Inventiones math. 45, 221 – 251 (1978)

Inventiones mathematicae

Collision Orbits in the Anisotropic Kepler Problem

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The anisotropic Kepler problem is a one-parameter family of classical mechanical systems with two degrees of freedom. When the parameter $\mu=1$, we have the well known Kepler or central force problem. As μ increases beyond 1, we introduce more and more anisotropy into the Kepler problem. As we show below, this changes the orbit structure of the system dramatically.

When $\mu=1$, the system is completely integrable and the orbit structure is well understood. With the exception of certain collision orbits, all orbits are closed and lie on two dimensional tori in the case of negative total energy.

For $\mu>1$, we keep the same potential energy, but make the kinetic energy anisotropic, i.e. the kinetic energy becomes

$$K(p)=\tfrac{1}{2}(\mu p_1^2+p_2^2).$$

This forces orbits to oscillate more and more rapidly about the y-axis as μ increases. By the time μ reaches 9/8, this behavior becomes highly random: one can find orbits which oscillate an arbitrarily large number of times about the y-axis before crossing the x-axis.

This motivates the introduction of symbolic dynamics into the problem. For each $\mu>9/8$, we isolate a closed subset M_μ of phase space having the following characterization: to each orbit remaining for all time in M_μ we may associate a doubly infinite sequence of symbols

$$(s)=(\dots s_{-2}, s_{-1}, s_0; s_1, s_2, \dots)$$

where each s_k is a non-zero integer. We determine (s) as follows: $|s_k|$ indicates the number of times the orbit crosses the y-axis between the k^{th} and the $(k+1)^{\text{st}}$ passage close to the x-axis. This will be made more precise in §7. The sign of s_k indicates whether the orbit oscillates about the positive of negative y-axis during the k^{th} passage. Let Σ_k denote the set of all such sequences where $|s_j|\geqq k$ for all j. Our main result is then:

* Partially supported by NSF Grant MPS 74-06731 A 01 at Northwestern University, Evanston, Illinois

Theorem A. *There is an open and dense subset of parameter values in* $(9/8, \infty)$ *for which the anisotropic Kepler problem satisfies: there is a closed subset* M_μ *of phase space and an integer* $k(\mu)$ *such that the set of orbits which remain for all time in* M_μ *is topologically conjugate to the suspension of the Bernoulli shift on the closure of* $\Sigma_{k(\mu)}$.

In particular, any periodic sequence in $\Sigma_{k(\mu)}$ corresponds to a hyperbolic closed orbit of the system. Hence there exist infinitely many long closed orbits for the system, and the associated string of symbols accurately describes the qualitative behavior of the trajectory in configuration space.

We remark that this theorem improves a result of Gutzwiller [8], who showed the existence of similar orbits for certain values of μ via different methods.

The main ingredient in the proof of Theorem A is the existence of *bi-collision* orbits in the anisotropic Kepler problem. These are orbits which begin and end in collision with the origin. For the ordinary Kepler problem, the bi-collision orbits fill a two dimensional submanifold of phase space. For $\mu > 1$, this submanifold breaks up. Some bi-collision orbits do persist: however they are generally much longer than the bi-collision orbits in the Kepler problem. In fact, as a Corollary of the above Theorem, we have:

Corollary B. *For an open and dense set of parameter values* $\mu > 9/8$, *there is a bi-collision orbit of the system associated as above with each finite string of symbols of the form* $(s_0, s_1, \ldots, s_n)$.

To study the bi-collision orbits, we employ a technique introduced by McGehee [10] to study triple collision in the collinear three body problem. By a change of time scale, the singularity at the origin in configuration space is removed, and in its place is pasted an invariant torus Λ. The new flow extends analytically over Λ, and on this torus, the flow is easily understood. The only non-wandering points on Λ are eight equilibria: two sources, two sinks, and four saddle points. Orbits which previously began or ended in collision with the origin now tend asymptotically to one of the equilibria. In this framework, bi-collision orbits can be interpreted as heteroclinic solutions of the new system.

There are four primary bi-collision orbits which are crucial for the existence of the Bernoulli shift. Two of these orbits connect the sinks to the sources in Λ. When $\mu > 9/8$, the characteristic exponents at the sinks and sources become complex. Nearby orbits then tend to spiral around these primary bi-collision orbits. This accounts for the oscillatory behavior described above.

The two other primary bi-collision orbits connect distinct saddle points in Λ, and, by symmetry, are trapped on the x-axis in configuration space. Each of these orbits lies in the intersection of two dimensional stable and unstable manifolds of distinct equilibria in Λ. One of our most important results is that these manifolds meet transversely along the primary bi-collision orbits.

Theorem C. *For all* $\mu > 1$, *the primary bi-collision orbits along the* x*-axis are transversal heteroclinic solutions of the system.*

The proof employs techniques introduced by Conley and Easton and is given in § 6.

The four primary bi-collision orbits together with certain heteroclinic solutions within Λ thus set up a chain of heteroclinic solutions. Using an idea of Easton [5], we set up a sequence of "windows" or transversals along this chain. Orbits which pass from one window to the next in a certain order then determine the subset M_μ. Orbits which continue to reintersect these windows for all time become the subsystem which is conjugate to the Bernoulli shift. This is treated in §7.

We should like to acknowledge several helpful conversations with R. McGehee and Z. Nitecki while this paper was being written.

§1. The Anisotropic Kepler Problem

In this section we discuss the basic properties of the planar anisotropic Kepler problem. The configuration space for the system is the plane $\mathbb{R}^2$ with Cartesian coordinates $\mathbf{q}+(q_1, q_2)$. The phase space is the tangent bundle of the plane $T\mathbb{R}^2$ with coordinates $\mathbf{p}=(p_1, p_2)$ in the fibers.

The anisotropic Kepler problem is then given as a first order system of ordinary differential equations, or a vector field on $T\mathbb{R}^2$ by

$$\begin{aligned}\dot{\mathbf{q}}&=M\mathbf{p}\\ \dot{\mathbf{p}}&=-\mathbf{q}/|\mathbf{q}|^3.\end{aligned} \tag{1.1}$$

Here M is the 2×2 matrix

$$\begin{pmatrix}\mu & 0\\ 0 & 1\end{pmatrix} \tag{1.2}$$

where $\mu \geqq 1$. When $\mu=1$, this system describes the ordinary Kepler or central force problem in Newtonian mechanics; when $\mu>1$, the system is no longer spherically symmetric, and μ measures how anisotropic the system is.

We henceforth denote by X_μ the vector field given by (1.1). Note that X_μ has a singularity at $\mathbf{q}=\mathbf{0}$.

This vector field is a Hamiltonian system on $T\mathbb{R}^2$. Let V be the usual central force potential

$$V(\mathbf{q})=\frac{1}{|\mathbf{q}|}. \tag{1.3}$$

And define the kinetic energy K by

$$K(\mathbf{p})=\tfrac{1}{2}\mathbf{p}^t M\mathbf{p}. \tag{1.4}$$

The total energy E is then given by

$$E=K-V. \tag{1.5}$$

Then (1.1) may be written in Hamiltonian form with E as the Hamiltonian:

$$\dot{\mathbf{q}} = \frac{\partial E}{\partial \mathbf{p}}$$
$$\dot{\mathbf{p}} = \frac{-\partial E}{\partial \mathbf{q}}. \tag{1.6}$$

Remark. One can equivalently define the system by making the potential anisotropic. More precisely, let

$$V'(\mathbf{q}) = \frac{1}{|M^{\frac{1}{2}}\mathbf{q}|},$$
$$K'(\mathbf{p}) = \tfrac{1}{2}|\mathbf{p}|^2,$$
$$E' = K' - V'.$$

The Hamiltonian system with Hamiltonian E' is then easily seen to be linearly equivalent to (1.6).

Since the system (1.6) is Hamiltonian, it is well known that the total energy E is an integral for the system. That is, E is constant along the solution curves of X_μ. Consequently, the level sets of E are invariant under the flow of X_μ. We denote the level set $E^{-1}(e)$ by Σ_e; such level sets are usually called *energy surfaces.* Henceforth, we consider only negative energy surfaces, i.e. the case where $e < 0$.

We wish to consider in more detail the topology of the various Σ_e. For this purpose we make the change of variables

$$\mathbf{q} = r\mathbf{s}$$
$$\mathbf{p} = r^{-\frac{1}{2}}\mathbf{u}, \tag{1.7}$$

where $\mathbf{s}$ is a point on the unit circle S^1 and where $\mathbf{u}$ is a vector in $\mathbb{R}^2$. The system (1.6) becomes

$$\dot{r} = r^{-\frac{1}{2}}\mathbf{s}^t M\mathbf{u}$$
$$\dot{\mathbf{s}} = r^{-\frac{3}{2}}(M\mathbf{u} - (\mathbf{s}^t M\mathbf{u})\mathbf{s}) \tag{1.8}$$
$$\dot{\mathbf{u}} = r^{-\frac{3}{2}}(\tfrac{1}{2}(\mathbf{s}^t M\mathbf{u})\mathbf{u} - \mathbf{s})$$

and the total energy relation becomes:

$$re = \tfrac{1}{2}\mathbf{u}^t M\mathbf{u} - 1. \tag{1.9}$$

(1.8) is an analytic vector field on the open manifold $(0, \infty) \times S^1 \times \mathbb{R}^2$.

Since e is negative, it follows that

$$0 \leqq r \leqq -1/e$$

for any solution curve in the energy surface Σ_e. That is, Σ_e projects onto the disk of radius $-1/e$ in $\mathbb{R}^2-\{0\}$. Over the interior of this disk, Σ_e intersects the fibers of $T\mathbb{R}^2$ along ellipses given by

$$\tfrac{1}{2}\mathbf{u}^t M\mathbf{u}=1+re.$$

Along the boundary $r=1/e$, Σ_e intersects the fibers only at the zero vector. For this reason, the curve

$$\begin{aligned} r&=-1/e\\ \mathbf{u}&=\mathbf{0}\end{aligned} \tag{1.10}$$

is called the *oval of zero velocity* in Σ_e. Henceforth, we denote this curve by Z.

Proposition 1.1. *For $e<0$, Σ_e is diffeomorphic to an open solid torus (i.e. a solid torus minus its boundary).*

Proof. Let B be the interior of the ellipse

$$\tfrac{1}{2}\mathbf{u}^t M\mathbf{u}=1$$

in the plane. Define Φ: $\Sigma_e\to S^1\times B$ by

$$\Phi(r,\mathbf{s},\mathbf{u})=(\mathbf{s},\mathbf{u})$$

Φ is clearly the required diffeomorphism. q.e.d.

Note that the diffeomorphism Φ maps the oval of zero velocity onto the core circle of the torus and that $r=0$ corresponds to the (missing) boundary of $S^1\times B$.

§2. The Collision Manifold

The system (1.8) defines an analytic vector field on the open manifold $(0,\infty)\times S^1\times\mathbb{R}^2$. The system does not extend over the boundary $r=0$ since, as $r\to 0$, the vector field X_μ "blows up". In this section we slow down the vector field so that the new system does extend to $r=0$.

To accomplish this, we change the time variable via

$$dt=r^{\frac{3}{2}}\,d\tau. \tag{2.1}$$

In the new time scale, (1.8) becomes

$$\begin{aligned} \dot r&=r(\mathbf{s}^t M\mathbf{u})\\ \dot{\mathbf{s}}&=M\mathbf{u}-(\mathbf{s}^t M\mathbf{u})\,\mathbf{s}\\ \dot{\mathbf{u}}&=\tfrac{1}{2}(\mathbf{s}^t M\mathbf{u})\,\mathbf{u}-\mathbf{s}\end{aligned} \tag{2.2}$$

while the total energy remains

$$re=\tfrac{1}{2}\mathbf{u}^t M\mathbf{u}-1. \tag{2.3}$$

We note several immediate consequences of this change of scale. First, (2.2) has no singularity at $r=0$; in fact, this system extends analytically to all of $[0,\infty]\times S^1\times\mathbb{R}^2$. Second, the submanifold $r=0$ is now invariant under the flow. Thus, this change of time scale has the effect of pasting an invariant boundary onto the phase space, and orbits of (1.8) are simply reparametrized. In the sequel we consider only the extended system (2.2), which we continue to denote by X_μ.

We now restrict attention to a single energy surface Σ_e with $e<0$. Via (2.3), Σ_e meets the boundary $r=0$ along the submanifold Λ defined by

$$\begin{aligned}&\tfrac{1}{2}\mathbf{u}^t M\,\mathbf{u}=1\\&\mathbf{s}\text{ arbitrary.}\end{aligned}\tag{2.4}$$

Λ is clearly diffeomorphic to a two-dimensional torus which we call the *collision manifold*. Note that Λ is independent of the total energy. Thus the change of time scale also has the effect of pasting an invariant boundary onto each Σ_e. X_μ extends over this boundary as before, and is given by

$$\begin{aligned}&\dot{\mathbf{s}}=M\,\mathbf{u}-(\mathbf{s}^t M\,\mathbf{u})\,\mathbf{s}\\&\dot{\mathbf{u}}=\tfrac{1}{2}(\mathbf{s}^t M\,\mathbf{u})\,\mathbf{u}-\mathbf{s}.\end{aligned}\tag{2.5}$$

We summarize this construction in the following proposition.

Proposition 2.1. *For the extended vector field X_μ in (2.2), the negative energy surfaces are diffeomorphic to solid tori. The boundary of Σ_e is the collision manifold Λ, and X_μ extends analytically over Λ.*

Orbits which previously began or ended in collision with the origin now tend asymptotically away from or toward Λ. Orbits which previously passed close to collision now come very close to Λ. How these orbits behave near the singularity is thus governed by the flow on Λ. We therefore discuss this flow in some detail in the next two sections.

3. The Collision Manifold for the Kepler Problem

The material in this section serves mainly as motivation for our treatment of the collision manifold for the case $\mu>1$ in the next section. Most of the material below is due to McGehee [11].

When $\mu=1$, the vector field X_1 is given by

$$\begin{aligned}&\dot{r}=r(\mathbf{s}^t\,\mathbf{u})\\&\dot{\mathbf{s}}=\mathbf{u}-(\mathbf{s}^t\,\mathbf{u})\,\mathbf{s}\\&\dot{\mathbf{u}}=\tfrac{1}{2}(\mathbf{s}^t\,\mathbf{u})\,\mathbf{u}-\mathbf{s}\end{aligned}\tag{3.1}$$

and the total energy is given by

$$re=\tfrac{1}{2}|\mathbf{u}|^2-1.\tag{3.2}$$

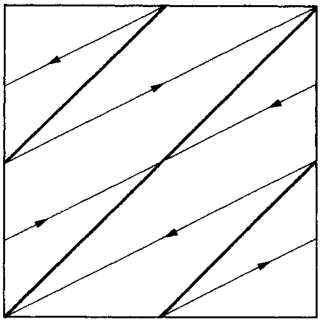

Fig. 1. The flow on the collision manifold for $\mu = 1$

We introduce new variables

$$\mathbf{s} = (\cos\theta, \sin\theta)$$
$$\mathbf{u} = \sqrt{2(re+1)}(\cos\psi, \sin\psi) \tag{3.3}$$

with θ, ψ defined mod 2π. The differential equation (3.1) is transformed into

$$\begin{aligned}\dot{r} &= r\sqrt{2(re+1)}\cos(\psi-\theta)\\ \dot{\theta} &= \sqrt{2(re+1)}\sin(\psi-\theta)\\ \dot{\psi} &= \frac{1}{\sqrt{2(re+1)}}\sin(\psi-\theta).\end{aligned} \tag{3.4}$$

On Λ, (3.4) reduces to

$$\begin{aligned}\dot{\theta} &= \sqrt{2}\sin(\psi-\theta)\\ \dot{\psi} &= \frac{1}{\sqrt{2}}\sin(\psi-\theta).\end{aligned} \tag{3.5}$$

This system is easily solved. For our purposes, however, it will be enough to note two features of this flow. First,

$$\theta - 2\psi \tag{3.6}$$

is constant along the orbits of (3.5). And secondly, the vector field vanishes along precisely two circles in Λ, namely the circles $\psi = \theta$ and $\psi = \theta + \pi$. All other orbits on Λ tend asymptotically away from $\psi = \theta + \pi$ and toward $\psi = \theta$. A sketch of this flow is given in Figure 1.

For the case $\mu > 1$, we show below that both of the circles of equilibria break up into isolated hyperbolic equilibrium points. However, some vestiges of the invariant circles remain: this is a consequence of the normal hyperbolicity studied by Sacker [14] and Hirsch, Pugh, Shub [9]. We take the liberty to adapt their definitions and theorems to our particular circumstance.

Definition 3.1. Let ϕ_t be a smooth flow on a manifold M and suppose C is a submanifold of M consisting entirely of equilibrium points for the flow. C is said to be normally hyperbolic if the tangent bundle to M over C splits into three subbundles TC, E^s, and E^u invariant under $d\phi_t$ and satisfying

a) $d\phi_t$ contracts E^s exponentially,
b) $d\phi_t$ expands E^u exponentially,
c) TC = tangent bundle of C.

For normally hyperbolic submanifolds one has the usual existence of smooth stable and unstable manifolds together with the persistence of these invariant manifolds under small perturbations. More precisely, we have the theorem:

Theorem 3.2. *Let C be a normally hyperbolic submanifold of equilibrium points for ϕ_t. Then there exist smooth stable and unstable manifolds tangent along C to $E^s \oplus TC$ and $E^u \oplus TC$ respectively. Furthermore, both C and the stable and unstable manifolds are permanent under small perturbations of the flow.*

For the proof of Theorem 3.2, we refer the reader to [9]. We emphasize that Theorem 3.2 does not say that a submanifold of zeroes persists; indeed all zeroes may be destroyed by a small perturbation. However, there must be some invariant manifold nearby.

As remarked above, we are primarily interested in applying Theorem 3.2 to the circles of equilibria in the Kepler problem. We thus need the following proposition.

Proposition 3.3. *The circles $\psi = \theta$ and $\psi = \theta + \pi$ are normally hyperbolic circles of equilibria for X_1.*

Proof. Using (3.4), one computes easily that

$$DX(0,\theta,\theta) = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

which has eigenvalues $\sqrt{2}, \frac{-1}{\sqrt{2}}, 0$. A similar computation shows that $DX(0,\theta,\theta+\pi)$ has eigenvalues $-\sqrt{2}, \frac{1}{\sqrt{2}}, 0$. q.e.d.

By Theorem 3.2, it follows that both circles of equilibria admit two dimensional stable and unstable manifolds. The stable manifold of $\psi = \theta$ lies entirely within Λ, and so does the unstable manifold of $\psi = \theta + \pi$ (see Fig. 1). On the other hand, the stable manifold of $\psi = \theta + \pi$ is a cylinder of orbits which tend asymptotically to Λ, while the unstable manifold corresponding to $\psi = \theta$ consists of orbits which tend asymptotically toward Λ in the negative time direction. Inspection of the flow on Λ shows that these are the only orbits which tend asymptotically toward Λ.

Definition 3.4. An orbit of the anisotropic Kepler problem which tends asymptotically to Λ in either forward or backward time is called a collision orbit. An orbit which tends to Λ in both directions is called a bi-collision orbit.

We have proven:

Proposition 3.5. *The set of collision orbits for negative energy in the Kepler problem consists (locally) of two smooth cylinders of orbits.*

Actually, much more can be said. Both the stable manifold of $\psi=\theta+\pi$ and the unstable manifold of $\psi=\theta$ meet along the oval of zero velocity Z. This means that orbits which leave Λ reach a point of zero velocity, and then fall back along an orbit heading for collision. A proof of this can be found in [13]. In our terminology, we have:

Proposition 3.6. *All collision orbits for the Kepler problem with negative energy are bi-collision orbits.*

§4. The Collision Manifold in the Anisotropic Problem

In this section we turn our attention to the flow on the collision manifold when $\mu>1$. As before we first introduce the variables

$$\begin{aligned} \mathbf{s} &= (\cos\theta, \sin\theta) \\ \mathbf{u} &= \sqrt{2(1+re)}\left(\frac{1}{\sqrt{\mu}}\cos\psi, \sin\psi\right) \\ dt &= \sqrt{2(1+re)}\,d\tau. \end{aligned} \tag{4.1}$$

The vector field (2.2) becomes

$$\begin{aligned} \dot r &= 2(1+re)(r)(\sqrt{\mu}\cos(\psi)\cos(\theta)+\sin(\psi)\sin(\theta)) \\ \dot\theta &= 2(1+re)(\sin(\psi)\cos(\theta)-\sqrt{\mu}\cos(\psi)\sin(\theta)) \\ \dot\psi &= \sqrt{\mu}\sin(\psi)\cos(\theta)-\cos(\psi)\sin(\theta). \end{aligned} \tag{4.2}$$

(4.2) is a vector field on $[0,-1/e]\times S^1\times S^1$. The boundary $r=0$ is the collision manifold Λ which we shall deal with here. The boundary $r=-1/e$ is the zero velocity manifold which we shall deal with in §6.

Restricted to Λ, the system is

$$\begin{aligned} \dot\theta &= 2(\sin(\psi)\cos(\theta)-\sqrt{\mu}\cos(\psi)\sin(\theta)) \\ \dot\psi &= \sqrt{\mu}\sin(\psi)\cos(\theta)-\cos(\psi)\sin(\theta). \end{aligned} \tag{4.3}$$

For $\mu=1$ we have the usual Kepler problem with two circles of equilibria on Λ. When $\mu>1$, each of these circles breaks up into four distinct equilibria: two sources (or sinks) and two hyperbolic saddle points. We single this fact out as a proposition.

Proposition 4.1. *The vector field X_μ admits exactly eight equilibrium solutions. The locations as well as the characteristic exponents of these equilibria are as displayed in Table 1.*

Proof. To see that these are the only equilibria on Λ, we first note that $\dot{\theta}=0$ iff both ψ and θ are multiples of π, or else

$$\cot\theta=\sqrt{\mu}\cot\psi.$$

On the other hand, $\dot{\psi}=0$ iff both ψ and θ are multiples of π, or else

$$\cot\psi=\sqrt{\mu}\cot\theta.$$

Since $\mu>1$, it follows that

$$\cot\psi=0=\cot\theta.$$

Examination of all of the possibilities then yields the result.

To compute the characteristic exponents of the various equilibria, one simply notes that

$$DX(0,\theta,\psi)=\begin{pmatrix} v & 0 \\ 0 & A \end{pmatrix}$$

where A is a 2×2 matrix giving the linearization of the restriction of X_μ to Λ, and where v is either ± 2 or $\pm\mu^{\frac{1}{2}}$, depending on the equilibrium point.

Table 1

Equilibrium point	Characteristic Exponents		Type on Λ
	on Λ	off Λ	
$\left(-\frac{\pi}{2},-\frac{\pi}{2}\right)$	$-\frac{1}{2}\pm\frac{1}{2}\sqrt{9-8\mu}$	2	Sink
$(0,0)$	$-\frac{\sqrt{\mu}}{2}\pm\frac{1}{2}\sqrt{9\mu-8}$	$2\sqrt{\mu}$	Saddle
$\left(\frac{\pi}{2},\frac{\pi}{2}\right)$	$-\frac{1}{2}\pm\frac{1}{2}\sqrt{9-8\mu}$	2	Sink
(π,π)	$-\frac{\sqrt{\mu}}{2}\pm\frac{1}{2}\sqrt{9\mu-8}$	$2\sqrt{\mu}$	Saddle
$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$\frac{1}{2}\pm\frac{1}{2}\sqrt{9-8\mu}$	-2	Source
$(0,\pi)$	$\frac{\sqrt{\mu}}{2}\pm\frac{1}{2}\sqrt{9\mu-8}$	$-2\sqrt{\mu}$	Saddle
$\left(\frac{\pi}{2},-\frac{\pi}{2}\right)$	$\frac{1}{2}\pm\frac{1}{2}\sqrt{9-8\mu}$	-2	Source
$(\pi,0)$	$\frac{\sqrt{\mu}}{2}\pm\frac{1}{2}\sqrt{9\mu-8}$	$-2\sqrt{\mu}$	Saddle

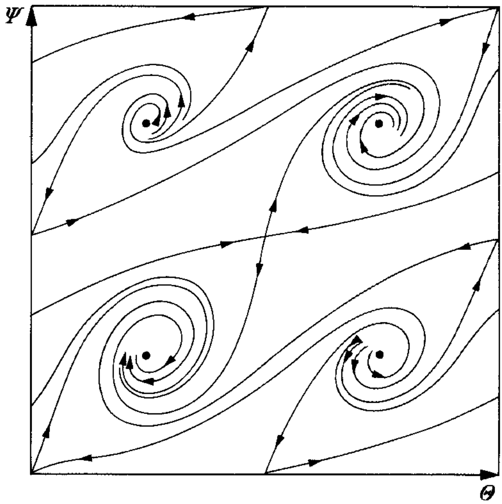

Fig. 2. The phase portrait of the flow on Λ for $\mu > 9/8$

Finally, since

$$\mu < 9\mu - 8$$

it follows that

$$\frac{\sqrt{\mu}}{2} < \frac{\sqrt{9\mu - 8}}{2} \tag{4.5}$$

and hence that each of the saddle points in Λ have one positive and one negative characteristic exponent, as required. We leave the rest of the details to the reader. q.e.d.

Note that the characteristic exponents of both the sinks and the sources have non-zero imaginary part when $\mu > 9/8$. This means that orbits of X_μ tend to spiral into and away from the corresponding sinks and sources for high anisotropy. This fact will have important consequences later on. For now we single it out as a corollary.

Corollary 4.2. *If $\mu > 9/8$, then each of the sinks and sources on the collision manifold have non-real characteristic exponents.*

We sketch the phase portrait of the flow on Λ in Figure 2.

One final qualitative feature of the flow on the collision manifold is the ultimate behavior of the stable and unstable manifolds of the saddle points. Recall that the stable (resp. unstable) manifold of a hyperbolic equilibrium point consists of all points which tend asymptotically toward (resp. away from) the equilibrium. It is well known that the stable and unstable manifolds are smooth immersed manifolds. In our case, each of the stable and unstable manifolds of the saddle points are analytic curves which consist of precisely two orbits tending toward or away from the equilibrium. We denote the stable (resp. unstable) manifold of an equilibrium point p by $W^s(p)$ (resp. $W^u(p)$).

Definition 4.3. Let p, q be distinct hyperbolic equilibria for a flow. An orbit γ is said to be heteroclinic if γ lies in $W^s(p) \cap W^u(q)$. If $p=q$, γ is said to be homoclinic. In the two-dimensional case, when p and q are both saddles, γ is also called a saddle connection.

Below we show that, for most values of μ, there are no saddle connections for the flow on Λ. Before proceeding with the proof, however, we need several additional facts.

Definition 4.4. Let X be a smooth vector field and let f be a smooth real-valued function. X is said to be gradient-like with respect to f if f increases along all non-equilibrium orbits.

Proposition 4.5. *Let $f_\mu: \Lambda \to \mathbb{R}$ be given by*

$$f_\mu(\mathbf{s}, \mathbf{u}) = |M^{-\frac{1}{2}}\mathbf{s}|^{-\frac{1}{2}}(\mathbf{s}^t\mathbf{u}) \tag{4.6}$$

where $M^{-\frac{1}{2}}$ is the 2×2 matrix given by

$$\begin{pmatrix} \mu^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then X_μ is gradient-like with respect to f_μ.

Proof. We first compute the time derivative of f_μ along an orbit.

$$\begin{aligned} \dot{f}_\mu &= -\tfrac{1}{2}|M^{-\frac{1}{2}}\mathbf{s}|^{-\frac{5}{2}}(\mathbf{s}^t\mathbf{u})(\mathbf{s}^t M^{-1}\dot{\mathbf{s}}) + |M^{-\frac{1}{2}}\mathbf{s}|^{-\frac{1}{2}}(\dot{\mathbf{s}}^t\mathbf{u} - \mathbf{s}^t\dot{\mathbf{u}}) \\ &= |M^{-\frac{1}{2}}\mathbf{s}|^{-\frac{5}{2}}\{-\tfrac{1}{2}(\mathbf{s}^t\mathbf{u})^2 + \mathbf{s}^t M^{-1}\mathbf{s}(\mathbf{u}^t M\mathbf{u} - 1)\} \\ &= |M^{-\frac{1}{2}}\mathbf{s}|^{-\frac{5}{2}}(-\tfrac{1}{2}(\mathbf{s}^t\mathbf{u})^2 + \mathbf{s}^t M^{-1}\mathbf{s}). \end{aligned}$$

Using (4.1) it follows that

$$|M^{-\frac{1}{2}}\mathbf{s}|^{\frac{5}{2}}\dot{f}_\mu = \left(\frac{1}{\sqrt{\mu}}\cos(\theta)\sin(\psi) - \sin(\theta)\cos(\psi)\right)^2. \tag{4.7}$$

Hence $\dot{f}_\mu \geqq 0$. If $\dot{f}_\mu = 0$, then it follows from (4.3) that $\dot{\theta} = 0$ also. On the other hand, X_μ is never tangent to $\dot{\theta} = 0$ in Λ, except at the equilibria. This implies that f_μ increases along all non-equilibrium orbits, and this completes the proof

Corollary 4.6. *There are no closed or recurrent orbits for X_μ on Λ.*

Proof. If so, $\dot{f}_\mu$ would vanish identically on such an orbit. q.e.d.

As a consequence, all orbits of X_μ must tend toward one of the equilibria.

Figure 3 shows the flow on Λ relative to the gradient function f_μ. Note that f_μ has maxima at two sinks for X_μ and minima at the sources. At the saddle

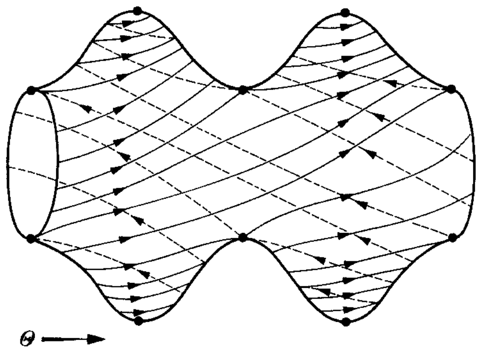

Fig. 3. The flow on the collision manifold relative to f_μ. We take f_μ to be projection onto a vertical axis (a "height" function)

points, one computes easily that

$$\begin{aligned} f_\mu(0,0) &= f_\mu(\pi,\pi) = (2/\mu^{\frac{1}{2}})^{\frac{1}{2}} \\ f_\mu(\pi,0) &= f_\mu(0,\pi) = -(2/\mu^{\frac{1}{2}})^{\frac{1}{2}}. \end{aligned} \tag{4.8}$$

Consequently the stable manifolds at $(\pi, 0)$ and $(0, \pi)$ must emanate directly from the sources, while the unstable manifolds at $(0, 0)$ and (π, π) must fall directly into sinks for all $\mu > 1$.

To show that there are no saddle connections for X_μ, it thus suffices to show that the remaining invariant manifolds do not match up. This is true for most values of μ.

Remark 4.7. The two branches of $W^s(\pi,0)$ each emanate from distinct sources in Λ. This follows immediately from the fact that (4.3) is invariant under the reflection

$$(\theta, \psi) \to (-\theta, -\psi).$$

Similar results hold for the other saddle points.

Proposition 4.8. *For an open and dense set of real numbers* $\mu > 1$, *the unstable manifolds at* $(\pi, 0)$ *and* $(0, \pi)$ *miss the stable manifolds at* $(0, 0)$ *and* (π, π).

Proof. First consider $W^u(\pi, 0) = W^u(-\pi, 0)$. Eliminating time from (4.3), we have

$$\frac{d\psi}{d\theta} = \frac{\sin(\psi-\theta) + \varepsilon\cos(\theta)\sin(\psi)}{2\sin(\psi-\theta) - 2\varepsilon\cos(\psi)\sin(\theta)} = F(\theta, \psi, \varepsilon) \tag{4.9}$$

where $\varepsilon = \mu^{\frac{1}{2}} - 1$. When $\varepsilon = 0$, $W^s(\pi, \pi)$ matches up exactly with $W^u(-\pi, 0)$. See Figure 1. Consider the branch of $W^u(-\pi, 0)$ which contains the point $(0, \pi/2)$. By (3.6), this curve lies along the line

$$2\psi - \theta = \pi. \tag{4.10}$$

When ε increases above 0, this branch of the unstable manifold varies smoothly in Λ. Let $\zeta(\theta, \varepsilon)$ denote the ψ-coordinate of this curve, with $\zeta(-\pi, \varepsilon) = 0$. Below we prove:

Lemma 4.9. $\frac{\partial}{\partial \varepsilon} \zeta(0,0) < 0$.

Consequently, $\zeta(0, \varepsilon) < \pi/2$ for ε small and positive. An easy computation then shows that $f_{1+\varepsilon}(0, \zeta(0, \varepsilon)) > 0$.

Now (4.3) is reversed by the transformation

$$(\theta, \psi) \to (-\theta, \pi - \psi). \tag{4.11}$$

In particular, the unstable manifold through $(-\pi, 0)$ is mapped onto the stable manifold through (π, π) by this map. Hence the stable manifold intersects the ψ-axis at some point $(0, \psi_0)$ with $\psi_0 > \pi/2$. At this point, however, $f_{1+\varepsilon}(0, \psi_0) < 0$. Consequently, the unstable manifold through $(\pi, 0)$ misses the stable manifold through (π, π), at least for ε small and positive.

Similar arguments show that all the unstable manifolds of the saddle points miss all the stable manifolds, at least for small $\varepsilon > 0$. Since these stable and unstable manifolds vary analytically with ε, it follows that they intersect only for a discrete set of values of ε. This completes the proof with the exception of Lemma 4.9.

Proof of Lemma 4.9. Note that ζ satisfies the equation

$$\zeta(\theta, \varepsilon) = \int_{-\pi}^{\theta} F(s, \zeta(s), \varepsilon)\, ds$$

where F is given by (4.9). Write

$$\zeta = \zeta_0(\theta) + \varepsilon \zeta_1(\theta) + O(\varepsilon^2).$$

We have shown (3.6) that

$$\zeta_0(\theta) = \tfrac{1}{2}\theta + \pi/2.$$

We now compute $\zeta_1(\theta)$:

$$\begin{aligned}
\zeta_1(\theta) &= \int_{-\pi}^{\theta} \left(\frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial \psi} \zeta_1(s) \right) ds \\
&= \int_{-\pi}^{\theta} \frac{\sin(\zeta_0 + s)}{2\sin(\zeta_0 - s)}\, ds \\
&= \tfrac{1}{2} \int_{-\pi}^{\theta} \frac{\sin(\tfrac{3}{2}s + \pi/2)}{\sin(-\tfrac{1}{2}s + \pi/2)}\, ds \\
&= \int_{-\pi/2}^{\theta/2} \frac{\cos(\tfrac{3}{2}s)}{\cos(\tfrac{1}{2}s)}\, ds \\
&= \sin(\theta) - \pi/2 - \theta/2.
\end{aligned}$$

Thus, when $\theta=0$, we have

$$\zeta_1(0)=-\pi/2=\frac{\partial}{\partial\varepsilon}\zeta(0,0).$$

This completes the proof of the lemma. q.e.d.

Combining Propositions 4.5 and 4.8, we have

Theorem 4.10. *For an open dense set of $\mu>1$, the flow of X_μ on Λ satisfies: all stable manifolds of saddle points emanate from sources, and all unstable manifolds of saddle points die in sinks.*

§ 5. Collision Orbits

In this section, we consider the special orbits of the anisotropic Kepler problem which begin and/or end in collision with the origin. The change of time scale (2.1) has the effect of slowing such orbits down so that they tend asymptotically toward or away from Λ.

Let $W^s(\Lambda)$ (resp. $W^u(\Lambda)$) denote the set of points in $\Sigma_e-\Lambda$ whose forward (resp. backward) orbits converge to Λ. Clearly, $W^s(\Lambda)$ is contained in the union of the stable manifolds of four of the equilibria in Λ, while $W^u(\Lambda)$ is contained in the unstable manifolds of the remaining four equilibria.

Recall that a bi-collision orbit is an orbit which begins and ends at collision. Such orbits lie in $W^s(\Lambda)\cap W^u(\Lambda)$, i.e., they are heteroclinic orbits. For $\mu=1$, all collision orbits are bi-collision orbits. When $\mu>1$, this is no longer true, as we shall prove below. There are several bi-collision orbits which do persist for all μ, however; these are called the *primary bi-collision orbits* and are given by the proposition below.

Proposition 5.1. *There are four bi-collision orbits for the anisotropic problem which persist for all μ. Each orbit leaves the origin and travels along the positive or negative q_i-axis to the oval of zero velocity and then returns to Λ.*

Proof. The proof follows immediately from the invariance of (1.1) under the reflections $(q_1,p_1)\to(-q_1,-p_1)$ and $(q_2,p_2)\to(-q_2,-p_2)$. q.e.d.

Notation. We denote the primary bi-collision orbit along the positive (resp. negative) q_i-axis by γ_i^+ (resp. γ_i^-). Also let

$$q_i^+=\gamma_i^+\cap Z,$$
$$q_i^-=\gamma_i^-\cap Z$$

where Z is the oval of zero velocity.

The proof of the next proposition is straightforward.

Proposition 5.2.

i) $\gamma_1^+\subset W^u(0,0)\cap W^s(0,\pi)$,

ii) $\gamma_1^-\subset W^u(\pi,\pi)\cap W^s(\pi,0)$,

iii) $\gamma_2^+=W^u(\pi/2,\pi/2)\cap W^s(\pi/2,-\pi/2)$,

iv) $\gamma_2^-=W^u(-\pi/2,-\pi/2)\cap W^s(-\pi/2,\pi/2)$.

From Table 1, both $W^u(\pm\pi/2, \pm\pi/2)$ and $W^s(\pm\pi/2, \mp\pi/2)$ are one-dimensional; this accounts for the equality in iii) and iv) above. This is, of course, a highly nongeneric situation. On the other hand, each of the invariant manifolds in i) and ii) are two-dimensional. It is natural to ask how these submanifolds meet along $\gamma_1^\pm$. In the next section, we will show that, in fact, $W^s(0,\pi)$ meets $W^u(0,0)$ transversely along γ_1^+, and similarly, $W^s(\pi,0)$ and $W^u(\pi,\pi)$ meet transversely along γ_1^-. For the remainder of this section, however, we confine our attention to the bi-collision orbits $\gamma_2^\pm$.

Let Λ be the annular submanifold of Σ_e defined by

$$\Lambda = \{(r, \mathbf{s}, \mathbf{u}) | r > 0,\ \mathbf{s}^t \mathbf{u} = 0\}. \tag{5.1}$$

Note that Λ contains the oval of zero velocity as its central circle.

From (2.2), we have that

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\mathbf{s}} &= \mathbf{0} \\ \dot{\mathbf{u}} &= -\mathbf{s} \end{aligned} \tag{5.2}$$

along Z. Hence

$$\frac{d}{dt}(\mathbf{s}^t \mathbf{u}) = -|\mathbf{s}|^2 = -1 \tag{5.3}$$

along the oval of zero velocity. This implies that orbits of X_μ cross Λ transversely in a neighborhood of Z. We wish to examine how $W^s(\Lambda)$ and $W^u(\Lambda)$ meet this neighborhood, at least near $q_2^\pm$.

Let F_μ denote the set of points in Λ whose forward orbits converge to Λ without reintersecting Λ. Note that $q_i^\pm$ belongs to F_μ for $i=1,2$. Also, $F_1 = Z$ identically, as we observed in Proposition 3.6. In Figure 4, we sketch F_μ for various values of μ.

Proposition 5.3. *Suppose $\mu > 9/8$. Then there is a neighborhood N of q_2^+ such that $F_\mu \cap N$ contains two smooth spirals converging to q_2^+.*

Proof. The bi-collision orbit γ_2^+ through q_2^+ is precisely the stable manifold of the point $(\pi/2, -\pi/2)$ in Λ. Nearby points in F_μ must therefore lie in the stable manifolds of either $(0,\pi)$ or $(\pi, 0)$. Hence we examine the behavior of these manifolds near γ_2^+.

By Remark 4.7, one branch of both $W^s(0,\pi)$ and $W^s(\pi,0)$ emanates from the source $(\pi/2, -\pi/2)$ in Λ. For $\mu > 9/8$, these orbits spiral away from the source. Let $\gamma = \gamma(s)$ be an arc in $W^s(\pi,0)$ which meets Λ transversely at $\gamma(0) \neq (\pi, 0)$. We follow the orbit of each point in γ backwards in time until it first hits Λ. This will give one of the spirals above.

Near $(\pi, 0)$, we make a smooth change of variables

$$(R, \Theta, z) = \psi(r, \theta, \psi) \tag{5.4}$$

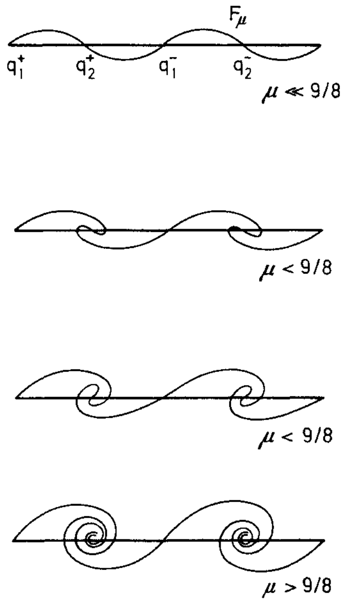

Fig. 4. F_μ denotes the set of points in A which converge directly to Λ

so that the system goes over to

$$\begin{aligned}\dot{R}&=-\alpha R+f_1(R,\Theta,z)\\ \dot{\Theta}&=\beta+f_2(R,\Theta,z)\\ \dot{z}&=\lambda z\end{aligned}\tag{5.5}$$

where $\alpha, \beta, \lambda > 0$, and where $f_i(R, \Theta, 0)=0$. Hence the plane $z=0$ represents the stable manifold of the equilibrium, while $R=0$ gives the unstable manifold.

Let

$$\Sigma(r_0,\delta)=\{(R,\Theta,z)\,|\,R=r_0,\ 0<z<\delta\}.\tag{5.6}$$

For r_0, δ small enough, $\Sigma(r_0, \delta)$ is transverse to the flow of (5.5). Let Σ' denote the plane $z=z^*$. Choosing r_0, δ smaller if necessary, there is defined a smooth Poincare map

$$\Phi\colon \Sigma(r_0,\delta)\to\Sigma'\tag{5.7}$$

obtained by following orbits forward in time until their first intersection with Σ'.

One computes immediately that Φ assumes the form

$$\begin{aligned}R_1&=R_1(\Theta,z,r_0,z^*)=Kz^{\alpha/\lambda}+A_1+B_1\\ \Theta_1&=\Theta_1(\Theta,z,r_0,z^*)=\Theta+(\beta/\lambda)\log(z^*/z)+A_2+B_2\end{aligned}\tag{5.8}$$

where A_i and B_i are analytic and satisfy $A_i(\Theta, z, 0, 0)=0$ and $B_i(\Theta, z, r_0, z^*)\to 0$ together with their first partial derivatives as $z\to 0$.

Using (5.8), one checks that if γ is any smooth curve in $\Sigma(r_0, \delta)$ which meets $z=0$ transversely, then $\Phi(\gamma(s))$ is a spiral in $z=z^*$ converging to $(0, -, z^*)$. One finally observes that the ordinary Poincare map from $z=z^*$ to A preserves this spiral. This completes the proof. q.e.d.

Corollary 5.4. *There are infinitely many bi-collision orbits in the anisotropic Kepler problem if* $\mu>9/8$.

Proof. It follows immediately from Proposition 5.3 that F_μ meets Z at infinitely many points near $q_2^\pm$. These points also lie in $R(F_\mu)$ where R is the reflection

$$R(r, \theta, \psi)=(r, \theta, \psi+\pi). \tag{5.9}$$

But R reverses X_μ, and so

$$R(F_\mu)\subset W^u(\Lambda).$$

Hence each intersection point above also lies in

$$W^s(\Lambda)\cap W^u(\Lambda). \quad \text{q.e.d.}$$

$W^s(\Lambda)$ consists of the two-dimensional stable manifolds of $(0, \pi)$ and $(\pi, 0)$ separated by the one-dimensional stable manifolds $\gamma_2^\pm$. When $\mu>9/8$, the two-dimensional stable manifolds spiral around $\gamma_2^\pm$ and hence $W^s(\Lambda)$ fails to be a smoothly immersed submanifold along these orbits. For $1<\mu<9/8$, however, the equilibria at $(\pm\pi/2, \mp\pi/2)$ have real and distinct characteristic exponents. This eliminates the spiralling and gives:

Proposition 5.5. *For all but a discrete set of* μ *in* $(1, 9/8)$, *both* $W^s(\Lambda)$ *and* $W^u(\Lambda)$ *are immersed submanifolds of* Σ_e.

Proof. We prove this only for $W^s(\Lambda)$ along γ_2^+. By (4.8), one branch of $W^s(0, \pi)\cap\Lambda$ and $W^s(\pi, 0)\cap\Lambda$ emanates directly from the source at $(\pi/2, -\pi/2)$. Thus there are two cases to consider: either these branches coincide exactly with the strong unstable manifold at $(\pi/2, -\pi/2)$ or else both branches emanate from the source in the direction of the weaker expanding eigenvalue. By symmetry, one of these possibilities holds simultaneously for all stable manifolds in Λ.

Perturbing away from the Kepler problem shows that the latter possibility occurs initially (this uses the normal hyperbolicity of Proposition 3.3). Using analyticity of these curves, it follows that $W^s(0, \pi)\cap\Lambda$ misses the strong unstable manifold for all but a discrete set of values in $(1, 9/8)$. For the non-exceptional values of μ, one may then construct a Poincare map as in the proof of Proposition 5.3 to show that $W^s(0, \pi)$ accumulates along γ_2^+ with a well defined tangent direction at each point. We leave the details to the reader. q.e.d.

Remark 5.6. Below we shall show that F_μ meets Z transversely at $q_1^\pm$. Via analyticity, it then follows that, for $1<\mu<9/8$, either $F_\mu\cap Z$ is a finite set of points, or else $F_\mu\cap Z$ contains infinitely many points which of necessity only accumulate at $q_2^\pm$. In the latter case, Proposition 5.5 implies that F_μ must be tangent to Z at $q_2^{\pm}$, and that furthermore, the angles of intersection must approach 0 as the points of intersection tend to $q_2^{\pm}$.

We wish to observe one final detail about the spiral given by Proposition 5.3. For each $\mu > 9/8$, F_μ meets Z at infinitely many points $\{x_i(\mu)\}$ converging monotonically to q_2^+ as $i \to \infty$. Let $\lambda(x_i)$ denote the angle formed by F_μ and Z relative to the coordinate system (θ, ψ) in A. Define

$$\lambda(\mu) = \lim_{(x_i \to q_2^+)} (\lambda(x_i(\mu))).$$

Using the Poincare map (5.8), it is not hard to show that this limit exists for all $\mu > 9/8$. Furthermore, since the change of coordinates (5.4) as well as $W^s(0, \pi)$ depend analytically on μ, it follows that λ is also real analytic for $\mu > 9/8$.

Proposition 5.7. *For all but a discrete subset of* $(9/8, \infty)$, $\lambda(\mu) \neq \pi/2$.

Proof. It suffices to show that $\lambda(\mu) \neq \pi/2$ for some $\mu \in (9/8, \infty)$. If $\lambda(\mu) = \pi/2$ identically, then it follows easily that $F_{9/8}$ meets Z transversely at infinitely many points near q_2^+. Furthermore, the angles of intersection of F_μ and Z at these points are bounded away from 0 in a neighborhood of q_2^+. This property then holds for all μ in a neighborhood of 9/8, in particular for an open set of $\mu < 9/8$. This, however, is impossible by Remark 5.6.

6. The Zero Velocity Manifold

The goal of this section is to prove that $W^s(0, \pi)$ meets $W^u(0, 0)$ transversely along the bi-collision orbit γ_1^+. The key idea in the proof is how orbits in these invariant manifolds approach the oval of zero velocity. To study such orbits we "blow up" the oval of zero velocity into a two-dimensional torus Ω. The original system extends analytically over Ω, and the behavior of orbits close to Z is governed by the flow on Ω.

Recall that the change of variables (4.1) converts the original system to

$$\begin{aligned} \dot{r} &= 2(1+er)(r)(\sqrt{\mu}\cos(\psi)\cos(\theta) + \sin(\psi)\sin(\theta)) \\ \dot{\theta} &= 2(1+er)(\sin(\psi)\cos(\theta) - \sqrt{\mu}\cos(\psi)\sin(\theta)) \\ \dot{\psi} &= \sqrt{\mu}\sin(\psi)\cos(\theta) - \cos(\psi)\sin(\theta). \end{aligned} \tag{6.1}$$

This system is an analytic vector field on $[0, -1/e] \times T^2$. The boundary $\{0\} \times T^2$ corresponds to the collision manifold Λ; the boundary $\{-1/e\} \times T^2$ corresponds to the oval of zero velocity in the old coordinates. We denote this component of the boundary by Ω and call it the *zero velocity manifold*. Clearly, Ω is invariant under the flow generated by (6.1).

On Ω, the induced system is given by

$$\begin{aligned} \dot{\theta} &= 0 \\ \dot{\psi} &= \sqrt{\mu}\sin(\psi)\cos(\theta) - \cos(\psi)\sin(\theta). \end{aligned} \tag{6.2}$$

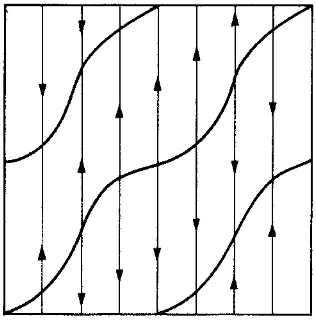

Fig. 5. The flow on the zero velocity manifold

This system is easily solved. For all μ there are two circles of equilibria in Ω given by

$$\sqrt{\mu}\sin(\psi)\cos(\theta)=\cos(\psi)\sin(\theta). \tag{6.3}$$

For $\mu=1$, (6.3) gives the circles $\psi=\theta$ and $\psi=\theta+\pi$ in Ω. For $\mu>1$, these circles are slightly skewed. See Figure 5. We denote by C_1 the circle of equilibria passing through the point $(-1/e,0,0)$ in Ω. Let C_2 be the other circle. Both C_1 and C_2 are normally hyperbolic as one sees by computing

$$DX(-1/e,\theta,\psi)=\begin{pmatrix}-2\nu(\theta,\psi) & 0 & 0\\ 0 & 0 & 0\\ 0 & \xi(\theta,\psi) & \nu(\theta,\psi)\end{pmatrix} \tag{6.4}$$

where

$$\nu(\theta,\psi)=\sqrt{\mu}\cos(\psi)\cos(\theta)+\sin(\psi)\sin(\theta),$$
$$\xi(\theta,\psi)=-\sqrt{\mu}\sin(\psi)\sin(\theta)-\cos(\psi)\cos(\theto).$$

We observe that $\nu\neq 0$ along C_i, since

$$\nu\cos(\psi)=\sqrt{\mu}\cos(\theta) \tag{6.5}$$

along C_i.

It also follows from (6.4) that C_1 is a repellor for the flow on Ω and that C_2 is an attractor. From (6.1) we have that the flow is everywhere tangent to $\theta=$constant. Thus the phase portrait of (6.2) is given as in Figure 5.

By normal hyperbolicity there is a two-dimensional cylinder of orbits tending toward C_1 in the direction normal to Ω. We denote this cylinder by $W^s(\Omega)$. Similarly, one has a two dimensional cylinder of orbits $W^u(\Omega)$ tending away from C_2. In the original system, these cylinders correspond to the set of orbits which cross the oval of zero velocity.

For later purposes, we observe that both circles of equilibria meet the circle $\theta=0$ at an angle $\beta(\mu)$ where, for $\mu>1$, $\beta(\mu)$ satisfies

$$\pi/4<\beta(\mu)<\pi/2. \tag{6.6}$$

This can be proved by implicit differentiation of (6.3).

The remainder of this section is devoted to proving the following:

Theorem C. *If* $\mu > 1$, *then* $W^s(0,\pi)$ *meets* $W^u(0,0)$ *(resp.* $W^s(\pi,0)$ *meets* $W^u(\pi,\pi)$*) transversely along the bi-collision orbit* γ_1^+ *(resp.* γ_1^-*).*

Proof. We prove that $W^s(0,\pi)$ meets $W^u(0,0)$ transversely along γ_1^+; the other case then follows by symmetry.

The idea of the proof is to examine the intersection of $W^s(0,\pi)$ and $W^u(0,0)$ with the torus $r=r_0$ in Σ_e. Denote this torus by $T(r_0)$. If r_0 is close enough to $-1/e$, $T(r_0)$ is an isolating block in the sense of (2) for the invariant set Ω.

For each r_0, $0 \leqq r_0 < -1/e$, $W^s(0,\pi)$ meets $T(r_0)$ transversely at the point $(r_0, 0, \pi)$ in $[0, -1/e] \times T^2$. Hence there is a smooth curve $\gamma_s(\mu, r_0)$ passing through $(r_0, 0, \pi)$ in $T(r_0) \cap W^s(0,\pi)$. Let $\alpha_s(\mu, r_0)$ denote the angle that this curve makes with the circle $\theta=0$ in T. Lemma 6.1 below shows that

$$0 < \alpha_s(\mu, r_0) < \pi/4 \tag{6.7}$$

if $\mu > 1$ and $0 \leqq r_0 < -1/e$.

Assuming this result for the moment, the proof is completed as follows. For each r_0, $0 < r_0 < -1/e$, $W^u(\Omega)$ also meets $T(r_0)$ transversely at $(r_0, 0, \pi)$. Denote the angle of intersection of $W^u(\Omega) \cap T(r_0)$ with $\theta=0$ by $\beta_u(\mu, r_0)$. By (6.6), it follows that

$$\pi/4 < \beta_u(\mu, r_0) < \pi/2 \tag{6.8}$$

for r_0 close enough to $-1/e$.

Now the system (6.1) is reversed by the symmetry

$$(r_0, \theta, \psi) \to (r_0, \theta, -\psi).$$

In particular, $W^s(\Omega)$ is mapped to $W^u(\Omega)$ and $W^s(0,\pi)$ is mapped to $W^u(0,0)$ by this map. Hence we have similar results near the point $(r_0, 0, 0)$ for $W^u(0,0) \cap T(r_0)$ and $W^s(\Omega) \cap T(r_0)$. Let $\alpha_u(\mu, r_0)$ and $\beta_s(\mu, r_0)$ denote the angles of intersection of these curves with $\theta=0$ at $(r_0, 0, 0)$. We have

$$\begin{aligned} 0 &< \alpha_u(\mu, r_0) < \pi/4 \\ \pi/4 &< \beta_s(\mu, r_0) < \pi/2 \end{aligned} \tag{6.9}$$

as above. Figure 6 shows the relative positions of these various curves in $T(r_0)$.

Via (6.1) we have

$$\dot{\theta} = 2(1+re)\sin\psi \tag{6.10}$$

along $\theta=0$. Hence $\dot{\theta} > 0$ for $\theta=0$, $0 < \psi < \pi$.

Now let $\gamma(s)$ be a small arc in $W^u(0,0) \cap T(r_0)$ passing through $\gamma(0) = (r_0, 0, 0)$. Suppose $\theta(\gamma(s)) > 0$ for $s > 0$. The forward orbits of all points $\gamma(s)$ with $s \neq 0$ eventually reintersect $r = r_0$. Let $\hat{\gamma}(s)$ denote the first such reintersection, for $s \neq 0$. Then $\hat{\gamma}(s)$ is a smooth curve in $T(r_0)$ satisfying

$$\lim_{s \to 0} \hat{\gamma}(s) = (r_0, 0, \pi) = \hat{\gamma}(0).$$

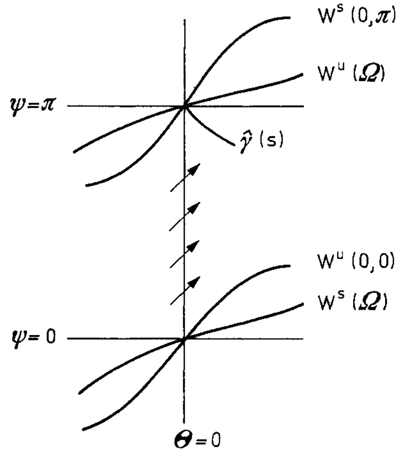

Fig. 6. The intersection of the invariant manifolds with $T(r_0)$

In fact, $\hat{\gamma}$ is smooth at $s=0$ since, in terms of the original system, $\hat{\gamma}$ may be obtained from γ by applying an ordinary Poincare map along the orbit γ_1^+.

Now (6.10) implies that $\theta(\hat{\gamma}(s))>0$ for $s>0$. Moreover, using the flow of (6.2), it follows immediately that $\hat{\gamma}(s)$ is contained for small s in the sector bounded by $\theta=0$ and $W^u(\Omega)$. This completes the proof with the exception of Lemma 6.1. q.e.d.

Lemma 6.1. *If* $\mu>1$ *and* $0\leqq r_0<-1/e$, *then* $0<\alpha_s(\mu, r_0)<\pi/4$.

Proof. The idea here is to construct a Wazewski set for the flow as in [3]. Using (4.3) one computes easily that the stable eigenspace at $(0,0,\pi)$ in Λ is given by the line

$$\psi'=\tfrac{1}{4}(3\sqrt{\mu}+\sqrt{9\mu-8})\,\theta'$$

in the tangent space to Λ. The slope of this line is greater than one, and so the result is true for $r_0=0$.

Now consider the submanifolds $\theta=0$ and $\psi=\theta+\pi$ near γ_1^+. Let D be the sector satisfying

$$0\leqq\theta\leqq\psi-\pi,$$
$$\pi<\psi<3\pi/2.$$

Along $\theta=0$ we have

$$\dot{r}\leqq 0,$$
$$\dot{\theta}=2(1+re)\sin\psi,$$
$$\dot{\psi}=\sqrt{\mu}\sin\psi.$$

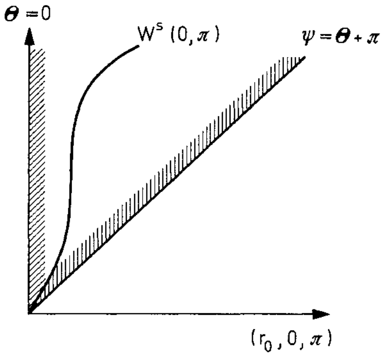

Fig. 7

Hence both $\dot{\theta}$ and $\dot{\psi}$ are negative along $\theta=0$, for $\pi<\psi<2\pi$.

On the other hand, along $\theta=\psi-\pi$, we have

$$\dot{r}\leqq 0,$$
$$\dot{\theta}=(1+re)(\mu^{\frac{1}{2}}-1)\sin 2\psi,$$
$$\dot{\psi}=\tfrac{1}{2}(1-\mu^{\frac{1}{2}})\sin 2\psi.$$

Hence $\dot{\theta}>0$ and $\dot{\psi}<0$ for $\pi<\psi<3\pi/2$. Thus orbits tend to leave the sector D in forward time, at least near γ_1^+. See Figure 7. It follows that $W^s(0,\pi)$ is trapped (at least locally) in the sector D together with its reflection about γ_1^+. q.e.d.

7. Proof of Theorem A

In this section we complete the proof of Theorem A. The basic idea is to construct a sequence of transversals or "windows" for the flow and then to isolate a collection of orbits which cross these transversals in a prescribed order.

Throughout this section, ε_i, $i=1,2,3,4$ denotes a small positive number depending only on μ. We first construct the local transversals in a sequence of steps.

Step 1. Construct annuli transverse to the stable and unstable manifolds of the sinks and sources in Λ as follows. First consider the sink at $(\pi/2,\pi/2)$ in Λ. $f_\mu(\pi/2,\pi/2)=\sqrt{2}$, where f_μ is the gradient function introduced in Proposition 4.5. Also

$$f_\mu\leqq 2^{\frac{1}{2}}\mu^{-\frac{1}{4}}$$

at any of the saddles in Λ as we see from (4.8). Choose v such that

$$2^{\frac{1}{2}}\mu^{-\frac{1}{4}}<v<2^{\frac{1}{2}}.$$

Note that the flow on Λ crosses the level set $f_\mu=v$ transversely.

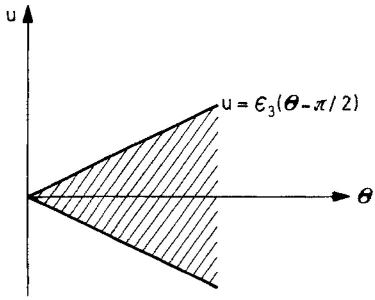

Fig. 8. The sector C^+ in A

Now define

$$A^+ = \{(r, \mathbf{s}, \mathbf{u}) \in \Sigma_e \mid f_\mu(\mathbf{s}, \mathbf{u}) = v,\ 0 < \theta < \pi,\ r \leqq \varepsilon_1\}. \tag{7.1}$$

For small enough ε_1, the flow is transverse to A^+.

A similar annular band B^+ may be constructed near the source at $(\pi/2, -\pi/2)$ by applying the symmetry

$$(r, \theta, \psi) \to (r, \theta, \psi + \pi) \tag{7.2}$$

to A^+. Also we define similar annuli A^-, B^- about $(-\pi/2, -\pi/2)$ and $(-\pi/2, \pi/2)$ respectively by reflecting A^+ and B^+ via the map

$$(r, \theta, \psi) \to (r, -\theta, -\psi). \tag{7.3}$$

All of these annuli are transverse to the flow if ε_1 is small enough, and each meets Λ in a circle which is completely contained in either the stable or unstable manifold of one of the sinks *ar* sources in Λ.

Denote by $a^\pm, b^\pm$ the intersections $A^\pm \cap \Lambda; B^\pm \cap \Lambda$.

Step 2. Recall that A is the annulus defined by $\mathbf{s}^t\mathbf{u} = \mathbf{0}$ in $\Sigma_e - \Lambda$. We choose coordinates (θ, u) on A via

$$\mathbf{s} = (\cos\theta, \sin\theta),$$
$$\mathbf{u} = u(\cos\psi, \sin\psi).$$

Note that $\mathbf{u} = \mathbf{0}$ corresponds to the oval of zero velocity, while $q_2^\pm$ correspond to $(\pm\pi/2, 0)$.

We now define sectors $C^\pm$ in A about $q_2^\pm$ as follows:

$$\begin{aligned} &0 < \theta \mp \pi/2 < \varepsilon_2 \\ &|\mathbf{u}| \leqq \varepsilon_3(\theta \mp \pi/2). \end{aligned} \tag{7.4}$$

Also, let $D^\pm$ denote the disk of radius ε_2 in A about $(\pi/2, 0)$. See Figure 8.

Now, if ε_1 is small enough, there is defined a Poincare map

$$\Phi_1 \colon A^\pm - a^\pm \to D^\pm - q_2^\pm \tag{7.5}$$

obtained by following orbits in the forward time direction. Similarly, let

$$\Psi_1\colon B^{\pm}-b^{\pm}\to D^{\pm}-q_2^{\pm} \tag{7.6}$$

be a Poincare map obtained by following orbits backwards in time. Observe that

$$R\Phi_1=\Psi_1 R \tag{7.7}$$

where R is the reflection (7.3).

Step 3. We now construct a square E in A about q_1^+. Choose coordinates (x, y) on A near q_1^+ so that the local stable manifold of $(0, \pi)$ is given by $y=0$, while the local unstable manifold of $(0, 0)$ is given by $x=0$. We may further assume that

$$R(x, y)=(y, x) \tag{7.8}$$

so that Z is given locally by the line $y=x$.

Now define E to be the open square centered at q_1^+ and having sides of length ε_4. We denote by A_x, A_y the x- and y-axes respectively in E. Now, in forward time, points in $E-A_x$ tend to follow the unstable manifold of $(0, \pi)$ in A. For most values of μ, each branch of $W^u(0, \pi)$ dies in a sink by Proportion 4.8. We henceforth consider only those values of μ.

Each branch of $W^u(0, \pi)$ crosses either a^+ or a^-, hence if ε_4 is small enough, there is defined a Poincare map

$$\Phi_2\colon E-A_x\to A^{\pm}. \tag{7.9}$$

Since each branch of $W^s(0, 0)$ must also emanate from sources for these values of μ, we also have a Poincare map

$$\Psi_2\colon E-A_y\to B^{\pm} \tag{7.10}$$

again obtained by following orbits backward in time.

As before, we define a square E^- about q_1^- together with Poincare maps Φ_2, Ψ_2 by applying the symmetry (7.3). Figure 9 gives a schematic of these maps and transversals.

Let $\Phi, \Psi\colon E\to D$ be given by

$$\begin{aligned}\Phi&=\Phi_1\circ\Phi_2\\ \Psi&=\Psi_1\circ\Psi_2.\end{aligned} \tag{7.11}$$

Now consider the set of points in E satisfying

$$\begin{aligned}&1)\ \ \Phi(x)\in C^{\pm}\\ &2)\ \ (\Psi_1)^{-1}(\Phi(x))\in\Psi_2(E).\end{aligned} \tag{7.12}$$

Condition 1) implies that the forward orbit of x meets $C^{\pm}$ after filtering through $A^{\pm}$. Condition 2) implies that the orbit returns again to E after filtering through $B^{\pm}$. Let Σ_1 denote the set of points in E satisfying 1) and 2) above.

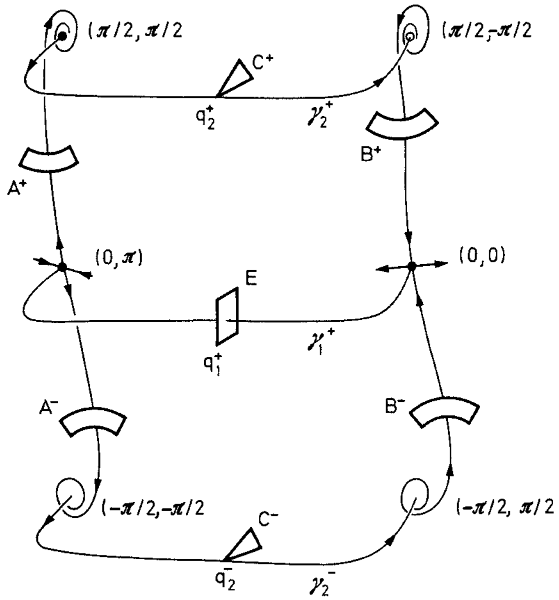

Fig. 9. Construction of the local transversals in Theorem A

Define $F: \Sigma_1 \to E$ by

$$F(x) = \Psi^{-1} \circ \Phi(x). \tag{7.13}$$

Below we show that F satisfies the axioms of a "horseshoe" mapping as in (12). Hence

$$\Sigma = \overline{\bigcap_{n=-\infty}^{\infty} F^n(\Sigma_1)} \tag{7.14}$$

is a Cantor set and the induced mapping $F: \Sigma \to \Sigma$ is topologically conjugate to a Bernoulli shift.

Proof of Theorem A. Let

$$\begin{aligned} \sigma_{x_0}(y) &= (x_0, y) \\ \sigma_{y_0}(x) &= (x, y_0) \end{aligned} \tag{7.15}$$

be lines parallel to A_y and A_x respectively. It suffices to show that, for x_0, y_0 close enough to zero,

1) $F(\sigma_{x_0}(y) \cap \Sigma_1)$ consists of an infinite collection of curves converging to A_y in the C^1 topology and satisfying

$$\left| \frac{d}{dy} F(\sigma_{x_0}(y)) \right| \to \infty \tag{7.16}$$

as $y \to 0$.

2) $F^{-1}(\sigma_{y_0}(x) \cap F(\Sigma_1))$ consists of an infinite collection of curves converging C^1 to A_x, and satisfying

$$\left|\frac{d}{dx} F^{-1}(\sigma_{y_0}(x))\right| \to \infty \tag{7.17}$$

as $x \to 0$.

Below we prove 2); 1) follows by symmetry.

Lemma 7.1 shows that $\Phi(\sigma_{x_0}(y))$ is a pair of smooth spirals converging to $q_2^{\pm}$, one in each of $D^{\pm}$, which, in addition, satisfy

$$\left|\frac{d}{dy} \Phi(\sigma_{x_0}(y))\right| \to \infty \tag{7.18}$$

as $y \to 0$. By symmetry, $\Psi(\sigma_{y_0}(x))$ has similar properties.

Now, $\Phi(A_y) = \Phi(\sigma_0(y))$ crosses $Z \cap D^{\pm}$ at infinitely many points. By the results of § 5, for most values of μ, the angles of these crossings are bounded away from $\pi/2$ as $y \to 0$. It follows that $\Phi(\sigma_{x_0}(y))$ meets each $\Psi(\sigma_{y_0}(x))$ transversely in $C^{\pm}$ as long as x_0, y_0 are close enough to 0.

In particular, $\Phi(E) \cap C^{\pm}$ consists of infinitely many disjoint strips each foliated by the $\Phi(\sigma_{x_0}(y))$. Also

$$\Psi(E) \cap C^{\pm} = R(\Phi(E) \cap C^{\pm}) \tag{7.19}$$

and so $\Psi(E) \cap C^{\pm}$ also consists of infinitely many strips, each of which meet at least one of the strips in $\Phi(E) \cap C^{\pm}$. By choosing $C^{\pm}$ smaller, we may assume that each strip in $\Phi(E) \cap C^{\pm}$ meets a unique strip in $\Psi(E) \cap C^{\pm}$, and that that intersection contains a subinterval of Z. By the above argument, the lines $\Psi(\sigma_{y_0}(x))$ are transverse to the foliation $\Phi(\sigma_{x_0}(y))$ in $\Phi(E)$, and thus it follows that $\Phi^{-1}(\Psi(\sigma_{y_0}(x)))$ consists of infinitely many arcs converging C^0 to A_x.

We remark at this juncture that one may index these strips as follows. To each strip D^+ (resp. D^-) we assign a positive (negative) integer k where $|k|$ is the number of times the orbit of any point in the strip crosses the q_2-axis in configuration space before reaching E. That this integer is constant along each strip is immediate from the definitions. Also, we note that this assignment is one-to-one and onto the set of integers of absolute value greater than some K. Here K depends on μ as well as all of the ε_i. Thus one may define the conjugacy in Theorem A exactly as in [12] or [16].

We now prove that the convergence in (7.17) is actually C^1.

Let $\xi = \xi(\sigma_{y_0}(x))$ be a unit tangent vector field along $\sigma_{y_0}(x)$ and let $\eta = \eta(\sigma_{x_0}(y))$ be a unit tangent vector field along $\sigma_{x_0}(y)$. Let $p, p' \in \Sigma$ be such that $F(p) = p'$. We suppose for simplicity that $\Phi(p) = q \in Z$. The general case is not much harder.

Let w_1 be a unit vector tangent at q to Z. Let

$$w_2 = d\Psi(\xi(p')). \tag{7.20}$$

Clearly, w_1 and w_2 form a basis of $T_q C^{\pm}$. In this basis, one checks using symmetry that

$$d\Phi(\eta(p)) = a(p)\, w_1 - w_2 \tag{7.21}$$

where $|a(p)| \to \infty$ as $p \to 0$.

Now Lemma 7.2 below shows that $\Psi^{-1}(Z \cap \Psi(E))$ is an infinite collection of curves converging to A_y in the C^1 sense. Furthermore,

$$|d\Psi^{-1}(w_1(q))| \to \infty \tag{7.22}$$

as $q \to q_1^{\pm}$. Hence

$$d(\Psi^{-1} \circ \Phi)(\eta(p)) = a(p)\, d\Psi^{-1}(w_1) - \xi(p') \tag{7.23}$$

so that

$$|dF(\eta(p))| \to \infty \tag{7.24}$$

as $p \to 0$. This completes the proof except for Lemmas 8.1 and 8.2. q.e.d.

Lemma 7.1. *If* $y > 0$, $\Phi(\sigma_{x_0}(y))$ *is a smooth spiral in* D^+ *which satisfies*

$$\left|\frac{d}{dy}\Phi(\sigma_{x_0}(y))\right| \to \infty$$

as $y \to 0$.

Proof. The proof of this lemma is similar to the proof of Lemma 7.2 and Proposition 5.3. Hence we omit the details. We simply remark that the result is obvious for $x_0' = 0$, since $\Phi_1(\sigma_0(y))$ approaches a^+ transversely. If $x_0 \neq 0$, however, the curve $\Phi_1(\sigma_{x_0}(y))$ approaches a^+ tangentially and one needs estimates similar to those below. q.e.d.

Lemma 7.2. *Let* $\lambda = \lambda(r)$ *be a ray in* $D^{\pm}$. *Then*

1) $\Phi(E) \cap \lambda(r)$ *consists of an infinite collection of open subintervals* Y_i *which accumulate only at* $q_2^{\pm}$ *as* $i \to \infty$.

2) $\Phi^{-1}(Y_i)$ *is a collection of smooth arcs in E which converge to* A_x *in the* C^1 *topology.*

3) *Let* $V_i(x)$ *be a unit tangent vector field along* Y_i. *Then*

$$|d\Phi^{-1}(V_i(x))| \to \infty$$

as $i \to \infty$.

Proof. The proofs of 1) and 3) are immediate; we prove only part 2). The basic idea is to follow the ray $\lambda(r)$ as it passes by each equilibrium point. For this purpose, we construct two local Poincare maps, one near each of the equilibria.

Near the source at $(\pi/2, -\pi/2)$, the vector field may be written in polar coordinates

$$\begin{aligned} \dot r &= \alpha r + \cdots \\ \dot\theta &= \beta + \cdots \\ \dot z &= -\gamma z \end{aligned} \tag{7.25}$$

where α, β, $\gamma > 0$. The exact values of these eigenvalues are given in Table 1. Of course, the variables r, θ chosen here have nothing to do with the previous variables (1.8), (3.3) of the same name.

We examine the solutions passing from the plane $z = z^*$ to the annulus $r = r^*$. For z^*, r^* sufficiently small, these submanifolds are locally transverse to the flow and there is a Poincare map between them which assumes the form

$$\begin{aligned} \theta_1 &= \theta_1(r_0, \theta_0, z^*, r^*) = \theta_0 + (\beta/\alpha)\log(r^*/r_0) + A + B \\ z_1 &= z_1(r_0, \theta_0, z^*, r^*) = z^* \exp((-\gamma/\alpha)\log(r^*/r_0). \end{aligned} \tag{7.26}$$

Here A and B are analytic and satisfy $A(r_0, \theta_0, 0, 0) = 0$ and B together with its first partial derivatives approaches 0 as $r_0 \to 0$. Call this mapping f_1 and let $v = v(r_0, \theta_0)$ be a unit vector field tangent to the ray $\theta_0 = \text{constant}$ at (r_0, θ_0). Also let

$$(\xi, \eta) = (\xi(\theta_1, z_1), \eta(\theta_1, z_1)) \tag{7.27}$$

be a tangent vector to the annulus $r = r^*$ at (r^*, θ_1, z_1). Consider the sector bundle

$$S_k(\theta_1, z_1) = \{(\xi, \eta) \,|\, |\eta| \leqq k z_1 \, |\xi|\} \tag{7.28}$$

where $k > 0$. Using (7.26), one checks easily that there exists $k = k(z^*, r^*)$ such that, for all r_0 sufficiently small,

$$d f_1(v(r_0, \theta_0)) \in S_k(\theta_1, z_1). \tag{7.29}$$

That is, vectors tangent to rays in $z = z^*$ are mapped into the sector bundle S_k.

From this, it follows immediately that f_1 maps rays in $z = z^*$ to curves which spiral toward $z = 0$ in $r = r^*$.

We also consider a mapping similar to f_1 near the saddle point at $(0, \pi)$. Near $(0, \pi)$, X_μ assumes the form

$$\begin{aligned} \dot{x} &= -a x + H(x, y, z) \\ \dot{y} &= -b y + G(x, y, z) \\ \dot{z} &= c z \end{aligned} \tag{7.30}$$

where a, b, $c > 0$, $b - a > 0$, and where again the coordinates chosen here have nothing to do with the previous coordinates. Also, H and G are analytic and contain terms of order two or more. Note that $z = 0$ gives the stable manifold of the equilibrium. We assume that $G(x, 0, z) = 0$ so that the plane $y = 0$ is invariant and corresponds to Λ in the original flow. We further assume that $H(0, y, 0) = 0$ so that the y-axis is also invariant (the strong stable manifold). This corresponds to γ_1^+ in the original variables.

As before, we consider a Poincare map obtained by following orbits backward in time from the plane

$$\begin{aligned} z &= z^* \\ y &\geqq 0 \end{aligned} \tag{7.31}$$

to the plane $y = y^*$ near $(0, y^*, 0)$. Call this mapping f_2.

For y^*, z^* sufficiently small, this mapping assumes the form

$$\begin{aligned} x_1 &= x_1(x_0, y_0, z^*, y^*) = x_0(y^*/y_0)^{a/b} + C_1 + D_1 \\ z_1 &= z_1(x_0, y_0, z^*, y^*) = z^*(y^*/y_0)^{-c/b} + C_2 + D_2 \end{aligned} \tag{7.32}$$

where again C_i and D_i are analytic for $i=1,2$. Also, $C(x_0, y_0, 0, 0)=0$ and $D \to 0$ together with its first partials as $y_0 \to 0$.

Let

$$(\xi, \eta) = (\xi(x_0, y_0), \eta(x_0, y_0)) \tag{7.33}$$

be a tangent vector to the plane $z=z^*$ at (x_0, y_0, z^*). Suppose

$$|\eta| \leqq y_0 |\xi|. \tag{7.34}$$

Let

$$(\xi_1, \zeta_1) = d f_2(\xi, \eta).$$

Using (7.32) together with (7.34), one computes that

$$|\zeta_1| \leqq (y_0)^{(c+a-\varepsilon)/b} |\xi_1| \tag{7.35}$$

where $\varepsilon > 0$ approaches 0 as z^*, $y^* \to 0$.

Now using (7.35) and (7.29) it then follows that the global mapping Φ^{-1} maps $\lambda(r)\ \Phi(E)$ to a collection of smooth curves which approach A_x in the C^1 topology. This completes the proof of the lemma.

References

1. Alekseev, V.M.: Quasirandom dynamical systems I, II, III. Math. USSR-Sb. **6**, 489–498 (1969)
2. Conley, C., Easton, R.: Isolated invariant sets and isolating blocks. Trans. Amer. Math. Soc. **158**, 35–61 (1971)
3. Conley, C.: Some applications of topology in differential equations. Preprint, University of Wisconsin, Madison, Wisconsin
4. Devaney, R.: Homoclinic orbits in Hamiltonian systems. J. Differential Equations **21**, 431–438 (1976)
5. Easton, R.: Isolating blocks and symbolic dynamics. J. Differential Equation **17**, 96–118 (1975)
6. Gutzwiller, M.C.: J. Mathematical Phys. **8**, 1979 (1967); **10**, 1004 (1969); **11**, 1971 (1970); and **12**, 343 (1971)
7. Gutzwiller, M.C.: The anisotropic Kepler problem in two dimensions. J. Mathematical Phys. **14**, 139–152 (1973)
8. Gutzwiller, M.C.: Bernoulli sequences and trajectories in the anisotropic Kepler problem. (To appear)
9. Hirsch, M., Pugh, C., Shub, M.: Invariant manifolds. (To appear)
10. McGehee, R.: Triple collision in the collinear three-body problem. Inventiones math. **27**, 191–227 (1974)
11. McGehee, R.: Double collisions for non-Newtonian potentials. (To appear)
12. Moser, J.: Stable and random motions in dynamical systems. Princeton, N.J.: University Press 1973

13. Pollard, H.: Mathematical Introduction to Celestial Mechanics. Prentice Hall: Englewood Cliffs, N.J. 1966
14. Sacker, R.: A perturbation theorem for invariant manifolds and Hölder continuity. J. Math. Mech. **18**, 705–762 (1969)
15. Silnikov, L.P.: Soviet Math. Dokl. **8**, 54–58 (1967)
16. Smale, S.: Differentiable dynamical systems. Bull. Amer. Math. Soc. **73**, 747–817 (1967)

Received June 15, 1977