+rC_3 \qquad \text{by Lemma 7.4}\\&<24\varepsilon_1 C_1+2rC_3\\&<\beta c_1/4+2C_3(\beta c_1/8C_3)=\tfrac{1}{2}\beta c_1.\end{aligned}$$

Likewise $\|\tilde{C}'-C'\|_\infty<\frac{1}{2}\beta c_1$. Combining this estimate with (2) yields (9).

7.7. *Some Comparison Surfaces*

P. Define

$$f_P: J\times[0,1]\to\mathbf{R}^3,$$
$$f_P:(t,\lambda)\mapsto(1-\lambda)\,B(t)+\lambda\,\tilde{C}(t).$$

Put

$$P=f_{P\#}(J\times[0,1]);$$
$$\partial P=f_{P\#}(\partial J\times[0,1])-(\tilde{C}_\#(J)-B_\#(J)).$$

To estimate $\mathbf{M}(P)$, we first estimate $J_2 f_P$.

$$\begin{aligned}J_2 f_P&\leqq|D_1 f_P||D_2 f_P| \qquad (\text{cf. } 2.5(3))\\&=|(1-\lambda)\,B'(t)+\lambda\,\tilde{C}'(t)|\,|\tilde{C}(t)-B(t)|\\&\leqq|B'(t)|\,|\tilde{C}(t)-B(t)|(1+\beta) \qquad \text{by 7.6(9) because } |B'(t)|\geqq c_1.\end{aligned}$$

$$\begin{aligned}\mathbf{M}(P)&\leqq\int_{J\times[0,1]} J_2 f_P\,dt\,d\lambda \qquad \text{by 2.5(1)}\\&\leqq(1+\beta)\int_J|\tilde{C}(t)-B(t)|\,|B'(t)|\,dt\\&\leqq\int_J|r\varphi(\mathbf{b}+\mathbf{c})(t)|\,|B'(t)|\,dt+6\beta r l \qquad \text{by 7.6(4, 7).}\end{aligned}$$

Q. Define

$$f_Q:\ \mathbf{S}^1\times[0,1]\to\mathbf{R}^3,$$
$$f_Q:\ (t,\lambda)\mapsto(1-\lambda)\,\tilde{B}_r(t)+\lambda\,C(t),$$

and put

$$Q=f_{Q\#}(\mathbf{S}^1\times[0,1]),$$
$$\bar{Q}=f_{Q\#}((\mathbf{S}^1-J)\times[0,1]).$$

We have that

$$\partial Q = \tilde{B}_{r\#}(\mathbf{S}^1) - C_\#(\mathbf{S}^1) = \tilde{B}_{r\#}(J) + B_\#(\mathbf{S}^1 - J) - C_\#(\mathbf{S}^1),$$
$$\partial \bar{Q} = \tilde{B}_{r\#}(\mathbf{S}^1 - J) - C_\#(\mathbf{S}^1 - J) - f_{Q\#}(\partial J \times [0,1])$$
$$= B_\#(\mathbf{S}^1 - J) - C_\#(\mathbf{S}^1 - J) - f_{P\#}(\partial J \times [0,1]).$$

Here we have used the fact that on $\bar{J}^c$, $B = \tilde{B}_r$, $C = \tilde{C}$, $f_P = f_Q$.

As for P we can compute

$$J_2 f_Q \leqq |(1-\lambda)\tilde{B}_r'(t) + \lambda C'(t)||\tilde{B}_r(t) - C(t)|$$
$$\leqq |B'(t)||\tilde{B}_r(t) - C(t)|(1+\beta) \quad \text{by 7.6(2, 8).}$$

$$\mathbf{M}(Q) \leqq \int_{\mathbf{S}^1 \times [0,1]} J_2 f \, dt \, d\lambda$$
$$\leqq (1+\beta) \int_{\mathbf{S}^1} |\tilde{B}_r(t) - C(t)||B'(t)| \, dt$$
$$\leqq 6\beta l r \quad \text{by 7.6(6).}$$

$\tilde{S}$ *and* $\tilde{R}$. We observe that

$$\tilde{S} = G_S \circ (B \times \mathrm{id})_\# (\mathbf{E}^2 \llcorner \{(t,s) \in J \times [0,r] : s \leqq r\varphi(t)\}),$$
$$\tilde{R} = G_R \circ (C \times \mathrm{id})_\# (\mathbf{E}^2 \llcorner \{(t,s) \in J \times [0,r] : s \leqq r\varphi(t)\}),$$
$$\partial \tilde{S} = B_\#(J) - \tilde{B}_{r\#}(J),$$
$$\partial \tilde{R} = C_\#(J) - \tilde{C}_\#(J).$$

$$\mathbf{M}(\tilde{S}) = \int_{t \in J} \int_{s=0}^{r\varphi(t)} (J_2 G_S \circ (B \times \mathrm{id})) \, ds \, dt$$
$$\geqq \int_{t \in J} \int_{s=0}^{r\varphi(t)} |B'(t)| \, ds \, dt - 84\varepsilon_1 r l \quad \text{by Lemma 7.4}$$
$$\geqq \int_{t \in J} r\varphi(t)|B'(t)| \, dt - \beta r l.$$

Similarly,

$$\mathbf{M}(R) \geqq \int_{t \in J} \int_{s=0}^{r\varphi(t)} |C'(t)| \, ds \, dt - 84\varepsilon_1 r l$$
$$\geqq \int_{t \in J} \int_{s=0}^{r\varphi(t)} |B'(t)| \, ds \, dt - \beta r c_1/2 - 84\beta r l/96$$

by 7.6(2) and choice of ε_1

$$\geqq \int_{t \in J} r\varphi(t)|B'(t)| \, dt - \beta r\, l/4\pi - 7\beta r\, l/8$$
$$\geqq \int_{t \in J} r\varphi(t)|B'(t)| \, dt - \beta r\, l.$$

X and Y. Put

$$X=(S-\tilde{S})-Q,$$
$$Y=(R-\tilde{R})+P+\tilde{Q}.$$
$$\begin{aligned}\partial X&=\partial S-\partial\tilde{S}-\partial Q\\&=B_{\#}(\mathbf{S}^1)-B_{\#}(J)+\tilde{B}_{r\#}(J)-\tilde{B}_{r\#}(J)-B_{\#}(\mathbf{S}^1-J)+C_{\#}(\mathbf{S}^1)\\&=C_{\#}(\mathbf{S}^1)=\partial R.\end{aligned}$$

Therefore, since R minimizes mass,

$$\mathbf{M}(R)\leqq\mathbf{M}(X)\leqq\mathbf{M}\,(S-\tilde{S})+\mathbf{M}(Q)=\mathbf{M}(S)-\mathbf{M}(\tilde{S})+\mathbf{M}(Q).$$

Similarly, $\partial Y=\partial S$ and therefore

$$\mathbf{M}(S)\leqq\mathbf{M}(Y)\leqq\mathbf{M}(R)-\mathbf{M}(\tilde{R})+\mathbf{M}(P)+\mathbf{M}(Q).$$

Adding the inequalities yields

$$\mathbf{M}(\tilde{S})+\mathbf{M}(\tilde{R})\leqq\mathbf{M}(P)+2\mathbf{M}(Q).$$

Now applying our previous estimates on these masses yields that

$$\int_{t\in J}(r\varphi(t)+r\varphi(t)-|r\varphi(t)(\mathbf{b}(t)+\mathbf{c}(t))|)|B'(t)|\,dt\leqq 20\beta r\,l.$$

Since the integrand is nonnegative,

$$\int_{t\in J_1} r\varphi(t)(2-|\mathbf{b}(t)+\mathbf{c}(t)|)|B'(t)|\,dt\leqq 20\beta r\,l.$$

Since $\varphi|J_1=1$,

$$\int_{t\in J_1}(2-|\mathbf{b}+\mathbf{c}|\,(t))|B'(t)|\,dt\leqq 20\beta l.$$

Since $|\mathbf{b}-\mathbf{c}|^2=(2+|\mathbf{b}+\mathbf{c}|)(2-|\mathbf{b}+\mathbf{c}|)\leqq 4(2-|\mathbf{b}+\mathbf{c}|)$,

$$\int_{t\in J_1}|\mathbf{b}(t)-\mathbf{c}(t)|^2|B'(t)|\,dt\leqq 80\beta l.$$

By choice of β,

$$\sup_{t\in J_1}|\mathbf{b}(t)-\mathbf{c}(t)|<\delta.$$

Since also by 7.6(2) $\|B-C\|<\frac{1}{2}\beta c_1<\delta$, it follows by choice of δ that

$$\mathrm{HM}(\operatorname{spt} S,\operatorname{spt} R)<\varepsilon/2.$$

We conclude that if $(R_1, C), (R_2, C) \in \mathscr{S}$, then $\mathrm{HM}(\operatorname{spt} R_1, \operatorname{spt} R_2) < \varepsilon$, so that $C \notin Z(B_0)$.

In summary, we have shown that

$$\mathbf{B}(\bar{B}_r, \beta r) \cap Z(B_0) = \varnothing,$$

while we also know that

$$\mathbf{B}(\bar{B}_r, \beta r) \subset \mathbf{B}(B, 2r C_3).$$

7.8. The Density Argument. $Z(B_0)$ is compact and hence measurable. For $E \in \mathscr{C}^N$, put

$$Z(B_0, E) = \{D \in \mathscr{C}_N \colon D + E \in Z(B_0)\}.$$

We want to apply the Density Lemma 2.4 to $Z(B_0, E)$ and the measure μ_N, which satisfies the hypotheses by Lemma 4.5.

Let $D \in Z(B_0, E)$. Put $B = D + E \in \mathbf{B}(B_0, r_5/2) \cap \mathscr{C}(M)$. By choice of N, we can choose $\bar{D} \in \mathscr{C}_N$ such that

$$\|\bar{D} - \varphi(\psi * \mathbf{b})\| < \tfrac{1}{2}\beta.$$

Then for $r < s_1$,

$$\|E + D + r\bar{D} - \bar{B}_r\| < \tfrac{1}{2}\beta r.$$

By the conclusion of 7.7,

$$\mathbf{B}(D + r\bar{D}, \tfrac{1}{2}\beta r) \cap Z(B_0, E) = \varnothing,$$
$$\mathbf{B}(D + r\bar{D}, \tfrac{1}{2}\beta r) \subset \mathbf{B}(D, 3r C_3).$$

$$\begin{aligned}\frac{\mu_N(\mathbf{B}(D, 3r C_3) \cap Z(B_0, E))}{\mu_N(\mathbf{B}(D, 3r C_3))} &\leqq 1 - \frac{\mu_N(\mathbf{B}(D + r\bar{D}, \tfrac{1}{2}\beta r))}{\mu_N(\mathbf{B}(D, 3r C_3))} \\ &\leqq 1 - \frac{k_0(\tfrac{1}{2}\beta r)^{3(2N+1)}}{k_1(3r C_3)^{3(2N+1)}} \qquad \text{by Lemma 4.5} \\ &\leqq 1 - \frac{k_0}{k_1}\left(\frac{\beta}{6C_3}\right)^{3(2N+1)}.\end{aligned}$$

Therefore,

$$\overline{\lim_{s \to 0}} \frac{\mu_N(\mathbf{B}(D, s) \cap Z(B_0, E))}{\mu_N(\mathbf{B}(D, s))} < 1.$$

Since this holds for all $D \in Z(B_0, E)$, it follows from Lemma 2.4 that

$$\mu_N(Z(B_0, E)) = 0.$$

Since this holds for all $E \in \mathscr{C}^N$, by Fubini's Theorem,

$$\mu(Z(B_0)) = 0.$$

This proves the Uniqueness Theorem.

7.9. *Remark. Almost every curve is an embedding,* i.e., $\mu(\mathscr{C}-\mathscr{E})=0$. Hence almost every curve in $\mathscr{C}$ or in $\mathscr{Z}_1(\mathbf{R}^3)$ *bounds a unique mass minimizing integral current.*

Proof. We use product decompositions and Fubini's Theorem as in the argument above (7.8). It is convenient to identify $\mathbf{R}^2$ with $\mathbf{C}$ and view

$$\Phi\colon (\mathbf{C}\times\mathbf{C}\times\mathbf{R}\times\mathbf{R}\times\prod_{n\neq 1}\mathbf{R}^3, \mathscr{G}^2\times\mathscr{G}^2\times\mathscr{G}^1\times\mathscr{G}^1\times\mathscr{L}^3\times\prod_{n\neq 0,1}\mathscr{G}^3)$$
$$\to \mathbf{R}^3\cong\mathbf{C}\times\mathbf{R},$$

$$B=\Phi(C_1,C_{-1},D_1,D_{-1},(B_0,B_2,B_{-2},\ldots))$$
$$=(2^{-\frac{1}{2}}[C_1e^{it}+C_{-1}e^{-it}],D_1\cos t+D_{-1}\sin t)$$
$$+B_0-\sum_{n=2}^{\infty}B_n n^{-3}\cos nt+B_{-n}n^{-3}\sin nt.$$

Denote by P projection of $\mathbf{R}^3$ onto $\mathbf{R}^2\times\{0\}$.

(1) For fixed C_{-1}, D_1, D_{-1}, $(B_0,B_2,\ldots)$, for $\mathscr{L}^2$ (or equivalently $\mathscr{G}^2$) almost all C_1, $|P\circ B'|>0$. That is, for fixed $C^{2,\alpha}$ $f\colon \mathbf{S}^1\to\mathbf{C}$, for almost all $C_1\in\mathbf{C}$, $f-2^{-\frac{1}{2}}C_1e^{it}$ has a nonvanishing derivative: indeed, it will unless $C_1\in\{-2^{-\frac{1}{2}}if'(t)e^{-it}\colon t\in\mathbf{S}^1\}$, which is a set of measure zero. Therefore almost every curve in $\mathscr{C}$ has a nonvanishing derivative.

(2) For $D_1=D_{-1}=0$, for fixed C_{-1}, $(B_0,B_2,\ldots)$, for almost all C_1, $P\circ B$ has only finitely many self intersections. That is, for fixed $C^{2,\alpha}f\colon \mathbf{S}^1\to\mathbf{C}$, for almost all $C_1\in\mathbf{C}$, $f-2^{-\frac{1}{2}}C_1e^{it}$ has only finitely many self intersections. As proof, consider the C^1 function

$$F\colon \mathbf{S}^1\times\mathbf{S}^1-\{(u,u)\colon u\in\mathbf{S}^1\}\to\mathbf{C}$$
$$F\colon (s,t)\mapsto\frac{f(t)-f(s)}{e^{it}-e^{is}}\sqrt{2}.$$

Then $f-2^{-\frac{1}{2}}C_1e^{it}$ has $\frac{1}{2}\operatorname{card}F^{-1}(C_1)$ self intersections. But by the area formula

$$\int_{\mathbf{C}}\operatorname{card}F^{-1}(C_1)\,d\mathscr{L}^2C_1=\int J_2F\,d\mathscr{L}^2<\infty$$

because J_2F is bounded (as one can easily check). Therefore $\operatorname{card}F^{-1}(C_1)<\infty$ for almost all C_1.

(3) Using the same argument as in (1), we deduce from (2) that almost every curve in $\mathscr{C}$ is injective.

We conclude from (1) and (3) that $\mu(\mathscr{C}-\mathscr{E})=0$.

7.10. *Remark.* The results of this paper hold equally well for flat chains modulo two. In fact, some of the arguments simplify in that case. We conclude that *almost every curve bounds a unique mass minimizing flat chain modulo two.*

7.11. *Remark.* The results of this paper also hold for the classical Plateau's problem (cf. [5, pp. 95ff]). *Almost every curve bounds a geometrically unique immersed disc of least mapping area.* However, there seem to be open sets of curves bounding more than one stable immersed disc: we only assert that for almost every such curve, one such surface will have less area than all the others.

7.12. Remark. The conclusion of 7.7 shows that $Z(B_0)$ is nowhere dense. It follows that *the set of curves bounding more than one area minimizing surface is a set of the first category.*

References

1. Allard, W.K.: On the first variation of a varifold, Ann. of Math. **95**, 417–491 (1972)
2. Allard, W.K.: On the first variation of a varifold: boundary behavior, Ann. of Math. **101**, 418–446 (1975)
3. Aronszajn, N.: Sur l'unicité du prolongement des solutions des équations aux dérivées partielles elliptiques du second ordre, C. R. Acad. Sci. Paris **242**, 723–725 (1956)
4. Bromwich, T.: An Introduction to the Theory of Infinite Series, London: Macmillan and Co., 1908
5. Courant, R.: Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces, New York: Interscience, 1950
6. Courant, R., Hilbert, D.: Methods of Mathematical Physics, New York: Interscience, 1962
7. Federer, H.: Geometric Measure Theory, Berlin, Heidelberg, New York: Springer 1969
8. Federer, H.: The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension, Bull. A.M.S. **76**, 767–771 (1970)
9. Fleming, W.H.: An example in the problem of least area, Proc. Amer. Math. Soc. **7**, 1063–1074 (1956)
10. Hunt, G.A.: Random Fourier transforms, Trans. Amer. Math. Soc. **71**, 38–69 (1951)
11. Kadota, T.K., Shepp, L.A.: Conditions for absolute continuity between a certain pair of probability measures, Z. Wahrscheinlichkeitstheorie Verw. Gebiete **16**, 250–260 (1970)
12. Lévy, P.: Le problème de Plateau, Mathematica **23**, 1–45 (1947–48)
13. Morgan, F.: A smooth curve in $\mathbb{R}^4$ bounding a continuum of area minimizing surfaces, Duke Math. J. **43**, 867–870 (1976)
14. Morrey, C.B., Jr.: Multiple Integrals in the Calculus of Variations, Berlin, Heidelberg, New York: Springer-Verlag 1966
15. Morrey, C.B., Jr.: Second order elliptic systems of differential equations, in Contributions to the Theory of Partial Differential Equations, edited by L. Bers, S. Bochner, and F. John, Annals of Mathematics Studies, Number 33, Princeton: University Press, 1954
16. Nitsche, J.C.C.: Concerning the isolated character of solutions of Plateau's Problem, Math. Z. **109**, 393–411 (1969)
17. Nitsche, J.C.C.: Contours bounding at least three solutions of Plateau's Problem, Arch. Rat. Mech. Analysis **30**, 1–11 (1968)
18. Nitsche, J.C.C.: A new uniqueness theorem for minimal surfaces, Arch. Rat. Mech. Analysis **52**, 319–329 (1973)
19. Radó, T.: On the Problem of Plateau, Berlin, Heidelberg, New York: Springer 1933, reprint 1971

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