Inventiones math. 38, 279–299 (1977)

Inventiones mathematicae

Locally Polynomial Algebras are Symmetric Algebras

H. Bass**, E. H. Connell, and D. L. Wright*

Department of Mathematics, Columbia University in the City of New York, New York, N.Y. 10027, USA

Introduction

We fix throughout a commutative ring K. Let A be a finitely presented K-algebra. Suppose that, for each maximal ideal $\mathfrak{m}$ of K, the $K_{\mathfrak{m}}$-algebra $A_{\mathfrak{m}}$ is isomorphic to a polynomial $K_{\mathfrak{m}}$-algebra. Then A is isomorphic to the symmetric algebra $S(P)$ of a finitely generated projective K-module P. This result, to which the title refers, is contained in Theorem (4.4) below. Geometrically it asserts that every locally trivial fibre space over spec(K) with affine space fibres arises from a vector bundle. The theorem is trivial if A is locally a polynomial algebra in one variable.[1] The only other case previously known to us is the case, treated in [W], when K is a principal ideal domain. The theorem solves a problem posed in [EH] p. 67, and in [W], § 6.[2]

The paper [ES] contains many results on locally polynomial algebras, but without our finite presentability assumption. The example (3.15) of [ES] furnishes a $\mathbb{Z}$-algebra A which is a noetherian UFD, locally a polynomial ring in one variable over $\mathbb{Z}$, yet not finitely generated over $\mathbb{Z}$, and, in particular, not the symmetric algebra of any $\mathbb{Z}$-module.

The above result is one of several "localization theorems" we prove here, by methods inspired by the proof of Quillen's localization theorem ([Q], Th. 1). The latter, along with a theorem of Horrocks, was the basis for Quillen's proof of Serre's conjecture in [Q]. We also use a technique developed in [BW].

Roughly speaking, these localization theorems say that two objects A and B over K which are locally isomorphic (i.e. $A_{\mathfrak{m}} \cong B_{\mathfrak{m}}$ over $K_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of K) are isomorphic over K. For example, in Theorem (4.1) A is a finitely presented (not necessarily commutative) K-algebra equipped with a "nicely

* Partially supported by NSF Grants MPS 75-08545, MCS 75-05265, and 76-06307, respectively

** To whom offprint requests should be sent

[1] One can easily reduce to the case when K is reduced (see [W], Lemma 6.7). Then A admits an ascending filtration $K = A_0 \subset A_1 \subset A_2 \subset \cdots$ such that locally A_n consists of the polynomials of degree $\leqq n$ for *any* choice of the variable. The K-module $P = A_1/K$ is then finitely generated and projective, so we obtain a homomorphism $S(P) \to A$ which must be an isomorphism since it is one locally

[2] The authors have recently learned from A. Suslin that he has also proved this result

behaved" filtration, and B is the associated graded algebra $\mathrm{gr}(A)$. In a particular case of Theorem (4.13), A is a finitely presented $K[T]$-algebra, and $B=(A/TA)[T]$. In a particular case of Theorem (4.14), A is a finitely presented left $E[T]$-module, where E is any K-algebra, and $B=(A/TA)[T]$. The special case of this when $E=K$ is just Quillen's localization theorem.

Another useful result obtained (from Theorem (4.14)) is the following. Call a K-algebra A invertible if A is a tensor factor of some polynomial algebra $K[X_1, \ldots, X_n]$. Theorem (4.9) asserts that a finitely presented K-algebra which is locally invertible is invertible.

These results were announced in [BCW].

The proofs systematically use an argument from [Q], which we have formalized in § 3, and christened "Quillen induction." Using the finite presentability assumptions, this permits one to pass from local isomorphisms to isomorphisms on a finite open covering of spec(K), but more importantly, to a covering by *two* principal affine open sets. In other words one is reduced to the case where one has $s_0, s_1 \in K$ such that $Ks_0+Ks_1=K$, and isomorphisms u_i: $B_{s_i} \to A_{s_i}$ over

$$K_{s_i} (i=0, 1).$$

To get an isomorphism $B \to A$ over K we wish to have $u_{0s_1}=u_{1s_0}$. We are only at liberty to modify u_i by an automorphism of B_{s_i}. For any K-algebra L let $G(L)$ denote the group of automorphisms of $L \otimes_K B$ over L. For example we have $u_{0s_1}^{-1} u_{1s_0} \in G(K_{s_0 s_1})$. The problem we pose requires more or less that

$$G(K_{s_0 s_1})=G(K_{s_0})_{s_1} \cdot G(K_{s_1})_{s_0}.$$

This is unreasonable to expect in general. However, in all the examples treated, B admits a *grading*, $B=B_0 \oplus B_1 \oplus \cdots$, relative to which we can define a natural action of the scalars in L on $G(L)$, and, in terms of this, a certain subgroup $G_0(L)$ of $G(L)$. The isomorphisms u_i above can be normalized so that $u_{0s_1}^{-1} u_{1s_0}$ belongs to $G_0(K_{s_0 s_1})$. Then, under a certain mild condition on G, which we call "Axiom Q", it can be shown that

$$(*) \quad G_0(K_{s_0 s_1})=G_0(K_{s_0})_{s_1} \cdot G_0(K_{s_1})_{s_0}.$$

The paper is organized as follows:

In § 1 we establish Axiom Q for the group functors of interest to us. They include all classical groups, as well as the automorphism groups of polynomial algebras. In § 2 we define scalar operations on a group functor G and prove $(*)$, assuming Axiom Q. In the brief § 3 we formulate Quillen induction abstractly. In § 4 the localization theorems are proved.

We gratefully acknowledge the hospitality of the University of Miami during an essential stage of this research.

Notational Conventions. All rings and algebras will be understood to be commutative (with unit) unless the contrary is indicated. If L is a ring and S is a multiplicative set in L we write L_S for the corresponding ring of fractions. If $S=\{s^n | n \geq 0\}$ for some $s \in L$ we also write L_s in place of L_S.

Let G be a functor from K-algebras to groups. Let L be a K-algebra and S a multiplicative set in L. The homomorphism $G(L) \to G(L_S)$ will be denoted

$u \to u_S$ for $u \in G(L)$. Its image is thus denoted $G(L)_S (\subset G(L_S))$. When S consists of powers of some $s \in L$ we also use s in the subscripts in place of S.

Let T be an indeterminate, let L' be an L-algebra, and let f be the L-algebra homomorphism $L[T] \to L'$ sending T to an element $t \in L'$. An element u of $G(L[T])$ will often be denoted also $u(T)$, and its image under $G(f)$: $G(L[T]) \to G(L')$ will then be denoted $u(t)$. When $L' = L[T]$ this defines $u(sT)$ for any $s \in L$; take $t = sT$. When $L' = L$ and $t = 0$ we denote the kernel of $G(f)$ by

$$G(TL[T]) = \{u(T) \in G(L[T]) | u(0) = 1\}.$$

§ 1. Axiom Q

(1.1) *Axiom Q.* Let G be a functor from K-algebras to groups. We say that G satisfies axiom Q if, given a K-algebra L, an element s of L and an element $u(T)$ of $G(TL_s[T])$, then there is an integer $r \geqq 0$ and an element $v(T)$ in $G(TL[T])$ such that $u(s^r T) = v(T)_s$.

The applications of this axiom will be made in § 2 and § 4. The balance of this section is devoted to verifying axiom Q for the examples of interest to us.

(1.2) *Remark.* Let K' be a K-algebra. For any K-algebra L put $L' = K' \otimes_K L$. If $s \in L$ then we can identify $(L_s)'$ with $L'_{s'}$ where $s' = 1 \otimes s \in L'$. Moreover we can identify $L[T]'$ with $L'[T]$. Let G' denote the functor on K-algebras L defined by $G'(L) = G(L')$. Then if G satisfies axiom Q so also does G'. For suppose $s \in L$ and $u(T) \in G'(TL_s[T]) = G(TL'_{s'}[T])$. Then there is an $r \geqq 0$ and a

$$v(T) \in G(TL'[T]) = G'(TL[T])$$

such that $v(T)_s (= v(T)_{s'})$ equals $u(s^r T) (= u(s'^r T))$.

(1.3) *Remark.* Let H be a subgroup functor of a functor G satisfying axiom Q. To deduce axiom Q for H it clearly suffices to have the following condition: Given $v(T) \in G(TL[T])$ such that $v(T)_s \in H(TL_s[T])$, there is an $m \geqq 0$ such that $v(s^m T) \in H(TL[T])$.

(1.4) *Remark.* Let H be a *normal* subgroup functor of G, and define $G' = G/H$ by $G'(L) = G(L)/H(L)$ for all K-algebras L. If G satisfies axiom Q one sees easily that G' does likewise. If, on the other hand, G' and H satisfy axiom Q then so also does G. In fact, suppose given $u(T) \in G(TL_s[T])$ as in (1.1). Applying axiom Q to the image $u'_1(T)$ of $u(T)$ in $G'(TL_s[T])$, we obtain an $r_1 \geqq 0$ and a $v'_1(T)$ in $G'(TL(T])$ such that $v'_1(T)_s = u'_1(s^{r_1} T)$. The decomposition of $G(L[T])$ into the semi-direct product of $G(L)$ with $G(TL[T])$ is canonical, so the surjectivity of

$$G(L[T]) \to G'(L[T])$$

implies that of $G(TL[T]) \to G'(TL[T])$, whence a lifting of $v'_1(T)$ to some $v_1(T) \in G(TL[T])$. Let $u_1(T) = u(s^{r_1} T) \, v_1(T)_s^{-1}$; its image in $G'(TL_s[T])$ is trivial, so it lies in $H(TL_s[T])$. By axiom Q for H we obtain an $r_2 \geqq 0$ and a $v_2(T) \in H(TL[T])$ such that $v_2(T)_s = u_1(s^{r_2} T) = u(s^{r_1 + r_2} T) \, v_1(s^{r_2} T)^{-1}$. Putting $r = r_1 + r_2$ and

$$v(T) = v_2(T) \, v_1(s^{r_2} T)$$

we thus verify axiom Q for G.

(1.5) For any not necessarily commutative ring E we denote its group of units by $E^{\times}$. If J is a two sided ideal we put

$$(1+J)^{\times}=\operatorname{Ker}(E^{\times}\to(E/J)^{\times}).$$

Lemma (Quillen, [Q]). *Let E be a not necessarily commutative ring, let s be a central element of E, and let T be an indeterminant. Given $u(T)$ in $(1+TE_s[T])^{\times}$, there is an $r\geqq 0$ and a $v(T)$ in $(1+TE[T])^{\times}$ such that $u(s^r T)=v(T)_s$.*

Write $u(T)=1+Tu_1(T)$ and $u(T)^{-1}=1+Tu_1'(T)$. For r_1 sufficiently large the elements $s^{r_1}u_1(s^{r_1}T)$ and $s^{r_1}u_1'(s^{r_1}T)$ lift back to elements $w_1(T)$ and $w_1'(T)$, respectively, in $E[T]$. Putting $w(T)=1+Tw_1(T)$ and $w'(T)=1+Tw_1'(T)$ we then have $w(T)_s=u(s^{r_1}T)$ and $w'(T)=u(s^{r_1}T)^{-1}$. Hence $w(T)w'(T)=1+Tx(T)$ and $w'(T)w(T)=1+Ty(T)$ with $x(T)_s=y(T)_s=0$. We can thus choose $r_2\geqq 0$ so that s^{r_2} annihilates $x(T)$ and $y(T)$, hence also $x(s^{r_2}T)$ and $y(s^{r_2}T)$, clearly. Put

$$v(T)=w(s^{r_2}T) \quad \text{and} \quad v'(T)=w'(s^{r_2}T).$$

We have $v(0)=v'(0)=1$ and $v(T)_s=w(s^{r_2}T)_s=u(s^{r_1+r_2}T)$. The lemma will be proved if we show that $v(T)$ is invertible. But $v(T)v'(T)=1+s^{r_2}Tx(s^{r_2}T)=1$ and $v'(T)v(T)=1+s^{r_2}Ty(s^{r_2}T)=1$.

(1.6) **Proposition.** *Let E be a not necessarily commutative K-algebra. Let G be the functor attaching to each K-algebra L the group $G(L)=(L\otimes_K E)^{\times}$ of units of $L\otimes_K E$. Then G satisfies axiom Q.*

This is immediate from Lemma (1.5).

(1.7) **Corollary** (Quillen, [Q]). *Let A be a not necessarily commutative K-algebra, and let M be a finitely presented left A-module. Let GL_M denote the functor attaching to each K-algebra L the group $GL_M(L)$ of $(L\otimes_K A)$-module automorphisms of $L\otimes_K M$. Then GL_M satisfies axiom Q.*

Let $E=\operatorname{End}_A(M)$. For each K-algebra L there is a canonical L-algebra homomorphism $L\otimes_K E\to\operatorname{End}_{L\otimes_K A}(L\otimes_K M)$ which, since M is finitely presented, is an isomorphism when L is flat. Thus we can identify GL_M with the functor G of Proposition (1.6) on the category of flat K-algebras. This verifies axiom Q in the special case $L=K$ in (1.1). The case of an arbitrary K-algebra L follows similarly, once we replace K by L and GL_M by $GL_{L\otimes_K M}$.

(1.8) *Remark.* Suppose, in (1.7), that M is equipped with a form $h\colon M\times M\to A$ which is sesquilinear relative to some antiautomorphism $a\mapsto\bar{a}$ of A, i.e. h is bi-additive and $h(ax, by)=a\,h(x,y)\,\bar{b}$ for $a,b\in A$, $x,y\in M$. This induces a similar form h_L on the $(L\otimes_K A)$-module $L\otimes_K M$, and we may consider the subgroup $U(L)$ of $GL_M(L)$ formed by those elements u leaving h_L invariant: $h_L(ux, uy)=h_L(x,y)$ for $x,y\in L\otimes_K M$. Then $U(L)$ likewise satisfies axiom Q. In view of (1.3) it suffices to show that if $u(T)\in GL_M(TL[T])$ and if $u(T)_s$ preserves $h_{L_s[T]}$, then $u(s^m T)$ preserves $h_{L[T]}$ for some $m\geqq 0$. For $x,y\in L[T]\otimes_K M=$ "$(L\otimes_K M)[T]$" put $d_{x,y}(T)=h_{L[T]}(ux,uy)-h_{L[T]}(x,y)$. This lies in $T(L[T]\otimes_K A)$ since $u(0)=1$, and it is annihilated by some s^m (m depending on x,y), so that $d_{x,y}(s^m T)=0$. In view of the sesquilinearity of $h_{L[T]}$ it suffices, for our purposes, to make $d_{x,y}(T)$ vanish when x,y run through a finite set of generators of M. Hence there is an $m\geqq 0$ such that $d_{x,y}(s^m T)=0$ for all $x, y \in L[T] \otimes_K M$, and so $u(s^m T)$ preserves $h_{L[T]}$, as required. Similarly, if A is commutative and Q is a quadratic form on M, then the orthogonal group $0(M, Q)$ satisfies axiom Q.

(1.9) *Finitely Presented Algebras.* Let A be a (not necessarily commutative) finitely presented K-algebra, and let $x=(x_1, \ldots, x_p) \in A^p$ be a sequence of elements generating A as K-algebra. Then A is the quotient of the free K-algebra on non commuting indeterminants $X_1, \ldots, X_p$ by a two sided ideal generated by some finite set $f_1, \ldots, f_q$, where $f_j=f_j(X)=f_j(X_1, \ldots, X_p)$. Let B be a K-algebra and let $u: A \to B$ be a K-algebra homomorphism. Then u is determined by $u(x)=(u(x_1), \ldots, u(x_p)) \in B^p$, and we thus obtain a bijection $u \mapsto u(x)$ from $\mathrm{Hom}_{K\text{-alg}}(A, B)$ to

$$H(A, B)=\{y \in B^p \mid f_j(y)=0,\ j=1, \ldots, q\}.$$

For later use we record the following lemma here.

(1.10) **Lemma.** *Let A and B be K-algebras as above with A finitely presented. Let S be a multiplicative set in K, which we order by divisibility. The canonical map*

$$(*) \quad \varinjlim_{s \in S} \mathrm{Hom}_{K_s\text{-alg}}(A_s, B_s) \to \mathrm{Hom}_{K_S}(A_S, B_S)$$

is bijective. If B is also finitely presented then (*) *is bijective also with* Isom *(the set of isomorphisms) in place of* Hom.

We can identify $\mathrm{Hom}_{K_S\text{-alg}}(A_S, B_S)$ with $\mathrm{Hom}_{K\text{-alg}}(A, B_S)$, and so, as in (1.9), with $H(A, B_S)$. To show injectivity of (*), let $s \in S$ and $y, y' \in H(A, B_s)$ be such that $y_S=y'_S$ in $H(A, B_S)$. Then clearly $y_t=y'_t$ in $H(A, B_{st})$ for some $t \in S$, whence the injectivity of (*). To show its surjectivity, given $y \in H(A, B_S)$, we must find $s \in S$ and $z \in H(A, B_s)$ such that $z_S=y$. First there is clearly a $t \in S$ and a $w \in B_t^p$ such that $w_S=y$ in B_S^p. The finitely many elements $f_j(w)$ in B_t vanish on passage to B_S, hence already in B_s, where $s=t\,t'$ for some $t' \in S$. Then $z=w_{t'}$ belongs to $H(A, B_s)$ and $z_S=y$, as required. To prove the last assertion it suffices to show that an isomorphism $u: A_S \to B_S$ can be lifted to an isomorphism $v: A_s \to B_s$ for some $s \in S$. By what has been proved we can find a $t \in S$ and homomorphism $w: A_t \to B_t$ and $w': B_t \to A_t$ such that $w_S=u$ and $w'_S=u^{-1}$. Then $w\,w'$ and $w'\,w$ become the identities after S-localization, hence already over K_s, where $s=t\,t'$ for some $t' \in S$. Then $v=w_{t'}$ is the required isomorphism lifting u.

(1.11) **Lemma.** *Let A be a (not necessarily commutative) finitely presented K-algebra, let $s \in K$, and let T be an indeterminate. Let $u(T)$ be a $K_s[T]$-algebra automorphism of $A_s[T]$ such that $u(0)$ is the identity automorphism of $A_s(=A_s[T]/TA_s[T])$. Then there is an $r \geqq 0$ and an automorphism $v(T)$ of the $K[T]$-algebra $A[T]$ such that $v(0)=1_A$ and $u(s^r T)=v(T)_s$.*

($u(s^r T)$ denotes the automorphism obtained from $u(T)$ via the base change $K_s[T] \to K_s[T]$ sending T to $s^r T$.)

With the notation of (1.9) we can identify $u(T)$ with the element $y(T)=u(T)(x)$ in $H(A, A_s[T]) \subset A_s[T]^p$. The condition $u(0)=1_{A_s}$ means that $y(T)=x_s+Ty_1(T)$ for some $y_1(T) \in A_s[T]^p$. Choose r_1 large enough so that $s^{r_1} y_1(s^{r_1} T)=w_1(T)_s$ for some $w_1(T) \in A[T]^p$, and put $w(T)=x+Tw_1(T)$ in $A[T]^p$, so that $w(T)_s=y(s^{r_1} T)$.

Now $f_j(w(T))=f_j(x+Tw_1(T))=f_j(x)+Tf_j'(w(T))=Tf_j'(w(T))$. Since $f_j(w(T))_s=0$ there is an $r_2\geqq 0$ such that $s^{r_2} f_j'(w(T))=0$, and so also $s^{r_2} f_j'(w(s^{r_2} T))=0$. We can choose one r_2 to work for all $j=1,\dots,q$. Replacing $w(T)$ by $w(s^{r_2} T)$ we then obtain $f_j(w(T))=0$ for all j, i.e. $w(T)\in H(A,A[T])$. Putting $r'=r_1+r_2$ we also have $w(T)_s=y(s^{r'}T)$. Similarly we can find $w'(T)=x+Tw_1'(T)$ in $H(A,A[T])$ such that $w'(T)_s=y'(s^{r''}T)$, where $y'(T)=u(T)=u(T)^{-1}(x)$. We can then adjust choices so that $r''=r'$. The endomorphisms of $A[T]$ corresponding to $w(T)$ and $w'(T)$ have composites corresponding in turn to elements of $H(A,A[T])$, which we shall denote $w(T)\circ w'(T)=x+Tz(T)$, and $w'(T)\circ w(T)=x+Tz'(T)$. On localizing to $A_s[T]$, $w(T)$ and $w'(T)$ correspond to inverse automorphisms, so $z(T)_s=z'(T)_s=0$. Choose $m\geqq 0$ so that $s^m z(s^m T)=s^m z'(s^m T)=0$. Then, with the notational conventions above, $w(s^m T)\circ w'(s^m T)=x=w'(s^m T)\circ w(s^m T)$. It follows that $w(s^m T)$ defines an automorphism $v(T)$ of $A[T]$, and we clearly have $v(0)=1_A$ and $v(T)_s=u(s^r T)$ where $r=m+r'$. This proves Lemma (1.11), from which the next result is now immediate.

(1.12) **Proposition.** *Let A be a (not necessarily commutative) finitely presented K-algebra. Let G denote the functor attaching to each K-algebra L the group $G(L)$ of L-algebra automorphisms of $L\otimes_K A$. Then G satisfies axiom Q.*

The preceeding proposition and lemmas are valid for nonassociative algebras, e.g. Lie algebras, as well. One need only interprete the "free algebra" in (1.9) in the sense appropriate to the category of algebras being considered.

The results are valid also for graded algebras and graded algebra homomorphisms. One need only take care to use homogeneous elements throughout.

The results also have analogues for filtered algebras, but these extensions of the results are not so straightforward. This setting, which is significant for our main applications, is treated next.

(1.13) *Filtered Algebras.* Let A be a (not necessarily commutative) K-algebra equipped with a (descending) filtration,

$$A=A_0\supset A_1\supset\cdots;\qquad A_p A_q\subset A_{p+q}.$$

For $a\in A$ put $\varphi(a)=\sup\{n\mid a\in A_n\}$. We call the filtration *separating* if $\bigcap_n A_n=0$, i.e. if $\varphi(a)<\infty$ for all $a\neq 0$. We call it *absolutely separating* if, for all K-algebras L, the filtration of the L-algebra $L\otimes_K A$ given by the images of the $L\otimes_K A_n$ is separating.

For each $n\geqq 0$ let A_n' $(\subset A_n)$ denote the sum of all the ideals $A_{p_1}\cdots A_{p_r}$, where $p_1+\cdots+p_r\geqq n$, but $p_i<n$ for each i. We say the filtration is of *finite type* of there is a *finite* subset X of A such that

$$A_n=A_n'+\sum_{x\in X\cap A_n} AxA$$

for all $n>0$. Note then that, for any K-algebra L, the filtration induced (as above) on the L-algebra $L\otimes_K A$ is still of finite type; one uses $1\otimes X$ to see this.

(1.14) *Examples.* 1. Suppose A is a K-algebra with a two sided ideal A_1 generated by a finite set X. Then the filtration defined by $A_n=A_1^n$ is clearly of finite type.

2. Suppose $A=A_0\oplus A_1\oplus\cdots$ is a graded K-algebra, with filtration defined by $A_{(n)}=A_n\oplus A_{n+1}\oplus\cdots$ $(n\geqq 0)$. This filtration is visibly absolutely separating. If X

is a finite set of homogeneous generators of A as K-algebra then this X serves to show that the filtration $(A_{(n)})_{n\geq 0}$ is also of finite type.

3. In order that a filtration of a K-algebra A be (absolutely) separating it suffices that, for each maximal ideal m of K, the induced filtration of the K_m-algebra A_m be (absolutely) separating. This is the case, for example, if A is locally isomorphic to a filtered algebra as in example 2 above.

(1.15) **Lemma.** *Let A be a (not necessarily commutative) K-algebra equipped with a filtration of finite type, and let X be as in the definition* (1.13). *Let B be a (not necessarily commutative) filtered K-algebra, and let $u: A \to B$ be a K-algebra homomorphism.*

(a) *In order that u preserve filtrations, i.e. that $u(A_n) \subset B_n$ for all n, it is (necessary and) sufficient that $u(x) \in B_{\varphi(x)}$ for all $x \in X$.* (We put $B_\infty = \cap B_n$.)

(b) *Suppose the filtration of A is separating. Let S be a multiplicative set in K, and suppose that $u_S: A_S \to B_S$ preserves filtrations. Then there is an $s \in S$ such that $u_s: A_s \to B_s$ preserves filtrations.*

(c) *Keep the assumptions of* (b) *and suppose A is a finitely presented K-algebra. The canonical map*

$$(*)\quad \varinjlim_{s\in S} \mathrm{Hom}^f_{K_s\text{-alg}}(A_s, B_s) \to \mathrm{Hom}^f_{K_S\text{-alg}}(A_S, B_S)$$

is bijective. (The superscript f designates that the algebra homomorphisms are filtration preserving.) *Suppose the filtered K-algebra B satisfies all the assumptions made on A. Then* $(*)$ *remains bijective with* Isom *(the set of isomorphisms) in place of* Hom.

The necessity in (a) is obvious. For sufficiency we prove $u(A_n) \subset B_n$ by induction on n, the case $n=0$ being trivial. For $n>0$ we have $u(A_n) \subset u(A'_n) + \sum_{x\in X\cap A_n} B\,u(x)\,B$. We have $u(x) \in B_n$ for $x \in X \cap A_n$ by hypothesis, and $u(A'_n) \subset B_n$ by the induction hypothesis, whence (a).

To prove (b), consider $x \neq 0$ in X. Then $\varphi(x) = n < \infty$ and $u(x)$ lands in $(B_S)_n = (B_n)_S$. Hence $s\,u(x) \in B_n$ for some $s \in S$. Since X is finite we can choose one s to work for all $x \neq 0$ in X. Then the composite homomorphism $A \xrightarrow{u} B \longrightarrow B_s$ preserves filtrations, by (a), and consequently, by localization, $u_s: A_s \to B_s$ does likewise. In view of (b), assertion (c) results immediately from Lemma (1.10).

(1.16) **Lemma.** *In the setting of Lemma* (1.11), *suppose A is equipped with a separating filtration of finite type. Let $u(T)$ be a filtration preserving automorphism of $A_s[T]$ such that $u(0) = 1_{A_s}$. Then there is an $r \geq 0$ and a filtration preserving automorphism $v(T)$ of $A[T]$ such that $v(0) = 1_A$ and $u(s^r T) = v(T)_s$.*

Lemma (1.11) furnishes an $r' \geq 0$ and a $v'(T)$ such that $v'(0) = 1_A$ and $u(s^{r'} T) = v'(T)_s$. Let X be as in the definition of finite type filtration (1.13). Let $x \neq 0$ be an element of X, and let $n = \varphi(x) < \infty$. We have $v'(T)(x) = x + T y(T)$ since $v'(0) = 1_A$. Since $v'(T)_s$ preserves filtration it follows that $y(T)$ lands in $A_s[T]_n = (A_n)_s[T]$. Therefore there is an $m \geq 0$ such that $s^m y(T) \in A_n[T]$, and so also $s^m y(s^m T) \in A_n[T]$. Since X is finite a single m will accomplish this for all $x \neq 0$ in X. Then if $v(T) = v'(s^m T)$ we have $v(0) = 1_A$ and $v(T)_s = u(s^r T)$ where $r = m + r'$. Further if $x \neq 0$

in X we have $v(T)(x)=x+s^m Ty(s^m T)\in A[T]_{\varphi(x)}$, so it follows from Lemma (1.15) (a) that $v(T)$ preserves filtrations.

Now, just as Proposition (1.12) was derived from Lemma (1.11), we obtain:

(1.17) **Proposition.** *Let A be a finitely presented K-algebra with an absolutely separating filtration of finite type* (see (1.13)). *Let G denote the functor attaching to each K-algebra L the group $G(L)$ of automorphisms of the filtered L-algebra $L\otimes_K A$. Then G satisfies axiom Q.*

§ 2. Scalar Operations on Group Functors

(2.1) Let G be a functor from K-algebras to groups. A *scalar operation* on G consists of an action $L\times G(L)\to G(L)$, denoted $(s,u)\mapsto {}^s u$, for each K-algebra L. Thus

$${}^1u=u;\quad {}^s({}^t u)={}^{st}u;\quad \text{and}\quad {}^s(uv)={}^s u\cdot {}^s v$$

for $s,t\in L$, $u,v\in G(L)$. Further these actions are to be natural, in the sense that if $f\colon L\to L'$ is a K-algebra homomorphism and if $G(f)\colon G(L)\to G(L')$ sends $u\in G(L)$ to $u'\in G(L')$ then it sends ${}^s u$ to ${}^{f(s)}u'$ for $s\in L$.

The action of L on $G(L)$ amounts to a multiplicative monoid homomorphism $L\to \mathrm{End}(G(L))$. In particular $u\mapsto {}^0u$ is an *idempotent* endomorphism of $G(L)$, whose image we denote ${}^0G(L)$, and whose kernel we denote

$$G_0(L)=\{u\in G(L)\mid {}^0u=1\}.$$

Thus $G(L)$ is the semi-direct product $G_0(L)\rtimes {}^0G(L)$, and this decomposition is functorial in L.

(2.2) *Example.* Let G be any functor from K-algebras to groups, and let T be an indeterminate. Define a new functor G' by $G'(L)=G(L[T])$. If $u=u(T)\in G'(L)$ and if $s\in L$ we can define ${}^s u=u(sT)$. This is easily seen to provide a scalar operation on G'. We then have ${}^0u=u(0)$ so ${}^0G(L)=G(L)$ and $G'_0(L)=G(TL[T])$.

More generally, let $H=H_0\oplus H_1\oplus\cdots$ be any graded K-algebra. If $s\in K$ define $\varepsilon_s\colon H\to H$ by $\varepsilon_s(a)=s^n a$ for $a\in H_n$. Then $\varepsilon_1=1_H$, $\varepsilon_s\circ\varepsilon_t=\varepsilon_{st}$, and ε_s is a K-algebra endomorphism of H. Similarly, if L is any K-algebra then $H\otimes_K L$ is a graded L-algebra equipped with an endomorphism ε_s for each $s\in L$. If $f\colon L\to L'$ is a K-algebra homomorphism sending s to s' then $(1_H\otimes f)\circ\varepsilon_s=\varepsilon_{s'}\circ(1_H\otimes f)\colon H\otimes_K L\to H\otimes_K L'$. Given G as above we can define G' now by $G'(L)=G(H\otimes_K L)$. Then G' admits the scalar operations such that $s\in L$ acts on $G'(L)$ as $G(\varepsilon_s)$. When $H=K[T]$ this is just the example above.

(2.3) *Example* (c.f. [BW]). We consider graded (not necessarily commutative) K-algebras $A=A_0\oplus A_1\oplus\cdots$, and equip them with the descending filtration $A_{(n)}=A_n\oplus A_{n+1}\oplus\cdots$ $(n=0,1,2,\ldots)$. Let $B=B_0\oplus B_1\oplus\cdots$ be another such graded K-algebra, and let $u\colon A\to B$ be a filtration preserving algebra homomorphism: $u(A_{(n)})\subset B_{(n)}$ for all n. As a linear map u can be decomposed into homogeneous components, $u=u_0+u_1+u_2+\cdots$, where u_n is a K-linear map $A\to B$ such that $u_n(A_p)\subset B_{p+n}$ for all n, $p\geqq 0$, and for a given $a\in A$, $u_n(a)=0$ for all but finitely many n. The fact that u is an algebra homomorphism is expressed by the fact that

$u(1)=1$ and

$$(*)\quad u_n(a\,b)=\sum_{p+q=n} u_p(a)\,u_q(b)$$

for all $n\geqq 0$, $a, b\in A$. It suffices to know (*) for homogeneous elements of A.

Let $s\in K$. Define ${}^s u$ by $({}^s u)_n = s^n\, u_n$, in other words

$${}^s u = u_0 + s\,u_1 + s^2\, u_2 + \cdots$$

Then ${}^s u(1)=1$, and $({}^s u)_n(a\,b)=s^n\,u_n(a\,b)=s^n\cdot\sum_{p+q=n} u_p(a)\,u_q(b)=\sum_{p+q=n}({}^s u)_p(a)\,({}^s u)_q(b)$, so ${}^s u$ is again a filtered algebra homomorphism from A to B. Visibly we have ${}^1 u=u$ and ${}^t({}^s u)={}^{ts}u$, for $s, t\in K$. Moreover ${}^0 u=u_0$, which is just the homomorphism of associated graded algebras induced by u. Suppose $v: B\to C$ is a second filtration preserving homomorphism of graded algebras. Then $(v\,u)_n = \sum_{p+q=n} v_p\, u_q$, so $s^n(v\,u)_n=\sum_{p+q=n}(s^p\,v_p)(s^q\,u_q)$, whence ${}^s(v\,u)={}^s v\,{}^s u$. It follows from this that if u is an isomorphism *and* if u^{-1} is also filtration preserving, then the same properties hold for ${}^s u$.

Let L be a K-algebra. Then $L\otimes_K A$ and $L\otimes_K B$ are graded L-algebras, so the scalars $s\in L$ operate as above on the filtration preserving L-algebra homomorphisms $u: L\otimes_K A\to L\otimes_K B$. If $f: L\to L'$ is a K-algebra homomorphism then one sees easily that, for $s\in L$, the following diagram is commutative.

$$\begin{array}{ccc} L\otimes_K A & \xrightarrow{\ {}^s u\ } & L\otimes_K B \\ {\scriptstyle f\otimes_K 1_A}\downarrow & & \downarrow{\scriptstyle f\otimes_K 1_B} \\ L'\otimes_K A & \xrightarrow[\ f(s)(1_{L'}\otimes_L u)\ ]{} & L'\otimes_K B. \end{array}$$

Now for a fixed graded algebra A as above, denote by G^A the functor attaching to each K-algebra L the group $G^A(L)$ of automorphisms of the *filtered* L-algebra $L\otimes_K A$. The discussion above shows then that the maps $u\to{}^s u$ ($s\in L$, $u\in G^A(L)$) define a scalar operation on the functor G^A. Thus $G^A(L)$ is the semi-direct product.

$$G^A(L)=G_0^A(L)\rtimes {}^0G^A(L)$$

where ${}^0G^A(L)$ is the group of automorphisms of the *graded* L-algebra $L\otimes_K A$, and such that $u\to{}^0u$ is the projection on the second factor with kernel the first factor.

(2.4) **Theorem.** *Let G be a functor from K-algebras to groups which satisfies axiom Q* (see (1.1)). *Let G be equipped with a scalar operation* (as in (2.1)). *For any K-algebra L let $G_0(L)=\{u\in G(L)\,|\,{}^0u=1\}$. Suppose $s_0, s_1\in L$ and $L=Ls_0+Ls_1$. Then*

$$G_0(L_{s_0 s_1})=G_0(L_{s_0})_{s_1}\cdot G_0(L_{s_1})_{s_0}.$$

We first prove a lemma, modeled after its analogue, Lemma 1 of [Q].

(2.5) **Lemma.** *Let G be as in Theorem* (2.4). *Let L be a K-algebra, $s\in L$, and $u\in G(L_s)$. There is an integer $r\geqq 0$ such that, if $a, b\in L$ satisfy $a\equiv b$* mod Ls^r *then there is a v in $G_0(L)$ such that $v_s=({}^a u)({}^b u)^{-1}$.*

Let Y, T be indeterminates and put $w=w(Y,T)=({}^{(Y+T)}u)({}^{Y}u)^{-1}\in G(L_s[Y,T])$ (where we identify $G(L_s)$ with a subgroup of $G(L_s[Y,T])$). Clearly ${}^0w=1$ and $w(Y,0)=1$. Applying axiom Q to $w\in G(TL_s[Y,T])$ and $s\in L[Y]$ we obtain an $r\geqq 0$ and a $v(Y,T)\in G(TL[Y,T])$ such that $v(Y,T)_s=w(Y,s^r T)$. Replacing v by $v\cdot({}^0v)^{-1}$, if necessary, which doesn't affect the above conditions, we can further arrange that ${}^0v=1$. Now suppose $a,b\in L$ and $a=b+s^r t$ for some $t\in L$. Then we have $v(b,t)\in G_0(L)$ and $v(b,t)_s=w(b,s^r t)=({}^{(b+s^r t)}u)({}^bu)^{-1}=({}^au)({}^bu)^{-1}$, whence the lemma.

(2.6) *Proof of Theorem* (2.4). Given $u\in G_0(L_{s_0 s_1})$, we apply Lemma (2.5) to the localizations $L_{s_i}\to L_{s_0 s_1}$ $(i=0,1)$ to obtain an $r\geqq 0$ such that if $a,b\in L_{s_i}$ satisfy, $a\equiv b \bmod L_{s_i}\cdot s_{1-i}^r$, then there is a v in $G_0(L_{s_i})$ such that $({}^au)({}^bu)^{-1}=v_{s_{1-i}}$. To apply this use the condition $Ls_0+Ls_1=L$ to obtain $a\in Ls_0^r$ such that $b=1-a$ lies in Ls_1^r, and write

$$u=[{}^1u\cdot({}^au)^{-1}]\,[{}^au\cdot({}^0u)^{-1}].$$

Since $1\equiv a \bmod Ls_1^r$ there is a v_0 in $G_0(L_{s_0})$ such that $v_{0\,s_1}={}^1u({}^au)^{-1}$. Similarly, since $a\equiv 0 \bmod Ls_0^r$ there is a $v_1\in G_0(L_{s_1})$ such that $v_{1\,s_0}={}^au({}^0u)^{-1}$. Thus $u=v_{0\,s_1}\cdot v_{1\,s_0}\in G_0(L_{s_0})_{s_1}\cdot G_0(L_{s_1})_{s_0}$, as was to be shown.

(2.7) **Corollary.** *Let G be a functor from K-algebras to groups satisfying axiom Q ((1.1)). Let $H=H_0\oplus H_1\oplus\cdots$ be a graded K-algebra with $H_0=K$, and let $\varepsilon\colon H\to K$ be the retraction with kernel $H_+=H_1\oplus H_2\oplus\cdots$. For any K-algebra L put*

$$G(H_+\otimes L)=Ker(G(H\otimes L)\xrightarrow{G(\varepsilon\otimes 1_L)}G(L)),$$

where $\otimes$ denotes $\otimes_K$. If $s_0, s_1\in L$ generate the unit ideal of L then

$$G(H_+\otimes L_{s_0 s_1})=G(H_+\otimes L_{s_0})_{s_1}\cdot G(H_+\otimes L_{s_1})_{s_0}.$$

In particular, if H is the graded polynomial algebra $K[T]$, we have

$$G(TL_{s_0 s_1}[T])=G(TL_{s_0}[T])_{s_1}\cdot G(TL_{s_1}[T])_{s_0}.$$

The functor $G'(L)=G(H\otimes L)$ satisfies axiom Q (Remark (1.2)), and it admits the scalar operations of Example (2.2) such that $G'_0(L)=G(H_+\otimes L)$. Thus the corollary is a special case of Theorem (2.4).

Corollary (2.7) applies notably to the functor $G=GL_M$ of (1.7), to the functor G of (1.12), and to the functor G of (1.17).

(2.8) **Corollary.** *Let $A=A_0\oplus A_1\oplus\cdots$ be a (not necessarily commutative)* graded *and finitely presented K-algebra. For each K-algebra L let $G^A(L)$ denote the group of L-algebra automorphisms of $L\otimes_K A$ preserving the filtration*

$$L\otimes_K A_{(n)},\qquad A_{(n)}=A_n\oplus A_{n+1}\oplus\cdots(n\geqq 0).$$

Let $G_0^A(L)$ denote the subgroup consisting of those automorphisms inducing the identity associated graded automorphism. Let L be a K-algebra and let s_0, $s_1\in L$ generate the unit ideal. Then

$$G_0^A(L_{s_0 s_1})=G_0^A(L_{s_0})_{s_1}\cdot G_0^A(L_{s_1})_{s_0}.$$

According to Example (1.14)2, the filtration of A is absolutely separating and of finite type, so Proposition (1.17) implies that the functor $L \mapsto G^A(L)$ satisfies axiom Q. On the other hand Example (2.3) shows that G^A admits a scalar operation such that $G_0^A(L) = \{u \in G(L) | {}^0u = 1\}$. Therefore Corollary (2.8) is a special case of Theorem (2.4).

For the applications to locally polynomial algebras we shall need a technical elaboration of Theorem (2.4), given in Proposition (2.10) below

(2.9) **Lemma.** *Let GA be a functor from K-algebras to groups which satisfies axiom Q. Let G be a subgroup functor of GA which admits scalar operations. Let L be a K-algebra, $s \in L$, $w \in GA(L_s)$, and $u \in G(L_s)$ be such that ${}^0u = 1$. There is an $r \geqq 0$ such that if $a \in L s^r$ then $w^{-1}\, {}^a u\, w = v_s$ for some $v \in GA(L)$.*

Let T be an indeterminate and put $u'(T) = (w^{-1})\,({}^T u)\, w \in GA(L_s[T])$. Since ${}^0u = 1$ we have $u'(T) \in GA(T L_s[T])$, so axiom Q furnishes an $r \geqq 0$ and a

$$v'(T) \in GA(TL[T])$$

such that $v'(T)_s = u'(s^r T)$. Suppose $a = s^r t$ with $t \in L$. Then with $v = v'(t) \in GA(L)$ we have $v_s = v'(t)_s = u'(s^r t) = u'(a) = w^{-1}\, {}^a u\, w$, whence the lemma.

(2.10) **Proposition.** *Let GA be a functor from K-algebras to groups which satisfies axiom Q. Let G and H be subgroup functors of GA such that G satisfies axiom Q and admits scalar operations, and such that $GA(L) = G_0(L) \cdot H(L)$ for any K-algebra L, where $G_0(L) = \{u \in G(L) | {}^0u = 1\}$. Let L be a K-algebra and let $s_0, s_1 \in L$ generate the unit ideal. Then*

$$GA(L_{s_0 s_1}) = G_0(L_{s_0})_{s_1} \cdot H(L_{s_0 s_1}) \cdot GA(L_{s_1})_{s_0}.$$

By assumption an element of $GA(L_{s_0 s_1})$ can be written as a product $u\,w$ with $u \in G_0(L_{s_0 s_1})$ and $w \in H(L_{s_0 s_1})$. Let $a \in L$ and write

$$u\,w = {}^1u({}^a u)^{-1} \cdot w \cdot w^{-1}\, {}^a u\, w.$$

Let r be a large positive integer. If $a \in L s_0^r$ then Lemma (2.9), applied to the localization $L_{s_1} \to L_{s_0 s_1}$, permits us to write $w^{-1}\, {}^a u\, w = v_{1\, s_0}$ for some $v_1 \in GA(L_{s_1})$. If, on the other hand, $a \equiv 1 \bmod L s_1^r$ then Lemma (2.5), applied to G and the localization $L_{s_0} \to L_{s_0 s_1}$, permits us to write ${}^1u({}^a u)^{-1} = v_{0\, s_1}$ for some $v_0 \in G_0(L_{s_0})$. Since

$$L s_0 + L s_1 = L$$

we can simultaneously solve the above congruences for a, and so write

$$u\,w = v_{0\, s_1}\, w\, v_{1\, s_0}$$

as required.

(2.11) *Example.* Let $A = A_0 \oplus A_1 \oplus \cdots$ be a (not necessarily commutative) graded K-algebra. In addition to the descending filtration $(A_{(n)} = A_n \oplus A_{n+1} \oplus \cdots)_{n \geqq 0}$, it admits the ascending filtration $(A^{(n)} = A_0 \oplus \cdots \oplus A_n)_{n \geqq 0}$. For any K-algebra L the group $GA(L)$ of all L-algebra automorphisms of $L \otimes_K A$ contains the subgroups $G(L)$ and $H(L)$ preserving the descending and ascending filtrations, respectively, of $L \otimes_K A$. The functor G admits scalar operations (Example (2.3)). If the K-algebra A is finitely presented then GA and G satisfy axiom Q (Propositions (1.12) and (1.17); see Example (1.14)2). Thus we can apply Proposition (2.10) above provided that

(*) $GA(L)=G_0(L)\cdot H(L)$

for all K-algebras L. We claim that (*) holds if A is the symmetric (or tensor) algebra of the K-module $M=A_1$. For in this case we can identify the group of graded algebra automorphisms of A with $GL_M(K)$, the group of linear automorphisms of M, and we have an embedding of $M^*=\operatorname{Hom}_K(M,K)$ into $H(K)$ sending $t\in M^*$ to the automorphism $\bar{t}$ of A determined by $\bar{t}(x)=t(x)+x\in K\oplus M=A_0\oplus A_1$ for $x\in M$. We then have semi-direct product decompositions

$$G(K)=G_0(K)\rtimes GL_M(K)$$

(c.f. Example (2.3)), and $H(K)=GL_M(K)\rtimes M^*$. Except for the presence of M^*. in place of M, the latter is just the affine group of M, so we shall write $Af_M(K)$ in place of $H(K)$. Now let u be any automorphism of A, i.e. $u\in GA(K)$. For $x\in M$ write $u(x)=t(x)+u_1(x)$ with $t(x)\in K$ and $u_1(x)\in A_{(1)}$. Then $t\in M^*$ and if $v=u\circ(\overline{-t})$ we have, for $x\in M$, $v(x)=u(-t(x)+x)=-t(x)+t(x)+u_1(x)=u_1(x)$, so $v\in G(K)$. Thus $u=v\circ\bar{t}\in G(K)\cdot M^*=G_0(K)\cdot GL_M(K)\cdot M^*=G_0(K)\cdot Af_M(K)$. This proves (*) for $L=K$. For any K-algebra L, the L-algebra $L\otimes_K A$ inherits all the hypotheses made on A over K, so (*) follows for all L.

(2.12) To apply the above discussion we now codify our notation a bit, to bring it more into conformity with that of [BW]. Let M be a K-module, and let A denote the symmetric algebra $S(M)$ or the tensor algebra $T(M)$, with the usual grading: $A_0=K$, $A_1=M,\ldots$ For any K-algebra L let $GA_M(L)$ denote the group of all L-algebra automorphisms of $L\otimes_K A$. Let $GA_M^0(L)$ denote its subgroup consisting of automorphisms preserving the descending filtration $(L\otimes_K A_{(n)})_{n\geqq 0}$, where $A_{(n)}=A_n\oplus A_{n+1}\oplus\cdots$, and let $GA'_M(L)$ denote its subgroup consisting of elements whose associated graded automorphism is the identity. Let $Af_M(L)$ denote the group of automorphisms of $L\otimes_K A$ preserving the ascending filtration

$$(L\otimes A^{(n)})_{n\geqq 0},\qquad \text{where } A^{(n)}=A_0\oplus\cdots\oplus A_n.$$

As in (2.11) above we canonically identify $Af_M(L)$ with the semi-direct product $GL_M(L)\times(L\otimes_K M)^*$, where $(L\otimes_K M)^*=\operatorname{Hom}_L(L\otimes_K M, L)$.

Now, by virtue of the discussion in Example (2.11), we obtain the following Corollary directly from Proposition (2.10).

Corollary. *Let M be a finitely presented K-module, and let A be the symmetric or the tensor algebra of M. For any K-algebra L the group $GA_M(L)$ of L-algebra automorphisms of $L\otimes_K A$ is the product*

$$GA_M(L)=GA'_M(L)\cdot Af_M(L),$$

where $Af_M(L)$ is isomorphic to $GL_M(L)\rtimes(L\otimes_K M)^*$, as above. Suppose s_0, $s_1\in L$ generate the unit ideal. Then

$$GA_M(L_{s_0s_1})=GA'_M(L_{s_0})_{s_1}\cdot Af_M(L_{s_0s_1})\cdot GA_M(L_{s_1})_{s_0}.$$

§ 3. Quillen Induction

Let Loc(K) denote the set of K-algebras of the form K_S, where S is a multiplicative set in K. By "Quillen induction" we refer to the following proposition (see the proof of Theorem 1 in [Q]).

(3.1) **Proposition.** *Let $P(L)$ be a property defined for K-algebras $L \in$ Loc(K). In order that $P(L)$ hold for all $L \in$ Loc(K) (in particular for $L=K$) it suffices that P satisfy the following conditions.*

1) *Specialization. $P(L)$ implies $P(L')$ whenever there is a K-algebra homomorphism $L \to L'$.*

2) *Finiteness. If S is a multiplicative set in K then $P(K_S)$ implies $P(K_s)$ for some $s \in S$.*

3) *Local validity. $P(K_{\mathfrak{m}})$ holds for all maximal ideals $\mathfrak{m}$ of K.*

4) *Sheaf condition. If $L \in$ Loc(K) and if $s_0, s_1 \in L$ generate the unit ideal then $P(L_{s_0})$ together with $P(L_{s_1})$ implies $P(L)$.*

Let S denote the set of $s \in K$ such that $P(K_s)$ holds. By specialization, it suffices to prove that $1 \in S$. By finiteness and local validity, S is contained in no maximal ideal of K. Thus it suffices to show that S is an ideal, or that, given $s_0, s_1 \in S$ and $s \in K s_0 + K s_1$ then $s \in S$. Let $L = K_s$, and let t_i denote the imgage of s_i in L. Then we have $L = L t_0 + L t_1$. Further $L_{t_i} = K_{s s_i}$ is a localization of K_{s_i}, so $P(L_{t_i})$ follows from $P(K_{s_i})$ by specialization (recall that $s_i \in S$). Now the sheaf condition gives $P(L)$, in the presence of $P(L_{t_i})$ $(i=0, 1)$.

Remark. This proposition is used typically in constructing global data on spec(K) from given local data over the various $K_{\mathfrak{m}}$. Finiteness permits passage to a finite open covering of spec$(K) = U_1 \cup \cdots \cup U_n$ where we may take each U_i to be affine, say spec(K_{s_i}). It is technically useful to be able to reduce to the case $n=2$. This can be done by arguing inductively on the open sets $X_i = U_1 \cup \cdots \cup U_i (1 \leqq i \leqq n)$. However the X_i need no longer be affine, and this is troublesome if the construction being performed is not valid in general on non-affine schemes. It is for bypassing this difficulty that Quillen induction is useful.

§ 4. Localization Theorems

(4.1) **Theorem.** *Let A be a (not necessarily commutative) K-algebra with an absolutely separating filtration of finite type* (see (1.13)). *Suppose that A and its associated graded algebra* gr(A) *are finitely presented K-algebras. If $A_{\mathfrak{m}}$ and* gr$(A)_{\mathfrak{m}}$ *are isomorphic filtered $K_{\mathfrak{m}}$-algebras for all maximal ideals $\mathfrak{m}$ of K, then A and* gr(A) *are isomorphic filtered K-algebras.*

Let $B = \mathrm{gr}(A)$. The theorem follows from Quillen induction (Proposition (3.1)) applied to the following proposition, where $L \in$ Loc(K) (see (3.1)).

$P(L)$: $L \otimes_K A$ and $L \otimes_K B$ are isomorphic filtered L-algebras.

Of the four properties in (3.1) to be verified, 1 (specialization) is trivial, and 3 (local validity) is our hypothesis. To verify 2 (finiteness) suppose $P(K_S)$ holds for some multiplicative set S in K, i.e. there is an isomorphism u: $A_S \to B_S$ of filtered

K_S-algebras. The filtration of B, like that of A, is absolutely separating of finite type, (Example (1.14)2), and both algebras are assumed finitely presented. It follows therefore from Lemma (1.15)(c) that u lifts to an isomorphism $v\colon A_s \to B_s$ of filtered K_s-algebras for some $s \in S$.

Finally we verify 4 (the sheaf condition). If $L \in \mathrm{Loc}(K)$ then L is K-flat so we can identify $L \otimes_K \mathrm{gr}(A)$ and $\mathrm{gr}(L \otimes_K A)$. Therefore, up to a change of notation, it suffices to prove 4 for $L = K$. We are given $s_0, s_1 \in K$ such that $Ks_0 + Ks_1 = K$, and isomorphisms $u_i\colon B_{s_i} \to A_{s_i}$ of filtered K_{s_i}-algebras ($i=0,1$). Multiplying by $\mathrm{gr}(u_i)^{-1}$, if necessary, we can assume $\mathrm{gr}(u_i) = 1_{B_{s_i}}$, where we identify

$$\mathrm{gr}(A) = \mathrm{gr}(B) = B.$$

For any K-algebra L let $G(L)$ denote the group of automorphisms of the filtered L-algebra $L \otimes_K B$, and let $G_0(L)$ denote the subgroup of all u such that $\mathrm{gr}(u) = 1_B$. The isomorphisms u_i above furnish two isomorphisms

$$B_{s_0 s_1} \underset{u_{1 s_0}}{\overset{u_{0 s_1}}{\rightrightarrows}} A_{s_0 s_1}$$

of filtered $K_{s_0 s_1}$-algebras, hence an element $u = u_{0 s_1}^{-1} u_{1 s_0} \in G_0(K_{s_0 s_1})$. By Corollary (2.8) we can write $u = v_{0 s_1} v_{1 s_0}^{-1}$ with $v_i \in G_0(K_{s_i})$ ($i=0,1$). Put $w_i = u_i v_i\colon B_{s_i} \to A_{s_i}$. Then $w_{0 s_1}^{-1} w_{1 s_0} = v_{0 s_1} u v_{1 s_0} = 1_{B_{s_0 s_1}}$, i.e. $w_{0 s_1} = w_{1 s_0}$. Thus w_0 and w_1 patch to form an isomorphism $w\colon A \to B$ of filtered K-algebras such that $w_{s_i} = w_i$ ($i=0,1$).

(The last step of the argument derives from the standard fact that, for any K-module M, the localization square

$$\begin{array}{ccc} M & \longrightarrow & M_{s_1} \\ \downarrow & & \downarrow \\ M_{s_0} & \longrightarrow & M_{s_0 s_1} \end{array}$$

is cartesian. Thus if N is another K-module, $\mathrm{Hom}_K(M, N)$ canonically identifies with the fibre product of the $\mathrm{Hom}_{K_{s_i}}(M_{s_i}, N_{s_i})$ ($i=0,1$) over $\mathrm{Hom}_{K_{s_0 s_1}}(M_{s_0 s_1}, N_{s_0 s_1})$.)

(4.2) *Remark.* In Theorem (4.1) the finite generation of $\mathrm{gr}(A)$ follows from the other assumptions on A, but not its finite presentability. For example suppose $a_1, \ldots, a_q$ generate an ideal in K that is not finitely presented as a K-module. In the polynomial ring $S = K[X, Y_1, \ldots, Y_q]$ let J be the ideal generated by

$$f_j = a_j X + Y_j^2 \; (j = 1, \ldots, q)$$

and put $A = S/J$. Give A the H-adic filtration, where H is the ideal generated by the images of $X, Y_1, \ldots, Y_q$. Then A is a finitely presented K-algebra, but $\mathrm{gr}(A)$ is not.

(4.3) **Corollary.** *Let A be a finitely presented (not necessarily commutative) K-algebra equipped with an augmentation $\varepsilon\colon A \to K$; let $\bar{A} = \mathrm{Ker}(\varepsilon)$. The K-module $M = \bar{A}/\bar{A}^2$ is finitely presented. Let $B = U(M)$, where $U = S$ (symmetric algebra), or T (tensor algebra) or Λ (exterior algebra); give B the augmentation vanishing on M. If $A_{\mathfrak{m}}$ and $B_{\mathfrak{m}}$ are isomorphic as augmented $K_{\mathfrak{m}}$-algebras for all maximal ideals $\mathfrak{m}$ of K then A and B are isomorphic as augmented K-algebras.*

Choose generators $x_1, \dots, x_p \in \bar{A}$ of the K-algebra A. Let T be the free K-algebra on generators $X_1, \dots, X_p$. Mapping X_i to x_i we represent A in the form T/I, and $\bar{A} = \bar{T}/I$, where $\bar{T}$ is the ideal generated by the X_i's. Then M is isomorphic to $\bar{T}/\bar{T}^2 + I \cong T_1/R$, where $T_1 = \sum_i KX_i$ and R is the module of linear parts of elements of I. Finite presentability of A implies that I is a finitely generated ideal, from which one sees easily that R is a finitely generated K-module. Thus M is finitely presented, and so $B = U(M)$ is a finitely presented K-algebra. The local isomorphisms $A_\mathfrak{m} \cong B_\mathfrak{m}$ plus the universal property of $B = U(M)$ imply that $\mathrm{gr}(A) = \bigoplus_{n \geq 0} \bar{A}/\bar{A}^n$ admits a graded algebra homomorphism $f\colon B \to \mathrm{gr}(A)$ inducing the identity on M in degree one. The local isomorphisms then further imply that f is an isomorphism, so permitting us to identify B with $\mathrm{gr}(A)$. Thus the corollary will follow from Theorem (4.1) once we confirm that the $\bar{A}$-adic filtration on A is absolutely separating of finite type. Since the ideal $\bar{A}$ is generated by the finite set $x_1, \dots, x_p$, these properties follow from Examples (1.14) 1 and 3.

Remark. Lemma (4.6) below shows that, when $U = S$ or T, any algebra isomorphism $A \to U(M)$ can be modified to be compatible with the augmentations.

(4.4) **Theorem.** *Let A be a finitely presented K-algebra. Suppose that, for all maximal ideals $\mathfrak{m}$ of K, the $K_\mathfrak{m}$-algebra $A_\mathfrak{m}$ is isomorphic to the symmetric algebra of some $K_\mathfrak{m}$-module. Then A is isomorphic to the symmetric algebra $S(M)$ of a finitely presented K-module M.*

(4.5) *Remarks.* 1. Theorem (4.4) contains the result that finitely presented locally polynomial algebras are symmetric algebras.

2. Theorem (4.4) has a non-commutative analogue in which the symmetric algebra is replaced by the tensor algebra. The proof completely parallels that given below for Theorem (4.4).

3. The module M in Theorem (4.4) is uniquely determined up to isomorphism by A, as the next (well known) lemma shows.

(4.6) **Lemma.** *Let M and N be K-modules, and let $U = S$ (for symmetric) or T (for tensor). If $U(M)$ and $U(N)$ are isomorphic K-algebras then M and N are isomorphic K-modules.*

Let $u\colon U(M) \to U(N)$ be an isomorphism. For $x \in M$ write $u(x) = v(x) - t(x)$ with $t(x) \in K$ and $v(x)$ in the augmentation ideal $U_+(N)$. Let $\bar{t}$ be the automorphism of $U(M)$ defined by $\bar{t}(x) = x + t(x)$ for $x \in M$, and put $w = u \cdot \bar{t}$. Then $w(x) = v(x)$ for $x \in M$ so w is an isomorphism of augmented K-algebras. We thus have K-module isomorphisms,

$$M \cong U_+(M)/U_+(M)^2 \cong U_+(N)/U_+(N)^2 \cong N.$$

(4.7) *Proof of Theorem* (4.4). The theorem follows from Quillen induction (Proposition (3.1)) applied to the following proposition, where $L \in \mathrm{Loc}(K)$.

$P(L)$: The L-algebra $L \otimes_K A$ is isomorphic to the symmetric algebra $S(M)$ of some finitely presented L-module M.

Of the four properties in (3.1) to be verified, 1, 2, and 3 follow just as in the proof of Theorem (4.1), where for 2 one appeals to Lemma (1.10) in place of Lemma

(1.15). One also uses the easily verified fact that every finitely presented K_S-module is the localization of one over K_s for some $s\in S$. To verify 4, moreover, it suffices, as in the proof of Theorem (4.1), to establish the implication: $P(K_{s_0})$ and $P(K_{s_1})$ imply $P(K)$ whenever $s_0, s_1\in K$ generate the unit ideal. Assume therefore that we are given a finitely presented K_{s_i}-module M_i and a K_s-algebra isomorphism from A_{s_i} to $S(M_i)$ $(i=0, 1)$. Then the $K_{s_0 s_1}$-algebras $S(M_{0\,s_1})$ and $S(M_{1\,s_0})$ are isomorphic, so Lemma (4.6) tells us that the $K_{s_0 s_1}$-modules $M_{0\,s_1}$ and $M_{1\,s_0}$ are isomorphic. Using such an isomorphism $f\colon M_{0\,s_1}\to M_{1\,s_0}$, the cartesian square

$$\begin{array}{ccccc} M & \longrightarrow & & & M_1 \\ \downarrow & & & & \downarrow \\ M_0 & \longrightarrow & M_{0\,s_1} & \xrightarrow[f]{} & M_{1\,s_0} \end{array}$$

furnishes a finitely presented K-module M such that $M_{s_i}\cong M_i$ over K_{s_i} $(i=0, 1)$. Let $B=S(M)$. For any K-algebra L let $GA_M(L)$ denote the group of L-algebra automorphisms of $L\otimes_K B=S(L\otimes_K M)$. Write

$$GA_M(L)=GA'_M(L)\cdot Af_M(L)$$

as in Corollary (2.12). By assumption we have isomorphisms of K_{s_i}-algebras $u_i\colon B_{s_i}\to A_{s_i}$ $(i=0, 1)$, whence an element $u=u_{0\,s_1}^{-1}\,u_{1\,s_0}\in GA_M(K_{s_0 s_1})$. According to Corollary (2.12) we can write $u=v_{0\,s_1}\,w\,v_{1\,s_0}^{-1}$ with $v_0\in GA'_M(K_{s_0})$, $w\in Af_M(K_{s_0 s_1})$, and $v_1\in GA_M(K_{s_1})$. Let $w_i=u_i\,v_i\colon B_{s_i}\to A_{s_i}$ $(i=0, 1)$. Then

$$w_{0\,s_1}^{-1}\,w_{1\,s_0}=v_{0\,s_1}^{-1}\,u\,v_{1\,s_0}=w\in Af_M(K_{s_0 s_1}).$$

Now $Af_P(L)$ is the group of automorphisms of $S(L\otimes_K M)$ preserving the ascending filtration of this graded algebra. It follows that A admits an ascending filtration, $K=A^{(0)}\subset A^{(1)}\subset\cdots$, so that w_0 and w_1 are isomorphisms of K_{s_i}-algebras with ascending filtrations. Put $N=A^{(1)}/A^{(0)}=A^{(1)}/K$. The sequence

$$0\to K\to A^{(1)}\to N\to 0$$

splits over K_{s_i} $(i=0, 1)$, so we can identify $A^{(1)}$ with $K\oplus N$. Let $t\colon S(N)\to A$ be the K-algebra homomorphism induced by the inclusion of N in A. Then it is clear that $w_i^{-1}\circ t_{s_i}$ is an isomorphism $(i=0, 1)$, whence t is an isomorphism. This proves Theorem (4.4).

(4.8) *Invertible Algebras* (see [BW] and [C]). For any K-algebra L and integer $n\geqq 0$ let $L^{[n]}$ denote the polynomial algebra $L[X_1, \ldots, X_n]$ in n variables. Call an L-algebra A invertible if $A\otimes_L B\cong L^{[n]}$ for some L-algebra B and some $n\geqq 0$. Note then that the augmentation $L^{[n]}\to L$ (sending X_i to 0) induces augmentations of A and of B. By augmenting B in $A\otimes_L B$ we see then that A is a retract of $L^{[n]}$, and that A is a finitely presented L-algebra and a projective K-module. If $\bar{A}$ is the augmentation ideal of A, one sees easily that the $\bar{A}$-adic filtration on A is absolutely separating and of finite type, in the sense of (1.13).

Consider an L-algebra C with augmentation $\varepsilon\colon C\to L$. By a *tensor decomposition* of C we understand a pair (α, β) of endomorphisms of the augmented L-algebra C such that $\alpha^2=\alpha$, $\beta^2=\beta$, $\alpha\beta=\beta\alpha=\varepsilon$ (viewing ε as an endomorphism of C), and such that the homomorphism $\alpha C\otimes_L \beta C\to C$ induced by the inclusions of αC and βC in C is an isomorphism. Thus, to say that an L-algebra A is invertible is to say that $A \cong \alpha C$ for some tensor decomposition (α, β) of some $C = L^{[n]}$.

Suppose S is a multiplicative set in L and that A is a finitely presented L-algebra such that A_S is an invertible L_S-algebra. Thus there is an $n \geqq 0$, a tensor decomposition (α, β) of C_S, where $C = L^{[n]}$, and an L_S-algebra isomorphism $u: A_S \to \alpha C_S$. Let ε denote the standard augmentation of C. By Lemma (1.10) there is an $s \in S$ and endomorphisms α', β' of the augmented L_s-algebra C_s such that $\alpha'_S = \alpha$ and $\beta'_S = \beta$. Replacing s by $s s'$ for some $s' \in S$, if necessary, we can further achieve that $\alpha'^2 = \alpha'$, $\beta'^2 = \beta'$, and $\alpha' \beta' = \beta' \alpha' = \varepsilon_s$. Let $f: \alpha' C_s \otimes_{L_s} \beta' C_s \to C_s$ be the homomorphism induced by the inclusions. Since f_S is an isomorphism, it follows that, f_t is an isomorphism of L_{st}-algebras for some $t \in S$[3]. In the same way now, the isomorphism $u: A_S \to \alpha\, C_S$ lifts to an isomorphism $v: A_{stt'} \to \alpha'_{tt'}\, C_{stt'}$ for some $t' \in S$. It follows therefore that the $L_{stt'}$-algebra $A_{stt'}$ is already invertible.

(4.9) **Theorem.** *Let A be a finitely presented K-algebra. If $A_{\mathfrak{m}}$ is an invertible $K_{\mathfrak{m}}$-algebra for all maximal ideals $\mathfrak{m}$ of K then A is an invertible K-algebra.*

The theorem follows from Quillen induction applied to the following proposition, where $L \in \mathrm{Loc}(K)$.

$P(L)$: $L \otimes_K A$ is an invertible L- algebra.

Of the four conditions in (3.1) to be checked, 1 is immediate, 3 is our hypothesis, and 2 was just verified in (4.8) above. For 4 it suffices to prove that $P(K_{s_0})$ and $P(K_{s_1})$ imply $P(K)$ if $s_0, s_1 \in K$ generate the unit ideal. Let $n \geqq 0$ and the K_{s_i}-algebra B_i be such that $A_{s_i} \otimes_{K_{s_i}} B_i \cong K_{s_i}^{[n_i]}$ $(i = 0, 1)$. Tensoring one of the B_i with a polynomial algebra we can arrange that $n_0 = n_1$; call this n, and put $C_i = B_i \otimes_{K_{s_i}} K_{s_i}^{[n]}$, so that

$$A_{s_i} \otimes_{K_{s_i}} C_i \cong K_{s_i}^{[2n]} \qquad (i = 0, 1).$$

Writing $\otimes$ for $\otimes_{K_{s_0 s_1}}$, we now have $K_{s_0 s_1}$-algebra isomorphisms

$$\begin{aligned} C_{0\, s_1} &\cong B_{0\, s_1} \otimes K_{s_0 s_1}^{[n]} \\ &\cong B_{0\, s_1} \otimes A_{s_0 s_1} \otimes B_{1\, s_0} \\ &\cong K_{s_0 s_1}^{[n]} \otimes B_{1\, s_0} \cong C_{1\, s_0}. \end{aligned}$$

Use such an isomorphism u: $C_{0\, s_1} \to C_{1\, s_0}$ to form the K-algebra C in the fibre product diagram.

$$\begin{array}{ccccc} C & & \longrightarrow & & C_1 \\ \downarrow & & & & \downarrow \\ C_0 & \longrightarrow & C_{0\, s_1} & \xrightarrow{u} & C_{1\, s_0} \end{array}$$

so that $C_{s_i} \cong C_i$ $(i = 0, 1)$. Put $D = A \otimes_K C$. Then $D_{s_i} \cong K_{s_i}^{[2n]}$ $(i = 0, 1)$. It follows therefore from Theorem (4.4) that $D \cong S(P)$ for some finitely generated projective

[3] To apply Lemma (1.10) we use the fact that a retract A of a finitely presented commutative K-algebra C is finitely presented. We must show that the kernel J of the retraction $C \to A$ is a finitely generated ideal. This follows since C is (obviously) a finitely generated A-algebra, hence generated by a finite subset X of J, and then X clearly generates J as an ideal

K-module P. We have $P \oplus Q \cong K^m$ for some Q and $m \geqq 0$, so $S(P) \otimes_K S(Q) \cong S(P \oplus Q) \cong K^{[m]}$. If follows that D, hence also A, is invertible, thus proving Theorem (4.9).

(4.10) *Stable Isomorphism.* Let A be a K-algebra with augmentation ε; write $\bar{A} = \operatorname{Ker}(\varepsilon)$ for the augmentation ideal, and let JA denote the K-module $\bar{A}/\bar{A}^2$. If B is another augmented K-algebra then so also is $A \otimes_K B$, and there is a canonical isomorphism $JA \oplus JB \to J(A \otimes_K B)$ of K-modules (c.f. [C] or [W]). We say A and B are *stably isomorphic* if $A \otimes_K K^{[n]}$ and $B \otimes_K K^{[n]}$ are isomorphic (augmented) K-algebras for some $n \geqq 0$. Analogously, K-modules M and N are said to be stably isomorphic if $M \oplus K^n \cong N \oplus K^n$ for some $n \geqq 0$.

(4.11) **Corollary.** *Let A and B be invertible augmented K-algebras. Suppose that JA and JB are stably isomorphic K-modules, and that $A_{\mathfrak{m}}$ and $B_{\mathfrak{m}}$ are stably isomorphic augmented $K_{\mathfrak{m}}$-algebras for all maximal ideals $\mathfrak{m}$ of K. Then A and B are stably isomorphic augmented K-algebras.*

Since A and B are finitely presented K-algebras, an isomorphism of $A \otimes_K K^{[n]}$ with $B \otimes_K K^{[n]}$ over $K_{\mathfrak{m}}$ lifts to an isomorphism over K_s for some $s \notin m$, by Lemma (1.15)(c). Since $\operatorname{spec}(K)$ is quasi-compact it follows that we can find a single $n \geqq 0$ such that $A' = A \otimes_K K^{[n]}$ and $B' = B \otimes_K K^{[n]}$ are isomorphic over $K_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of K. Choose a K-algebra B'' so that $B' \otimes_K B'' \cong K^{[m]}$ for some $m \geqq 0$. Then $A' \otimes_K B''$ is locally a polynomial algebra, hence (Theorem (4.4)) isomorphic to $S(P)$ for some finitely generated projective module P. We have $P \cong JS(P) \cong JA \oplus K^n \oplus JB''$, and $K^m \cong JK^{[m]} \cong JB \oplus K^n \oplus JB''$. Since JA and JB are stably isomorphic it follows that P and K^m are likewise, say $P \oplus K^q \cong K^{m+q}$. Putting $C = K^{[n]} \otimes_K B'' \otimes_K K^{[q]}$ we then have

$$A \otimes_K C \cong S(P) \otimes_K K^{[q]} \cong K^{[m+q]} \cong B \otimes_K C.$$

Since C is invertible this proves that A and B are stably isomorphic, as claimed.

(4.12) *Remark.* With the K-theoretic notation of [BW], Corollary (4.11) is equivalent to the injectivity of the canonical homomorphism

$$KA'_0(K) \to \prod_{\mathfrak{m}} KA'_0(K_{\mathfrak{m}}).$$

(4.13) **Theorem.** *Let $H = H_0 \oplus H_1 \oplus \cdots$ be a graded K-algebra with $H_0 = K$. Let $\varepsilon\colon H \to K$ be the retraction with kernel $H_+ = H_1 \oplus H_2 \oplus \cdots$. Let A be a finitely presented H-algebra. Let $A_0 = K \otimes_H A = A/H_+ A$ and $B = H \otimes_K A_0$. If $A_{\mathfrak{m}}$ and $B_{\mathfrak{m}}$ are isomorphic $H_{\mathfrak{m}}$-algebras for all maximal ideals $\mathfrak{m}$ of K then A and B are isomorphic H-algebras.*

The theorem follows from Quillen induction (3.1) applied to the following proposition, where $L \in \operatorname{Loc}(K)$.

$P(L)$: $L \otimes_K A$ and $L \otimes_K B$ are isomorphic $(L \otimes_K H)$-algebras. Of the four properties in (3.1) to be checked, 1 is immediate, and 3 is our hypothesis. Since A and B are finitely presented H-algebras 2 follows from Lemma (1.10). For 4 it suffices to prove that $P(K_{s_0})$ and $P(K_{s_1})$ imply $P(K)$ if $s_0, s_1 \in K$ generate the unit ideal. Suppose then that we are given H_{s_i}-algebra isomorphisms u_i: $B_{s_i} \to A_{s_i}$ $(i = 0, 1)$.

We can canonically identify $K_{s_i} \otimes_{H_{s_i}} A_{s_i}$ with $K_{s_i} \otimes_{H_{s_i}} B_{s_i}$, and so identify

$$^0u_i = 1_{K_{s_i}} \otimes_{H_{s_i}} u_i$$

with an automorphism of $K_{s_i} \otimes_{H_s} B_{s_i}$. This gives an automorphism

$$1_{H_{s_i}} \otimes_{K_{s_i}} {}^0u_i \quad \text{of } H_{s_i} \otimes_{K_{s_i}} (K_{s_i} \otimes_{H_{s_i}} B_{s_i}) = B_{s_i}.$$

Replacing u_i by $u_i \circ (1_{H_{s_i}} \otimes_{K_{s_i}} {}^0u_i)^{-1}$, we can then arrange that ^0u_i equals the identity. For any K-algebra L let $G(L)$ denote the group of $(L \otimes_K H)$-algebra automorphisms of $L \otimes_K B$, and let $G_0(L)$ denote its subgroup consisting of automorphisms inducing the identity modulo $(L \otimes_K H_+) \cdot (L \otimes_K B)$. We have the element $u = u_{0\,s_1}^{-1} u_{1\,s_0} \in G_0(K_{s_0 s_1})$. By Corollary (2.7) we can write $u = v_{0\,s_1} v_{1\,s_0}^{-1}$ with $v_i \in G_0(K_{s_i})$ $(i=0, 1)$. Put $w_i = u_i v_i$; $B_{s_i} \to A_{s_i}$. Then $w_{0\,s_1}^{-1} w_{1\,s_0} = v_{0\,s_1}^{-1} u v_{1\,s_0} = 1_{B_{s_0 s_1}}$, i.e. $w_{0\,s_1} = w_{1\,s_0}$. Thus w_0 and w_1 patch to form an H-algebra isomorphism w: $B \to A$ such that $w_{s_i} = w_i$ $(i=0, 1)$. This proves Theorem (4.13).

(4.14) **Theorem.** *Let H be as in Theorem* (4.13). *Let A be a (not necessarily commutative) K-algebra. Let M be a finitely presented left $(H \otimes_K A)$-module. Let M_0 denote the A-module $K \otimes_H M = M/H_+ M$, and N the $(H \otimes_K A)$-module $H \otimes_K M_0$. If $M_{\mathfrak{m}}$ and $N_{\mathfrak{m}}$ are isomorphic $(H_{\mathfrak{m}} \otimes_K A)$-modules for all maximal ideals $\mathfrak{m}$ of K, then M and N are isomorphic $(H \otimes_K A)$-modules.*

For $L \in \mathrm{Loc}(K)$ consider the proposition,

$P(L)$: $L \otimes_K M$ and $L \otimes_K N$ are isomorphic $(L \otimes_K H \otimes_K A)$-modules.

This is proved by Quillen induction, exactly following the lines of the proof of Theorem (4.13). In verifying (3.1)2 (finiteness) one uses the fact that, for S a multiplicative set in K,

$$\varinjlim_{s \in S} \mathrm{Hom}_{H_s \otimes A}(M_s, N_s) \to \mathrm{Hom}_{H_S \otimes A}(M_S, N_S)$$

is bijective, because M is finitely presented, and similarly with M and N reversed. In verifying 4 one considers the functor G attaching to each K-algebra L the group $G(L)$ of $(L \otimes_K A)$-automorphisms of $L \otimes_K M_0$. Then G satisfies axiom Q (Corollary (1.7)) so we can apply Corollary (2.7) to the group $G(L \otimes_K H)$ of $(L \otimes_K H \otimes_K A)$-automorphisms of $L \otimes_K H \otimes_K M_0 = L \otimes_K N$, to conclude that, with the notation of (2.7),

$$G(L_{s_0 s_1} \otimes H_+) = G(L_{s_0} \otimes H_+)_{s_1} \cdot G(L_{s_1} \otimes H_+)_{s_0}$$

whenever $Ls_0 + Ls_1 = L$. This is precisely the information necessary to carry out the patching argument in verifying the sheaf condition 4 of (3.1).

(4.15) *Remarks.* 1. Theorems (4.13) and (4.14) are of particular interest when H is the graded polynomial algebra, $K[T]$. For example (4.13) then says that if A and $A_0[T]$ are isomorphic algebras over $K_{\mathfrak{m}}[T]$ for all maximal ideals $\mathfrak{m}$ of K then they are isomorphic over $K[T]$. Here A is finitely presented $K[T]$-algebra and $A_0 = A/TA$.

2. If $A = K$ in Theorem (4.14) then Theorem (4.14) follows by applying Theorem (4.13) to the symmetric algebra $S(M)$ over H. (One uses Lemma (4.6) for this.)

3. In case $A=K$, and $H=K[T]$ then Theorem (4.14) becomes Quillen's localization theorem ([Q] Th. 1), which was the inspiration of everything above.

4. The analogue of Theorem (4.14) for modules equipped with some quadratic structure (e.g. a hermitian form when A is equipped with an involution) is also valid. The proof is similar. One uses Remark (1.8) to verify axiom Q for the functor G that intervenes.

5. The weary reader will have noted an evident repetitiveness in the proofs above. One method of axiomatizing them goes as follows. Let $\mathscr{C}(L)$ denote a category attached to a K-algebra L, with base change functors $A \mapsto L' \otimes_L A$ from $\mathscr{C}(L)$ to $\mathscr{C}(L')$ for each K-algebra homomorphism $L \to L'$. We require that $\mathscr{C}$ be "localizable" in the sense that it satisfies properties (F) and (Sh) below.

(F). If S is a multiplicative set in a K-algebra L, and if $A, B \in \mathscr{C}(L)$, then the canonical map

$$\varinjlim_{s\in S} \mathrm{Hom}_{\mathscr{C}(L_s)}(A_s, B_s) \to \mathrm{Hom}_{\mathscr{C}(L_S)}(A_S, B_S)$$

is bijective.

(Sh). If L is a K-algebra, $s_0, s_1 \in L$ generate the unit ideal, and $A, B \in \mathscr{C}(L)$, then the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}(L)}(A, B) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}(L_{s_1})}(A_{s_1}, B_{s_1}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{C}(L_{s_0})}(A_{s_0}, B_{s_0}) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}(L_{s_0 s_1})}(A_{s_0 s_1}, B_{s_0 s_1}) \end{array}$$

is cartesian.

Now let $L \mapsto \mathscr{C}_0(L)$ be a second such category, and suppose we are given functors

$$\mathscr{C}_0(L) \underset{\gamma}{\overset{\varepsilon}{\rightleftarrows}} \mathscr{C}(L)$$

that commute with base change and such that $\gamma \circ \varepsilon$ is naturally isomorphic to the identity functor of $\mathscr{C}_0$.

Let $A \in \mathscr{C}(K)$, and put $B = \varepsilon \gamma A$. We wish to prove that $A \cong B$ in $\mathscr{C}(K)$ provided that $A_{\mathfrak{m}} \cong B_{\mathfrak{m}}$ in $\mathscr{C}(K_{\mathfrak{m}})$ for all maximal ideals $\mathfrak{m}$ of K. We apply Quillen induction to the following proposition, where $L \in \mathrm{Loc}(K)$.

$$P(L)\colon L \otimes_K A \cong L \otimes_K B \text{ in } \mathscr{C}(L).$$

Of the four conditions in (3.1) to be verified, 1 is clear, 3 is our hypothesis, and 2 follows, as in the proofs above, from property (F) of $\mathscr{C}$. Condition (Sh) on $\mathscr{C}$ is used in trying to establish 4. For any K-algebra L, put

$$G(L) = \mathrm{Aut}_{\mathscr{C}(L)}(L \otimes_K B),$$

and

$$G_0(L) = \mathrm{Ker}\,(G(L) \xrightarrow{\ \gamma\ } \mathrm{Aut}_{\mathscr{C}_0(L)}(\gamma(L \otimes_K B))).$$

Then, as in the proofs above, condition 4 of (3.1) can be established, so completing the proof, provided we know that: With L, s_0, s_1 as in $(S\,h)$ above, we have

$$G_0(L_{s_0 s_1}) = G_0(L_{s_0})_{s_1} \cdot G_0(L_{s_1})_{s_0}.$$

The example corresponding to Theorem (4.1) is where $\mathscr{C}(L)$ is the category of finitely presented L-algebras A with absolutely separating filtrations of finite type such that $\gamma A = gr(A)$ is also finitely presented, where $\mathscr{C}_0(L)$ is the category of finitely presented graded L-algebras $B = B_0 \oplus B_1 \oplus \cdots$, and where $\varepsilon B \in \mathscr{C}(L)$ is B equipped with the filtration by the $B_{(n)} = B_n \oplus B_{n+1} \oplus \cdots$ for $n \geqq 0$.

In Theorems (4.13) and (4.14) we have $\mathscr{C}(L) = \mathscr{C}_0(L \otimes_K H)$, and ε and γ are induced by the inclusion $L \to L \otimes_K H$ and the augmentation $L \otimes_K H \to L$, respectively. In (4.13) $\mathscr{C}_0(L)$ is the category of finitely presented L-algebras. In (4.14) $\mathscr{C}_0(L)$ is the category of finitely presented left $(L \otimes_K A)$-modules.

References

[BCW] Bass, H., Connell, E.H., Wright, D.L.: Locally polynomial algebras are symmetric. Bull. AMS **82**, 719-720 (1976)

[BW] Bass, H., Wright, D.L.: Localization in the K-theory of invertible algebras. J. Pure and Appl. Algebra **9**, 89-105 (1976)

[C] Connell, E.H.: A K-theory for the category of projective algebras. J. Pure and Appl. Algebra **5**, 281-292 (1974)

[EH] Eakin, P., Heinzer, W.: A cancellation problem for rings. Conf. on Comm. Alg., Lawrence, Kansas (1972), Lecture Notes in Math. **311**, pp. 61-77. Berlin-Heidelberg-New York: Springer 1973

[ES] Eakin, P., Silver, J.: Rings which are almost polynomial rings. Trans. AMS **174**, 425-449 (1972)

[Q] Quillen, D.: Projective modules over polynomial rings. Inventiones math. **36**, 167-171 (1976)

[W] Wright, D.: Algebras which resemble symmetric algebras. Thesis, Columbia University, 1975

Received November 2, 1976