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# Triangulation of Subanalytic Sets and Proper Light Subanalytic Maps

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## Introduction

The smallest nonempty class $\mathscr{A}$ of subsets of a real analytic space $M$ which is closed under the formation of locally finite unions, intersections, complements, and connected components and which contains $A \cap g^{-1}\{0\}$ for any $A$ in $\mathscr{A}$ and real-valued function $g$ analytic in a neighborhood of Clos $A$ is the class of *semianalytic subsets* of $M$ (see [6, § 1] for a list of references, the most basic being [11]). The smallest such class also containing all proper real analytic images of semianalytic sets is the (strictly larger, for dim $M \geqq 3$) class of *subanalytic subsets* of $M$ ([6, 8, 14, 15]).

A continuous function from a subset of real analytic space $M$ into a real analytic space $N$ is called a *subanalytic map* if its graph is a subanalytic subset of $M \times N$. Here triangulations of subanalytic sets are constructed using only subanalytic maps throughout. A map is *proper* (respectively, *light*) if its inverse image preserves compact (respectively, discrete) sets. Our main results are:

**Theorem 2.** *For any locally finite family $\mathscr{R}$ of closed subanalytic sets in $\mathbb{R}^n$, there exists a simplicial decomposition $\mathscr{F}$ of $\mathbb{R}^n$ and a subanalytic map $f$: $[0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ such that $f_t = f(t, \cdot)$ is a homeomorphism for each $t \in [0, 1]$, $f_0$ is the identity, and $f_1^{-1}(R)$ is a subcomplex of $\mathscr{F}$ for every $R \in \mathscr{R}$.*

**Theorem 3.** *If $P$ is a closed finite-dimensional subanalytic set, $f$: $P \to N$ is a proper light subanalytic map, and $\mathscr{Q}$ and $\mathscr{R}$ are locally finite families of closed subanalytic sets in $P$ and $f(P)$, respectively, then $f = h \circ p \circ g^{-1}$ for some simplicial map $p$ between simplicial complexes $\mathscr{G}$ and $\mathscr{H}$ and homeomorphisms $g$: $\bigcup \mathscr{G} \to P$ and $h$: $\bigcup \mathscr{H} \to f(P)$ such that $g^{-1}(Q)$ is a subcomplex of $\mathscr{G}$ for every $Q \in \mathscr{Q}$ and $h^{-1}(R)$ is a subcomplex of $\mathscr{H}$ for every $R \in \mathscr{R}$.*

An arbitrary proper real analytic map, as Hironaka has kindly pointed out, may not admit a triangulation in the sense of Theorem 3. For example, in

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any triangulation of the map,

$$f\colon \mathbb{R}\times\mathbb{S}^1 \to \mathbb{R}^2, \quad f(r,\xi)=r\cdot\xi \quad \text{for } (r,\xi)\in\mathbb{R}\times\mathbb{S}^1,$$

at least one 1-simplex but no 2-simplex in the triangulation of $\mathbb{R}\times\mathbb{S}^1$ would have to be collapsed.

For semianalytic sets, use of the Weierstrass Preparation Theorem and elimination theory as in [11, pp. 150–153] has classically provided partitions $\mathscr{S}$, called semianalytic stratifications, into semianalytic, real analytic submanifolds such that if $R$ and $S$ are distinct sets in $\mathscr{S}$ and $R\cap\operatorname{Clos} S$ is nonempty, then $R$ is contained in $\operatorname{Clos} S$ and $\dim R$ is strictly less than $\dim S$. The triangulability of semianalytic sets was not fully established until ten years ago when it was proven by Lojasiewicz in [10]. Independently Giesecke obtained simplicial decompositions in [5]. Recently it was shown in [8] (first) and in [6] that subanalytic sets admit stratifications into subanalytic, real analytic submanifolds. The referee has also noted the earlier work of Gabriélov, [14] and [15], treating the Whitney stratification of subanalytic sets; the author apologizes for being unaware of and not mentioning Gabriélov's work previously. While the present paper was being written, Hironaka obtained subanalytic triangulations of subanalytic sets in his interesting Arcata lectures [9]. Triangulations of Whitney stratified sets have been treated in the theses of Johnson [16], Hendricks [17], and Ullman [18]. (See [9] for references to older triangulation literature.) Because of differences of proofs and our construction of an isotopy of the ambient Euclidean space, the present paper may be of independent interest. The conjecture of [13, p. 167] on the triangulability of the mapping cylinder of a proper analytic map follows from [9, 3.7] or Theorem 2, the Embedding Lemma, and [7, 4.8].

The stratification of subanalytic sets is here used to construct the isotopy of Theorem 2 and the triangulations of Theorem 3 in roughly the following six steps:

I. In case the union of the sets in $\mathscr{R}$ is bounded, choose stratifications $\mathscr{S}_i$ in $\mathbb{R}^i$ and orthogonal projections $p_i\colon \mathbb{R}^i\to\mathbb{R}^{i-1}$ for $i\in\{n, n-1, \ldots, 1\}$ so that $\mathscr{S}_n$ stratifies each $R$ in $\mathscr{R}$, $\mathscr{S}_{i-1}$ stratifies $p_i(S)$, and $p_i|\operatorname{Tan}(S,x)$ is injective whenever $x\in S\in\mathscr{S}_i$ and $\dim S<i$.

II. With $\mathscr{C}_0=\{\mathbb{R}^0\}$, the family $\mathscr{C}_i$ of components of sets $p_i^{-1}(C)\cap S$ for $C\in\mathscr{C}_{i-1}$ and $S\in\mathscr{S}_i$ forms, as in [7, 3.1], a $CW$ decomposition for $i\in\{1, 2, \ldots, n\}$.

III. An analogue of barycentric subdivision of each $\mathscr{C}_i$ provides in Theorem 1 the desired triangulation for the bounded case.

IV. Inasmuch as this triangulation maps any set in $\mathscr{R}$ which is already a simplex onto itself, triangulations and homeomorphisms for the unbounded case may, in Theorem 2, be built up from those of the bounded case.

V. Since there exist subanalytic partitions of unity, any $n$-dimensional paracompact real analytic space may, by well-known arguments in our Embedding Lemma, be mapped into $\mathbb{R}^{2n+1}$ by a proper subanalytic homeomorphism. (Thus even for analytic manifolds, the analytic embedding theorem of [4, Theorem 3] is not needed here because the nonsingularity of the subanalytic embedding is not required.)

VI. Theorem 3 follows from the argument of III because the triangulations of Theorem 1 commute with the projections $p_i$.

All of our proofs and results carry over for any collection $\mathcal{M}$ of manifolds and class $\mathcal{A}$ of subsets satisfying the axioms of [6, 2.2], in particular, for the class of real semialgebraic subsets of paracompact real algebraic manifolds. An elementary proof of the triangulability of semianalytic sets not involving the subanalytic stratification theory of [6] or [8] results by using generic projections in Theorem 1 in the manner of [7, 3.2, 3.3].

An important remaining question is whether two subanalytically (respectively, semialgebraically, semianalytically) homeomorphic simplicial complexes are $PL$ isomorphic. Thus, even though by [9, 3.7] or Theorem 2, there exists, for any two subanalytic (respectively, semialgebraic, semianalytic) triangulations

$$f: \bigcup \mathcal{F} \to A \quad \text{and} \quad g: \bigcup \mathcal{G} \to A$$

of a subanalytic (respectively, semialgebraic, semianalytic) set $A$, a third triangulation $h: \bigcup \mathcal{H} \to A$ so that each image $h(H)$ of a simplex $H \in \mathcal{H}$ is contained in $f(F) \cap g(G)$ for some simplices $F \in \mathcal{F}$ and $G \in \mathcal{G}$, it is not obvious that $\mathcal{F}$ and $\mathcal{G}$ are $PL$ isomorphic (see the interesting discussion and example in [12]). The question reduces to the weakened $PL$ Schoenfleiss problem of whether an $n$-dimensional subanalytic (respectively, semialgebraic, semianalytic) closed ball in $\mathbb{R}^n$ which is bounded by a locally flat $PL$ sphere is a $PL$ ball. The latter is known for all positive integers $n \neq 4$ [19, 3.37].

**Preliminaries.** (Our notations follow those of [3, pp. 669–671].)

For any family $\mathcal{A}$ of sets, $\bigcup \mathcal{A}$ is the set $\bigcup_{A \in \mathcal{A}} A$.

For any two families $\mathcal{B}$ and $\mathcal{C}$ of subsets of the same set, $\mathcal{B}$ is said to be *compatible with* $\mathcal{C}$ if

$$B \subset C \quad \textit{whenever } B \in \mathcal{B},\ C \in \mathcal{C}, \text{ and } B \cap C \neq \emptyset.$$

For any subset $D$ of a topological space, the *frontier of* $D$, Fron $D$, is the set $(\operatorname{Clos} D) \sim D$.

For any subset $E$ of a real analytic manifold, a *subanalytic stratification* $\mathcal{S}$ *of* $E$ is a locally finite partition of $E$ into subanalytic, real analytic (properly embedded) submanifolds such that

$$R \subset \operatorname{Fron} S \text{ and } \dim R < \dim S \quad \text{whenever } R \in \mathcal{S},\ S \in \mathcal{S}, \text{ and } R \cap \operatorname{Fron} S \neq \emptyset.$$

For any stratification $\mathcal{S}$ of a closed subset of a Euclidean space such that each member of $\mathcal{S}$ is bounded and convex, the family

$$\mathcal{T} = \{\operatorname{Clos} S \colon S \in \mathcal{S}\}$$

is called a *convex cell complex* or *convex cell decomposition of* $\bigcup \mathcal{S}$. Then, for any $T = \operatorname{Clos} S$ in $\mathcal{T}$. ,

$$V_T = \{v \colon v \in T \text{ and } \{v\} \in \mathcal{T}\} \quad \text{is finite,}$$

$$T = \{\sum_{v \in V_T} t_v v \colon t_v \geqq 0 \text{ for } v \in V_T \text{ and } \sum_{v \in V_T} t_v = 1\},$$

and

$$\mathring{T} = \{\sum_{v \in V_T} t_v v \colon t_v > 0 \text{ for } v \in V_T \text{ and } \sum_{v \in V_T} t_v = 1\} = S.$$

For any member $T$ of a convex cell complex, the point

$$b_T = \sum_{v \in V_T} (\text{card } V_T)^{-1} v$$

in $T$ is called the *barycenter of* $T$.

Any member $T$ of a convex cell complex for which card $V_T = 1 + \dim T$ is called a *simplex*. For a point $x$ in a simplex $T$, the representation $x = \sum_{v \in V_T} t_v v$ is unique. A convex cell complex $\mathscr{T}$ consisting entirely of simplices is called a *simplicial complex* or *simplicial decomposition* of $\bigcup \mathscr{T}$.

A simplicial complex $\mathscr{G}$ is a *simplicial subdivision* of a convex cell complex $\mathscr{H}$ if $\bigcup \mathscr{G} = \bigcup \mathscr{H}$ and $\{\mathring{G}: G \in \mathscr{G}\}$ is compatible with $\{\mathring{H}: H \in \mathscr{H}\}$. By [9, 1.1], there exist a simplicial subdivision $\mathscr{G}$ of $\mathscr{H}$ with $\bigcup_{G \in \mathscr{G}} V_G = \bigcup_{H \in \mathscr{H}} V_H$. Another simplicial subdivision of $\mathscr{H}$ is the *first barycentric subdivision of* $\mathscr{H}$ consisting of the convex hulls of $\{b_{H_0}, b_{H_1}, \ldots, b_{H_k}\}$ corresponding to finite sequences $H_0, H_1, \ldots, H_k$ in $\mathscr{H}$ with

$$H_0 \supsetneqq H_1 \supsetneqq \cdots \supsetneqq H_k.$$

**Theorem 1.** *(Triangulation of bounded subanalytic sets in* $\mathbb{R}^n$.*) If*

$$\mathscr{S}_0 = \mathscr{C}_0 = \mathscr{D}_0 = \{\mathbb{R}^0\}, \quad b(\mathbb{R}^0) = 0 \in \mathbb{R}^0, \quad h_0 = 0: \mathbb{R}^0 \to \mathbb{R}^0.$$

*$K_n$ is the union of a finite simplicial complex in $\mathbb{R}^n$, and, for each*

$$i \in \{1, 2, \ldots, n\}, \quad p_i: \mathbb{R}^i \to \mathbb{R}^{i-1}$$

*is an orthogonal projection, $K_{i-1} = p_i(K_i)$, $\mathscr{S}_i$ is a subanalytic stratification of $K_i$, $\mathscr{S}_{i-1}$ is compatible with $\{p_i(S)\}$ and $p_i|\mathrm{Tan}\,(S, x)$ is injective whenever $x \in S \in \mathscr{S}_i$ and* $\dim S < i$, *and*

$$\mathscr{C}_i = \{\textit{components of } p_i^{-1}(C) \cap S: C \in \mathscr{C}_{i-1} \textit{ and } S \in \mathscr{S}_i\},$$

*then*

(1) *$\mathscr{C}_i$ is a subanalytic CW decomposition* ([2,V, 2.1]) *of $K_i$,*

(2) *each $C \in \mathscr{C}_i$ contains a point $b(C)$ such that $b[p_i(C)] = p_i[b(C)]$ and the collection $\mathscr{D}_i$ of convex hulls $D(C_0, \ldots, C_k)$ of $\{b(C_0), \ldots, b(C_k)\}$ corresponding to finite sequences $C_0, \ldots, C_k$ in $\mathscr{C}$ with*

$$\text{Fron } C_0 \supset C_1, \text{ Fron } C_1 \supset C_2, \ldots, \text{Fron } C_{k-1} \supset C_k$$

*is a simplicial decomposition of $K_i$,*

(3) *there exists a subanalytic map $h_i: K_i \to K_i$ such that*

(4) *$(1-t)\mathbb{1}_{K_i} + t h_i$ is a homeomorphism for each $t \in [0, 1]$,*

(5) *$p_i \circ h_i = h_{i-1} \circ (p_i|K_i)$,*

(6) *$h_i[b(C)] = b(C)$ for each $C \in \mathscr{C}_i$,*

(7) *$h_i(T) = T$ for each member $T$ of any simplicial complex in $K_i$ with which $\mathscr{S}_i$ is compatible,*

(8) *$h_i|\mathring{D}$ is an analytic isomorphism for each $D \in \mathscr{D}$, and*

(9) *$\{h_i(\mathring{D}): D \in \mathscr{D}_i\}$ is compatible with $\mathscr{C}_i$ and hence with $\mathscr{S}_i$.*

*Proof.* We use induction on $n$. Since $\mathscr{S}_1$ consists of singleton sets and open intervals, the case $n=1$ is easily treated. We now assume conclusions (1) through (9) hold for $i=n-1$ and abbreviate $K=K_n$, $p=p_n$, $\mathscr{C}=\mathscr{C}_n$, and $\mathscr{D}=\mathscr{D}_n$.

As in [7, 3.1], we divide $\mathscr{C}$ into the two families

$$\mathscr{A}=\{\text{components of } p^{-1}(C)\cap S\colon\ C\in\mathscr{C}_{n-1},\ S\in\mathscr{S}_n, \text{ and } \dim S<n\},$$
$$\mathscr{B}=\{\text{components of } p^{-1}(C)\cap S\colon\ C\in\mathscr{C}_{n-1},\ S\in\mathscr{S}_n, \text{ and } \dim S=n\},$$

and observe that the argument of [7, 3.1] establishes (1) and shows that, for each $A\in\mathscr{A}$,

$p(A)\in\mathscr{C}_{n-1}$ and $p|\text{Clos } A$ is a homeomorphism, and, for each $B\in\mathscr{B}$, the two sets,

$$B^-=p^{-1}[p(B)]\cap\{(x_1,\ldots,x_n\}\colon x_n=\inf\{z\colon(x_1,\ldots,x_{n-1},z)\in B\}\},$$
$$B^+=p^{-1}[p(B)]\cap\{(x_1,\ldots,x_n)\colon x_n=\sup\{z\colon(x_1,\ldots,x_{n-1},z)\in B\}\},$$

belong to $\mathscr{A}$ and

$$B=\{(1-t)\,x+t\,y\colon 0<t<1,\ x\in B^-,\ y\in B^+, \text{ and } p(x)=p(y)\}.$$

For each sequence $C_0,\ldots,C_k$ as in (2), there exists, therefore, an integer $j(C_0,\ldots,C_k)$ in $\{0,1,\ldots,k+1\}$ so that

$$C_j\in\mathscr{B} \quad \text{for } j<j(C_0,\ldots,C_k) \quad \text{and}$$
$$C_j\in\mathscr{A} \quad \text{for } j\geqq j(C_0,\ldots,C_k).$$

Let

$$b(A)=(p|\text{Clos } A)^{-1}(b[p(A)]) \qquad \text{for } A\in\mathscr{A},$$
$$b(B)=\tfrac{1}{2}[b(B^-)+b(B^+)] \qquad \text{for } B\in\mathscr{B},$$

and

$$\mathscr{E}=\mathscr{D}\cap\{D(C_0,\ldots,C_k)\colon 1\leqq j(C_0,\ldots,C_k)\leqq k \ \text{ and } \ C_{j(C_0,\ldots,C_k)}=C^{\pm}_{j(C_0,\ldots,C_k)-1}\}.$$

If $D(C_0,\ldots,C_k)\in\mathscr{D}$, $0\leqq j<l\leqq k$, and $p[b(C_j)]=p[b(C_l)]$, then $p(C_j)=p(C_l)$, $D(C_0,\ldots,C_k)\in\mathscr{E}$, and $j+1=j(C_0,\ldots,C_k)=l$. Thus, for each $D\in\mathscr{D}\sim\mathscr{E}$,

$$p(D)\in\mathscr{D}_{n-1} \quad \text{and} \quad p|\text{Clos } D \quad \text{is a homeomorphism},$$

and, for each $E=D(C_0,\ldots,C_k)\in\mathscr{E}$, the two sets,

$$E^-=D(C_0,\ldots,C_{j(C_0,\ldots,C_k)-2},C_{j(C_0,\ldots,C_k)},\ldots,C_k),$$
$$E^+=D(C_0,\ldots,C_{j(C_0,\ldots,C_k)-1},C_{j(C_0,\ldots,C_k)+1},\ldots,C_k),$$

belong to $\mathscr{D}\sim\mathscr{E}$, and

$$E=\{(1-t)\,x+t\,y\colon 0\leqq t\leqq 1,\ x\in E^-,\ y\in E^+, \text{ and } p(x)=p(y)\}.$$

Applying (2) for $i=n-1$ and the above observations, we conclude that each $D(C_0,\ldots,C_k)$ in $\mathscr{D}$ is a $k$ dimensional simplex, that $\{\mathring{D}\colon D\in\mathscr{D}\}$ is disjointed, and that (2) holds for $i=n$.

Next, to obtain $h_n$, we define $h_D\colon D \to K$ for $D = D(C_0, \ldots, C_k) \in \mathscr{D}$ by induction on $k$ as follows:

I. *For* $C_0 \in \mathscr{A}$, we let $h_D = (p|\mathrm{Clos}\ C_0)^{-1} \circ h_{n-1} \circ (p|D)$.

II. *For* $C_0 \in \mathscr{B}$ *and* $k=0$, we note that $D = D(C_0) = \{b(C_0)\}$ and let $h_D = b(C_0)$.

III. *For* $C_0 \in \mathscr{B}$, $k \geqq 1$, $C_1 \neq C_0^{\pm}$, *and* $h_{D(C_1,\ldots,C_k)}$ *already defined*, we let

$$h_D[b(C_0)] = b(C_0)$$

and for $x \in D \sim \{b(C_0)\}$, choose $0 < r \leqq 1$, $y \in D(C_1, \ldots, C_k)$, and $0 \leqq s \leqq 1$ so that

$$x = (1-r)\, b(C_0) + r\, y,$$

$$h_{D(C_1,\ldots,C_k)}(y) = [(1-s)(p|\mathrm{Clos}\ C_0^-)^{-1} + s(p|\mathrm{Clos}\ C_0^+)^{-1}] \circ h_{n-1} \circ p(y)$$

$$\text{in case } [(p|\mathrm{Clos}\ C_0^-)^{-1} - (p|\mathrm{Clos}\ C_0^+)^{-1}] \circ h_{n-1} \circ p(y) \neq 0,$$

$$s = \tfrac{1}{2} \quad \text{in case } [(p|\mathrm{Clos}\ C_0^-)^{-1} - (p|\mathrm{Clos}\ C_0^+)^{-1}] \circ h_{n-1} \circ p(y) = 0,$$

and let

$$h_D(x) = ([\tfrac{1}{2}(1-r) + r(1-s)]\,(p|\mathrm{Clos}\ C_0^-)^{-1} + [\tfrac{1}{2}(1-r) + r\,s]\,(p|\mathrm{Clos}\ C_0^+)^{-1}) \circ h_{n-1} \circ p(x).$$

IV. *For* $C_0 \in \mathscr{B}$, $k \geqq 1$, $C_1 = C_0^{\pm}$ *and* $h_{D^+}$ *and* $h_{D^-}$ *already defined*, we choose $0 \leqq t \leqq 1$ so that

$$x = [(1-t)(p|D^-)^{-1} + t\,(p|D^+)^{-1}] \circ p(x)$$

and let

$$h_D(x) = [(1-t)\, h_{D^-} \circ (p|D^-)^{-1} + t\, h_{D^+} \circ (p|D^+)^{-1}] \circ p(x).$$

Clearly each function $h_D$, for $D \in \mathscr{D}$, is a continuous subanalytic map whose restriction to $\mathring{D}$ is an analytic isomorphism. To see that $h_n = \bigcup_{D \in \mathscr{D}} h_D$ is a well-defined continuous function we will verify, for each $D = D(C_0, \ldots, C_k) \in \mathscr{D}$ and $j \in \{0, \ldots, k\}$, by induction on $k$, that

$$h_D | E = h_E \quad \text{where } E = D(C_0, \ldots, C_{j-1}, C_{j+1}, \ldots, C_k)$$

in each of the following eight cases.

*Case 1.* $C_0 \in \mathscr{A}$ *and* $j=0$. Here $\mathrm{Clos}\ C_1 \subset \mathrm{Clos}\ C_0$, $C_1 \in \mathscr{A}$, and, by I,

$$h_E = (p|\mathrm{Clos}\ C_1)^{-1} \circ h_{n-1} \circ (p|E) = h_D | E.$$

*Case 2.* $C_0 \in \mathscr{A}$ *and* $j \geqq 1$. Here, by *I*,

$$h_E = (p|\mathrm{Clos}\ C_0)^{-1} \circ h_{n-1} \circ (p|E) = h_D | E.$$

*Case 3.* $C_0 \in \mathscr{B}$, $k \geqq 1$, $C_1 \neq C_0^{\pm}$, *and* $j=0$. Here for $x \in E = D(C_1, \ldots, C_k)$ we choose $r=1$ and $y=x$ in III to compute $h_D(x) = h_E(x)$.

*Case 4.* $C_0 \in \mathscr{B}$, $k=1$, $C_1 \neq C_0^{\pm}$, *and* $j=1$. Here $E = D(C_0)$, and, by II and III, $h_E = b(C_0) = h_D | E$.

*Case 5.* $C_0 \in \mathscr{B}$, $k \geqq 2$, $C_1 \neq C_0^{\pm}$, *and* $j \geqq 1$. Here $C_2 \neq C_0^{\pm}$ and $h_E[b(C_0)] = h_D[b(C_0)]$ by III. For $x \in E \sim \{b(C_0)\}$ and $0 < r \leqq 1$ and $y \in D(C_1, \ldots, C_k)$ chosen as in III, we see that $y \in D(C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_k)$, infer by induction that

$$h_{D(C_1, \ldots, C_k)}(y) = h_{D(C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_k)}(y),$$

and conclude by III that $h_E(x) = h_D(x)$.

*Case 6.* $C_0 \in \mathscr{B}$, $k \geqq 1$, $C_1 = C_0^{\pm}$ *and* $j = 0$. Here $E = D^-$, and, for $x \in E$, we choose $t = 0$ in IV and infer from I and IV that $h_E(x) = h_D(x)$.

*Case 7.* $C_0 \in \mathscr{B}$, $k \geqq 1$, $C_1 = C_0^{\pm}$, *and* $j = 1$. Here $E = D^+$, and, for $x \in E$, we choose $t = 1$ in IV and infer from I and IV that $h_E(s) = h_D(x)$.

*Case 8.* $C_0 \in \mathscr{B}$, $k \geqq 1$, $C_1 = C_0^{\pm}$ *and* $j \geqq 2$. Here $E^- \subset D^-$, $E^+ \subset D^+$, and, by IV, $h_E = h_D | E$.

From the definition of $h$ we now readily obtain conclusions (3), (5), (6), (8), and (9) for $i = n$ and deduce, for each $D \in \mathscr{D} \sim \mathscr{E}$, that $p[h_n(D)] \in \{h_{n-1}(F) \colon F \in \mathscr{D}_{n-1}\}$ that $p | h_n(D)$ is a homeomorphism, and, for each $E \in \mathscr{E}$, that

$$h_n(E) = \{(1-t)x + ty \colon 0 \leqq t \leqq 1, x \in h_n(E^-), y \in h_n(E^+), \text{ and } p(x) = p(y)\}.$$

Thus $\{h_n(\mathring{D}) \colon D \in \mathscr{D}\}$ is a disjointed covering of $K$, and $h_n$, being continuous and bijective, is a homeomorphism.

Conclusion (4) for $i = n$ now follows from (4) for $i = n-1$, (5), and the observation that, by (6), $\mathbf{e}_n \bullet h_n(x_1, \ldots, x_{n-1}, \cdot)$ is increasing whenever $(x_1, \ldots, x_{n-1}) \in K_{n-1}$.

To establish (7) for $i = n$, we note, by (7) for $i = n-1$, that $h_{n-1}[p(T)] = p(T)$ and use induction on dim $T$. If $p | T$ is bijective, then every $C \in \mathscr{C}$ contained in $T$ belongs to $\mathscr{A}$, and (7) follows from (6) and the definitions of $\mathscr{D}$ in (2) and $h_n$ in I. If $p | T$ is not bijective, then dim $p(T) = (\dim T) - 1$, $h_n(T)$ is contained in the dim $T$ dimensional cylinder $p^{-1}[p(T)]$, $h_n(\text{Fron } T) = \text{Fron } T$ by induction, and $h_n(T) = T$ by the connectedness of $T$.

**Theorem 2.** *(Triangulation of subanalytic sets.) For any locally finite family $\mathscr{R}$ of subanalytic subsets of $\mathbb{R}^n$, there exist a simplicial decomposition $\mathscr{F}$ of $\mathbb{R}^n$ and a subanalytic map $f \colon [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ such that $f_t = f(t, \cdot)$ is a homeomorphism for each $t \in [0,1]$, $f_0$ is the identity, and $\{f_1(\mathring{F}) \colon F \in \mathscr{F}\}$ is compatible with $\mathscr{R}$.*

*Proof.* For $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $j \in \{-1, 0, 1, 2, \ldots\}$, let

$$\|(x_1, \ldots, x_n)\| = \sup\{|x_1|, \ldots, |x_n|\},$$
$$L_j = \mathbb{R}^n \cap \{x \colon \|x\| \leqq j\}, \quad \text{and} \quad \mathscr{R}_j = \{R \cap L_j \colon R \in \mathscr{R}\}.$$

With $\mathscr{E}_0 = \{L_0\} = \{\{0\}\}$ and $e_0 \colon [0,1] \times L_0 \to L_0$, we will choose inductively for $j \in \{1, 2, \ldots\}$ a simplicial decomposition $\mathscr{E}_j$ of $L_j$ and a subanalytic map $e_j \colon [0,1] \times L_j \to L_j$ such that

(1) $e_{j,t} = e_j(t, \cdot)$ is a homeomorphism for each $t \in [0,1]$,
(2) $e_{j,0} = \mathbb{1}_{L_j}$,
(3) $\{e_{j,1}(\mathring{E}) \colon E \in \mathscr{E}_j\}$ is compatible with $\mathscr{R}_j$,
(4) $\{E \cap L_{j-2} \colon E \in \mathscr{E}_j\} = \{E \cap L_{j-2} \colon E \in \mathscr{E}_{j-1}\}$,
(5) $e_j | [0,1] \times L_{j-2} = e_{j-1} | [0,1] \times L_{j-2}$, and
(6) for each $E \in \mathscr{E}_j$, $E \cap \text{Bdry } L_j \subset \{(x_1, \ldots, x_n) \colon |x_m| = j\}$ for some $m \in \{1, \ldots, n\}$.

Assuming $e_0, \mathscr{E}_0, \ldots, e_{j-1}, \mathscr{E}_{j-1}$ have been chosen, we first observe that the equations

$$d(t,x)=x \quad \text{for } (t,x)\in[0,1]\times L_1 \text{ in case } j=1,$$
$$d(t,x)=e_{j-1}(t,x) \quad \text{for } (t,x)\in[0,1]\times L_{j-1}$$

and

$$d(t,x)=(j-1)^{-1}\|x\|\, e_{j-1}[t,(j-1)\|x\|^{-1}x]$$
$$\text{for } (t,x)\in[0,1]\times(L_j\sim L_{j-1}) \text{ in case } j\geqq 2,$$

define a subanalytic map $d\colon [0,1]\times L_j\to[0,1]\times L_j$ extending $e_{j-1}$ such that $d_t=d(t,\cdot)$ is a homeomorphism for each $t\in[0,1]$ and $d_0=\mathbb{1}_{L_j}$.

Second we let $\mathscr{S}_0=\{R^0\}$ and use [7, 3.2] to select for $i\in\{n,n-1,\ldots,1\}$ orthogonal projections $p_i\colon \mathbb{R}^i\to\mathbb{R}^{i-1}$ and subanalytic stratifications $\mathscr{S}_i$ of

$$p_{i+1}(\ldots(p_n(L_j))\ldots)$$

so that $\mathscr{S}_n$ is compatible with $\mathscr{E}_{j-1}\cup\{d_1^{-1}(R)\colon R\in\mathscr{R}_j\}$, $\mathscr{S}_{i-1}$ is compatible with $\{p_i(S)\}$, and $p_i|\mathrm{Tan}(S,x)$ is injective whenever $x\in S\in\mathscr{S}_i$ and $\dim S<i$.

Third, we apply Theorem 1 with $K_n=L_j$ to obtain a simplicial decomposition $\mathscr{D}$ of $L_j$ and a subanalytic map $c\colon [0,1]\times L_j\to L_j$ such that $c_t=c(t,\cdot)$ is a homeomorphism for each $t\in[0,1]$, $c_0=\mathbb{1}_{L_j}$, $c_t(E)=E$ for each $E\in\mathscr{E}_{j-1}$, and $\{c_1(\mathring{D})\colon D\in\mathscr{D}\}$ is compatible with $\mathscr{S}_n$.

Fourth, we let

$$V=\bigcup_{E\in\mathscr{E}_{j-1}} V_E, \qquad W=\mathrm{Bdry}\, L_{j-1},$$
$$t_v\colon L_{j-1}\to[0,1] \quad \text{for } v\in V,$$
$$t_v(x)=0 \quad \text{whenever } x\in E\in\mathscr{E}_{j-1} \text{ and } v\notin E,$$
$$x=\sum_{v\in V} t_v(x)\,v \quad\text{and}\quad \sum_{v\in V} t_v(x)=1 \quad \text{whenever } x\in E\in\mathscr{E}_{j-1} \text{ and } v\in E,$$
$$s\colon L_{j-1}\to[0,1], \qquad s(x)=\sum_{v\in V\cap W} t_v(x),$$
$$r\colon \mathbb{R}^n\cap\{x\colon s(x)>0\}\to\mathbb{R}^n, \qquad r(x)=\sum_{v\in V\cap W} t_v(x)\,v/s(x),$$
$$e_j(t,x)=d(t,x)=e_{j-1}(t,x) \quad \text{for } (t,x)\in[0,1]\times L_{j-1} \text{ and } s(x)=0,$$
$$e_j(t,x)=d(t,x+s(x)[c(t,r(x))-r(x)]) \quad \text{for } (t,x)\in[0,1]\times L_{j-1} \text{ and } s(x)>0,$$
$$e_j(t,x)=d[t,c(t,x)] \quad \text{for } (t,x)\in[0,1]\times(L_j\sim L_{j-1}),$$

$\mathscr{E}_j$ be a simplicial subdivision of $\mathscr{E}=$(first barycentric subdivision of

$$\mathscr{D}\cap\{D\colon D\subset L_j\sim \mathrm{Int}\, L_{j-1}\})\cup(\mathscr{E}_{j-1}\cap\{E\colon \mathring{E}\subset\mathrm{Int}\, L_{j-1}\})$$

with

$$\bigcup_{E\in\mathscr{E}_j} V_E=\bigcup_{E\in\mathscr{E}} V_E,$$

and observe that $e_j$ is a (continuous) subanalytic map and that $\mathscr{E}$ is a convex cell decomposition of $L_j$ because $\mathscr{D}$, being compatible with $\{c_1^{-1}(S)\colon S\in\mathscr{S}_n\}$ is compatible with $\mathscr{E}_{j-1}=\{c_1^{-1}(E)\colon E\in\mathscr{E}_{j-1}\}$. Moreover, for each $E\in\mathscr{E}_{j-1}$ with $E\cap W\neq\emptyset$, we infer, from (6) with $j$ replaced by $j-1$, that

$$F=r(E\cap\{x: s(x)>0\})\subset E\cap W,\ F\in\mathscr{E}_{j-1},\ c_t(F)=F,\ e_{j,t}(E)=e_{j-1,t}(E),$$

and $e_{j,t}|E$ is injective for $t\in[0,1]$. Statements (1), (2), (4), (5), and (6) are now readily verified.

To prove (3) we assume $E\in\mathscr{E}_j$, $R\in\mathscr{R}_j$, and $e_{j,1}(\mathring{E})\cap R\neq\emptyset$, and verify that $e_{j,1}(\mathring{E})\subset R$ in each of the following three cases.

*Case* 1. $E\subset\operatorname{Int} L_{j-1}$. Here $s|\mathring{E}=0$, $E\in\mathscr{E}_{j-1}$, $e_{j,1}|\mathring{E}=e_{j-1,1}|\mathring{E}$, and

$$e_{j-1,1}(\mathring{E})\subset R\cap L_{j-1}$$

by (3) applied with $j$ replaced by $j-1$.

*Case 2.* $\mathring{E}\subset\operatorname{Int} L_{j-1}$ *and* $E\cap W\neq\emptyset$. Here $s|\mathring{E}>0$, $\mathring{E}\subset\mathring{F}$ for some $F\in\mathscr{E}_{j-1}$, and $e_{j,1}(\mathring{E})\subset e_{j-1,1}(\mathring{F})\subset R\cap L_{j-1}$ by (3) applied with $j$ replaced by $j-1$.

*Case 3.* $E\subset L_j\sim\operatorname{Int} L_{j-1}$. Here $E$ is contained in some member of $\mathscr{D}$, and

$$e_{j,1}(\mathring{E})=d_1[c_1(\mathring{E})]\subset R$$

because $\{c_1(\mathring{D})\colon D\in\mathscr{D}\}$ is compatible with $\mathscr{S}_n$ and $\mathscr{S}_n$ is compatible with $\{d_1^{-1}(R)\}$.

Having obtained $e_0, \mathscr{E}_0, e_1, \mathscr{E}_1, e_2, \mathscr{E}_2, \ldots$, the proof is completed by observing that

$$\mathscr{F}=\bigcup_{j=1}^{\infty}\mathscr{E}_j\cap\{E\colon E\subset L_{j-2}\}$$

is a simplicial decomposition of $\mathbb{R}^n$ and letting

$$f(t,x)=\lim_{j\to\infty} e_j(t,x) \qquad \text{for } (t,x)\in[0,1]\times\mathbb{R}^n.$$

**Embedding Lemma.** *For any $n$ dimensional subanalytic subset $V$ of a paracompact analytic space $M$, there exists a proper subanalytic homeomorphism mapping $V$ into $\mathbb{R}^{2n+1}$.*

*Proof.* For $i\in\{1,2,\ldots\}$, there are open subsets $G_i$ and $U_i$ of $M$, an integer $n_i\geqq n$, and subanalytic functions $\rho_i\colon M\to[0,1]$, $\sigma_i\colon G_i\to\mathbb{R}^{n_i}$ such that

$$\operatorname{Clos} U_i\subset G_i\subset M\sim\operatorname{Fron} V,\ \{U_i\cap V\colon i\in\{1,2,\ldots\}\}$$

is a locally finite relatively open cover of $V$, $\operatorname{spt}\rho_i\subset G_i$, $U_i\cap\operatorname{spt}(1-\rho_i)=\emptyset$, and $\sigma_i$ is a homeomorphism. Defining

$$\rho_{i,j}(y)=\rho_i(y)\,\mathrm{e}_j\bullet\sigma_i(y) \quad \text{for } y\in G_i \quad \text{and} \quad \rho_{i,j}(y)=0 \quad \text{for } y\in M\sim G_i,$$

we now need only modify the proof of [1, pp. 13–16] by using a subanalytic partition of unity, omitting the arguments involving tangent vectors and smoothness, and observing that, on [1, pp. 13–14], $\varphi(D)$ has measure zero in $P^{2n+1}$ because both $\varphi$ and $D$ are subanalytic and $\dim D=2n$.

**Theorem 3.** *(Triangulation of proper light subanalytic maps.) If $M$ and $N$ are finite-dimensional paracompact real analytic spaces, $P$ is a closed subanalytic subset of $M$, $f\colon P\to N$ is a proper light subanalytic map, and $\mathscr{Q}$ and $\mathscr{R}$ are locally finite families* *of subanalytic subsets of P and f(P) respectively, then there exist positive integers m and n, an orthogonal projection* $p\colon \mathbb{R}^m \to \mathbb{R}^n$, *a simplicial complex* $\mathscr{G}$ *in* $\mathbb{R}^m$ *and subanalytic homeomorphisms*

$$g\colon \bigcup \mathscr{G} \to P \quad \text{and} \quad h\colon p(\bigcup \mathscr{G}) \to f(P),$$

*such that*

$M = \{p(G)\colon G \in \mathscr{G}\}$ *is a simplicial complex in* $\mathbb{R}^n$,
$\{g(\mathring{G})\colon G \in \mathscr{G}\}$ *is compatible with* $\mathscr{Q}$,
$\{h(\mathring{H})\colon H \in \mathscr{H}\}$ *is compatible with* $\mathscr{R}$, *and* $f = h \circ p \circ g^{-1}$.

*Proof.* First we assume, by the Embedding Lemma that $M \subset \mathbb{R}^l$ and $N \subset \mathbb{R}^n$, let

$$p\colon \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n, \quad q\colon \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^l, \quad r\colon P \to \mathbb{R}^n \times \mathbb{R}^l,$$

$p(x, y) = x$ and $q(x, y) = y$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$, $r(z) = (f(z), z)$ for $z \in P$, identify $\mathbb{R}^n \times \mathbb{R}^l$ with $\mathbb{R}^m$ where $m = n + l$, and let $p_1\colon \mathbb{R}^1 \to \mathbb{R}^0 = \{0\}$ and

$$p_i\colon \mathbb{R}^i \to \mathbb{R}^{i-1}, \quad p_i(x_1, \dots, x_i) = (x_1, \dots, x_{i-1})$$

for $i \in \{2, \dots, m\}$ and $(x_1, \dots, x_i) \in \mathbb{R}^i$.

Second we use [6, 4.1, 4.2, 4.3, 2.1] to choose subanalytic stratifications

$$\mathscr{S}_m \text{ of } K_m = r(P), \quad \mathscr{S}_{m-1} \text{ of } K_{m-1} = p_m(K_m),$$
$$\mathscr{S}_{m-2} \text{ of } K_{m-2} = p_{m-1}[p_m(K_m)], \dots, \mathscr{S}_n \text{ of } p(K_m) = f(P)$$

such that, for $i \in \{m, m-1, \dots, n+1\}$, $\mathscr{S}_m$ is compatible with $\{r(Q)\colon Q \in \mathscr{Q}\}$, $\mathscr{S}_n$ is compatible with $\mathscr{R}$, $\mathscr{S}_{i-1}$ is compatible with $\{p_i(S)\}$, and $p_i|\operatorname{Tan}(S, x)$ is injective whenever $x \in S \in \mathscr{S}_i$.

Third we apply Theorem 2 to obtain a simplicial complex $\mathscr{F}$ in $\mathbb{R}^n$ and a subanalytic homeomorphism $h_n\colon \bigcup \mathscr{F} \to p(K_m)$ such that $\mathscr{C}_n = \{h_n(\mathring{F})\colon F \in \mathscr{F}\}$ is compatible with $\mathscr{S}_n$, and then let $\mathscr{D}_n$ be the first barycentric subdivision of $\mathscr{F}$,

$$b[h_n(F)] = h_n(b_F) \quad \text{for } F \in \mathscr{F},$$

and

$$\mathscr{C}_i = \{\text{components of } p_i^{-1}(C) \cap S\colon C \in \mathscr{C}_{i-1} \text{ and } S \in \mathscr{S}_i\} \quad \text{for } i \in \{n+1, \dots, m\}.$$

Fourth we observe that each member of $\mathscr{C}_n$ is simply connected and that each point in $\mathbb{R}^n$ has an arbitrarily small neighborhood whose intersection with each member of $\mathscr{C}_n$ is connected. Thus we may repeat the arguments in the proofs of [7, 3.1] and Theorem 1 to obtain inductively for $i \in \{n+1, \dots, m\}$ a point

$$b(C) \text{ in } (h_n \times \mathbb{1}_{\mathbb{R}^{i-n}})^{-1}(C) \quad \text{for each } C \in \mathscr{C}_i$$

and a subanalytic homeomorphism

$$h_i\colon \bigcup \mathscr{D}_i \to K_i$$

where the collection $\mathscr{D}_i$ of convex hulls of $\{b(C_0), \dots, b(C_k)\}$, corresponding to $C_0, \dots, C_k$ in $\mathscr{C}_i$ with

$$\operatorname{Fron} C_0 \supset C_1, \operatorname{Fron} C_1 \supset C_2, \dots, \operatorname{Fron} C_{k-1} \supset C_k,$$

is a simplicial complex,

$$p_i \circ h_i = h_{i-1} \circ (p_i | \bigcup \mathscr{D}_i),$$

and $\{h_i(\mathring{D}): D \in \mathscr{D}_i\}$ is compatible with $\mathscr{S}_i$. Then

$$\mathscr{D}_n = \{p(D): D \in \mathscr{D}_m\}, \quad p \circ h_m = h_n \circ (p | \bigcup \mathscr{D}_m),$$

and the proof is completed with

$$\mathscr{G} = \mathscr{D}_m, \quad g = g \circ h_m, \quad \text{and} \quad h = h_n.$$

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