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Limits of Hodge Structures

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§ 0. Introduction

If X is a compact Kähler manifold, by a construction of Hodge [10] the complex cohomology groups of X admit a so-called Hodge decomposition

$$\mathrm{H}^n(X, \mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X)$$

where for $p, q \geqq 0$ $\mathrm{H}^{p,q}(X)$ is the vectorspace of harmonic forms of type (p, q) on X. Under complex conjugation with respect to $\mathrm{H}^n(X, \mathbb{R})$ one has

$$\mathrm{H}^{p,q}(X) = \overline{\mathrm{H}^{q,p}(X)}.$$

The couple consisting of the abelian group $\mathrm{H}^n(X, \mathbb{Z})$ and the Hodge decomposition of $\mathrm{H}^n(X, \mathbb{C}) = \mathrm{H}^n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ is called the canonical Hodge structure on $\mathrm{H}^n(X, \mathbb{Z})$. Deligne [3, 4] has extended this construction by attaching to every separated scheme X of finite type over $\mathbb{C}$ a canonical and functorial mixed Hodge structure. Its ingredients are a weight filtration W on $\mathrm{H}^n(X, \mathbb{Q})$ and a Hodge filtration F on $\mathrm{H}^n(X, \mathbb{C})$, inducing a Hodge decomposition of $\mathrm{Gr}_k^W \mathrm{H}^n(X, \mathbb{C})$ for every $k \in \mathbb{Z}$.

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The power of these mixed Hodge structures arises from the fact that every morphism $f: X \to Y$ induces a morphism of mixed Hodge structures

$$f^*: \mathrm{H}^n(Y) \to \mathrm{H}^n(X)$$

and that every such map is strictly compatible with both filtrations.

Here is one example where functoriality seems to fail. Suppose $f: X \to S$ is a projective holomorphic map from a complex manifold X to the unit disk S. Suppose that $Y = f^{-1}(0)$ is a divisor with normal crossings on X and that f is smooth on $X \setminus Y$. For $t \in S \setminus \{0\}$ denote $X_t = f^{-1}(t)$. Then X_t is a nonsingular projective variety for all t. Because Y is a strong deformation retract of X, for all n the natural map

$$\alpha: \mathrm{H}^n(X) \to \mathrm{H}^n(Y)$$

is an isomorphism. For $t \neq 0$ denote $\beta_t: \mathrm{H}^n(X) \to \mathrm{H}^n(X_t)$ the restriction map. The composed map

$$\beta_t \alpha^{-1}: \mathrm{H}^n(Y) \to \mathrm{H}^n(X_t)$$

plays an important role in local Lefschetz theory. However in general it is not a morphism of mixed Hodge structures. This problem is related to the study of the behaviour of the Hodge decomposition on $\mathrm{H}^n(X_t, \mathbb{C})$ when t tends to zero, as carried out by Schmid [15].

In this paper we show that the Hodge structure on $\mathrm{H}^n(X_t, \mathbb{C})$ tends to a mixed Hodge structure $\lim \mathrm{H}^n$ which can be expressed in terms of the cohomology of certain intersections of components of Y by means of a spectral sequence. Moreover one has a morphism of mixed Hodge structures

$$\beta \alpha^{-1}: \mathrm{H}^n(Y) \to \lim \mathrm{H}^n.$$

The monodromy action T on $\mathrm{H}^n(X_t)$ induces an automorphisms T_0 of $\lim \mathrm{H}^n$ which determines completely the weight filtration on $\lim \mathrm{H}^n$. Conversely our construction permits in some cases direct computation of the weight filtration; this gives then information about the Jordan type of T_0. The precise structure of $\lim \mathrm{H}^n$ was conjectured by Deligne (cf. [8], conjecture 9.17) and determined by Schmid [15]. We give a different proof, using algebraic methods. From this result we derive a proof of the invariant cycle theorem, which states that the image of $\beta \alpha^{-1}$ coincides with the subspace of invariants of T_0.

In §§ 1 and 2 we construct a geometric realization of the canonical extension of the relative De Rham cohomology sheaf. The main result is Theorem (2.18); it was proved by Katz in the case of a reduced special fibre.

In § 3 we recall some basic facts about mixed Hodge structures and compute the canonical mixed Hodge structure for a variety with normal crossings.

§ 4 contains the construction of $\lim \mathrm{H}^n$ under very general hypotheses. To show that it has reasonable properties we consider the projective case in § 5.

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Notations. We denote $\mathbb{C}\{Z_1, \ldots, Z_n\}$ the local ring of convergent power series in the variables $Z_1, \ldots, Z_n$.

If $f\colon X\to Y$ is a continuous map of topological spaces and $\mathfrak{F}$ is a sheaf on Y, we denote $f^{\cdot}\mathfrak{F}$ the sheaf on X with $\Gamma(U,f^{\cdot}\mathfrak{F})=\varinjlim_{V\supset f(U)\text{ open}}\Gamma(V,\mathfrak{F})$.

We denote $\mathbb{R}^i F$ the hyperderived functors of a functor F (cf. [9]).

If $K^{\cdot}$ is a complex and $n\in\mathbb{Z}$, one denotes $K^{\cdot}[n]$ the complex $K^{\cdot}$ with a shift of n places, i.e.

$$(K[n])^p=K^{p+n}\qquad\text{for } p\in\mathbb{Z}.$$

The symbol □ means: end of proof or absence of proof.

§ 1. Complexes of Holomorphic Differentials and Their Cohomology Sheaves

(1.1) Let X be a complex manifold. The sheaf Ω_X^1 of holomorphic 1-forms on X is a locally free sheaf and its exterior powers $\Omega_X^p=\Lambda^p\Omega_X^1$ for $p\in\mathbb{Z}$ form the *holomorphic De Rham complex* on X.

(1.2) An analytic subvariety Y of X is called a *divisor with normal crossings* if for every point $P\in Y$ there exist local coordinates $(z_1,\ldots,z_n)$ in a neighborhood U of P in X such that

$$Y\cap U=\left\{z\in U\,\middle|\,\prod_{i=1}^{\nu}z_i=0\right\}$$

for some ν with $1\leqq\nu\leqq n$.

In this case one defines the holomorphic De Rham complex on X with logarithmic poles along Y, notation $\Omega_X^{\cdot}(\log Y)$, as follows: a section of $\Omega_X^p(\log Y)$ over an open set $U\subset X$ is a holomorphic p-form ω on $U\setminus Y$ which is meromorphic along Y such that ω and $d\omega$ have at most a simple pole along Y. In local coordinates as above, $\Omega_X^1(\log Y)_P$ is a free $\mathcal{O}_{X,P}$-module with generators $dz_1/z_1,\ldots,dz_\nu/z_\nu,dz_{\nu+1},\ldots,dz_n$ and $\Omega_X^p(\log Y)=\Lambda_{\mathcal{O}_X}^p\Omega_X^1(\log Y)$ (cf. [5], p. 72).

(1.3) Let X and S be complex manifolds and let $f\colon X\to S$ be a smooth holomorphic map, i.e. everywhere of maximal rank. Then $f^*\Omega_S^1$ is a locally free subsheaf of Ω_X^1 and $\Omega_{X/S}^1=\Omega_X^1/f^*\Omega_S^1$ is locally free on X. Its exterior powers $\Omega_{X/S}^p(p\in\mathbb{Z})$ form the *relative De Rham complex of X over S*.

(1.4) We define a complex $\Omega_{X/S}^{\cdot}(\log Y)$ in the following situation. Let X be a complex manifold and let S be a smooth curve. Let $f\colon X\to S$ be a holomorphic map. Let $T\subset S$ be a finite set of points such that $Y=f^{-1}(T)$ is a divisor with normal crossings on X and such that f is smooth on $X\setminus Y$. Then $f^*\Omega_S^1(\log T)$ is a locally free subsheaf of $\Omega_X^1(\log Y)$ and $\Omega_{X/S}^1(\log Y)=\Omega_X^1(\log Y)/f^*\Omega_S^1(\log T)$ is locally free on X. Its exterior powers $\Omega_{X/S}^p(\log Y)$ form the *relative De Rham complex of X over S with logarithmic poles along Y.*

If $P\in X$, there exists a coordinate neighborhood U of P in X with coordinates $(z_0,\ldots,z_n)$ and integers $\nu,e_0,\ldots,e_\nu$ with $0\leqq\nu\leqq n$, $e_i\geqq 1$ such that $P=(0,\ldots,0)$ and $f(z_0,\ldots,z_n)=z_0^{e_0}\ldots z_\nu^{e_\nu}$ is a coordinate on S at $f(P)$. Then $\Omega_{X/S}^1(\log Y)_P$ is the $\mathcal{O}_{X,P}$-module with generators $\{dz_0/z_0,\ldots,dz_\nu/z_\nu,dz_{\nu+1},\ldots,dz_n\}$, subject to the relation $\sum_{i=0}^{\nu}e_i\,dz_i/z_i=0$.

(1.5) The differentials in the complexes above are induced by the usual differentiation in the complex $\Omega^{\bullet}_X$. These differentials are $\mathbb{C}$-linear in the case of $\Omega^{\bullet}_X(\log Y)$ and even $f^{\bullet}\mathcal{O}_S$-linear in the case of $\Omega^{\bullet}_{X/S}$ and $\Omega^{\bullet}_{X/S}(\log Y)$.

(1.6) For $q \in \mathbb{Z}$ and $K^{\bullet}$ a complex of sheaves on a space M, we denote $\mathcal{H}^q(K^{\bullet})$ the q-th cohomology sheaf of $K^{\bullet}$, i.e. the sheaf associated to the presheaf $U \mapsto H^q(\Gamma(U, K^{\bullet}))$. The remainder of this chapter describes the cohomology sheaves of the above defined complexes.

(1.7) Let X be a complex manifold and let $Y \subset X$ be a divisor with normal crossings. One defines an increasing filtration W, called the *weight filtration*, on the complex $\Omega^{\bullet}_X(\log Y)$ by $W_k \Omega^p_X(\log Y) = \Omega^k_X(\log Y) \wedge \Omega^{p-k}_X$. Assume that $Y = Y_1 \cup \cdots \cup Y_N$ is a union of smooth divisors. Denote $Y^{(p)}$ the union inside X of all intersections $Y_{i_1} \cap \cdots \cap Y_{i_p}$ for $1 \leqq i_1 < \cdots < i_p \leqq N$ and denote $\tilde{Y}^{(p)}$ their disjoint union. One has a natural projection map $a_p \colon \tilde{Y}^{(P)} \to X$.

Choose $P \in X$ and let $(z_1, \ldots, z_n)$ be a coordinate system in a neighborhood of P in X such that Y has the local equation $z_1 \ldots z_l = 0$. We assume that $(z_1, \ldots, z_n)$ are chosen in such an order that the indices of the components of Y corresponding to $z_i = 0$ $(i = 1, \ldots, l)$ form an increasing sequence $\sigma(1), \ldots, \sigma(l)$ in $\{1, \ldots, N\}$. The Poincaré residue map

$$R\colon\; W_k \Omega^p_X(\log Y) \to (a_k)_* \Omega^{p-k}_{\tilde{Y}^{(k)}}$$

(cf. [5], p. 76) is defined as follows. A section α of $W_k \Omega^p_X(\log Y)$ has the form

$$\alpha = \sum_{1 \leqq i_1 < \cdots < i_k \leqq l} \alpha_{i_1 \ldots i_k} \xi_{i_1} \wedge \cdots \wedge \xi_{i_k},$$

where we have put $\xi_i = dz_i / z_i$, and $\alpha_{i_1 \ldots i_k}$ is a section of Ω^{p-k}_X. Then

$$R\alpha = \sum_{1 \leqq i_1 < \cdots < i_k \leqq l} (a_{i_1 \ldots i_k})^* (\alpha_{i_1 \ldots i_k})$$

where

$$a_{i_1 \ldots i_k} \colon\; Y_{\sigma(i_1)} \cap \cdots \cap Y_{\sigma(i_k)} \to X$$

is the inclusion. One can verify that the definition of R does not depend on the choice of the coordinates.

(1.8) **Proposition.** *For every $k \geqq 0$ R induces an isomorphism*

$$\mathrm{Gr}^W_k \Omega^{\bullet}_X(\log Y) \xrightarrow{\sim} (a_k)_* \Omega^{\bullet}_{\tilde{Y}^{(k)}}[-k].$$

Cf. [5], *p.* 76. □

(1.9) **Corollary.**

$$\mathcal{H}^p(\mathrm{Gr}^W_k \Omega^{\bullet}_X(\log Y)) = 0 \quad \textit{for } p \neq k;$$

$$\mathcal{H}^p(\mathrm{Gr}^W_p \Omega^{\bullet}_X(\log Y)) \cong (a_p)_* \mathbb{C}_{\tilde{Y}^{(p)}}. \quad □$$

(1.10) **Corollary.** $\mathcal{H}^p(\Omega^{\bullet}_X(\log Y)) \cong (a_p)_* \mathbb{C}_{\tilde{Y}^{(p)}}$ *for all* $p \in \mathbb{Z}$.

Proof. Use the spectral sequence of the complex $\Omega^{\bullet}_X(\log Y)$ with its filtration W. This gives

$$E_1^{pq} = \mathcal{H}^{p+q}(\mathrm{Gr}^W_{-p}\,\Omega^{\bullet}_X(\log Y)) \Rightarrow \mathcal{H}^{p+q}(\Omega^{\bullet}_X(\log Y)).$$

So $E_1^{pq}=0$ for $q \neq -2p$ and the sequence degenerates at E_1. Hence $E_\infty^p = E_1^{-p,2p}$. □

(1.11) **Proposition** (relative Poincaré lemma). *Let f: $X \to S$ be a smooth holomorphic map between smooth complex varieties. Then $\Omega^{\bullet}_{X/S}$ is a resolution of the sheaf $f^{\bullet}\mathcal{O}_S$. Cf.* [5], p. 15–16 *for a proof.* □

(1.12) For the sequel it is convenient to have the notion of a Koszul complex. If A is a commutative $\mathbb{C}$-algebra and $D_1, \ldots, D_k$ are $\mathbb{C}$-linear mutually commuting maps from A to A, the *Koszul complex* on A with operators $D_1, \ldots, D_k$ is the complex $\Lambda^{\bullet}_A(A^k)$ where d: $\Lambda^p_A(A^k) \to \Lambda^{p+1}_A(A^k)$ is given by

$$d(f e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{i=1}^{k} D_i(f)\, e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_p}$$

for $f \in A$ and $e_1, \ldots, e_k$ the natural basis of A^k. If one of the operators is bijective, one easily shows that this complex is acyclic, i.e. has zero cohomology.

(1.13) **Proposition.** *Let $X = \mathbb{C}^{m+1}$ with coordinates $(z_0, \ldots, z_n)$. Let $S = \mathbb{C}$ and let v, $e_0, \ldots, e_v$ be integers with $0 \leqq v \leqq n$ and $e_i \geqq 1$. Define f: $X \to S$ by $f(z_0, \ldots, z_n) = z_0^{e_0} \ldots z_v^{e_v}$. Let $e = \gcd(e_0, \ldots, e_v)$ and define $n_i = e_i/e$. Put $y = z_0^{n_0} \ldots z_v^{n_v}$. Let Y be the analytic subset of X with ideal (y^e). Then:*

$$\mathcal{H}^0(\Omega^{\bullet}_{X/S}(\log Y))_0 \cong \mathbb{C}\{y\};$$

$\mathcal{H}^1(\Omega^{\bullet}_{X/S}(\log Y))_0$ *is the $\mathbb{C}\{y\}$-module with generators $\xi_0, \ldots, \xi_v$ and relation* $\sum_{i=0}^{v} e_i \xi_i = 0$;

$$\mathcal{H}^p(\Omega^{\bullet}_{X/S}(\log Y))_0 = \Lambda^p_{\mathbb{C}\{y\}}\, \mathcal{H}^1(\Omega^{\bullet}_{X/S}(\log Y))_0.$$

Proof. The complex $\Omega^{\bullet}_{X/S}(\log Y)_0$ is isomorphic to $K^{\bullet} \otimes L^{\bullet}$ where $K^{\bullet}$ is the Koszul complex on $\mathbb{C}\{z_0, \ldots, z_n\}$ with operators

$$D_i = z_i\, \partial/\partial z_i - (e_i/e_0)\, z_0\, \partial/\partial z_0 \qquad (i = 1, \ldots, v)$$

and $L^{\bullet}$ is the Koszul complex on $\mathbb{C}\{z_0, \ldots, z_n\}$ with operators $\partial/\partial z_j$ $(j = v+1, \ldots, n)$. It follows from the relative Poincaré lemma that $H^p(L^{\bullet}) = 0$ for $p \neq 0$. From [7], Theorem I.4.8.1 it follows that $\Omega^{\bullet}_{X/S}(\log Y)_0$ is quasi-isomorphic to the Koszul complex on $\mathbb{C}\{z_0, \ldots, z_v\}$ with operators D_i $(i = 1, \ldots, v)$. The cohomology of this complex may be computed monomial by monomial, because the operators are homogeneous. One only gets a non-zero contribution from those monomials on which the D_i are all zero. Because

$$D_i(z_0^{\alpha_0} \ldots z_v^{\alpha_v}) = (\alpha_i - \alpha_0\, e_i/e_0)\, z_0^{\alpha_0} \ldots z_v^{\alpha_v},$$

this amounts to saying that $z_0^{\alpha_0} \ldots z_v^{\alpha_v}$ is a power of y. □

(1.14) **Corollary.** *For all* $p \geqq 0$ *one has*

$$\mathscr{H}^p(\Omega^{\cdot}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)_0 \cong \mathscr{H}^p(\Omega^{\cdot}_{X/S}(\log Y))_0 \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{y\}/(y^e);$$

$$\mathscr{H}^p(\Omega^{\cdot}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}})_0 \cong \mathscr{H}^p(\Omega^{\cdot}_{X/S}(\log Y))_0 \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{y\}/(y). \quad \square$$

§ 2. Analytic De Rham Cohomology

(2.1) In this chapter we study the cohomology of the fibers of a proper holomorphic map f from a connected complex manifold X onto the unit disk S. We assume that f is smooth over the punctured disk S^* and that $Y = f^{-1}(0)$ is a divisor with normal crossings on X. Then the complex $\Omega^{\cdot}_{X/S}(\log Y)$ is well-defined. We use its relative hypercohomology sheaves $\mathbb{R}^p f_* \Omega^{\cdot}_{X/S}(\log Y)$ to glue the complex cohomology groups of the fibers of f together. For $s \in S$ denote $X_s = f^{-1}(s)$.

(2.2) **Proposition.** *If* $s \neq 0$ *then* $\mathbb{H}^p(X_s, \Omega^{\cdot}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s}) \cong \mathrm{H}^p(X_s, \mathbb{C})$.

Proof. For $s \neq 0$, we have $\Omega^{\cdot}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s} \cong \Omega^{\cdot}_{X_s}$, which is a resolution of the constant sheaf $\mathbb{C}$ on X_s. $\square$

We are going to deduce an interpretation of $\mathbb{H}^p(Y, \Omega^{\cdot}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$ as well.

(2.3) Let $\tilde{S}^* = \{u \in \mathbb{C} \mid \mathrm{Im}(u) > 0\}$ be the upper half plane. The map $u \mapsto \exp(2\pi i u) = t$ makes $\tilde{S}^*$ into a universal covering of S^*. Denote $\tilde{X}^* = X \times_S \tilde{S}^*$ and let $k\colon \tilde{X}^* \to X$ be the projection map. Denote $i\colon Y \to X$ the inclusion. One has the commutative diagram

$$\begin{array}{ccccc} \tilde{X}^* & \xrightarrow{k} & X & \xleftarrow{i} & Y \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \tilde{S}^* & \longrightarrow & S & \longleftarrow & \{0\} \end{array}$$

in which both squares are cartesian.

(2.4) **Lemma.** $\mathrm{H}^p(\tilde{X}^*, \mathbb{C}) \cong \mathbb{H}^p(X, k_* \Omega^{\cdot}_{\tilde{X}^*})$ *for all* $p \in \mathbb{Z}$.

Proof. If $U \subset X$ is an open subset which is a Stein manifold, then $k^{-1}(U)$ is a closed subvariety of $U \times \tilde{S}^*$; hence $k^{-1}(U)$ is Stein too. This implies that $\mathrm{R}^q k_* \mathfrak{F} = 0$ for all $q > 0$ and for every coherent sheaf $\mathfrak{F}$ on X. The complex $\Omega^{\cdot}_{\tilde{X}^*}$ is a resolution of the constant sheaf $\mathbb{C}$ on $\tilde{X}^*$ with sheaves which are acyclic for the functor k_*. Hence for all $q \geqq 0$ one has $\mathrm{R}^q k_* \mathbb{C}_{\tilde{X}^*} \cong \mathscr{H}^q(k_* \Omega^{\cdot}_{\tilde{X}^*})$. $\square$

(2.5) **Lemma.** $\mathbb{H}^p(X, k_* \Omega^{\cdot}_{\tilde{X}^*}) \cong \mathbb{H}^p(Y, i^* k_* \Omega^{\cdot}_{\tilde{X}^*})$ *for all* $p \in \mathbb{Z}$.

Proof. Shrinking S a little bit does not change the homotopy type of $\tilde{X}^*$. Hence

$$\mathbb{H}^p(X, k_* \Omega^{\cdot}_{\tilde{X}^*}) \cong \varinjlim_{0 \in U \subset S} \mathbb{H}^p(f^{-1}(U), k_* \Omega^{\cdot}_{\tilde{X}^*}|_{f^{-1}(U)}).$$

If U runs over a fundamental system of neighborhoods of $0 \in S$, then $f^{-1}(U)$ runs over a fundamental system of neighborhoods of Y in X. So

$$\mathbb{H}^p(X, k_* \Omega^{\bullet}_{\tilde{X}^*}) \cong \varinjlim_{Y \subset V \subset X} \mathbb{H}^p(V, k_* \Omega^{\bullet}_{\tilde{X}^*}|_V)$$

$$\cong \mathbb{H}^p(Y, i^{\bullet} k_* \Omega^{\bullet}_{\tilde{X}^*}). \quad \square$$

(2.6) It has been shown in [6] that the complex $i^{\bullet} k_* \Omega^{\bullet}_{\tilde{X}^*}$ is quasi-isomorphic to its subcomplex consisting of germs of meromorphic quasi-unipotent sections. The idea of the following proof is due to Katz. For $\alpha \in \mathbb{Q}$ with $0 \leqq \alpha < 1$ denote $L^{\bullet}_{\alpha}$ the complex of sheaves on Y whose sections have the form

$$\omega = t^{-\alpha} \sum_{i=0}^{s} \omega_i (\log t)^i$$

with $\omega_0, \ldots, \omega_s$ sections of $i^{\bullet} \Omega^{\bullet}_X(\log Y)$. The rules $t^{-\alpha} = \exp(2\pi i \alpha u)$ and $\log t = 2\pi i u$ make $L^{\bullet}_{\alpha}$ into a subcomplex of $i^{\bullet} k_* \Omega^{\bullet}_{\tilde{X}^*}$. The natural map

$$\varphi\colon \bigoplus_{\alpha} L^{\bullet}_{\alpha} \to i^{\bullet} k_* \Omega^{\bullet}_{\tilde{X}^*}$$

is injective.

One defines a map

$$\psi\colon \bigoplus_{\alpha} L^{\bullet}_{\alpha} \to \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

as follows. If $\omega = t^{-\alpha} \sum_{i=0}^{s} \omega_i (\log t)^i$ is a section of $L^{\bullet}_{\alpha}$, then $\psi(\omega)$ is the image of ω_0 under the natural map $i^{\bullet} \Omega^{\bullet}_X(\log Y) \to \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. One easily checks that ψ is a homomorphism of complexes. We will show that φ and ψ are quasi-isomorphisms. This can be checked locally on Y.

(2.7) Fix a point $Q \in Y$ and let $(z_0, \ldots, z_n)$ be a coordinate system on a neighborhood U of Q in X centered at Q; let $v, e_0, \ldots, e_v \in \mathbb{Z}$ with $0 \leqq v \leqq n$, $e_i \geqq 1$ such that $f(z_0, \ldots, z_n) = z_0^{e_0} \ldots z_v^{e_v}$. Denote $e = \gcd(e_0, \ldots, e_v)$ and $n_i = e_i/e$. Put

$$\xi_i = dz_i/z_i \qquad (i = 0, \ldots, v).$$

The fundamental relation is

$$f^*(dt/t) = \sum_{i=0}^{v} e_i \xi_i.$$

By abuse of notation we write dt/t instead of $f^*(dt/t)$.

(2.8) **Lemma.** *$\mathcal{H}^q(i^{\bullet} k_* \Omega^{\bullet}_{\tilde{X}^*})_Q$ is generated over $\mathbb{C}$ by the classes of the forms*

$$t^{-a/e} \left(\prod_{i=0}^{v} z_i^{n_i} \right)^a \xi_{i_1} \wedge \cdots \wedge \xi_{i_q} \Big|_{0 \leqq i_1 < \cdots < i_q \leqq v}^{a = 0, 1, \ldots, e-1;}$$

which are subject to the relation $\sum_{i=0}^{v} e_i \xi_i = 0$.

Proof. For every $\varepsilon>0$ choose $\eta>0$ with $\eta\ll\varepsilon$ such that if $B_\varepsilon=\{z\in U\mid \|z\|<\varepsilon\}$, the intersection of δB_ε and X_t is transversal for all $t\in S_\eta=\{t\in S\mid |t|<\eta\}$. The sets $V_\varepsilon=B_\varepsilon\cap f^{-1}(S_\eta)$ for $\varepsilon\to 0$ form a fundamental system of neighborhoods of Q in X. Hence

$$\mathscr{H}^q(i^{\cdot} k_* \Omega^{\cdot}_{\tilde X^*})_Q \cong \lim_{\varepsilon\to 0} H^q(\Gamma(k^{-1}(V_\varepsilon), \Omega^{\cdot}_{\tilde X^*})).$$

Moreover

$$k^{-1}(V_\varepsilon)=\left\{(z,u)\in B_\varepsilon\times \tilde S^*_\eta \,\middle|\, \prod_{i=0}^{v} z_i^{e_i}=\exp(2\pi i u)\right\}.$$

Denote $F=\left\{y\in\mathbb{C}^{n+1}\,\middle|\,\prod_{i=0}^{v} y_i^{e_i}=1\right\}$. Choose integers $a_0,\dots,a_v$ with $\sum_{j=0}^{v} a_j e_j=e$. Then the map $g\colon k^{-1}(V_\varepsilon)\to F\times\tilde S^*_\eta$ defined by $g(z_0,\dots,z_n;u)=(y_0,\dots,y_n;u)$ where $y_j=z_j\exp(-2\pi i a_j u/e)$ if $j\leqq v$ and $y_j=z_j$ for $j>v$, is a homotopy equivalence, as well as the projection $p_1\colon F\times\tilde S^*_\eta\to F$.

The manifold F is the disjoint union of e components, each of which is isomorphic to $(\mathbb{C}^*)^v$. Denote $\tau=y_0^{n_0}\dots y_v^{n_v}\in \mathrm{H}^0(F,\mathbb{C})$. Then τ generates $\mathrm{H}^0(F,\mathbb{C})$ as a $\mathbb{C}$-algebra and $\tau^e=1$. Moreover $\mathrm{H}^1(F,\mathbb{C})$ is the $\mathrm{H}^0(F,\mathbb{C})$-module generated by the classes of dy_j/y_j $(j=0,\dots,v)$ with the relation $\sum_{j=0}^{v} e_j\,dy_j/y_j=0$. Finally $\mathrm{H}^q(F,\mathbb{C})$ is the q-th exterior power of $\mathrm{H}^1(F,\mathbb{C})$ as a $\mathrm{H}^0(F,\mathbb{C})$-module. One gets the desired representatives of $\mathscr{H}^q(i^{\cdot} k_*\Omega^{\cdot}_{\tilde X^*})_Q$ by applying $g^* p_1^*$ to the usual basis of $\mathrm{H}^q(F,\mathbb{C})$. □

(2.9) *Remark.* In the same way one shows that $(i^{\cdot} R^q k_* \mathbb{Z}_{\tilde X^*})_Q$ (resp. $(i^{\cdot} R^q k_* \mathbb{Q}_{\tilde X^*})_Q$) is generated over $\mathbb{Z}(\mathbb{Q})$ by the classes of the forms

$$\frac{1}{e}\sum_{a=0}^{e-1}\zeta^{-a}t^{-a/e}\left(\prod_{i=0}^{v} z_i^{n_i}\right)^a (2\pi i)^{-q}\,\xi_{i_1}\wedge\dots\wedge\xi_{i_q},\qquad (\zeta=\exp(2\pi i/e),$$

subject to the relation $\sum_{i=0}^{v} e_i\,\xi_i=0$. □

(2.10) **Lemma.** *If* $0\leqq\alpha<1$ *and* $\alpha\notin\{0,1/e,\dots,1-1/e\}$ *then the complex* $i^{\cdot} t^{-\alpha}\Omega^{\cdot}_X(\log Y)_Q$ *is acyclic.*

Proof. This complex is isomorphic to the Koszul complex on $\mathbb{C}\{z_0,\dots,z_n\}$ with operators $D_0,\dots,D_n$ where $D_i=z_i\,\partial/\partial z_i-\alpha e_i$ if $0\leqq i\leqq v$ and $D_i=\partial/\partial z_i$ for $i>v$. If $\alpha e_i\notin\mathbb{Z}$ then D_i is bijective $(i\leqq v)$. □

(2.11) **Lemma.** *If* $\alpha=a/e$ *with* $a\in\{0,1,\dots,e-1\}$, *then* $\mathscr{H}^q(i^{\cdot} t^{-\alpha}\Omega^{\cdot}_X(\log Y))_Q$ *has as a basis over* $\mathbb{C}$ *the classes of the forms*

$$\left\{t^{-\alpha}\left(\prod_{i=0}^{v} z_i^{n_i}\right)^a \xi_{i_1}\wedge\dots\wedge\xi_{i_q}\,\middle|\,0\leqq i_1<\dots<i_q\leqq v\right\}.$$

Proof. As in the proof of proposition (1.12) one reduces to the case $n=v$. So $i^{\cdot} t^{-\alpha}\Omega^{\cdot}_X(\log Y)_Q$ is quasi-isomorphic to the Koszul complex on $\mathbb{C}\{z_0,\dots,z_v\}$ with operators $D_i=z_i\,\partial/\partial z_i-an_i$ $(i=0,\dots,v)$. Again these are homogeneous operators and the only non-zero contribution to the cohomology comes from those monomials on which the D_i are all zero, i.e. from $(z_0^{n_0} \dots z_\nu^{n_\nu})^a$ only. □

(2.12) **Lemma.** *Let $M^\cdot$ and $L^\cdot$ be bounded complexes of $\mathbb{C}$-vectorspaces with increasing filtrations F resp. G such that $M^\cdot = \bigcup_{n \geq 0} F_n M^\cdot$ and $L^\cdot = \bigcup_{n \geq 0} G_n L^\cdot$. If $\varphi: (M^\cdot, F) \to (L^\cdot, G)$ is a morphism of filtered complexes and $\mathrm{Gr}_n(\varphi): \mathrm{Gr}_n^F(M^\cdot) \to \mathrm{Gr}_n^G(L^\cdot)$ is a quasi-isomorphism for all n, then φ is a quasi-isomorphism.*

Proof. Left to the reader. □

(2.13) Denote H_α^q the subspace of $i^\cdot t^{-\alpha} \Omega_X^q(\log Y)_Q$ generated over $\mathbb{C}$ by the representatives of the cohomology as given in lemma (2.11). Denote $H_\alpha^q[\log t]$ the subspace of $L_{\alpha,Q}^q$ consisting of elements of the form $\sum_{i=0}^{s} \omega_i (\log t)^i$ with $\omega_i \in H_\alpha^q$. Then $H_\alpha^\cdot[\log t]$ is a subcomplex of $L_{\alpha,Q}^\cdot$.

(2.14) **Lemma.** *The inclusion $H_\alpha^\cdot[\log t] \to L_{\alpha,Q}^\cdot$ is a quasi-isomorphism.*

Proof. Denote F the filtration on $H_\alpha^\cdot[\log t]$ and $L_{\alpha,Q}^\cdot$ by the degree in $\log t$. Then the map

$$\mathrm{Gr}_n^F H_\alpha^\cdot[\log t] \to \mathrm{Gr}_n^F L_{\alpha,Q}^\cdot$$

is just the inclusion $H_\alpha^\cdot \to i^\cdot t^{-\alpha} \Omega_X^\cdot(\log Y)$, which is clearly a quasi-isomorphism. Next use lemma (2.12). □

(2.15) **Lemma.** *The injection $H_\alpha^\cdot \to H_\alpha^\cdot[\log t]$ induces surjective maps*

$$H_\alpha^q \to H^q(H_\alpha^\cdot[\log t])$$

with kernels formed by the elements $\eta = dt/t \wedge \omega$ for $\omega \in H_\alpha^q$.

Proof. Let $\omega = \sum_{i=0}^{s} \omega_i (\log t)^i$ with $\omega_i \in H_\alpha^q$ for $i = 0, \dots, s$. Then $d\omega = 0$ if and only if $dt/t \wedge \omega_i = 0$ for $i = 1, \dots, s$. By a lemma of De Rham (cf. [1], p. 8) this implies that $\omega_i = dt/t \wedge \eta_i$ for some $\eta_i \in H_\alpha^{q-1}$. Put $\eta = \sum_{k=1}^{s} (k+1)^{-1} \eta_k (\log t)^{k+1}$. Then $\omega = \omega_0 + d\eta$. This proves that $H_\alpha^q \to H^q(H_\alpha^\cdot[\log t])$ is surjective. Moreover ω_0 is mapped to zero if and only if there exists $\xi \in H_\alpha^{q-1}$ with $\omega = d(\xi \log t) = dt/t \wedge \xi$.

(2.16) **Proposition.** $\mathrm{H}^p(\tilde{X}^*, \mathbb{C}) \cong \mathbb{H}^p(Y, \Omega_{X/S}^\cdot(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$ *for all $p \geq 0$.*

Proof. The above lemmas show that the maps φ and ψ from (2.6) are quasi-isomorphisms. □

(2.17) *Warning.* Unlike the isomorphism in Proposition (2.2), the map in Proposition (2.16) is not canonical but depends on the choice of a parameter t on the disk S. We return to this problem later.

(2.18) **Theorem.** *For all $p \geq 0$, the sheaf $\mathbb{R}^p f_*(\Omega_{X/S}^\cdot(\log Y))$ is locally free on S and for all $s \in S$ the canonical map*

$$\mathbb{R}^p f_*(\Omega_{X/S}^\cdot(\log Y)) \otimes_{\mathcal{O}_S} (\mathcal{O}_{S,s}/\mathfrak{m}_{S,s}) \to \mathbb{H}^p(X_s, \Omega_{X/S}^\cdot(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s})$$

is an isomorphism.

Proof. Propositions (2.2) and (2.16) together with the fact that $X^* \to S^*$ is a locally trivial C^∞-fibre bundle, show that for any $p \geqq 0$ the function

$$s \mapsto \dim_{\mathbb{C}} \mathbb{H}^p(X_s, \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s})$$

is constant on S. Moreover f is flat because f is surjective, X is connected and $\dim S = 1$. Conclude by [14], Corollary 2, p. 50. □

(2.19) We conclude this chapter with some facts concerning the Gauss-Manin connection. We refer to [11] and [12] for more details and proofs. Keeping the same notations, the Gauss-Manin connection can be constructed as the connecting homomorphism

$$\nabla: \mathbb{R}^p f_* \Omega^{\bullet}_{X/S}(\log Y) \to \Omega^1_S(\log 0) \otimes_{\mathcal{O}_S} \mathbb{R}^p f_* \Omega^{\bullet}_{X/S}(\log Y)$$

in the long exact sequence of hypercohomology, associated to the exact sequence of complexes

$$0 \to f^* \Omega^1_S(\log 0) \otimes_{\mathcal{O}_X} \Omega^{\bullet}_{X/S}(\log Y)[-1] \xrightarrow{A} \Omega^{\bullet}_X(\log Y) \to \Omega^{\bullet}_{X/S}(\log Y) \to 0$$

on X. Its sheaf of horizontal sections on S^* is the local system $R^p f_* \mathbb{C}_{X^*}$. For $s \in S^*$ the fundamental group $\pi_1(S^*, s) = \mathbb{Z}$ acts on $H^p(X_s, \mathbb{C})$. The action of a generator of this extends to an automorphism T of the sheaf $\mathbb{R}^p f_* \Omega^{\bullet}_{X/S}(\log Y)$, called the monodromy. Denote T_0 the induced automorphism of

$$\mathbb{H}^p(Y, \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y).$$

Define $R_0: \Omega^1_S(\log 0) \to \mathbb{C}$ by $R_0(f\,dt/t) = f(0)$. The map

$$(R_0 \otimes \mathrm{id}) \circ \nabla: \mathbb{R}^p f_* \Omega^{\bullet}_{X/S}(\log Y)_0 \to \mathbb{H}^p(Y, \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

is zero on $t\,\mathbb{R}^p f_* \Omega^{\bullet}_{X/S}(\log Y)_0$. The induced endomorphism of

$$\mathbb{H}^p(Y, \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

is denoted by $\mathrm{Res}_0(\nabla)$. It can also be seen as induced by the $\mathbb{C}$-linear endomorphism (derivation) $\nabla_{t\partial/\partial t}$ of $\mathbb{R}^p f_* \Omega^{\bullet}_{X/S}(\log Y)_0$.

(2.20) **Proposition.** *If* α *is an eigenvalue of* $\mathrm{Res}_0(\nabla)$, *then* $\alpha \in \mathbb{Q}$ *and* $0 \leqq \alpha < 1$.

Proof. The Gauss-Manin connection acts on the spectral sequence

$$E_2^{pq} = R^p f_*(\mathscr{H}^q(\Omega^{\bullet}_{X/S}(\log Y)) \Rightarrow \mathbb{R}^{p+q} f_*(\Omega^{\bullet}_{X/S}(\log Y))$$

(cf. [13]) hence its residue acts on the spectral sequence

$$E_2^{pq} = H^p(Y, \mathscr{H}^q(\Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)) \Rightarrow H^{p+q}(Y, \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y).$$

If α is an eigenvalue of $\mathrm{Res}_0(\nabla)$, it has to appear as an eigenvalue of the action on some E_2^{pq}. This action is induced by maps $\delta_q \in \mathrm{End}(\mathscr{H}^q(\Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y))$ which are defined as the connecting homomorphisms in the long exact sequence of cohomology sheaves, associated to the exact sequence

$$0 \to \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y[-1] \xrightarrow{\rho} \Omega^{\bullet}_X(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to 0$$

of complexes on Y, where $\rho(\omega)=\omega\wedge dt/t$. Computation of δ_q in local coordinates (notations as in Proposition (1.12)) gives

$$\delta_q(y^a\,\xi_{i_1}\wedge\cdots\wedge\xi_{i_q})=\frac{a}{e}\cdot y^a\,\xi_{i_1}\wedge\cdots\wedge\xi_{i_q},$$

so $\alpha=a/e$ for some a, $e\in\mathbb{Z}$ and $0\leq a<e$. □

The relation between T_0 and $\mathrm{Res}_0(\nabla)$ is given by

(2.21) **Theorem.** $T_0=\exp(-2\pi i\,\mathrm{Res}_0(\nabla))$. *Cf.* [5], Theorem II.3.11. □

Because the characteristic polynomial of T_s is constant on S, these two statements prove that the eigenvalues of the monodromy are roots of unity.

§ 3. Mixed Hodge Structures

(3.1) We list some notions from Hodge theory. We refer to [3] and [4] for details and proofs.

Let A be a noetherian subring of $\mathbb{R}$ such that $A\otimes_{\mathbb{Z}}\mathbb{Q}$ is a field. An *A-Hodge structure of weight n* is an A-module $\mathbf{H}_A$ of finite type together with a finite decreasing filtration F on $\mathbf{H}_{\mathbb{C}}=\mathbf{H}_A\otimes_A\mathbb{C}$ such that for all p, $q\in\mathbb{Z}$ with $p+q=n+1$:

$$F^p\,\mathbf{H}_{\mathbb{C}}\oplus\overline{F^q\,\mathbf{H}_{\mathbb{C}}}\cong\mathbf{H}_{\mathbb{C}}.$$

This is equivalent to the splitting of $\mathbf{H}_{\mathbb{C}}$ into a direct sum

$$\mathbf{H}_{\mathbb{C}}=\bigoplus_{p+q=n}\mathbf{H}^{pq}$$

such that $\mathbf{H}^{pq}=\overline{\mathbf{H}^{qp}}$. One takes $\mathbf{H}^{pq}=F^p\mathbf{H}_{\mathbb{C}}\cap\overline{F^q\mathbf{H}_{\mathbb{C}}}(p+q=n)$ and

$$F^p\mathbf{H}_{\mathbb{C}}=\bigoplus_{r\geq p}\mathbf{H}^{r,n-r}.$$

(3.2) *Example.* If X is a compact Kähler manifold and $p,q\in\mathbb{Z}$, let $\mathbf{H}^{pq}(X)$ be the space of harmonic forms on X of type (p,q). Hodge [10] has shown that

$$\mathrm{H}^n(X,\mathbb{C})\cong\bigoplus_{p+q=n}\mathbf{H}^{pq}(X)$$

and $\mathbf{H}^{pq}(X)=\overline{\mathbf{H}^{qp}(X)}$.

A similar statement holds for X a complete nonsingular algebraic variety over $\mathbb{C}$:

$$\mathrm{H}^n(X,\mathbb{C})\cong\mathbb{H}^n(X,\Omega_X^\bullet)\cong\bigoplus_{p+q=n}\mathrm{H}^q(X,\Omega_X^p)$$

and

$$F^p\mathrm{H}^n(X,\mathbb{C})\cong\mathbb{H}^n(X,\sigma_{\geq p}\Omega_X^\bullet)$$

where $\sigma_{\geq}$ denotes the stupid filtration ("filtration bête") which is defined for any complex $K^\bullet$ by $\sigma_{\geq p}K^q=K^q$ if $q\geq p$ and $\sigma_{\geq p}K^q=0$ if $q<p$.

(3.3) The *Hodge structure of Tate* $\mathbb{Z}(1)$ is the Hodge structure of weight-2, purely of type $(-1,-1)$ with $\mathbf{H}_{\mathbb{Z}}=2\pi i\mathbb{Z}\subset\mathbb{C}=\mathbf{H}_{\mathbb{C}}^{-1,-1}$.

If $(\mathbf{H}_{\mathbb{Z}}, F)$ is a Hodge structure of weight n and $k \in \mathbb{Z}$, one denotes $\mathbf{H}_{\mathbb{Z}}(k)$ the tensor product $\mathbf{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}(1)^{\otimes k}$ which is a Hodge structure of weight $n-2k$.

(3.4) Let A be a noetherian subring of $\mathbb{R}$ such that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field. A *mixed A-Hodge structure* consists of the following data:

(i) a finitely generated A-module $\mathbf{H}_A$;

(ii) a finite increasing filtration W on $\mathbf{H}_A \otimes_{\mathbb{Z}} \mathbb{Q}$, called the weight filtration;

(iii) a finite decreasing filtration F on $\mathbf{H}_A \otimes_A \mathbb{C}$, called the Hodge filtration,

such that $\mathrm{Gr}_n^W(\mathbf{H}_A \otimes_{\mathbb{Z}} \mathbb{Q})$ together with the filtration, induced by F on $\mathrm{Gr}_n^W(\mathbf{H}_A \otimes_A \mathbb{C})$, is a $A \otimes_{\mathbb{Z}} \mathbb{Q}$-Hodge structure of weight n for all $n \in \mathbb{Z}$.

(3.5) *Example.* Let Y be a complete complex algebraic variety with irreducible components $Y_1, \ldots, Y_N$ which are nonsingular of the same dimension, such that for all p the intersections $Y_p \cap Y_q$ for $q \neq p$ form a smooth divisor on Y_p with only normal crossings. Then $H^n(Y, \mathbb{Z})$ carries for all $n \in \mathbb{Z}$ a canonical and functorial mixed Hodge structure [4] which can be constructed as follows.

Denote for $k \geq 1$

$$\tilde{Y}^{(k)} = \coprod_{i_1 < \cdots < i_k} Y_{i_1} \cap \cdots \cap Y_{i_k}$$

and denote $a_k: \tilde{Y}^{(k)} \to Y$ the natural map. Let $\delta_j: \tilde{Y}^{(k)} \to \tilde{Y}^{(k-1)}$ be the map with as components the inclusions

$$Y_{i_1} \cap \cdots \cap Y_{i_k} \to Y_{i_1} \cap \cdots \cap Y_{i_{j-1}} \cap Y_{i_{j+1}} \cap \cdots \cap Y_{i_k}.$$

On Y one defines a double complex $K^{\cdot\cdot}$ as follows. Put $K^{pq} = (a_{q+1})_* \Omega^p_{\tilde{Y}^{(q+1)}}$ $(p, q \geq 0)$. Define $d': K^{pq} \to K^{p+1,q}$ to be the differentiation in the complex $(a_{q+1})_* \Omega^{\cdot}_{\tilde{Y}^{(q+1)}}$ and let $d'': K^{pq} \to K^{p,q+1}$ be defined by $d'' = \sum_{j=1}^{q+1} (-1)^{q+j} \delta_j^*$.

One defines filtrations F and W on $K^{\cdot\cdot}$ by $F^p K^{\cdot\cdot} = \bigoplus_{r \geq p} K^{r\cdot}$ and $W_q K^{\cdot\cdot} = \bigoplus_{s \geq -q} K^{\cdot s}$. Denote $K^{\cdot}$ the associated single complex. This is a resolution of $\mathbb{C}_Y$. Moreover W and F induce filtrations on the hypercohomology groups $\mathbb{H}^n(Y, K^{\cdot})$, which give a mixed Hodge structure.

The weight filtration on $\mathrm{H}^n(Y, \mathbb{Q})$ is obtained as follows: for $\mathbb{Q}_Y$ one has the resolution

$$0 \to a_* \mathbb{Q}_{\tilde{Y}} \xrightarrow{d''} (a_2)_* \mathbb{Q}_{\tilde{Y}^{(2)}} \xrightarrow{d''} \cdots.$$

Hence the hypercohomology of this complex is just $\mathrm{H}^n(Y, \mathbb{Q})$. The spectral sequence of hypercohomology of this complex with its stupid filtration gives

$$\mathrm{E}_1^{pq} = \mathrm{H}^q(\tilde{Y}^{(p+1)}, \mathbb{Q}) \Rightarrow \mathrm{H}^{p+q}(Y, \mathbb{Q}),$$

and

$$\mathrm{E}_2^{pq} = \mathrm{E}_\infty^{pq} = \mathrm{Gr}_q^W \mathrm{H}^{p+q}(Y, \mathbb{Q}).$$

The map $d_1: \mathrm{E}_1^{pq} \to \mathrm{E}_1^{p+1,q}$ is just $\sum_{j=1}^{q+1} (-1)^{q+j} \delta_j^*$.

(3.6) The preceeding example shows a situation, where the Hodge and weight filtrations on $\mathbf{H}_{\mathbb{C}}$ are induced from filtrations on a complex of sheaves by passage to hypercohomology. This leads to the following definition.

Let A be a noetherian subring of $\mathbb{R}$ such that $A\otimes_{\mathbb{Z}}\mathbb{Q}$ is a field, and let X be a topological space. A *cohomological mixed A-Hodge complex* on X consists of the data:

(i) an object $K^{\cdot}_A$ of the derived category $D^+(X, A)$ (cf. [9]) such that for all $n\in\mathbb{Z}$ the A-module $\mathbb{H}^n(X, K^{\cdot}_A)$ is of finite type;

(ii) an object $(K^{\cdot}_{A\otimes\mathbb{Q}}, W)$ in the filtered derived category $D^+F(X, A\otimes\mathbb{Q})$ where W is an increasing filtration, such that $K^{\cdot}_{A\otimes\mathbb{Q}}\cong K^{\cdot}_A\otimes_{\mathbb{Z}}\mathbb{Q}$ in $D^+(X, A\otimes\mathbb{Q})$;

(iii) an object $(K^{\cdot}_{\mathbb{C}}, W, F)$ of the bifiltered derived category $D^+F_2(X, \mathbb{C})$ where W is increasing and F is decreasing, such that $(K^{\cdot}_{\mathbb{C}}, W)\cong(K^{\cdot}_{A\otimes\mathbb{Q}}\otimes\mathbb{C}, W)$ in $D^+F(X,\mathbb{C})$.

One postulates that for all $n, k\in\mathbb{Z}$ the couple

$$(\mathbb{H}^n(X, \mathrm{Gr}^W_k(K^{\cdot}_{A\otimes\mathbb{Q}})), (\mathbb{H}^n(X, \mathrm{Gr}^W_k K^{\cdot}_{\mathbb{C}}), F))$$

is an $A\otimes\mathbb{Q}$-Hodge structure of weight $n+k$.

For example (3.5) these data are:

(i) the sheaf $\mathbb{Z}_Y$;

(ii) the complex $0\to(a_1)_*\mathbb{Q}_{\tilde{Y}}\xrightarrow{d''}(a_2)_*\mathbb{Q}_{\tilde{Y}^{(2)}}\xrightarrow{d''}\cdots$ with its stupid filtration;

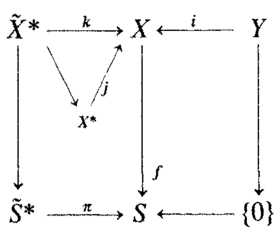
(iii) the complex $K^{\cdot\cdot}$ with the filtrations W and F.

§ 4. A Mixed Hodge Structure on the Limit

(4.1) Let X be a connected complex manifold and let S be the unit disk. Let $f: X\to S$ be a proper surjective holomorphic map. We assume that $Y=f^{-1}(0)$ is a union of smooth divisors on X with normal crossings and that f is smooth at every point of $X^*=X\setminus Y$. We choose a parameter t on S. For $t\neq 0$ denote $f^{-1}(t)=X_t$.

Under the assumptions that the monodromy automorphism T of $\mathrm{H}^q(X_t, \mathbb{C})$ is unipotent and that Y is an algebraic variety, we put a mixed Hodge structure on $\mathbb{H}^q(Y, \Omega^{\cdot}_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_Y)$. In this chapter we investigate how it depends on the choice of the parameter t and how it is related to the canonical mixed Hodge structure on $\mathrm{H}^q(Y, \mathbb{C})$. In § 5 we will consider the case of a projective morphism.

Throughout § 4 and § 5 we fix the following notations. We put $S^*=S\setminus\{0\}$, let $\tilde{S}^*$ be its universal covering and $\tilde{X}^*=X\times_S\tilde{S}^*$. Denote k: $\tilde{X}^*\to X$, j: $X^*\to X$, i: $Y\to X$ and π: $\tilde{S}^*\to S$ the natural maps.



Let $Y=Y_1\cup\dots\cup Y_N$, and for $k\geqq 1$ denote

$$\tilde{Y}^{(k)}=\coprod_{i_1<\dots<i_k} Y_{i_1}\cap\dots\cap Y_{i_k}.$$

Denote $a_k\colon \tilde{Y}^{(k)}\to Y$ and $\delta_j\colon \tilde{Y}^{(k)}\to\tilde{Y}^{(k-1)}$ $(j=1,\dots,k)$ as in Example (3.5).

(4.2) If $\mathbb{G}$ is a constant sheaf of abelian groups on $\tilde{X}^*$, a generator of $\pi_1(S^*)$ acts on the sheaf $k_*\mathbb{G}$. By functoriality one gets an action of $\pi_1(S^*)$ on the object $i^*\mathbb{R}k_*\mathbb{G}$ of the derived category $D^+(Y,\mathbb{Z})$. For $\mathbb{G}$ any of the sheaves $\mathbb{Z}_{\tilde{X}^*}$, $\mathbb{Q}_{\tilde{X}^*}$ or $\mathbb{C}_{\tilde{X}^*}$ we denote $(i^*\mathbb{R}k_*\mathbb{G})_{\mathrm{un}}$ the maximal subobject of $i^*\mathbb{R}k_*\mathbb{G}$ on which $\pi_1(S^*)$ acts with unipotent automorphisms.

(4.3) **Lemma.**

$$\mathrm{H}^n(\tilde{X}^*,\mathbb{Q})\cong\mathbb{H}^n(Y,(i^*\mathbb{R}k_*\mathbb{Q}_{\tilde{X}^*})_{\mathrm{un}}),$$

$$\mathrm{H}^n(\tilde{X}^*,\mathbb{C})\cong\mathbb{H}^n(Y,(i^*\mathbb{R}k_*\mathbb{C}_{\tilde{X}^*})_{\mathrm{un}}).$$

Proof. This follows from the fact that the monodromy T is unipotent and from Lemmas (2.4) and (2.5). □

(4.4) If Z is a topological space and $\mathbb{G}$ is an object of $D^+(Z,\mathbb{Q})$, for $k\in\mathbb{Z}$ we denote $\mathbb{G}(k)$ the object $(2\pi\mathrm{i})^k\,\mathbb{G}$ of $D^+(Z,\mathbb{Q})$.

(4.5) Define $\theta=f^*(dt/t)$. Because the form dt/t on S^* has period $2\pi\mathrm{i}$, one should consider θ as an element of $\mathrm{H}^1(X^*,\mathbb{Q}(1))$.

Cupproduct with θ defines for every $K\in\mathrm{Ob}\,D^+(X^*,\mathbb{Q})$ a morphism

$$\theta\colon K\to K(1)\,[1].$$

Because $\theta\wedge\theta=0$, this morphism has square zero. By funtoriality one obtains a morphism

$$\theta\colon i^*\mathbb{R}j_*K\to i^*\mathbb{R}j_*K(1)\,[1].$$

(4.6) **Lemma.** *The sequence*

$$i^*\mathbb{R}j_*\mathbb{Q}_{X^*}(-1)\,[-1]\xrightarrow{\theta} i^*\mathbb{R}j_*\mathbb{Q}_{X^*}\xrightarrow{\theta} i^*\mathbb{R}j_*\mathbb{Q}_{X^*}(1)\,[1]$$

is exact in $D^+(Y,\mathbb{Q})$.

Proof. With notations as in (2.7) the stalk of $i^*\mathrm{R}^q j_*\mathbb{Q}_{X^*}$ at Q is generated over $\mathbb{Q}$ by the classes

$$\{(2\pi\mathrm{i})^{-q}\,\xi_{i_1}\wedge\dots\wedge\xi_{i_q}\,|\,0\leqq i_1<\dots<i_q\leqq v\}.$$

The lemma follows from the lemma of De Rham ([1], p. 8). □

(4.7) Denote $k'\colon\tilde{X}^*\to X^*$ the projection. Then for all $q\neq 0$ one has $\mathrm{R}^q k'_*\mathbb{Q}_{\tilde{X}^*}=0$. Hence

$$k'_*\mathbb{Q}_{\tilde{X}^*}=\mathbb{R}k'_*\mathbb{Q}_{\tilde{X}^*}.$$

The isomorphism of derived functors

$$\mathbb{R}k_*\cong\mathbb{R}j_*\circ\mathbb{R}k'_*$$

(cf. [9], p. 59) induces a canonical isomorphism in $D^+(Y, \mathbb{Q})$:

$$i^{\cdot}\mathbb{R}k_*\mathbb{Q}_{\tilde{X}^*} \cong i^{\cdot}\mathbb{R}j_*(k'_*\mathbb{Q}_{\tilde{X}^*}).$$

Denote ψ: $i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*} \to i^{\cdot}\mathbb{R}k_*\mathbb{Q}_{\tilde{X}^*}$ the map induced from the inclusion $\mathbb{Q}_{X^*} \to k'_*\mathbb{Q}_{\tilde{X}^*}$ by functoriality.

(4.8) **Lemma.** *Im* ψ *is contained in* $(i^{\cdot}\mathbb{R}k_*\mathbb{Q}_{\tilde{X}^*})_{\mathrm{un}}$ *and the sequence*

$$i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*}(-1)[-1] \xrightarrow{\theta} i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*} \xrightarrow{\psi} (i^{\cdot}\mathbb{R}k_*\mathbb{Q}_{\tilde{X}^*})_{\mathrm{un}} \to 0$$

is exact in $D^+(Y, \mathbb{Q})$.

Proof. The first assertion follows from the fact that $\pi_1(S^*)$ acts trivially on $i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*}$. The exactness follows from Lemma (4.6) and Remark (2.9). □

(4.9) One obtains an injective map

$$\bar{\theta}: (i^{\cdot}\mathbb{R}k_*\mathbb{Q}_{\tilde{X}^*})_{\mathrm{un}} \to i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*}(1)[1].$$

Denote W the canonical filtration on $i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*}$ (cf. [3], (1.4.6)). Because θ maps $W_k(i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*})$ to $W_{k+1}(i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*}(1)[1]) = (W_k i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*})(1)[1]$, it induces maps

$$\theta: H^k_{\mathbb{Q}} \to H^{k+1}_{\mathbb{Q}} \qquad (k \geqq 0)$$

where $H^k_{\mathbb{Q}} = i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*}(k+1)[k+1]/(W_k i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*})(k+1)[k+1]$.

(4.10) **Lemma.** *The sequence*

$$0 \to (i^{\cdot}\mathbb{R}k_*\mathbb{Q}_{\tilde{X}^*})_{\mathrm{un}} \xrightarrow{\theta} H^0_{\mathbb{Q}} \xrightarrow{\theta} H^1_{\mathbb{Q}} \to \cdots$$

is exact in $D^+(Y, \mathbb{Q})$.

Proof. This just means that for all $q \geqq 0$ the sequence

$$0 \to (i^{\cdot}R^q k_*\mathbb{Q}_{\tilde{X}^*})_{\mathrm{un}} \xrightarrow{\theta} i^{\cdot}R^{q+1}j_*\mathbb{Q}_{X^*}(1) \xrightarrow{\theta} i^{\cdot}R^{q+2}j_*\mathbb{Q}_{X^*}(2) \to \cdots$$

is exact. This is an immediate consequence of Lemma (4.6). □

(4.11) *Definition.* Denote $A^{\cdot}_{\mathbb{Q}}$ the associated single complex of the double complex $H^0_{\mathbb{Q}} \to H^1_{\mathbb{Q}} \to \cdots$. Define the filtration W on $A^{\cdot}_{\mathbb{Q}}$ as follows

$$W_r H^k_{\mathbb{Q}} := W_{r+2k+1}(i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*})(k+1)[k+1]/W_k(i^{\cdot}\mathbb{R}j_*\mathbb{Q}_{X^*})(k+1)[k+1]$$

$W_r A^{\cdot}_{\mathbb{Q}}$ is the associated single complex of the double complex $W_r H^0_{\mathbb{Q}} \to W_r H^1_{\mathbb{Q}} \to \ldots$.

Define $A^{\cdot}_{\mathbb{Z}} = (i^{\cdot}\mathbb{R}k_*\mathbb{Z}_{\tilde{X}^*})_{\mathrm{un}}$.

(4.12) **Lemma.** $\theta: A^{\cdot}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} A^{\cdot}_{\mathbb{Q}}$ in $D^+(Y, \mathbb{Q})$. *This is a translation of Lemma* (4.10). □

(4.13) **Lemma.** *For* $r \in \mathbb{Z}$ *one has*

$$\mathrm{Gr}_r(A^{\cdot}_{\mathbb{Q}}) \cong \bigoplus_{\substack{k \geqq 0 \\ k \geqq -r}} (a_{r+2k+1})_* \mathbb{Q}_{\tilde{Y}^{(r+2k+1)}}(-r-k)[-r-2k].$$

Proof. The map $\theta: H^k_{\mathbb{Q}} \to H^{k+1}_{\mathbb{Q}}$ induces the zero map $\mathrm{Gr}^W_r H^k_{\mathbb{Q}} \to \mathrm{Gr}^W_r H^{k+1}_{\mathbb{Q}}$. Hence

$$\mathrm{Gr}^W_r A^{\cdot}_{\mathbb{Q}} \cong \bigoplus_{k \geqq 0} \mathrm{Gr}^W_r(H^k_{\mathbb{Q}})[-k].$$

Moreover $\mathrm{Gr}_r^W(H_{\mathbb{Q}}^k) \cong \mathrm{Gr}_{r+2k+1}^W(i^* \mathbb{R} j_* \mathbb{Q}_{X^*})\,(k+1)\,[k+1]$ if $k \geq 0$ and $r \geq -k$, and $\mathrm{Gr}_r^W(H_{\mathbb{Q}}^k) = 0$ otherwise. The lemma now follows from the isomorphism (see [3], (3.1.4.)):

$$\mathrm{Gr}_q^W(i^* \mathbb{R} j_* \mathbb{Q}_{X^*}) \cong (a_q)_* \, \mathbb{Q}_{\tilde{Y}^{(q)}}(-q)\,[-q] \; (q \geq 1). \qquad \square$$

(4.14) We pass to the $\mathbb{C}$-level. The results of §2 easily show that the isomorphism in $D^+(Y, \mathbb{C})$

$$i^* \mathbb{R} k_* \mathbb{C}_{\tilde{X}^*} \xrightarrow{\sim} \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

composed with the canonical map

$$\Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y\mathrm{red}}$$

induce an isomorphismin $D^+(Y, \mathbb{C})$:

$$(i^* \mathbb{R} k_* \mathbb{C}_{\tilde{X}^*})_{\mathrm{un}} \to \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y\mathrm{red}}.$$

Denote $A^{pq} = \Omega_X^{p+q+1}(\log Y)/W_q \Omega_X^{p+q+1}(\log Y)$.

Define $d'\colon A^{pq} \to A^{p+1,q}$ by $d'(\omega) = cl(d\omega)$.

Define $d''\colon A^{pq} \to A^{p,q+1}$ by $d''(\omega) = cl(\theta \wedge \omega)$ where $\theta = f^*(dt/t)$. Then $d'd'' + d''d' = 0$, so $(A^{\bullet\bullet}, d', d'')$ is a double complex.

Define $A^\bullet_{\mathbb{C}}$ to be its associated single complex.

(4.15) **Lemma.** *One has an exact sequence of coherent sheaves on* Y^{red}:

$$0 \to \Omega^p_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y\mathrm{red}} \xrightarrow{\theta} A^{p0} \xrightarrow{\theta} A^{p1} \xrightarrow{\theta} \cdots.$$

Proof. One has to check that θ is strictly compatible with the filtration W on $\Omega^\bullet_X(\log Y)$, i.e. that

$$\mathrm{Gr}_{k-1}^W \Omega_X^{p-1}(\log Y) \xrightarrow{\theta} \mathrm{Gr}_k^W \Omega_X^p(\log Y) \xrightarrow{\theta} \mathrm{Gr}_{k+1}^W \Omega_X^{p+1}(\log Y)$$

is exact for $k \geq 2$ and that

$$\mathrm{Ker}\,(\theta\colon \mathrm{Gr}_1^W \Omega_X^p(\log Y) \to \mathrm{Gr}_2^W \Omega_X^{p+1}(\log Y))$$
$$\cong W_0 \Omega_X^{p-1}(\log Y)/I(Y^{\mathrm{red}}) \cdot \Omega_X^{p-1}(\log Y).$$

This can be done by taking Poincaré residues. The resulting sequences are

$$0 \to \Omega^p_{Y\mathrm{red}} \to (a_1)_* \, \Omega^p_{\tilde{Y}} \xrightarrow{d''} (a_2)_* \Omega^p_{\tilde{Y}^{(2)}} \xrightarrow{d''} \cdots.$$

Here d'' is the same map as in example (3.5) except for a factor e_{i_j} before δ_j^* in the restriction maps

$$\Omega^p_{Y_{i_1} \cap \cdots \cap Y_{i_{j-1}} \cap Y_{i_{j+1}} \cap \cdots \cap Y_{i_k}} \to \Omega^p_{Y_{i_1} \cap \cdots \cap Y_{i_k}}.$$

This does not spoil the exactness of the sequence. $\square$

(4.16) **Corollary.** *The map* $\tilde{\theta}\colon \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y\mathrm{red}} \to A^\bullet_{\mathbb{C}}$, *defined by* $\tilde{\theta}(\omega) = (-1)^p \, \theta \wedge \omega$ *for* ω *section of* $\Omega^p_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y\mathrm{red}}$, *is a quasi-isomorphism.*

Proof. After Lemma (4.15) one only has to check that $\tilde{\theta}$ is a homomorphism of complexes. One has

$$\tilde{\theta}(d\omega) = (-1)^{p+1} \, \theta \wedge d\omega = (-1)^p \, d(\theta \wedge \omega) = d\tilde{\theta}(\omega)$$

for ω a section of $\Omega^p_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y\mathrm{red}}$. $\square$

(4.17) We define filtrations F and W, F decreasing and W increasing, on $A^{\bullet}_{\mathbb{C}}$ as follows.

Let $F^p A^{\bullet}_{\mathbb{C}} = \bigoplus_{\substack{p' \geq p \\ q \geq 0}} A^{pq}$.

Let $W_r A^{\bullet}_{\mathbb{C}} = \bigoplus_{p,q \geq 0} W_r A^{pq}$ where

$$W_r A^{pq} = W_{2q+r+1}\, \Omega_X^{p+q+1}(\log Y)/W_q\, \Omega_X^{p+q+1}(\log Y).$$

One checks that d' and d'' are compatible with the filtrations F and W.

(4.18) **Lemma.** $\mathrm{Gr}_r^W A^{\bullet}_{\mathbb{C}} \cong \bigoplus_{\substack{k \geq 0 \\ k \geq -r}} (a_{2k+r+1})_* \,\Omega^{\bullet}_{\tilde{Y}^{(2k+r+1)}}[-r-2k]$.

Proof. Analogous to the proof of Lemma (4.13). □

(4.19) **Theorem.** *The data* $A^{\bullet}_{\mathbb{Z}}$, $(A^{\bullet}_{\mathbb{Q}}, W)$ *and* $(A^{\bullet}_{\mathbb{C}}, F, W)$ *together with the isomosphisms*

$$A^{\bullet}_{\mathbb{Z}} \otimes \mathbb{Q} \xrightarrow{\sim} A^{\bullet}_{\mathbb{Q}} \quad \text{in } D^+(Y, \mathbb{Q})$$

and

$$(A^{\bullet}_{\mathbb{Q}} \otimes \mathbb{C}, W_{\mathbb{C}}) \xrightarrow{\sim} (A^{\bullet}_{\mathbb{C}}, W) \quad \text{in } D^+F(Y, \mathbb{Q})$$

determine a cohomological mixed Hodge complex on Y.

Proof. That the natural map $A^{\bullet}_{\mathbb{Q}} \otimes \mathbb{C} \to A^{\bullet}_{\mathbb{C}}$ is a quasi-isomorphism follows from (4.14) and (4.16). The formula's in (4.13) and (4.18) show that it is strictly compatible with the filtration W.

The Hodge-theory for complete nonsingular algebraic varieties guarantees that for all $q \geq 0$, $r \in \mathbb{Z}$ the couple

$$(\mathbb{H}^q(Y, \mathrm{Gr}_r^W(A^{\bullet}_{\mathbb{Q}})), (\mathbb{H}^q(Y, \mathrm{Gr}_r^W A^{\bullet}_{\mathbb{C}}), F))$$

is a $\mathbb{Q}$-Hodge structure of weight $q+r$, namely:

$$\mathbb{H}^q(Y, \mathrm{Gr}_r^W(A^{\bullet}_{\mathbb{Q}})) = \bigoplus_{\substack{k \geq 0 \\ k \geq -r}} \mathrm{H}^{q-r-2k}(\tilde{Y}^{(2k+r+1)}, \mathbb{Q})(-r-k).$$

This has indeed weight $q-r-2k-2(-r-k) = q+r$. □

(4.20) **Corollary.** (*cf.* [4], *scholie* 8.19)

(i) *The spectral sequence of hypercohomology for the filtered complex* $(A^{\bullet}_{\mathbb{Q}}, W)$

$${}_W\mathrm{E}_1^{-r,q+r} = \mathbb{H}^q(Y, \mathrm{Gr}_r^W A^{\bullet}_{\mathbb{Q}}) \Rightarrow \mathbb{H}^q(Y, A^{\bullet}_{\mathbb{Q}})$$

degenerates at E_2, *i.e.* $\mathrm{E}_2 = \mathrm{E}_\infty$;

(ii) *The spectral sequence of hypercohomology for the filtered complex* $(A^{\bullet}_{\mathbb{C}}, F)$

$${}_F\mathrm{E}_1^{pq} = \mathrm{H}^q(Y, \Omega^p_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}}) \Rightarrow \mathbb{H}^{p+q}(Y, \Omega^{\bullet}_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}})$$

degenerates at E_1. □

(4.21) **Proposition.** $\mathrm{Gr}_r^W \mathbb{H}^q(Y, A^{\bullet}_{\mathbb{Q}}) \neq 0 \Rightarrow 0 \leq r \leq 2q$.

Proof. $\mathrm{Gr}_r^W \mathbb{H}^q(Y, A_{\mathbb{Q}}^{\bullet}) = {}_W E_2^{q-r,r} \neq 0$ implies that ${}_W E_1^{q-r,r} \neq 0$, i.e. there exists an integer k with $k \geqq 0$ and $k \geqq q-r$ such that

$$H^{2q-r-2k}(\tilde{Y}^{(2k+r-q+1)}, \mathbb{Q}) \neq 0, \qquad \text{i.e. } 2q-r-2k \geqq 0$$

and so: $2q-r \geqq 2k$. Thus $2q-r \geqq 0$ and $2q-r \geqq 2q-2r$ i.e. $0 \leqq r \leqq 2q$. □

(4.22) The space $\mathbb{H}^q(Y, A_{\mathbb{C}}^{\bullet})$ has an additional structure: it has a nilpotent endomorphism $\mathrm{N} = \mathrm{Res}_0(\nabla)$. To see how it behaves with respect to the mixed Hodge structure, we construct an endomorphism $\tilde{\nu}$ of the complex $A_{\mathbb{C}}^{\bullet}$ which induces N if one takes hypercohomology.

Analogous to the proof of (2.20) one obtains N as the connecting homomorphism in the long exact sequence of hypercohomology associated to

$$0 \to \Omega_{X/S}^{\bullet-1}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}} \xrightarrow{\theta} \Omega_X^{\bullet}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}}$$
$$\to \Omega_{X/S}^{\bullet}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}} \to 0.$$

Define a double complex $B_{\mathbb{C}}^{\bullet\bullet}$ as follows. Denote $B^{pq} = A^{p-1,q} \oplus A^{pq}$ $(p, q \geqq 0)$. Denote $\nu\colon A^{pq} \to A^{p-1,q+1}$ the canonical projection

$$\Omega_X^{p+q+1}(\log Y)/W_q\, \Omega_X^{p+q+1}(\log Y) \to \Omega_X^{p+q+1}(\log Y)/W_{q+1}\, \Omega_X^{p+q+1}(\log Y).$$

The differentiation in $B_{\mathbb{C}}^{\bullet}$ is given by

$$d'(\omega_1, \omega_2) = (d'\,\omega_1, d'\,\omega_2),$$

$$d''(\omega_1, \omega_2) = (d''\,\omega_1 + (-1)^{p+q+1}\,\nu(\omega_2), d''\,\omega_2),$$

for ω_1 and ω_2 sections of $A^{p-1,q}$ resp. A^{pq}.

Define $\eta\colon \Omega_X^{\bullet}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}} \to B_{\mathbb{C}}^{\bullet}$ by

$$\eta(\omega) = (\omega \bmod W_0, (-1)^p\, \theta \wedge \omega)$$

for ω a section of $\Omega_X^p(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}}$. Then one has a commutative diagram of complexes with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega_{X/S}^{\bullet}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}}[-1] & \xrightarrow{\tilde{\theta}} & A_{\mathbb{C}}^{\bullet}[-1] \\
\downarrow & & \downarrow \\
\Omega_X^{\bullet}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}} & \xrightarrow{\eta} & B_{\mathbb{C}}^{\bullet} \\
\downarrow & & \downarrow \\
\Omega_{X/S}^{\bullet}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}} & \xrightarrow{\tilde{\theta}} & A_{\mathbb{C}}^{\bullet} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Because $\tilde{\theta}$ is a quasi-isomorphism, one can use the five-lemma to show that η is a quasi-isomorphism. The construction of connecting homomorphisms shows that N: $\mathbb{H}^q(Y, A^\bullet_{\mathbb{C}}) \to \mathbb{H}^q(Y, A^\bullet_{\mathbb{C}})$ is induced by the endomorphism $\tilde{\nu}$ of $A^\bullet_{\mathbb{C}}$ which is $(-1)^{p+q+1}\nu$ on A^{pq}. It anti-commutes with the differentiation.

(4.23) **Proposition.** *$\tilde{\nu}(W_r A^\bullet_{\mathbb{C}}) \subset W_{r-2} A^\bullet_{\mathbb{C}}$ and the induced map*

$$\tilde{\nu}^r\colon \mathrm{Gr}^W_r A^\bullet_{\mathbb{C}} \to \mathrm{Gr}^W_{-r} A^\bullet_{\mathbb{C}}$$

is an isomorphism for all $r \geqq 0$.

Proof. For $r \in \mathbb{Z}$, $p, q \geqq 0$ one has

$$W_r A^{pq} = W_{2q+r+1}\, \Omega^{p+q+1}_X(\log Y)/W_q\, \Omega^{p+q+1}_X(\log Y)$$

and

$$W_{r-2} A^{p-1,q+1} = W_{2q+r+1}\, \Omega^{p+q+1}_X(\log Y)/W_{q+1}\, \Omega^{p+q+1}_X(\log Y).$$

For $r \geqq 0$ one has

$$\mathrm{Gr}^W_r A^{pq} = \mathrm{Gr}^W_{2q+r+1}\, \Omega^{p+q+1}_X(\log Y) = \mathrm{Gr}^W_{-r} A^{p-r,q+r};$$

these are identified by $\tilde{\nu}^r$. □

One deduces from this proposition that N acts on the spectral sequence

$${}_W\mathrm{E}_1^{-r,q+r} = \mathbb{H}^q(Y, \mathrm{Gr}^W_r A^\bullet_{\mathbb{Q}}) \Rightarrow \mathbb{H}^q(Y, A^\bullet_{\mathbb{Q}}),$$

mapping ${}_W\mathrm{E}_1^{-r,q+r}$ to ${}_W\mathrm{E}_1^{-r+2,q+r-2}$ and inducing isomorphisms

$$\mathrm{N}^r\colon {}_W\mathrm{E}_1^{-r,q+r} \xrightarrow{\sim} {}_W\mathrm{E}_1^{r,q-r}$$

for all $q, r \geqq 0$.

Unfortunately this is not sufficient to deduce that N^r maps $\mathrm{Gr}^W_{q+r}\mathbb{H}^q(Y, \mathbb{Q})$ isomorphically to $\mathrm{Gr}^W_{q-r}\mathbb{H}^q(Y, \mathbb{Q})$. We will prove this in Chapter 5 under more restrictive conditions.

Because $\tilde{\nu}(F^p A^\bullet_{\mathbb{C}}) \subset F^{p-1} A^\bullet_{\mathbb{C}}$, the induced map N is a *morphism of mixed Hodge structures* of type $(-1, -1)$ (cf. [15], p. 217).

(4.24) *Dependence on choices.*

In our construction we have chosen a parameter t on the disk S. This choice fixes an isomorphism in $D^+(Y, \mathbb{C})$

$$\psi_t\colon\ i^\bullet \Omega^\bullet_X(\log Y)[\log t] \to \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y\mathrm{red}}$$

(see (2.6)), by $\psi_t\left(\sum_{k=0}^{s} \omega_k (\log t)^k\right) = \bar{\omega}_0$: the image ω_0 under the natural map

$$i^\bullet \Omega^\bullet_X(\log Y) \to \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y\mathrm{red}}.$$

Let $t' = \alpha t$ be an other parameter on S, such that α is invertible on S. Then $\log t' - \log t = \log \alpha$ is a holomorphic function on S. Choose a Stein covering $\mathfrak{U}$ of Y. Then for $q \geqq 0$

$$\mathbb{H}^q(Y, i^\bullet \Omega^\bullet_X(\log Y)[\log t]) = H^q(\mathrm{C}^\bullet(\mathfrak{U}, i^\bullet \Omega^\bullet_X(\log Y)[\log t]))$$

where $C^n(\mathfrak{U}, i^{\cdot}\Omega^{\cdot}_X(\log Y)[\log t])$ is the Čech-bicomplex of $i^{\cdot}\Omega^{\cdot}_X(\log Y)[\log t]$ associated to the covering $\mathfrak{U}$. A similar expression holds for $\Omega^{\cdot}_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_{Y_{\mathrm{red}}}$. Let $\bar{\omega}_0\in C^q(\mathfrak{U}, \Omega^{\cdot}_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_{Y_{\mathrm{red}}})$ represent the class $\bar{x}_0$ and let $\psi_t^{-1}(\bar{x}_0)$ be represented by the cocycle $\omega=\sum_{k=0}^{s}\omega_k(\log t)^k$ with $\omega_k\in C^n(\mathfrak{U}, i^{\cdot}\Omega^{\cdot}_X(\log Y))$. The cocycle condition for ω gives the relations:

$$d\omega_s=0;$$

$$d\omega_k=-\frac{k+1}{2\pi i}\frac{dt}{t}\wedge\omega_{k+1}\qquad(k=0,\ldots,s-1).$$

Hence the images $\bar{\omega}_k\in C^n(\mathfrak{U}, \Omega^{\cdot}_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_{Y_{\mathrm{red}}})$ satisfy:

$$d\bar{\omega}_k=0\qquad(k=0,\ldots,s).$$

Denoting $\bar{x}_k$ the cohomology class of $\bar{\omega}_k$, one has by definition of N:

$$\mathrm{N}\bar{x}_s=0;$$

$$\mathrm{N}\bar{x}_k=-\frac{k+1}{2\pi i}\bar{x}_{k+1}\qquad(k=0,\ldots,s-1).$$

Hence

$$\bar{x}_k=\frac{(-2\pi i\,\mathrm{N})^k}{k!}\bar{x}_0\qquad(k=0,\ldots,s).$$

$$\mathrm{N}^{s+1}\bar{x}_0=0.$$

This shows that

$$\psi_{t'}\psi_t^{-1}(\bar{x}_0)=\sum_{k=0}^{s}\overline{\omega_k(\log\alpha)^k}=\sum_{k=0}^{s}(\log\alpha(0))^k\bar{x}_k=\exp(-2\pi i\,\mathrm{N}\log\alpha(0))\,\bar{x}_0$$

so

$$\boxed{\psi_{t'}\psi_t^{-1}=\exp(-2\pi i\,\mathrm{N}\log\alpha(0))}.$$

We make the following observations.

1. For every choice of the parameter t, the morphism of complexes ψ_t is compatible with cup-product. This implies that the induced map

$$\psi_t\colon \bigoplus_q \mathbb{H}^q(Y, i^{\cdot}\Omega^{\cdot}_X(\log Y))\to\bigoplus_q \mathbb{H}^q(Y, \Omega^{\cdot}_{X/S}(\log Y)\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y_{\mathrm{red}}})$$

is a ring homomorphism with respect to the cup-product. Because it is bijective, it is even a ring isomorphism.

2. For every pair (t, t') as above, $\psi_{t'}\psi_t^{-1}$ preserves the weight filtration on $\mathbb{H}^q(Y, \Omega^{\cdot}_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_{Y_{\mathrm{red}}})$, and induces the identity on the graded vectorspace associated to the weight filtration. This follows from

$$\exp(-2\pi i\,\mathrm{N}\log\alpha(0))=I-2\pi i\,\mathrm{N}\log\alpha(0)+\ldots=I+R$$

where $R(W_r)\subset W_{r-2}$. Our construction thus provides a one-parameter family of mixed Hodge structures with fixed Hodge and weight filtrations but a varying integral lattice. If one takes the inverse image under one ψ_t, one gets a fixed integral lattice $H^q(\tilde{X}^*, \mathbb{Z})$, a fixed weight filtration on $H^q(\tilde{X}^*, \mathbb{Q})$ and a varying Hodge filtration. This is the same ambiguity as in Schmid's construction. Cf. [15], p. 255. One can ask the question of independence from a different point of view. If we are given a proper and smooth family X^* over the punctured disk with unipotent monodromy, does there exist an intrinsic characterization of the sheaf

$$\mathbb{R}^q f_* \Omega^{\cdot}_{X/S}(\log Y) \quad \text{on } S,$$

independent of the choice of an extension X of the family over the whole disk? The answer is yes. The sheaf $\mathbb{R}^q f_* \Omega^{\cdot}_{X/S}(\log Y)$ is the unique locally free sheaf on S such that its restriction to S^* is $\mathbb{R}^q f_* \Omega^{\cdot}_{X^*/S^*}$ (by abuse of language we denote the restriction of f to X^* again by f) and such that the Gauss-Manin connection extends to a connection with a logarithmic pole at 0 with a nilpotent residue. Cf. [5], Proposition II.5.2.

(4.25) The functorial properties of our mixed Hodge structure can be expressed by making some standard exact sequences into exact sequences of mixed Hodge structures. This is a powerful tool in many kinds of problems, because every morphism of mixed Hodge structures is strictly compatible with the weight filtration. We will apply this in the next chapter.

The key role is played by $\mathrm{H}^*(X^*)$. It occurs in the Wang sequence

$$\cdots \to \mathrm{H}^q(X^*) \to \mathrm{H}^q(X_t) \xrightarrow[T_0 - I]{} \mathrm{H}^q(X_t) \to \cdots \qquad (t \in S^*)$$

and in the sequence

$$\cdots \to \mathrm{H}^q(X) \to \mathrm{H}^q(X^*) \to \mathrm{H}^{q+1}_Y(X) \to \mathrm{H}^{q+1}(X) \to \cdots$$

which is dual to the usual exact sequence on homology:

$$\cdots \to \mathrm{H}_q(\partial X) \to \mathrm{H}_q(X) \to \mathrm{H}_q(X, \partial X) \to \cdots$$

A mixed Hodge structure on $\mathrm{H}^q(X^*)$ can be obtained as follows. First observe that $j\colon X^* \to X$ induces an isomorphism $\mathrm{H}^q(X^*, \mathbb{Z}) \cong \mathbb{H}^q(X, \mathbb{R} j_* \mathbb{Z}_{X^*})$. As in [3] one shows that

$$\mathrm{H}^q(X^*, \mathbb{C}) \cong \mathbb{H}^q(X, \Omega^{\cdot}_X(\log Y)).$$

With the same argument as in Lemma (2.5) one shows that $\mathbb{H}^q(X, \Omega^{\cdot}_X(\log Y)) \cong \mathbb{H}^q(Y, i^{\cdot} \Omega^{\cdot}_X(\log Y))$. Moreover the maps

$$i^{\cdot} \Omega^{\cdot}_X(\log Y) \to \Omega^{\cdot}_X(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{\mathrm{red}}} \to B^{\cdot}_{\mathbb{C}}$$

are quasi-isomorphisms. Hence $\mathrm{H}^q(X^*, \mathbb{C}) \cong \mathbb{H}^q(Y, B^{\cdot}_{\mathbb{C}})$. We give $B^{\cdot}_{\mathbb{C}}$ a Hodge filtration F and a weight filtration W such that in the exact sequence

$$0 \to A^{\cdot}_{\mathbb{C}}[-1] \xrightarrow{\lambda} B^{\cdot}_{\mathbb{C}} \xrightarrow{\mu} A^{\cdot}_{\mathbb{C}} \to 0$$

λ induces a morphism of mixed Hodge structures of type $(1,1)$ and μ induces a morphism of Hodge structures of type $(0,0)$. Define $W_r B^{pq} = W_{r-2} A^{p-1,q} \oplus W_r A^{p,q}$ and denote $F^p B^n_{\mathbb{C}} = F^{p-1} A^{n-1}_{\mathbb{C}} \oplus F^p A^n_{\mathbb{C}}$. Then W and F are filtrations of $B^{\cdot}_{\mathbb{C}}$ by subcomplexes, because the map $\tilde{\nu}: A^{\cdot}_{\mathbb{C}} \to A^{\cdot}_{\mathbb{C}}$ satisfies

$$\tilde{\nu}(W_r A^{\cdot}_{\mathbb{C}}) \subset W_{r-2} A^{\cdot}_{\mathbb{C}} \quad \text{and} \quad \tilde{\nu}(F^p A^{\cdot}_{\mathbb{C}}) \subset F^{p-1} A^{\cdot}_{\mathbb{C}}.$$

One checks easily that μ is compatible with both filtrations, that

$$\lambda(F^p A^{\cdot}_{\mathbb{C}}[-1]) \subset F^{p+1} B^{\cdot}_{\mathbb{C}} \quad \text{and} \quad \lambda(W_r A^{\cdot}_{\mathbb{C}}[-1]) \subset W_{r+2} B^{\cdot}_{\mathbb{C}}.$$

Moreover $\mathrm{Gr}_r^W B^{\cdot}_{\mathbb{Q}}$ is the complex

$$\mathrm{Gr}_r^W A^{\cdot}_{\mathbb{Q}} \xrightarrow{\tilde{\nu}} \mathrm{Gr}_{r-2}^W A^{\cdot}_{\mathbb{Q}}(-1),$$

if $B^{\cdot}_{\mathbb{Q}}$ is defined analogous to $A^{\cdot}_{\mathbb{Q}}$. Resuming:

(4.26) **Proposition.** *The "Wang sequence"*

$$\cdots \to \mathbb{H}^q(Y, B^{\cdot}_{\mathbb{Q}}) \to \mathbb{H}^q(Y, A^{\cdot}_{\mathbb{Q}}) \xrightarrow{\mathrm{N}} \mathbb{H}^q(Y, A^{\cdot}_{\mathbb{Q}})(-1) \to \cdots$$

is an exact sequence of mixed Hodge structures. □

(4.27) Because Y is a strong deformation retract of X (cf. [2]) the restriction map $\mathrm{H}^q(X) \to \mathrm{H}^q(Y)$ is an isomorphism for all $q \geqq 0$. The canonical mixed Hodge structure on $\mathrm{H}^q(Y)$ induces a mixed Hodge structure on $\mathrm{H}^q(X)$.

Claim. *The natural map $\mathrm{H}^q(X) \to \mathrm{H}^q(\tilde{X}^*)$ is a morphism of mixed Hodge structures.*

Proof. Denote $A^{\cdot}_{\mathbb{C}}(Y) = \mathrm{Ker}(\tilde{\nu})$. Then $A^{\cdot}_{\mathbb{C}}(Y)$ is a resolution of $\mathbb{C}_Y$ and the filtrations W and F on $A^{\cdot}_{\mathbb{C}}$ induce filtrations W and F on $A^{\cdot}_{\mathbb{C}}(Y)$. The resulting mixed Hodge structure on $\mathrm{H}^q(Y)$ is the canonical one, as one may see comparing the complex $A^{\cdot}_{\mathbb{C}}(Y)$ and the complex $K^{\cdot\cdot}$ from Example (3.5). □

In $D^+(Y, \mathbb{C})$ the sequence

$$0 \to \mathrm{Ker}(\tilde{\nu}) \to B^{\cdot}_{\mathbb{C}} \to \mathrm{Coker}(\tilde{\nu})[-1] \to 0$$

is exact, for $B^{\cdot}_{\mathbb{C}} \cong \mathfrak{s}(A^{\cdot}_{\mathbb{C}} \xrightarrow{\tilde{\nu}} A^{\cdot}_{\mathbb{C}})$ where $\mathfrak{s}$ denotes: take the associated single complex. This induces a long exact sequence

$$\cdots \to \mathrm{H}^q(Y, \mathbb{C}) \to \mathrm{H}^q(X^*, \mathbb{C}) \to \mathbb{H}^{q-1}(\mathrm{Coker}(\tilde{\nu})) \to \mathrm{H}^{q+1}(Y, \mathbb{C}) \to \cdots.$$

This shows that $\mathbb{H}^{q-1}(\mathrm{Coker}(\tilde{\nu})) \cong \mathrm{H}^{q+1}_Y(X, \mathbb{C})$ and gives $\mathrm{H}^{q+1}_Y(X)$ a mixed Hodge structure.

(4.28) Because Y is complete, $\mathrm{Gr}_r^W \mathrm{H}^q(Y) = 0$ if $r > q$. This can also be concluded from the fact that $\mathrm{Gr}_r^W A^{\cdot}_{\mathbb{C}}(Y) = 0$ if $r > 0$. For Coker $(\tilde{\nu})$ one has $\mathrm{Gr}_r^W(\mathrm{Coker}(\tilde{\nu})) = 0$ if $r < 2$ (mind that $\tilde{\nu}$ decreases weight by two or, equivalently, preserves weight in $A^{\cdot}_{\mathbb{Q}} \xrightarrow{\tilde{\nu}} A^{\cdot}_{\mathbb{Q}}(-1)$; hence $\mathrm{Coker}(\tilde{\nu})$ has to be considered as a quotient of $A^{\cdot}_{\mathbb{Q}}(-1)$). This implies that $\mathrm{Gr}_r^W \mathrm{H}^{q+1}_Y(X) = 0$ for $r < q+1$.

(4.29) **Proposition.** *The sequence*

$$\cdots \to \mathrm{H}^q(Y) \xrightarrow{\alpha} \mathrm{H}^q(X^*) \xrightarrow{\beta} \mathrm{H}^{q+1}_Y(X) \to \mathrm{H}^{q+1}(Y) \to \cdots$$

is an exact sequence of mixed Hodge structures. In particular for $r \leqq q$: $\mathrm{Gr}_r^W(\alpha)$ is surjective, and $\mathrm{Gr}_r^W(\beta)$ is injective for $r \geqq q+1$. □

§ 5. The Projective Case

(5.1) In this chapter we preserve the notations and assumptions of (4.1). We make the additional assumption that X is a closed subset of $\mathbb{P}^m \times S$ for some m and f is the restriction to X of the projection on the second factor. Let $\dim X = n+1$. Let $H \subset \mathbb{P}^m$ be a hyperplane which intersects all components of Y transversally such that $((H\times S)\cap X)\cup Y$ is a divisor with normal crossings on X. For $t\in S$ sufficiently small H intersects X_t transversally. After possibly shrinking S we may assume that H intersects all $X_t (t\in S^*)$ transversally. Denote $H_t = X_t \cap H$ and let $L_t \in \mathrm{H}^2(X_t, \mathbb{Z})$ be the cohomology class of H_t.

Remind the "hard Lefschetz theorem" ([16], Cor. to Th. IV.5):

$$L_t^q\colon \mathrm{H}^{n-q}(X_t, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^{n+q}(X_t, \mathbb{Q}) \qquad (q \geqq 0)$$

where $L_t^q(\omega) = L_t^q \wedge \omega$. The map L_t can be decomposed into 2 maps as follows. Denote

$$\rho_t\colon \mathrm{H}^i(X_t, \mathbb{Q}) \to \mathrm{H}^i(H_t, \mathbb{Q})$$

the restriction map and let

$$\gamma_t\colon \mathrm{H}^i(H_t, \mathbb{Q}) \to \mathrm{H}^{i+2}(X_t, \mathbb{Q})(1)$$

be its dual, i.e. the Gysin map. Then $L_t = (2\pi i)^{-1}\gamma_t \rho_t$. Because ρ_t and γ_t are morphisms of Hodge structures, L_t is a morphism of Hodge structures of type $(1,1)$, as is also clear from its definition.

Denote $L \in \mathrm{H}^2(\mathbb{P}^m, \mathbb{Z})$ the cohomology class of H, and let L_0 be the image of L under the composed map $\mathrm{H}^2(\mathbb{P}^m, \mathbb{Z}) \xrightarrow{\text{res}} \mathrm{H}^2(Y, \mathbb{Z}) \to \mathbb{H}^2(Y, A^\bullet_{\mathbb{Q}})$.

(5.2) **Proposition.** *For all $q \geqq 0$ cupproduct with L_0 induces*

$$L_0^q\colon \mathbb{H}^{n-q}(Y, A^\bullet_{\mathbb{Q}}) \xrightarrow{\sim} \mathbb{H}^{n+q}(Y, A^\bullet_{\mathbb{Q}}).$$

Proof. Choose a parameter t on S, a point $u \in \tilde{S}^*$ and let $\lambda = \pi(u) \in S^*$. One has an injection $i_u\colon X_\lambda \to \tilde{X}^*$ which is a homotopy equivalence. The diagram

$$\begin{array}{ccccc}
\mathrm{H}^2(\mathbb{P}^m, \mathbb{Q}) & & \longrightarrow & & \mathrm{H}^2(Y, \mathbb{Q}) \\
\big\downarrow \text{res} & & & & \big\downarrow \\
\mathrm{H}^2(X_\lambda, \mathbb{Q}) & \xrightarrow{(i_u^*)^{-1}} & \mathrm{H}^2(\tilde{X}^*, \mathbb{Q}) & \xrightarrow{\psi_t} & \mathbb{H}^2(Y, A^\bullet_{\mathbb{Q}})
\end{array}$$

is commutative, so $L_0 = \psi_t (i_u^*)^{-1} L_\lambda$. The proposition follows now from the hard Lefschetz theorem for X_λ and the fact that ψ_t and i_u^* are ring isomorphisms for cup-product. □

(5.3) We determine the behaviour of the weight filtration under cup-product with L_0. Remark that the restriction of f to $(H\times S)\cap X$ satisfies all conditions for our construction. Hence one disposes of morphisms of mixed Hodge structures

$$\rho\colon \mathbb{H}^i(Y, A^\bullet_{\mathbb{Q}}) \to \mathbb{H}^i(H\cap Y, A^\bullet_{\mathbb{Q}|H\cap Y})$$

and

$$\gamma\colon \mathbb{H}^i(H\cap Y, A^{\bullet}_{\mathbb{Q}|H\cap Y})\to \mathbb{H}^{i+2}(Y, A^{\bullet}_{\mathbb{Q}})(1)$$

such that $L_0=(2\pi i)^{-1}\gamma\rho$.

Both ρ and γ induce morphisms of the spectral sequences of hypercohomology of the complex $A^{\bullet}_{\mathbb{Q}}$ (resp. $A^{\bullet}_{\mathbb{Q}|H\cap Y}$) with the filtration W. On the E_1-terms they induce the restriction map

$$\rho\colon H^i(Y_{i_1}\cap\cdots\cap Y_{i_p}, \mathbb{Q})\to H^i(Y_{i_1}\cap\cdots\cap Y_{i_p}\cap H, \mathbb{Q})$$

resp. the Gysin map

$$\gamma\colon H^i(Y_{i_1}\cap\cdots\cap Y_{i_p}\cap H, \mathbb{Q})\to H^{i+2}(Y_{i_1}\cap\cdots\cap Y_{i_p}, \mathbb{Q})(1).$$

From the hard Lefschetz theorem for every $\tilde{Y}^{(p)}$, one deduces that for all $r\in\mathbb{Z}$, $q\geqq 0$ the map L_0^q induces an isomorphism between

$$E_1^{-r,n-q+r}=\bigoplus_{\substack{k\geqq 0\\ k\geqq -r}} H^{n-q-r-2k}(\tilde{Y}^{(2k+r+1)}, \mathbb{Q})(-r-k)$$

and

$$E_1^{-r,n+q+r}=\bigoplus_{\substack{k\geqq 0\\ k\geqq -r}} H^{n+q-r-2k}(\tilde{Y}^{(2k+r+1)}, \mathbb{Q})(-r-k),$$

for $\tilde{Y}^{(2k+r+1)}$ is nonsingular projective of dimension $n-2k-r$.

Because ρ and γ are morphisms of mixed Hodge structures of type $(0,0)$, $L_0=(2\pi i)^{-1}\gamma\rho$ is a morphism of mixed Hodge structures of type $(1,1)$. In particular L_0 is strictly compatible with the filtration W. Recall that the spectral sequence of hypercohomology for $(A^{\bullet}_{\mathbb{Q}}, W)$ degenerates at E_2 and that (cf. (4.20)):

$$E_2^{-r,n-q+r}=\mathrm{Gr}^W_{n-q+r}\mathbb{H}^{n-q}(Y, A^{\bullet}_{\mathbb{Q}}).$$

Hence for all $r\in\mathbb{Z}$, $q\geqq 0$ the map L_0^q induces an isomorphism between $E_2^{-r,n-q+r}$ and $E_2^{-r,n+q+r}$.

(5.4) For q, $r\in\mathbb{Z}$, $i=1,2$ we define the *primitive part* $P_i^{-r,n-q+r}\subset E_i^{-r,n-q+r}$ by $P_i^{-r,n-q+r}=\operatorname{Ker} L_0^{q+1}$ if $q\geqq 0$ and $P_i^{-r,n-q+r}=0$ if $q<0$.

Then for $E_i^{-r,n-q+r}$ one has the *Lefschetz decomposition*

$$E_i^{-r,n-q+r}=\bigoplus_{k\geqq 0} L_0^k\, P_i^{-r,n-q+r-2k}.$$

Cf. [16], Theorem IV.5. A primitive cohomology class is by definition an element of $P_i^{-r,n-q+r}$ for some $q, r\in\mathbb{Z}$.

(5.5) The primitive cohomology groups of a smooth projective variety V of dimension n carry a positive definite quadratic form, which is defined as follows. For $q\geqq 0$ denote $P^{n-q}(V,\mathbb{Q})\subset H^{n-q}(V,\mathbb{Q})$ the primitive part. Because L is a morphism of Hodge structures (of type $(1,1)$), $P^{n-q}(V,\mathbb{Q})$ is a sub-Hodge structure of $H^{n-q}(V,\mathbb{Q})$.

One defines the operator C as being multiplication with i^{r-s} on $H^{r,s}$. Then

$$Q(x,y)=(-1)^{(n-q)(n-q-1)/2}\,Cx\wedge L^q\bar{y}[V]$$

defines a positive definite quadratic form on $P^{n-q}(V,\mathbb{C})$.

(5.6) Returning to the weight spectral sequence, we define $Z(E_1^{rs})=\operatorname{Ker}(d_1^{rs})$ and $B(E_1^{rs})=\operatorname{Im}(d_1^{r-1,s})$. The following lemmas show the relation between d_1 and the primitive decomposition. If $\xi\in Z(E_1^{rs})$, we denote $[\xi]$ its class in E_2^{rs}.

(5.7) **Lemma.** *If $\xi\in Z(E_1^{pq})$, there exist $\eta\in B(E_1^{pq})$ and $\xi_k\in Z(E_1^{p,q-2k})$ $(k\geqq 0)$ with ξ_k primitive and*

$$\xi=\eta+\sum_{k\geqq 0}L_0^k\,\xi_k.$$

Moreover

$$[\xi]=\sum_{k\geqq 0}L_0^k[\xi_k]$$

is then the primitive decomposition of $[\xi]\in E_2^{pq}$.

Proof. Write $[\xi]=\sum_{k\geqq 0}L_0^k[\xi_k']$ with $[\xi_k']\in P_2^{p,q-2k}$. First assume that ξ_k' is primitive too. Then put $\eta=\xi-\sum_{k\geqq 0}L_0^k\,\xi_k'$. Because $[\eta]=0$ we have indeed $\eta\in B(E_1^{pq})$.

Remains to be shown that every primitive element of some E_2^{pq} can be represented by an element of P_1^{pq}.

Let $\zeta\in E_1^{r,n-q-r}$ be such that $d_1\zeta=0$ and $[\zeta]\in P_2^{r,n-q-r}$. This implies that $L_0^{q+1}\zeta\in B(E_1^{r,n+q-r+2})$. Choose $\gamma\in E_1^{r-1,n+q-r+2}$ with $L_0^{q+1}\zeta=d_1\gamma$. There exists a unique γ' with $\gamma'\in E_1^{r-1,n-q-r}$ and $\gamma=L_0^{q+1}\gamma'$. Then $L_0^{q+1}(\zeta-d_1\gamma')=L_0^{q+1}\zeta-d\gamma=0$, so $\zeta-d\gamma'$ is primitive. Moreover $[\zeta-d\gamma']=[\zeta]$. □

(5.8) **Lemma.** *If $\xi=\sum_{k\geqq 0}L_0^k\,\xi_k\in B(E_1^{pq})$ with ξ_k primitive and $\xi_k\in Z(E_1^{p,q-2k})$ for all $k\geqq 0$, then for each k $\xi_k\in B(E_1^{p,q-2k})$.*

Proof. Because $\xi_k\in Z(E_1^{p,q-2k})$ one has the relation

$$0=[\xi]=\sum_{k\geqq 0}L_0^k[\xi_k]$$

and $[\xi_k]\in P_2^{p,q-2k}$ for all k. Because the primitive decomposition is a direct sum, one obtains $[\xi_k]=0$ for all $k\geqq 0$. □

Now we have all ingredients to prove (cf. (4.23)):

(5.9) **Theorem.** *The endomorphism N of $\mathbb{H}^q(Y,A_{\mathbb{Q}}^{\bullet})$ induces for all $q,r\geqq 0$ an isomorphism of Hodge structures of weight $q+r$*

$$N^r\colon\ \mathrm{Gr}^W_{q+r}\mathbb{H}^q(Y,A_{\mathbb{Q}}^{\bullet})\xrightarrow{\ \sim\ }\mathrm{Gr}^W_{q-r}\mathbb{H}^q(Y,A_{\mathbb{Q}}^{\bullet})(-r).$$

Proof. Consider again the weight spectral sequence. One has the diagram, which is commutative up to sign

$$\begin{array}{ccccc} E_1^{-r-1,q+r} & \xrightarrow{d_1} & E_1^{-r,q+r} & \xrightarrow{d_1} & E_1^{-r+1,q+r} \\ \cap\big\downarrow \tilde{v}^r & & \simeq \big\downarrow \tilde{v}^r & & \big\downarrow \tilde{v}^r \\ E_1^{r-1,q-r} & \xrightarrow{d_1} & E_1^{r,q-r} & \xrightarrow{d_1} & E_1^{r+1,q-r}. \end{array}$$

Hence the theorem is equivalent to the conjunction of the statements

(A) $\tilde{v}^r Z(E_1^{-r,q+r}) + B(E_1^{r,q-r}) = Z(E_1^{r,q-r})$;

(B) $\tilde{v}^r Z(E_1^{-r,q+r}) \cap B(E_1^{r,q-r}) = \tilde{v}^r B(E_1^{-r,q+r})$.

We first prove (B).

We deduce from (4.13) that

$$E_1^{r-1,q-r} = \tilde{v}^r E_1^{-r-1,q+r} \oplus H^{q-r}(\tilde{Y}^{(r)}, \mathbb{Q}).$$

The map, induced by d_1 on the second factor, is just

$$\theta\colon H^{q-r}(\tilde{Y}^{(r)}, \mathbb{Q}) \to H^{q-r}(\tilde{Y}^{(r+1)}, \mathbb{Q}) \subset E_1^{r,q-r}.$$

Take $\xi \in Z(E_1^{-r,q+r})$ with $\tilde{v}^r \xi \in B(E_1^{r,q-r})$. Write $\xi = d_1 \eta + \sum_{k \geqq 0} L_0^k \xi_k$ with ξ_k primitive and $d_1 \xi_k = 0$ for all k. We first show: $\tilde{v}^r \xi_k \in B(E_1^{r,q-r-2k})$ for every k.

Because L_0 and $\tilde{v}$ commute, the elements $\tilde{v}^r \xi_k$ are primitive. Clearly

$$\tilde{v}^r \xi_k \in Z(E_1^{r,q-r-2k}) \quad \text{and} \quad \sum_{k \geqq 0} L_0^k \tilde{v}^r \xi_k = \tilde{v}^r(\xi - d_1 \eta)$$

which is an element of $B(E_1^{r,q-r})$. By Lemma (5.8) one concludes that

$$\tilde{v}^r \xi_k \in B(E_1^{r,q-r-2k}) \qquad \text{for all } k.$$

To show $\xi \in B(E_1^{-r,q+r})$, we may therefore assume that ξ is primitive. In particular $q \leqq n$.

Because $B(E_1^{r,q-r}) = \tilde{v}^r B(E_1^{-r,q+r}) + \theta H^{q-r}(\tilde{Y}^{(r)}, \mathbb{Q})$ we may assume that $\tilde{v}^r \xi = \theta \eta$ for $\eta \in H^{q-r}(\tilde{Y}^{(r)}, \mathbb{Q})$. This implies that $\xi \in P^{q-r}(\tilde{Y}^{(r+1)}, \mathbb{Q})(-r)$. Denote

$$\gamma\colon H^{q-r}(\tilde{Y}^{(r+1)}, \mathbb{Q})(-r) \to H^{q-r+2}(\tilde{Y}^{(r)}, \mathbb{Q})(-r+1)$$

the map induced by d_1. Then γ is nothing but a generalized Gysin map. It is the dual map to

$$\rho = \sum (-1)^j \delta_j^*\colon H^{2n-q-r}(\tilde{Y}^{(r)}, \mathbb{Q}) \to H^{2n-q-r}(\tilde{Y}^{(r+1)}, \mathbb{Q}).$$

Cf. Example (3.5). Hence for all $\zeta \in H^{2n-q-r}(\tilde{Y}^{(r)}, \mathbb{Q})$ one has the projection formula:

$$\zeta \wedge \gamma(\xi) = \rho(\zeta) \wedge \xi = 0.$$

If Y is reduced we have $\theta = \rho$ (cf. (4.15)). Moreover

$$\begin{aligned} Q(\xi, \xi) &= \pm L_0^{n-q} C\xi \wedge \bar{\xi}[\tilde{Y}^{(r+1)}] \\ &= \pm \rho(L_0^{n-q} C\eta) \wedge \xi[\tilde{Y}^{(r+1)}] = 0. \end{aligned}$$

Hence $\xi = 0$.

If Y is not reduced, denote $\eta = \sum \eta_{i_1 \dots i_r}$ with $\eta_{i_1 \dots i_r} \in \mathrm{H}^{q-r}(Y_{i_1} \cap \dots \cap Y_{i_r}, \mathbb{Q})$, and define

$$\tilde{\eta} = \sum_{i_1 < \dots < i_r} (e_{i_1} \cdots e_{i_r})^{-1} \eta_{i_1 \dots i_r}.$$

Then from (4.15) one deduces for $\tilde{\xi} = \rho(\tilde{\eta})$ that

$$(\tilde{\xi})_{i_0 \dots i_r} = (e_{i_0} \dots e_{i_r})^{-1} \xi_{i_0 \dots i_r} \in \mathrm{H}^{q-r}(Y_{i_0} \cap \dots \cap Y_{i_r}, \mathbb{Q}).$$

Moreover $\xi_{i_0 \dots i_r}$ is primitive for all $(i_0, \dots, i_r)$. Again

$$0 = Q(\tilde{\xi}, \xi) = \sum_{i_0 < \dots < i_r}^{r} (e_{i_0} \dots e_{i_r})^{-1} Q(\xi_{i_0 \dots i_r}, \xi_{i_0 \dots i_r}).$$

Hence $\xi_{i_0 \dots i_r} = 0$ for all $(i_0, \dots, i_r)$. This proves (B). If Y is reduced one deduces (A) from (B) by a duality argument. If Y is not reduced, one can use a duality argument with respect to the modified pairing

$$\tilde{Q}(x, y) = \sum_{i_0 < \dots < i_r} (e_{i_0} \dots e_{i_r})^{-1} x_{i_0 \dots i_r} \wedge y_{i_0 \dots i_r} [Y_{i_0} \cap \dots \cap Y_{i_r}]$$

$(x \in \mathrm{H}^{q-r}(\tilde{Y}^{(r+1)}, \mathbb{Q}),\ y \in \mathrm{H}^{2n-q-r}(\tilde{Y}^{(r+1)}, \mathbb{Q}))$ to conclude (A). □

(5.10) **Corollary.** *The filtration W coincides with the weight filtration, constructed by Schmid* [15].

Proof. Schmid's definition of W is, that it is the unique decreasing filtration on $\mathbb{H}^q(Y, A^\cdot_{\mathbb{Q}})$ such that $\mathrm{N}(W_i) \subset W_{i-2}$ for all i and such that

$$\mathrm{N}^r \colon \mathrm{Gr}^W_{q+r} \mathbb{H}^q(Y, A^\cdot_{\mathbb{Q}}) \to \mathrm{Gr}^W_{q-r} \mathbb{H}^q(Y, A^\cdot_{\mathbb{Q}})(-r)$$

is an isomorphism for every $r \geqq 0$. Cf. [15], Lemma (6.4). □

Remark. In a future paper we will show that $\mathrm{R}^q f_* \Omega^p_{X/S}(\log Y)$ is locally free for all $p, q \geqq 0$. This implies that our Hodge filtration coincides with the one of Schmid.

(5.11) **Corollary.** Ker $\mathrm{N} \subset \mathbb{H}^q(Y, A^\cdot_{\mathbb{Q}})$ *is a mixed Hodge substructure and*

$$\mathrm{Gr}^W_r(\mathrm{Ker\ N}) = 0 \quad \textit{for } r > q.$$

Proof. Theorem (5.9) implies that $\mathrm{N} \colon \mathrm{Gr}^W_r \mathbb{H}^q(Y, A^\cdot_{\mathbb{Q}}) \to \mathrm{Gr}^W_{r-2} \mathbb{H}^q(Y, A^\cdot_{\mathbb{Q}})(-1)$ is injective for $r \geqq q+1$. □

The following theorem was proved by Katz with l-adic methods. It will be treated in a forthcoming joint paper of Schmid and Clemens. An analogous version of it for an algebraic base scheme has been proved by Deligne ([3], Th. (4.1.1.)).

(5.12) **Theorem.** (*Local invariant cycle theorem*). *For all $q \geqq 0$ the sequence*

$$\mathrm{H}^q(Y, \mathbb{Q}) \to \mathbb{H}^q(Y, A^\cdot_{\mathbb{Q}}) \xrightarrow{\mathrm{N}} \mathbb{H}^q(Y, A^\cdot_{\mathbb{Q}})(-1)$$

is exact.

Proof. (Deligne) Consider the diagram

$$\begin{array}{l} \mathrm{H}^q(Y,\mathbb{Q}) \\ \quad\downarrow \alpha \\ \mathrm{H}^q(X^*,\mathbb{Q}) \xrightarrow{\beta} \mathrm{Ker}(N) \\ \quad\downarrow \\ \mathrm{H}_Y^{q+1}(X) \end{array}$$

Proposition (4.26) gives that β is surjective.

Proposition (4.29) gives that $\mathrm{Gr}_r^W(\alpha)$ is surjective for $r \leqq q$. Hence $\mathrm{Gr}_r^W(\beta\alpha)$ is surjective for all $r \leqq q$. Finally Corollary (5.11) implies that $\beta\alpha$ is surjective. □

(5.13) *Remark.* Theorem (5.12) gives a strong restriction on the possible Hodge filtrations on $\mathbb{H}^q(Y, A_{\mathbb{C}}^\bullet)$. It fixes the Hodge filtration on Ker N. Moreover one has a filtration of $\mathbb{H}^q(Y, A_{\mathbb{Q}}^\bullet)$ by mixed Hodge substructures

$$(0) \subset \mathrm{Ker\,N} \subset \cdots \subset \mathrm{Ker\,N}^q \subset \mathrm{Ker\,N}^{q+1} = \mathbb{H}^q(Y, A_{\mathbb{Q}}^\bullet).$$

The map $\mathrm{N}^k (k \geqq 0)$ induces an isomorphism (of type $(-k, -k)$) between $\mathrm{Ker\,N}^{k+1}/\mathrm{Ker\,N}^k$ and $\mathrm{Ker\,N} \cap \mathrm{Im\,N}^k \subset \mathrm{Ker\,N}$. This fixes the Hodge filtration on $\mathrm{Ker\,N}^{k+1}/\mathrm{Ker\,N}^k$ for every $k \geqq 0$. In particular there is no choice for the numbers $h^{rs} = \dim \mathbb{H}^{s+r}(Y, \mathrm{Gr}_F^r A^\bullet)$.

The pure Hodge structures $\mathrm{Gr}_{q+r}^W \mathbb{H}^q(Y, A_{\mathbb{Q}}^\bullet)$ are also completely determined by (5.9) and (5.12). If one denotes

$$\mathfrak{R}^{q,r} = \mathrm{Ker}(\mathrm{N}^{r+1}: \mathrm{Gr}_{q+r}^W \mathbb{H}^q(Y, A_{\mathbb{Q}}^\bullet) \to \mathrm{Gr}_{q-r-2}^W \mathbb{H}^q(Y, A_{\mathbb{Q}}^\bullet)(-r-1)) \qquad \text{if } r \geqq 0$$

and $\mathfrak{R}^{q,r} = 0$ for $r < 0$, then analogous to the Lefschetz decomposition one has for all $r \in \mathbb{Z}$:

$$\mathrm{Gr}_{q+r}^W \mathbb{H}^q(Y, A_{\mathbb{Q}}^\bullet) = \bigoplus_{k \geqq 0} \mathrm{N}^k \mathfrak{R}^{q,r+2k}(k)$$

This is a direct sum in the category of mixed Hodge structures. Moreover it is clear that

$$\mathrm{N}^r: \mathfrak{R}^{q,r} \to \mathrm{Gr}_{q-r}^W (\mathrm{Ker\,N})(-r)$$

is an isomorphism of Hodge structures for every $r \geqq 0$. So finally:

(5.14) **Proposition.** *For every $q, r \in \mathbb{Z}$ the pure Hodge structures of weight $q+r$ on $\mathrm{Gr}_{q+r}^W \mathbb{H}^q(Y, A_{\mathbb{Q}}^\bullet)$ as constructed above and by Schmid* [15] *coincide.*

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