Invent math. 75, 143–178 (1984)

Inventiones mathematicae

Local invariants of spectral asymmetry

M. Wodzicki

Steklov Mathematical Institute, USSR Academy of Sciences, Moscow 117333, USSR
Department of Mathematics, Warsaw University, PKiN, 9p., 00-901 Warsaw, Poland

0. Introduction

0.1. The purpose of this paper is to prove a certain strong uniqueness property of the spectral asymmetry local invariants. The results were motivated by the following question. Let A be an elliptic pseudo-differential operator (ψDO) of positive order on a closed manifold X. Assume that there are two cuttings in the spectral plane $\operatorname{Arg}\lambda=\theta'$, $\operatorname{Arg}\lambda=\theta''$ $(0<\theta'<\theta''\leqq 2\pi)$ whose conical neighbourhoods contain no eigenvalues of A. This allows us to define:

$$\rho(s;A)=\zeta_{\theta'}(s;A)-\zeta_{\theta''}(s;A)=\operatorname{Tr}A_{\theta'}^{-s}-\operatorname{Tr}A_{\theta''}^{-s}\quad\left(\operatorname{Re}s>\frac{\dim X}{\operatorname{ord}A}\right). \tag{1}$$

Here A_θ^{-s} $(\theta=\theta',\theta'')$ denotes the complex powers of A with respect to the cutting $\operatorname{Arg}\lambda=\theta$ (see e.g. [8, 11]). It is known that $\rho(s;A)$ analytically continues to the whole complex plane as a meromorphic function. Moreover, $\rho(s;A)$ is regular at all integral points. Since $A_\theta^{-l}=A^{-l}$ does not depend on θ for $l\in\mathbb{Z}$, we must have $\rho(l;A)=0$ at least when A^{-l} has a finite trace (i.e. when $l>\dim X/\operatorname{ord}A$). Assume for a moment that A has a very special form:

$$A=A'\oplus A'':\mathscr{C}^\infty(X,E'\oplus E'')\to\mathscr{C}^\infty(X,E'\oplus E''), \tag{2}$$

where $\operatorname{Spec}A'\subset\{\theta'<\operatorname{Arg}\lambda<\theta''\}$, $\operatorname{Spec}A''\subset\{\theta''<\operatorname{Arg}\lambda<\theta'+2\pi\}$. Later, after making precise basic notions connected with spectral asymmetry, it will be reasonable to say that such operators have the *trivial spectral asymmetry* (cf. 1.9).

Of course we have:

$$\rho(s;A)=\rho(s;A')+\rho(s;A'')=(1-e^{-2\pi i s})\,\zeta_{\theta'}(s;A')$$

(the asymmetry between A' and A'' is caused by our choice $0<\theta'<\theta''\leqq 2\pi$).
In particular, $\rho(l;A)=2\pi i\operatorname{Res}_{s=l}\zeta_{\theta'}(s;A')$. Thus the residue of $\zeta(s)$ is an obvious obstruction to the vanishing of $\rho(s)$. It appears that this is the *only* obstruction. More precisely, for $l\leqq\dim X/\operatorname{ord}A$ $\operatorname{Res}_{s=l}\zeta(s;A)\equiv 0$ in two cases:

1° $l \cdot \mathrm{ord} A \notin \mathbb{Z}$, 2° $l=0$. The vanishing (even microlocally) of $\rho(l)$ in the first case has been proved in [11]. For the second case see 1.24 of the present paper (it is closely related to the regularity of the eta-function of Atiyah-Patodi-Singer, see e.g. [4, 5, 11]).

Actually the described phenomenon proves to have a more general nature. Out of all properties of $\rho(l)$ the following are essential:

(a) $\rho(l)$ is *local*, i.e. $\rho(l)$ is an integral of some 1-density written down locally in terms of the complete symbol of an operator;

(b) $\rho(l)$ is *spectral*, i.e. $\rho(l)$ is constant on classes of iso-spectral operators;

(c) $\rho(l)$ is *additive*, i.e. $\rho(l; A \oplus B) = \rho(l; A) + \rho(l; B)$, where A, B are any elliptic ψDOs admitting of the same cuttings θ', θ''.

We prove in this paper that:

any spectral asymmetry invariant satisfying the conditions (a)–(c) *is uniquely determined by its restriction to operators with trivial spectral asymmetry, in case:*

(A) *X is an even-dimensional closed manifold, or*

(B) *$X = \Sigma \times Y$, where Σ is an odd-dimensional rational homology sphere, and Y is an arbitrary closed manifold.*

For the precise meaning of the word "local invariant of the spectral asymmetry" and for more detailed (and more general) statements we refer to Sect. 1.

Of course it is equivalent to prove the corresponding vanishing properties for invariants, whose restriction to trivial asymmetry operators is zero. And we deal only with this case further on.

0.2. A few words about the contents of the paper. Section 1 contains basic definitions and statements. Besides, in Sect. 1 we prove that the adjoint action orbits of the group of invertable ψDOs of order zero which intersect the set of operators with trivial spectral asymmetry generate, in some sense, all *topologically trivial* operators (for the definition see 1.13). "In some sense" means the following: for any topologically trivial ψDO A there exist operators D, T (D – with trivial spectral asymmetry, T – smoothing) such that $A \oplus D + T$ lies on one of the orbits mentioned above. Any spectral invariant must be constant along the adjoint action orbits. On the other hand topologically trivial operators define the equivalence relation on the class of elliptic ψDOs with "fixed" spectral asymmetry (see 1.17), and the corresponding set of equivalence classes with the additive structure induced by the direct sum $\oplus$ is naturally isomorphic to $K(S^*X)/\pi^* K(X) \simeq K^1(T^*X)$, where $S^*X \xrightarrow{\pi} X$ is the co-sphere bundle. Therefore our invariants must factor through K-theory. We end Sect. 1 with the corrigenda to the preceding paper [11]. In Sect. 2 we introduce certain K-theoretical classes which turn out to provide a sort of generators for our invariants. The corresponding "reductions" are the object of Sects. 5 and 6. Part of these reductions is much in spirit of the Gilkey's paper [5]. Localness of the invariants is essential. Section 3 contains auxiliary technical constructions connected with simultaneous modifications of ψDOs together with vector bundles on the co-sphere bundle, and of a base manifold itself. This material applies repeatedly later on. In Sect. 4 we prove the vanishing of our invariants on the generators introduced in Sect. 2. The essential ingredients of the proof are as follows:

– *spectral* invariants are constant on the adjoint action orbits in $K(S^*X)/\pi^*K(X)$ of the group of diffeomorphisms of a base manifold;

– by performing surgery one can provide *any* oriented odd-dimensional manifold with an orientation reversing diffeomorphism;

– the generators of Sect. 2 behave well with respect to surgery;

– invariants under the consideration are assumed to be local.

At last in Sect. 7, which is independent of the preceding, we deal again with the original question of the coincidence at the origin of zeta-functions defined by various cuttings in the spectral plane. We prove that this is equivalent to the vanishing of the "non-commutative residue" for *any* pseudo-differential projector. This "non-commutative residue" is a higher-dimensional analogue of the one-dimensional non-commutative residue in the Adler-Lebiediev-Manin scheme in the integrable systems theory. Recently it has become clear that this (higher-dimensional) residue is an obstruction to representing a ψDO as a finite sum of commutators (up to smoothing operators). Moreover, this turns to be the only obstruction. Since the coincidence of the zero values of the zeta-functions is actually proved, we obtain this way that any pseudo-differential projector (on a closed manifold) must be a finite sum of commutators up to a smoothing operator.

0.3. It is likely that the italicized statement of 0.1 remains true for an arbitrary odd-dimensional manifold, at least under certain additional assumptions on the invariant. Indeed, $K(S^*X)/\pi^*K(X)\otimes\mathbb{Q}$ for an oriented odd-dimensional X is generated by special differential operators A_E of order one. They are the boundary components of the signature with coefficients operators (cf. [2, p. 83]), and they depend on a connection on the vector bundle E and on the riemannian metric. Let Δ_E be the laplacian (with respect to the same data) acting on sections of E. Let $B_A^{(m)}=A_E(1+\Delta_E)^{\frac{m-1}{2}}$. $B_E^{(m)}$ also generate $K(S^*X)/\pi^*K(X)\otimes\mathbb{Q}$, and their order equals m. Let R be the considered invariant. By the localness assumption it is of the form $R=\int R(x)\,|dx|$, where $R(x)\,|dx|$ is a 1-density. If one requires in addition that $R(x;B_E^{(m)})$ is *a smooth function of finite jets of the connection and of the metric* then one obtains, *a smooth invariant* (in the sense of Gilkey) and by applying the results of [3] one ought to have $\int R(x)\,|dx|\equiv 0$.

0.4. *Notation.* If X is any compact space then by $K(X)$ we denote its (ordinary) K-group, by $H^*(X)$ – its rational cohomology, and by θ_X^k – the canonical trivial bundle with the total space $X\times\mathbb{C}^k$. Below we usually distinguish between *trivializable* and *trivial* vector bundles.

I would like to thank F.A. Bogomolov, O.R. Musin and A.N. Tyurin for stimulating conversations. I wish also to acknowledge the invaluable support of Yu.I. Manin.

1. Basic statements

1.1. Recall the definition of a morphism of vector bundles. If $E \to X$, $F \to Y$ are two vector bundles then by a morphism $\varphi\colon F \to E$ we mean a pair $\varphi = (f, r)$, where $f\colon Y \to X$ is a smooth mapping, $r \in \operatorname{Hom}(f^{\cdot}E, F)$.

Any such morphism induces the natural mapping $\varphi^*\colon \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(Y, F)$:

$$\text{for } u \in \mathscr{C}^\infty(X, E) \quad (\varphi^* u)(y) = r_y u(f(y)).$$

1.2. In case f is an open embedding (in particular – diffeomorphism), and $r\colon f^{\cdot}E \xrightarrow{\sim} F$ an isomorphism one can define also the arrow in opposite side $\varphi_!\colon \mathscr{C}_0^\infty(Y, F) \to \mathscr{C}_0^\infty(X, E)$:

$$\text{for } v \in \mathscr{C}_0^\infty(Y, F) \quad (\varphi_! v)(x) = \begin{cases} r_y^{-1} v(y) & \text{if } x = f(y) \text{ for some } y \in Y \\ 0 & \text{otherwise.} \end{cases}$$

1.3. In this case one can define also the natural mapping

$$\varphi^{\#}\colon \mathscr{L}(X, E) \to \mathscr{L}(Y, F),$$

where $\mathscr{L}(X, E)$ denotes the space of all linear continuous operators $A\colon \mathscr{C}_0^\infty(X, E) \to \mathscr{C}^\infty(X, E)$, and analogously $\mathscr{L}(Y, F)$. Namely, $\varphi^{\#} A\colon \mathscr{C}_0^\infty(Y, F) \to \mathscr{C}^\infty(Y, F)$ is given by the commutativity of the diagram

$$\begin{array}{ccc} \mathscr{C}_0^\infty(Y, F) & \dashrightarrow & \mathscr{C}^\infty(Y, F) \\ {\scriptstyle \varphi_!}\downarrow & & \uparrow{\scriptstyle \varphi^*} \\ \mathscr{C}_0^\infty(X, F) & \longrightarrow & \mathscr{C}^\infty(X, F) \end{array}$$

In other words, the correspondence $(X, E) \rightsquigarrow \mathscr{L}(X, E)$ establishes the contravariant functor $\mathscr{L}\colon \mathscr{V}ect_0 \to \mathscr{S}ets$. Here and further on $\mathscr{V}ect_0$ stands for the category of (smooth) vector bundles with morphisms as in 1.2. In case Y is an open piece of X, $i\colon Y \hookrightarrow X$ is a tautological embedding, $F = i^{\cdot}E$, and $r = \mathrm{id}$ we shall write briefly $A|_Y$.

Let $\mathscr{M}\colon \mathscr{V}ect_0 \to \mathscr{S}ets$ be any subfunctor of the functor $\mathscr{L}$, i.e. in particular, $\mathscr{M}(X, E) \subseteq \mathscr{L}(X, E)$ for every $(X, E) \in Ob\, \mathscr{V}ect_0$. Suppose, in addition, there is given a family of mappings

$$\mathscr{R}_{X,E}\colon \mathscr{M}(X, E) \to \mathscr{C}^\infty(X, |\wedge|),$$

where $|\wedge|$ denotes a linear bundle of 1-densities.

1.4. **Definition.** The family $\{\mathscr{R}_{X,E}\}$ will be called *a local invariant* of the functor $\mathscr{M}$, if for an arbitrary $\varphi = (f, r) \in \operatorname{Hom}_{\mathscr{V}ect_0}(F, E)$ the following diagram:

$$\begin{array}{ccc} \mathscr{M}(Y, F) & \xrightarrow{\mathscr{R}_{Y,F}} & \mathscr{C}^\infty(Y, |\wedge|) \\ {\scriptstyle \varphi^{\#}}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathscr{M}(X, E) & \xrightarrow{\mathscr{R}_{X,E}} & \mathscr{C}^\infty(X, |\wedge|) \end{array} \tag{1}$$

is commutative.

For any local invariant $\mathcal{R}$ by $\int\mathcal{R}$ will be denoted the family of functionals $\int_X \mathcal{R}_{X,E}\colon \mathcal{M}(X,E)\to\mathbb{C}$ obtained by integration of 1-densities over *closed* manifolds X.

1.5. *Remark.* The correspondence $(X,E)\rightsquigarrow X\rightsquigarrow\mathscr{C}^\infty(X,|\wedge|)$ establishes the contravariant functor $\mathscr{V}ect_0\to\mathscr{S}ets$. The condition of commutativity of the diagram (1) says then that local invariants are exactly natural transformations $\mathcal{R}\colon\mathcal{M}\to\mathscr{C}^\infty(\cdot,|\wedge|)$ of the corresponding functors.

In the language of Grothendieck topologies this could be said equivalently as: by local invariants we mean (arbitrary) morphisms $\mathcal{R}\colon\mathcal{M}\to p^*|\wedge|$ of (pre-) sheaves on the category $\mathscr{V}ect_0$. Here

a) $|\wedge|$ is the sheaf of 1-densities on the category $\mathcal{M}an_0$ of (smooth) manifolds with open embeddings as morphisms;

b) $p\colon\mathscr{V}ect_0\to\mathcal{M}an_0$ is the forgetting functor.

1.6. *Remark.* This notion of a local invariant admits natural generalization to the case of manifolds and vector bundles with extra structure.

1.7. *Spectral asymmetry.* Let $\mathscr{D}',\mathscr{D}''$ be two open (non-empty) sectors in the complex plane such that $\bar{\mathscr{D}}'\cap\bar{\mathscr{D}}''=\{0\}$. By $Ell^m_{\mathscr{D}',\mathscr{D}''}(X,E)$ we denote the set of all classical elliptic ψDOs of order m, for which the spectrum of a principal symbol lies in $\mathscr{D}'\cup\mathscr{D}''$. This is clearly invariant with respect to changing coordinates on base and in a fibre, so we have the contravariant functor $Ell^m_{\mathscr{D}',\mathscr{D}''}\colon\mathscr{V}ect_0\to\mathscr{S}ets$. It will be useful also to consider its restriction to $\mathscr{V}ect_0/X$ denoted by $Ell^m_{\mathscr{D}',\mathscr{D}''}(X)$.

1.8. Let $\pi\colon T_0^*X\to X$ be the natural projection ($T_0^*X=T^*X\setminus X$). If $a\colon\pi^*E\to\pi^*E$ is a principal symbol of $A\in Ell^m_{\mathscr{D}',\mathscr{D}''}(X,E)$ then the so called *sectorial splitting* is uniquely defined as:

$$\pi^*E=\mathscr{E}'\oplus\mathscr{E}'',\qquad\text{where }\mathscr{E}',\mathscr{E}''\subset\pi^*E. \tag{2}$$

This is characterized by two properties:

a) $a(\mathscr{E}')\subseteq\mathscr{E}'$, $a(\mathscr{E}'')\subseteq\mathscr{E}''$;

b) $\operatorname{Spec}a(x,\xi)|_{\mathscr{E}'}\subset\mathscr{D}'$, $\operatorname{Spec}a(x,\xi)|_{\mathscr{E}''}\subset\mathscr{D}''$ for large $|\xi|$. In view of invariance of the splitting (2) with respect to the action of $\mathbb{R}_+^\times$ on T_0^*X, $\mathscr{E}'$ and $\mathscr{E}''$ can be seen as vector bundles on $S^*X:=T_0^*X/\mathbb{R}_+^\times$.

For arbitrary operators $A\in Ell^m_{\mathscr{D}',\mathscr{D}''}(X,E)$, $B\in Ell^m_{\mathscr{D}',\mathscr{D}''}(X,F)$ their direct sum $A\oplus B$ belongs to $Ell^m_{\mathscr{D}',\mathscr{D}''}(X,E\oplus F)$, and moreover,

$$\mathscr{E}'_{A\oplus B}=\mathscr{E}'_A\oplus\mathscr{E}'_B,\qquad\mathscr{E}''_{A\oplus B}=\mathscr{E}''_A\oplus\mathscr{E}''_B.$$

1.9. It is natural to introduce two subfunctors $Ell^m_{\mathscr{D}'}$, $Ell^m_{\mathscr{D}''}$ of the functor $Ell^m_{\mathscr{D}',\mathscr{D}''}$ defined as

$$Ell^m_{\mathscr{D}'}(X,E)=Ell^m_{\mathscr{D}',\mathscr{D}''}(X,E)\cap\{A'|\operatorname{Spec}a'\subset\mathscr{D}'\}$$

and analogously for $Ell^m_{\mathscr{D}''}(X,E)$. It would be reasonable to say that operators representable in the form $A'\oplus A''$ ($A'\in Ell^m_{\mathscr{D}'}$, $A''\in Ell^m_{\mathscr{D}''}$) have the *trivial spectral asymmetry*.

1.10. *K-condition.* All local invariants under the consideration in this paper are assumed to satisfy the following condition:

for an arbitrary closed manifold X:

a) $\left.\begin{array}{l} A\in Ell^m_{\mathscr{D}',\mathscr{D}''}(X,E), \quad R\in CL^0(X,E)^\times \\ (\textit{i.e. } R \textit{ is invertible } \psi\text{DO} \textit{ of order } 0) \end{array}\right\} \Rightarrow \int_X \mathscr{R}(R^{-1}AR)=\int_X \mathscr{R}(A);$

b) $\left.\begin{array}{l} A-A'\oplus A''\in L^{-\infty}, \quad \textit{where} \\ A'\in Ell^m_{\mathscr{D}'}(X,E'), \quad A''\in Ell^m_{\mathscr{D}''}(X,E'') \end{array}\right\} \Rightarrow \int_X \mathscr{R}(A)=0;$

c) $A_i\in Ell^m_{\mathscr{D}',\mathscr{D}''}(X,E_i) \quad (i=1,2) \Rightarrow \int_X \mathscr{R}(A_1)+\int_X \mathscr{R}(A_2)=\int_X \mathscr{R}(A_1\oplus A_2).$

Now we are ready to formulate our main result.

1.11. **Theorem.** *Let* $\mathscr{R}$: $Ell^m_{\mathscr{D}',\mathscr{D}''}\to p^\bullet|\wedge|$ *be a local K-invariant whose integral is invariant with respect to the adjoint action of the group of diffeomorphisms, i.e.*

$$\int_X \mathscr{R}(f^{\#}A)=\int_X \mathscr{R}(A), \tag{3}$$

where X is a closed manifold, and $f\in \mathrm{Diff}\,X$ *(and this holds for all X and f).*

Then

$$\int_X \mathscr{R}\equiv 0$$

in case:

(A) *X is a (closed) even-dimensional manifold, or*

(B) $X=\Sigma^l\times Y$, *where* Σ^l *is a rational homology odd-dimensional sphere (i.e. a closed odd-dimensional orientable manifold with* $\tilde{H}^{\mathrm{ev}}(\Sigma;\mathbb{Q})=0$), *and Y is an arbitrary (closed) manifold.*

Vanishing of $\int_X \mathscr{R}$ in the cases (A), (B) will be called further Theorem A and, respectively, Theorem B. It will be convenient to separate the case when Σ^l is a *standard* sphere into the independent Theorem B_0. Proofs consist of two parts: a chain of reductions (Sects. 5, 6) and establishing that $\int \mathscr{R}$ vanishes on operators with some special sectorial splittings, which turn to be, in some sense, "generators".

1.12. We would like at last to emphasize that *any* spectral invariant *R* (i.e. *R* is constant on the class of iso-spectral operators) automatically satisfies conditions (1.10a) and (3). As a corollary of Theorem 1.11 we obtain, thus, the italicized statement of the introduction.

For brevity, any invariant of spectral asymmetry satisfying (3) will be called further *admissible.*

1.13. Recall that $A\in Ell^m_{\mathscr{D}',\mathscr{D}''}(X,E)$ is called *topologically trivial* (cf. 4.1 of [11]), if its sectorial splitting $\pi^\bullet E=\mathscr{E}'\oplus\mathscr{E}''$ is isomorphic to a direct sum $\pi^\bullet E'\oplus\pi^\bullet E''$ for some vector bundles E', E'' on X. Applying results of the preceding paper [11] we shall prove now the following assertion.

1.14. **Proposition.** *For an arbitrary topologically trivial operator $A \in Ell^m_{\mathcal{D}',\mathcal{D}''}(X,E)$ on a closed manifold X there exist a vector bundle F with the property:*

for any $D \in Ell^m_{\mathcal{D}''}(X,F)$ there exist:

a) *operators $B' \in Ell^m_{\mathcal{D}'}(X,G')$, $B'' \in Ell^m_{\mathcal{D}''}(X,G'')$, where G', G'' are subbundles of $E \oplus F$, and $E \oplus F = G' \oplus G''$;*

b) *an invertable operator of order zero $P \in CL^0(X,E)^\times$, such that*

$$P^{-1}(A \oplus D)P - B' \oplus B'' \in L^{-\infty}.$$

Proof. According to the definition of topological triviality (1.13) there exist vector bundles E', E'' and a suitable isomorphism $\varphi: \pi^* E \xrightarrow{\sim} \pi^*(E' \oplus E'')$. Put $F = E' \oplus E''$, $G' = E'$, $G'' = E'' \oplus E$. Obviously

$$\psi = \begin{pmatrix} 0 & \varphi^{-1} \\ \varphi & 0 \end{pmatrix}$$

defines an automorphism of $\pi^*(E \oplus F)$ such that $\psi|_{\pi^* G'}: \pi^* G' \xrightarrow{\sim} \mathscr{E}'$, $\psi|_{\pi^* G''}: \pi^* G'' \xrightarrow{\sim} \mathscr{E}'' \oplus \pi^* F$. Let S_0 be any ψDO of order zero whose principal symbol is ψ. Observe that index $S_0 = 0$. Indeed, if

$$Q_1: \mathscr{C}^\infty(X,E) \to \mathscr{C}^\infty(X, E' \oplus E''), \qquad Q_2: \mathscr{C}^\infty(X, E' \oplus E'') \to \mathscr{C}^\infty(X,E)$$

be any ψDOs of order zero with principal symbols φ and φ^{-1} respectively, then by multiplicativity of index we obtain:

$$\operatorname{index} Q_1 + \operatorname{index} Q_2 = \operatorname{index} Q_1 Q_2 = \operatorname{index}(I + \text{compact}) = 0. \tag{4}$$

On the other hand the homotopy invariance of the index implies

$$\operatorname{index} S_0 = \operatorname{index}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(Q_1 \oplus Q_2)\right\} = \operatorname{index} Q_1 + \operatorname{index} Q_2 = 0,$$

in view of (4).

Therefore, there exists an isomorphism of finite dimensional subspaces in $L^2(X, E \oplus F)$ (it is assumed that a positive 1-density $d\mu$ and an hermitian metric on E are chosen):

$$\operatorname{Ker} S_0 \xrightarrow{\sim} (\operatorname{Im} S_0)^\perp.$$

This is, of course, an operator with smooth kernal. By adding any such isomorphism to S_0 we obtain an *invertible* ψDO S of order zero such that for an arbitrary $D \in Ell^m_{\mathcal{D}''}(X,F)$ $S^{-1}(A \oplus D)S$ has the form

$$\begin{pmatrix} C' & T \\ U & C'' \end{pmatrix} + W: \mathscr{C}^\infty(X, G' \oplus G'') \to \mathscr{C}^\infty(X, G' \oplus G''),$$

where

1) $C' \in Ell^m_{\mathcal{D}'}(X,G')$, $C'' \in Ell^m_{\mathcal{D}''}(X,G'')$;
2) $\operatorname{ord} T$, $\operatorname{ord} U \leq m-1$;
3) $W \in L^{-\infty}(X, G' \oplus G'')$.

Proposition 4 of the paper [11] asserts existence of operators

$$K_-: \mathscr{C}^\infty(X, G'') \to \mathscr{C}^\infty(X, G'), \qquad K_+: \mathscr{C}^\infty(X, G') \to \mathscr{C}^\infty(X, G'')$$

such that

$$H := \begin{pmatrix} 1 & K_- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K_+ & 1 \end{pmatrix}$$

is an invertible ψDO of order zero, and $H^{-1}\begin{pmatrix} C' & T \\ U & C'' \end{pmatrix} H - \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix} \in L^{-\infty}$ for some $B' \in Ell^m_{\mathscr{D}'}(X, G')$, $B'' \in Ell^m_{\mathscr{D}''}(X, G'')$. Putting $P = SH$ provides the required operator. □

1.15. *Remark.* As it is seen from the proof one could take for F any vector bundle admitting of action of a ψDO with index equal to $-\operatorname{index} \varphi$. For example, this is the case when F admits a ψDO with index ± 1.

1.16. **Corollary.** *Let X be a closed manifold. Then an arbitrary K-invariant (not necessarily local) vanishes on topologically trivial operators.*

Proof. This is immediately implied by Proposition 1.14 and the K-condition. □

1.17. From now up to the end of this section X is assumed to be *closed*. We say that operators A_1 and A_2 are stable equivalent iff there exist such topologically trivial operators B_1, B_2 that $A_1 \oplus B_1 = A_2 \oplus B_2$ (equality involves an identification of vector bundles $E_1 \oplus F_1$ and $E_2 \oplus F_2$). The simple reasoning (similar to that of [11, Sect. 5.1]) shows that the $\oplus$-additive correspondence

$$Ell^m_{\mathscr{D}', \mathscr{D}''}(X) \ni A \mapsto [\mathscr{E}'_A] \in K(S^* X)/\pi^* K(X)$$

induces an isomorphism of abelian monoids:

$$\left\{\begin{matrix} \text{stable equivalence} \\ \text{classes on } X \end{matrix}\right\} = Ell^m_{\mathscr{D}', \mathscr{D}''}(X)/Ell^m_{\mathscr{D}', \mathscr{D}''}(X)_{\text{top. triv.}} \xrightarrow{\sim} K(S^* X)/\pi^* K(X).$$

According to Corollary 1.16, therefore, any K-invariant factors through $K(S^* X)/\pi^* K(X) \simeq K^1(T^* X)$. In particular, for a local K-invariant one has the commutative diagram:

$$\begin{array}{ccccc} A & & Ell^m_{\mathscr{D}', \mathscr{D}''}(X) & \xrightarrow{\mathscr{R}_X} & \mathscr{C}^\infty(X, |\wedge|) \\ \downarrow & & \downarrow & & \downarrow \int_X \\ [\mathscr{E}'_A] & & K(S^* X)/\pi^* K(X) & \dashrightarrow & \mathbb{C}. \end{array}$$

The dashed arrow will often be denoted by a letter R. By linearity this extends uniquely to the $\mathbb{Q}$-linear mapping $K(S^* X)/\pi^* K(X) \otimes \mathbb{Q} \to \mathbb{C}$ denoted by the same letter R. Instead of $R([\mathscr{E}'_A])$ we shall occasionally write simply $R(A)$.

Hereby, the problem reduces to examination of some topological invariants $K(S^* X)/\pi^* K(X) \to \mathbb{C}$.

1.18. *Action of diffeomorphisms.* Now, we are going to consider an action of diffeomorphisms of a closed manifold in the context of spectral asymmetry.

Let $A \in Ell^m_{\mathscr{D}',\mathscr{D}''}(X, E)$, $\pi^{\cdot} E = \mathscr{E}' \oplus \mathscr{E}''$ – be its sectorial splitting, and $f \in \operatorname{Diff} X$. Then, by the definition

$$f^{\#} A: \mathscr{C}^{\infty}(X, f^{\cdot} E) \to \mathscr{C}^{\infty}(X, f^{\cdot} E).$$

Its exact properties are described by the following lemma.

1.19. **Lemma (on the diffeomorphism).** *$f^{\#} A \in Ell^m_{\mathscr{D}',\mathscr{D}''}(X, f^{\cdot} E)$ and its sectorial splitting is*

$$\pi^{\cdot} f^{\cdot} E = (S^* f)^{\cdot} \mathscr{E}' \oplus (S^* f)^{\cdot} \mathscr{E}'',$$

where $S^* f$: $S^* X \to S^* X$ is induced by the cotangent map

$$T^* f: (x, \xi) \mapsto (f(x), (f^*_x)^{-1}(\xi)).$$

Proof. By the standard formula for change of variables we have

$$a_{f^{\#} A}(x, \xi) = a(f(x), (f^*_x)^{-1} \xi)$$

for a principal symbol of $f^{\#} A$. The assertion follows. □

1.20. The embedding $\operatorname{Diff} X \hookrightarrow \operatorname{Diff} S^* X$ given by $f \mapsto S^* f$ is a group homomorphism and defines an action of $\operatorname{Diff} X$ on $K(S^* X)$. If we take $K(X)$ with its standard $\operatorname{Diff} X$-action then the natural arrow π^*: $K(X) \to K(S^* X)$ turns to be $\operatorname{Diff} X$-equivariant. This gives rise to the action of $\operatorname{Diff} X$ on $K(S^* X)/\pi^* K(X)$. Lemma 1.19 and results of 1.17 together imply the following corollary.

1.21. **Corollary.** *Any admissible K-invariant determines the unique* $\operatorname{Diff} X$-*invariant homomorphism*

$$R: K(S^* X)/\pi^* K(X) \to \mathbb{C}. \quad \square$$

In particular, such homomorphism must kill those elements $e \in K(S^* X)/\pi^* K(X)$, for which one can find a diffeomorphism f: $X \to X$ with property $(S^* f)^* e = -e$. This will be one of crucial arguments in proof of Theorems A and B.

1.22. *Values of zeta-functions at integral points.* Let $0 < \theta' < \theta'' \leqq 2\pi$ be two cuttings in the spectral plane separating the sectors $\mathscr{D}', \mathscr{D}''$. As a corollary of Theorem 1.11 we obtain:

1.23. **Theorem.** *Let $\mathscr{R}$: $Ell^m_{\mathscr{D}',\mathscr{D}''} \to p^{\cdot} |\wedge|$ $(m > 0)$ be a local invariant such that for any closed manifold one has:*

a) $A \in Ell^m_{\mathscr{D}',\mathscr{D}''}(X, E)$, $R \in CL^0(X, E)^{\times} \Rightarrow \int_X \mathscr{R}(R^{-1} A R) = \int_X \mathscr{R}(A)$;

b) $A \in Ell^m_{\mathscr{D}',\mathscr{D}''}(X, E)$, $f \in \operatorname{Diff} X \Rightarrow \int_X \mathscr{R}(f^{\#} A) = \int_X \mathscr{R}(A)$;

c) $A_i \in Ell^m_{\mathscr{D}',\mathscr{D}''}(X, E_i)$ $(i = 1, 2) \Rightarrow \int_X \mathscr{R}(A_1 \oplus A_2) = \int_X \mathscr{R}(A_1) + \int_X \mathscr{R}(A_2)$;

d) $\left.\begin{array}{l} A - A' \oplus A'' \in L^{-\infty}, \quad \text{where} \\ A' \in Ell^m_{\mathscr{D}'}(X, E_1),\ A'' \in Ell^m_{\mathscr{D}''}(X, E_2) \end{array}\right\} \Rightarrow \int_X \mathscr{R}(A)$

$$= \alpha' \operatorname{Res}_{s=k} \zeta(s; A') + \alpha'' \operatorname{Res}_{s=k} \zeta(s; A''),[1]$$

where α', α'' *are some constants.*

Then

$$\int_X \mathscr{R}(A) = \frac{\alpha' - \alpha''}{2\pi i} \rho(k; A) + \alpha'' \operatorname{Res}_{s=k} \zeta(s; A),$$

if X *is:* (1) *even-dimensional or* (2) $X = \Sigma \times Y$, *where* Σ *is an odd-dimensional rational homology sphere. Here, as usual,*

$$\rho(k; A) := \lim_{s \to k} \{\zeta_{\theta'}(s; A) - \zeta_{\theta''}(s; A)\}.$$

Proof. At first, one ought to show that both $\rho(k; A)$ and $\operatorname{Res}_{s=k} \zeta(s; A)$ satisfy properties a)–c), but this is obvious in view of the fact that they depend only on $\operatorname{Spec} A$. $\rho(k; A)$ and $\operatorname{Res}_{s=k} \zeta(s; A)$ are both local:

$$\rho(k; A) = \int_X \rho_k(x; A) |dx|, \qquad \operatorname{Res}_{s=k} \zeta(s; A) = \int_X \gamma_k(x; A) |dx|,$$

and for $A - A' \oplus A'' \in L^{-\infty}$:

$$\rho(k; A) = \rho(k; A' \oplus A'') = 2\pi i \cdot \operatorname{Res}_{s=k} \zeta(s; A').$$

Therefore the difference $R - \frac{\alpha' - \alpha''}{2\pi i} \rho(k; A) - \alpha'' \operatorname{Res}_{s=k} \zeta(s; A)$ satisfies the assumptions of Theorem 1.11. Hence the assertion results. □

1.24. Note that $\mathscr{R} \equiv 0$ satisfies the conditions a)–d) when $k = 0$. Therefore we must have $0 = \int_X \rho_0(x; A) |dx|$ for X even-dimensional or $\Sigma^{\text{odd}} \times Y$. In fact this implies that $0 = \int_X \rho_0(x; A) |dx|$ for *any* closed X. Indeed, let A be a self-adjoint elliptic differential operator on (X, E), B – be any elliptic differential operator on $S^1 \times S^1$ with $\operatorname{ord} B = \operatorname{ord} A$. Then, by an observation of Atiyah et al. [2, p. 84] we have

$$\rho_0\left(\begin{pmatrix} A \otimes 1 & 1 \otimes B^* \\ 1 \otimes B & -A \otimes 1 \end{pmatrix}\right) = \operatorname{index} B \cdot \rho_0(A). \tag{5}$$

Here $\begin{pmatrix} A \otimes 1 & 1 \otimes B^* \\ 1 \otimes B & -A \otimes 1 \end{pmatrix}$ is a differential operator on $S^1 \times S^1 \times X$, and $\rho_0 := \zeta_\uparrow(0) - \zeta_\downarrow(0)$. Taking for B the $\bar{\partial}$-operator with coefficients in a line bundle of degree $d \neq 0$ we have $\operatorname{index} B = d \neq 0$. Thus, applying Theorem 1.23 to $X' = S^1 \times Y$, where $Y = S^1 \times X$, we obtain in case $\operatorname{ord} A = 1$

$$\rho_0(A) = \frac{1}{d} \rho_0\left(\begin{pmatrix} A \otimes 1 & 1 \otimes B^* \\ 1 \otimes B & -A \otimes 1 \end{pmatrix}\right) = 0.$$

[1] Recall that the residues of $\zeta_\theta(s; A)$ at integral points do not depend (even locally) on the cutting θ

Since $K(S^*X)/\pi^* K(X) \otimes \mathbb{Q}$, for oriented odd-dimensional X, as is well known (see [2, Prop. 4.4]), is generated by self-adjoint first order differential operators, the assertion results.

1.25. *Corrigenda to* [11]. Unfortunately, an error crept into the statement of the main result of my previous paper [11, p. 117]. "$l=-1,-2,\ldots$" ought to be deleted from the last sentence of the statement of the theorem. This error was caused by my uncritical relying on the assertion from [8, p. 290] on the vanishing of residues of $\zeta(s)$ at non-positive integers. And this is not quite correct[2]. If so the method of proof of the main theorem clearly exhibits that $\rho_l(A)=\lim_{s\to l}\{\zeta_{\theta'}(s;A)-\zeta_{\theta''}(s'';A)\}$ *cannot* be identically zero when $l\in\mathbb{Z}\cap\mathbb{Z}\langle 1/\operatorname{ord}A\rangle\cap\left(-\infty,\frac{\dim X}{\operatorname{ord}A}\right]$ and $l\neq 0$.

In view of this one ought to make the following changes in the text of [11]:
- delete "$l=-1,-2.\ldots$" from p. 117 (lines 4 top, 1 bottom), p. 120 (line 1 top), p. 128 (lines 6, 10 top), p. 129 (lines 2, 5, 6 top), p. 130 (line 7 top);
- in the statement of Prop. 5 to write $\rho(l;A)=2\pi i\cdot\operatorname{Res}_{s=l}\zeta(s;A')$ ($l\in\mathbb{Z}$) instead of $\rho(l;A)=0$ ($l=0,-1,-2,\ldots$);
- to rewrite (5.1) and the preceding formula as

$$(c^{-l}-1)\,\rho(l;A)=2\pi i\operatorname{Res}_{s=l}\{\zeta(s;A'_c)-\zeta(s;A'_1)\},$$

where $cA\oplus D$ ($c\in\mathbb{C}^*$) is conjugated $\operatorname{mod}L^{-\infty}$ to $A'_c\oplus A''_c$ ($\operatorname{Spec}a'_c\subset\mathscr{D}'$, $\operatorname{Spec}a''_c\subset\mathscr{D}''$).

In the light of the changes made the argument of 5.2 does not give the vanishing of $\rho(0;A)$ automatically[3]. The gap which arises is filled in 1.24 of the present paper. Except for 5.2 no proofs are affected by the above changes.

In addition, a few omissions and misprints occur in [11]. This is their list:
- add "order $A=1$" on p. 117 (line 6 bottom), p. 132 (line 7 bottom);
- the line 14 top on p. 126 read: "$-\frac{1}{2\pi i}\int(a-\lambda)^{-1}\,d\lambda$";
- add "and to be regular at all negative integers" on p. 132 (line 11 top).

My thanks are due to Howard D. Fegan and Peter B. Gilkey for alerting me to incorrect statement of the vanishing of $\rho(l)$.

2. The canonical (Q, ε)-class

2.1. Let X^d be a compact *orientable* manifold, which we temporarily assume to be connected. The standard Gysin exact sequence of the vector bundle

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[2] Nevertheless, an analytic continuation of $\zeta(s)$ to the *whole* complex plane for general ψDOs is not in danger. The correct formulae for the residues at integral points are given by

$$\operatorname{Res}_{s=l}\zeta(s;A)=\frac{1}{m}\int_X\int_{|\xi|=1}a^{(-l)}_{-\dim X}(x,\xi)\,d\xi'\,dx \qquad (m=\operatorname{ord}A), \tag{6}$$

where $a^{(-l)}(x,\xi)\sim\sum_{j=0}^{\infty}a^{(-l)}_{-lm-j}(x,\xi)$ is the complete symbol of A^{-l} (cf. 7.19 of the present paper)

[3] However, the method of relating the behaviour of $\zeta(s)$ at different integer points by means of its dependence on the spectral shift proves to be very fruitful

$\pi\colon S^*X \xrightarrow{S} X$ has the form:

$$\dots\to H^{j+d-1}(X)\xrightarrow{\pi^*}H^{j+d-1}(S^*X)\xrightarrow{\delta}H^j(X)\otimes H^{d-1}(S)\xrightarrow{\chi}H^{j+d}(X)\to\dots \tag{1}$$

Here the arrow χ must vanish for $j\neq 0$, by dimensional argument.[4] By the symbol 1_X we denote the canonical generator of the group $H^0(X)$. A choice of a generator ω_S in $H^{d-1}(S)$ is equivalent to choosing orientation of X. Then, for a closed X: $H^0(X)\otimes H^{d-1}(S)\to H^d(X)$ is defined as $1_X\otimes\omega_S\mapsto e_X$, where e_X is the Euler class of the manifold X orientated by the class ω_S. Therefore, for manifolds with non-empty boundary or with vanishing Euler characteristics, (1) breaks up into a set of short exact sequences:

$$0\to H^{j+d-1}(X)\xrightarrow{\pi^*}H^{j+d-1}(S^*X)\xrightarrow{\delta}H^j(X)\otimes H^{d-1}(S)\to 0. \tag{2}$$

2.2. In any case δ induces an isomorphism

$$\delta\colon\ H^{\text{ev}}(S^*X)/\pi^*H^{\text{ev}}(X)\xrightarrow{\sim}H^{\varepsilon}(X)\otimes H^{d-1}(S),$$

where

$$\varepsilon=\begin{cases}\text{even}, & \text{if } d=\text{odd};\\ \text{odd}, & \text{if } d=\text{even}.\end{cases}$$

2.3. Notice that the Gysin sequence in the form (1) is equivariant with respect to automorphisms of the fibre bundle S^*X, i.e. pairs of diffeomorphisms φ: $S^*X\to S^*X$, $f\colon X\to X$ such that the diagram:

$$\begin{array}{ccc} S^*X & \xrightarrow{\varphi} & S^*X\\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi}\\ X & \xrightarrow{f} & X\end{array}$$

is commutative. In particular, (1) is Diff$X$-equivariant (cf. 1.20).

2.4. Let us fix:

a) a connected oriented closed manifold Q^q,

b) "integral" cohomology class $\varepsilon\in H^p(Q;\mathbb{Q})$ (i.e. lying in the image of $H^p(Q;\mathbb{Z})$), where $p\equiv q\,(\text{mod}\,2)$.

Let N^n be an arbitrary odd-dimensional oriented compact manifold (eventually with boundary or not connected). We are going to define a certain functorial element $\lambda(N;Q,\varepsilon)\in\tilde{K}(S^*(Q\times N))/\text{Tors}$.

Since the Euler characteristics of $Q\times N$ is zero (in case N closed) we know that for all N under consideration the Gysin sequence breaks up into short exact sequences (2), in particular, the following sequence is exact:

$$\begin{aligned}0\longrightarrow H^{p+q+n-1}(Q\times N)\xrightarrow{\pi^*}H^{p+q+n-1}(S^*(Q\times N))\xrightarrow{\ \delta\ }\\ H^p(Q\times N)\otimes H^{q+n-1}(S)\longrightarrow 0.\end{aligned} \tag{3}$$

2.5. **Lemma.** *The sequence* (3) *has a canonical splitting.*

[4] If X has a non-empty boundary the same holds also for $j=0$

Proof. Consider at any point $(s,x)\in Q\times N$ a mirror reflection of $T^*_{(s,x)}(Q\times N)$ with respect to "the mirror" $T^*_sQ\subset T^*_{(s,x)}(Q\times N)$. The induced automorphism of $S^*(Q\times N)$ we denote by τ. Since Q is of odd co-dimension in $Q\times N$, then τ must reverse orientation and, moreover, it possesses the following properties:

a) $\tau^2=1$;

b) $\tau=1$ on $H^{p+q+n-1}(Q\times N)$;

c) $\tau=-1$ on $H^p(Q\times N)\otimes H^{q+n-1}(S)$.

So, having in mind (2.3), the decomposition of $H^{p+q+n-1}(S^*(Q\times N))$ into ±1-eigenspaces carries out the splitting of the sequence (3):

$$H^{p+q+n-1}(S^*(Q\times N))_1 \xrightarrow[\sim]{\pi^{*-1}} H^{p+q+n-1}(Q\times N),$$

$$H^{p+q+n-1}(S^*(Q\times N))_{-1} \xrightarrow[\sim]{\delta} H^p(Q\times N)\otimes H^{q+n-1}(S). \quad \square$$

Thus, we may think of $H^p(Q\times N)\otimes H^{q+n-1}(S)$ as of a linear subspace in $H^{p+q+n-1}(S^*(Q\times N))$. Remark that this embedding does not depend on orientation of $Q\times N$.

Put $\mathbb{1}_N=\sum_{[N_\iota]\in\pi_0 N} 1_{N_\iota}\in H^0(N)$. The rational Chern character provides an isomorphism $ch_{\mathbb{Q}}\colon K(\cdot)\otimes\mathbb{Q}\xrightarrow{\sim} H^{\mathrm{ev}}(\cdot)$ on the category of compact CW-complexes. Now, we are ready to give the following definition.

2.6. **Definition.**

$$\lambda(N;Q,\varepsilon):=\frac{1}{[(p+q+n-1)/2]!}\,ch_{\mathbb{Q}}^{-1}(\varepsilon\otimes\mathbb{1}_N\otimes\omega_S)\in\tilde{K}(S^*(Q\times N))\otimes\mathbb{Q}.$$

In fact the element $\lambda(N;Q,\varepsilon)$ lies in the $\mathbb{Z}$-submodule

$$\tilde{K}(S^*(Q\times N))/\mathrm{Tors}\subset\tilde{K}(S^*(Q\times N))\otimes\mathbb{Q}.$$

The factor $[(p+q+n-1)/2]!$ is chosen in order to avoid superfluous integral multiplicities. This element of $\tilde{K}(S^*(Q\times N))/\mathrm{Tors}$ will be called the (Q,ε)-class of N.

The following properties of the (Q,ε)-class arise immediately from the definition:

2.7. $\lambda(-N;Q,\varepsilon)=-\lambda(N;Q,\varepsilon)$.

2.8. $\lambda(N;-Q,\varepsilon)=\lambda(N;Q,-\varepsilon)=-\lambda(N;Q,\varepsilon)$.

2.9. *(Functoriality).* Let $j\colon N_0\hookrightarrow N$ be an oriented embedding of a compact co-dimension 0 submanifold (j could be eventually a diffeomorphism) and $\iota\colon S^*(Q\times N_0)\hookrightarrow S^*(Q\times N)$ an induced embedding of co-spheres. Then

$$\iota^*\lambda(N;Q,\varepsilon)=\lambda(N_0;Q,\varepsilon)\quad\text{in }\tilde{K}(S^*(Q\times N))\otimes\mathbb{Q}.$$

One of crucial points in proof of Theorems A and B is establishing the following assertion.

2.10. **Proposition** Λ. *Let* $\mathcal{R}$ *be an admissible local K-invariant* (*cf.* 1.10, 1.12). *Then*

$$\int_{Q \times N} \mathcal{R}\{\lambda(N;Q,\varepsilon)\}=0, \tag{4}$$

for any closed oriented odd-dimensional manifold N.

We postpone a proof to Sect. 4. The rest of the proof of Theorems A and B consists of a sequence of reductions leading at length to some (Q,ε)-classes associated with special manifolds Q.

2.11. Make once more remark. Proposition 2.10 in case Q = *product of standard spheres* we shall call Proposition Λ_0. This will be proved in Sect. 4 equally as the implication

$$\text{Theorems A and } B_0 \Rightarrow \text{Proposition } \Lambda. \tag{5}$$

In Sect. 5 we prove the implication

$$\text{Proposition } \Lambda_0 \Rightarrow \text{Theorem A}. \tag{6}$$

In Sect. 6 we prove the implications

$$\text{Proposition } \Lambda_0 \Rightarrow \text{Theorem } B_0, \tag{7}$$

$$\text{Proposition } \Lambda \Rightarrow \text{Theorem B}, \tag{8}$$

and this will end the proof of Theorems A, B and, simultaneously, of Proposition Λ.

2.12. At the end of this section we prove the vanishing of any admissible K-invariant R on the manifold $X=S^{k_1}\times\ldots\times S^{k_r}$. This fact serves as an ingredient of the proof of Proposition Λ_0.

2.13. **Lemma** *(on the spheres). Let* R: $Ell^m_{\mathcal{D}',\mathcal{D}''}(S^{k_1}\times\ldots\times S^{k_r})\to\mathbb{C}$ *be an admissible K-invariant (not necessarily local). Then* $R\equiv 0$.

Proof. Put $X=S^{k_1}\times\ldots\times S^{k_r}$, $d=\sum_{\nu=1}^{r} k_\nu$, all $k_\nu\geqq 1$. $K(S^*X)/\pi^*K(X)$ is generated over $\mathbb{Q}$ by elements which correspond via the isomorphism $\delta\circ ch_{\mathbb{Q}}$ to the classes

$$\varepsilon_1\otimes\ldots\otimes\varepsilon_r\otimes\omega_S\in H^e(X)\otimes H^{d-1}(S),$$

where $e=\sum\deg\varepsilon_\nu$, $e+d-1$ = even, and each ε_ν is $1_{S^{k_\nu}}$ or $\omega_{S^{k_\nu}}$. The condition of pairity of $e+d-1$ implies existence of at least one $\varepsilon_\mu=1_{S^{k_\mu}}$. Let f_μ: $S^{k_\mu}\to S^{k_\mu}$ be any orientation reversing diffeomorphism, f_ν: $S^{k_\nu}\to S^{k_\nu}(\nu\neq\mu)$ be identical maps, and $f:=f_1\times\ldots\times f_r$. Then

$$(S^*f)^*(\varepsilon_1\otimes\ldots\otimes\varepsilon_r\otimes\omega_S)=\varepsilon_1\otimes\ldots\otimes\varepsilon_r\otimes(-\omega_S)=-\varepsilon_1\otimes\ldots\otimes\varepsilon_r\otimes\omega_S$$

since f reverses the orientation of X, and $\varepsilon_\mu=1_{S^{k_\mu}}$. In view of Remark 2.3 $\delta\circ ch_{\mathbb{Q}}$ is Diff$X$-equivariant, therefore, letting $\lambda=ch_{\mathbb{Q}}^{-1}\circ\delta^{-1}(\varepsilon_1\otimes\ldots\otimes\varepsilon_r\otimes\omega_S)$, we obtain

$$R((S^*f)^*\lambda)=R(-\lambda)=-R(\lambda).$$

On the other hand $R((S^*f)^*\lambda)=R(\lambda)$, due to Corollary 1.21. Hence $R(\lambda)=0$. The lemma has been proven. □

3. Lemmas on gluing together

3.1. Let us fix basic notions and notation. In this section X will stand for a closed manifold represented as a sum of two compact submanifolds X_- and X_+ glued together along their common boundary Z:

$$X=X_-\cup_Z X_+.$$

In particular, a tubular neighbourhood of Z in X is diffeomorphic to $Z\times[0,1]$, so we can assume X to be represented in the form

$$X=\mathscr{U}_0\cup Z\times[0,1]\cup\mathscr{U}_1, \qquad (1)$$

where $\mathscr{U}_0\cup Z\times[0,1/2]=X_-$, $Z\times[1/2,1]\cup\mathscr{U}_1=X_+$. We fix any such decomposition further on. Restrictions of the co-sphere bundles to X_-, X_+ or Z will be denoted by S^*X_-, S^*X_+ and S_Z^*, respectively. S^*X_-, S^*X_+ are compact manifolds with common boundary S_Z^*.

3.2. Let Y be a manifold with a boundary ∂Y, and f: $\partial Y\to\partial Y$ be a diffeomorphism. Glue two copies of Y together along their boundaries with the aid of f. Then one obtains the closed manifold D_fY. If Y is oriented then putting

$$D_fY=\begin{cases} Y\cup_f -Y & \text{if } f \text{ preserves orientation,}\\ Y\cup_f Y & \text{if } f \text{ reverses orientation,}\end{cases}$$

equips D_fY with induced orientation. For $f=\mathrm{id}$ we shall briefly write DY and call it *the double* of Y.

3.3. Decomposition (1) determines an involution of S_Z^* induced by a reflection

$$\sigma\colon\ (z,0;\zeta,\tau)\mapsto(z,0;\zeta,-\tau) \qquad ((z,t;\zeta,\tau)\in T^*(Z\times[0,1])).$$

If one forms the doubles DX_- and DX_+ then one can easily check that $S^*(DX_\pm)=D_\sigma(S^*X_\pm)$. If, moreover, there is given a vector bundle $\mathscr{E}$ on S^*X such that there exists an isomorphism φ: $\mathscr{E}|_{S_Z^*}\xrightarrow{\sim}\sigma^{\cdot}(\mathscr{E}|_{S_Z^*})$, then one can naturally form "doubles" of $\mathscr{E}_\pm:=\mathscr{E}|_{S^*X_\pm}$ denoted by $D_\varphi\mathscr{E}_\pm$ (since they depend on φ). These are vector bundles on $S^*(DX_\pm)$. In case $\mathscr{E}|_{S_Z^*}$ is trivial (recall, this means: with chosen trivialization), then one has a canonical isomorphism $\mathscr{E}|_{S_Z^*}\xrightarrow{\sim}\sigma^{\cdot}(\mathscr{E}|_{S_Z^*})$, and $D_\varphi\mathscr{E}_\pm$ will be denoted simply by $D\mathscr{E}_\pm$.

3.4. **Lemma** *(on the double). Let $\mathscr{E}$ be a vector bundle on S^*X such that its restriction to $S^*\mathscr{U}$ is trivial (here $\mathscr{U}\subset X$ denotes some neighbourhood of X_-). Then there exist* ψDOs $A\in Ell^m_{\mathscr{D}',\mathscr{D}''}(X)$, $B_\pm\in Ell^m_{\mathscr{D}',\mathscr{D}''}(DX_\pm)$ *acting on sections of trivial vector bundles such that*

(a) $\mathscr{E}'_A=\mathscr{E}$, $\mathscr{E}'_{B_\pm}=\mathrm{D}\mathscr{E}_\pm$

(b) *if* $DX_{\pm}=X_{\pm}^{(1)}\cup X_{\pm}^{(2)}$ *then*

$$B_{-}|_{X_{-}^{(i)}}=A|_{X_{-}},\qquad B_{+}|_{X_{+}^{(i)}}=A|_{X_{+}}\quad (i=1,2)$$

(exact equality of operators). In particular, for every local invariant $\mathscr{R}$

$$\int_X \mathscr{R}(A)=\tfrac{1}{2}\int_{DX_{+}}\mathscr{R}(B_{+}). \tag{2}$$

Proof. We can safely assume $\mathscr{E}$ to be trivial on $S^*(\mathscr{U}_0\cup Z\times[0,1])$. In the course of the proof we shall use the following convenient notation:

$$\mathscr{U}_t=\begin{cases}\mathscr{U}_0\cup Z\times[0,t], & \text{if } t<1/2;\\ Z\times[t,1]\cup\mathscr{U}_1, & \text{if } t>1/2.\end{cases}$$

We are going to construct operators A, $B_{\pm}$ satisfying required conditions in an explicit form.

Step 1. Choose some complement $\mathscr{F}$ for $\mathscr{E}$:

$$\mathscr{E}\oplus\mathscr{F}=\theta^r_{S^*X},$$

such that $\mathscr{F}|_{S^*(Z\times[0,1])}$ is also trivial. Liftings of $\mathscr{E}$ and $\mathscr{F}$ onto T_0^*X will be denoted by the same letters $\mathscr{E}$ and $\mathscr{F}$.

Choose a section

$$S^*X\hookrightarrow T_0^*X \textit{ such that } S^*(Z\times[0,1])\hookrightarrow T_0^*(Z\times[0,1])$$
$$\textit{does not depend on } t\in[0,1]. \tag{3}$$

For $(x,\xi)\in S^*X$ we put

$$a_m^0(x,\xi)=\begin{pmatrix} d'\cdot\mathrm{id}_{\mathscr{E}} & 0\\ 0 & d''\cdot\mathrm{id}_{\mathscr{F}}\end{pmatrix}\in\mathrm{Aut}(\mathscr{E}\oplus\mathscr{F}), \tag{4}$$

where $d'\in\mathscr{D}'$, $d''\in\mathscr{D}''$ are fixed complex numbers, and extend this by homogeneity of degree m to the whole T_0^*X, using the chosen embedding (3).

This would be an (unsmoothed) principal symbol of the operator A.

Step 2. Choose any locally finite covering by coordinate charts $\{Z^\kappa\}$ of the manifold Z. Put $X^\kappa=Z^\kappa\times(1/4,3/4)$. In this manner we get covering by charts $\{X^\kappa\}$ of the (open) manifold $Z\times(1/4,3/4)$.

Let $\{X^\lambda\}$, $\{X^\mu\}$ be such locally finite families of charts, that

$$\mathscr{U}_{1/4}\subset\bigcup_\lambda X^\lambda\subset\mathscr{U}_{1/3},$$

$$\mathscr{U}_{3/4}\subset\bigcup_\mu X^\mu\subset\mathscr{U}_{2/3}.$$

$\{X^\kappa\}$, $\{X^\lambda\}$, $\{X^\mu\}$ together form a certain covering $\{X^\nu\}$ of the whole X.

Let $\{\varphi^\nu;\psi^\nu\}$ be a subordinated partition of unity together with cut-off functions, i.e.

(a) $\varphi^\nu, \psi^\nu \in \mathscr{C}_0^\infty(X^\nu)$;
(b) $0 \leqq \varphi^\nu \leqq 1$; $0 \leqq \psi^\nu \leqq 1$;
(c) $\sum_\nu \varphi^\nu \equiv 1$;
(d) $\psi^\nu \equiv 1$ on $\operatorname{supp}\varphi^\nu$.

We require in addition, that:
(e) $\varphi^\kappa, \psi^\kappa$ are invariant with respect to symmetry $t \mapsto 1-t$ $(t \in [0,1])$;
(f) $\operatorname{supp}\psi^\lambda \subset \mathscr{U}_{2/5}$, $\operatorname{supp}\psi^\mu \subset \mathscr{U}_{3/5}$.

Small changes in the ordinary construction of a partition of unity always allow to fulfil these additional conditions.

Step 3. Let $\omega(\xi)$ be the standard smoothing function in $\mathbb{R}^n$:

$$\omega(\xi) = \begin{cases} 0 & \text{for } |\xi| \leqq \frac{1}{2}, \\ 1 & \text{for } |\xi| \geqq 1. \end{cases}$$

By A^ν we denote an operator $\mathscr{C}_0^\infty(X^\nu, \mathbb{C}^r) \to \mathscr{C}^\infty(X^\nu, \mathbb{C}^r)$ with a kernel which in local coordinates looks like

$$\int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} \omega(\xi) a_m^0(x, \xi) |d\xi'| \qquad (x, y \in X^\nu). \tag{5}$$

Let Φ^ν, Ψ^ν be operators of multiplication by φ^ν and ψ^ν, respectively. Put, at last,

$$A = \sum_\nu \Phi^\nu A^\nu \Psi^\nu. \tag{6}$$

Obviously, $A \in Ell^m_{\mathscr{D}', \mathscr{D}''}(X, \theta^r_X)$.

Step 4. Form a covering $\{X^\alpha\}$ of the manifold DX_- by taking the covering $\{X^\kappa\}$ and "twice" the covering $\{X^\lambda\}$:

$$\{X^\alpha\} := \{X^\lambda_{(1)}\} \cup \{X^\kappa\} \cup \{X^\lambda_{(2)}\}.$$

This is a covering of DX_- by coordinate charts. Due to condition (e) (see Step 2) the family of functions

$$\{\varphi^\alpha\} := \{\varphi^\lambda_{(1)}\} \cup \{\varphi^\kappa\} \cup \{\varphi^\lambda_{(2)}\}$$

is automatically a partition of unity subordinated to $\{X^\alpha\}$, equipped with the corresponding family of cut-off functions $\{\psi^\alpha\}$. Put, as before,

$$B_- = \sum_\alpha \Phi^\alpha A^\alpha \Psi^\alpha. \tag{7}$$

Remember that φ^κ, ψ^κ are invariant with respect to symmetry $t \mapsto 1-t$.

Moreover, A^κ by its definition and the requirement on the embedding $S^*X \hookrightarrow T_0^*X$ (see (3)) is also invariant with respect to $t \mapsto 1-t$. Hence restrictions of B_- to $X_-^{(1)}$ or $X_-^{(2)}$ are equal.

On the other hand, by comparing (6) and (7), we get

$$A|_{X_-} = \sum_\lambda \Phi^\lambda A^\lambda \Psi^\lambda + \sum_\kappa (\Phi^\kappa A^\kappa \Psi^\kappa)|_{Z \times (0,1/2)} = B_-|_{X_-^{(1)}},$$

i.e. $A|_{X_-}=B_-|_{X^{(1)}_-}=B_-|_{X^{(2)}_-}$. If one defines (in exactly the same manner) the operator B_+ in terms of the covering $\{X^\beta\}:=\{X^\mu_{(2)}\}\cup\{X^\kappa\}\cup\{X^\mu_{(1)}\}$, then one obtains

$$A|_{X_+}=B_+|_{X^{(1)}_+}=B_+|_{X^{(2)}_+}.$$

Furthermore, by the very definition,

$$\mathscr{E}'_{B_-}=D\mathscr{E}_-, \qquad \mathscr{E}'_{B_+}=D\mathscr{E}_+.$$

Step. 5. For any local invariant $\mathscr{R}$ we can write:

$$\int_X \mathscr{R}(A)=\int_{X_-}\mathscr{R}(A)+\int_{X_+}\mathscr{R}(A)=\int_{X_-}\mathscr{R}(B_-)+\int_{X_+}\mathscr{R}(B_+)$$
$$=\tfrac{1}{2}\Big\{\int_{DX_-}\mathscr{R}(B_-)+\int_{DX_+}\mathscr{R}(B_+)\Big\}.$$

By the assumption, $\mathscr{E}'_{B_-}=D\mathscr{E}_-$ has to be trivial, so $\int_{DX_-}\mathscr{R}(B_-)=0$ (see Corollary 1.16). Therefore we obtain finally

$$\int_X \mathscr{R}(A)=\tfrac{1}{2}\int_{DX_+}\mathscr{R}(B_+).$$

The lemma has been proven. □

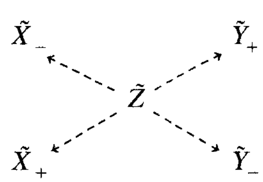
3.4. And now, let us consider the more general situation. Assume that there are given two manifolds each represented as a sum of its submanifolds glued together along a common boundary:

$$X=X_-\cup X_+, \qquad Y=Y_-\cup Y_+.$$

We assume, moreover, ∂X_- and ∂Y_- to be diffeomorphic to the same closed manifold Z, and so X, Y can be cut and glued anew into another pair of manifolds:

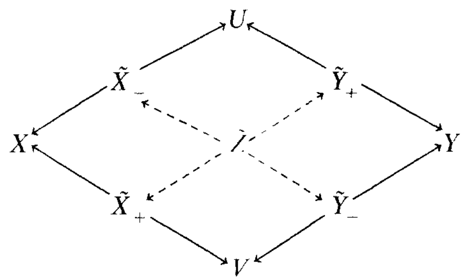
$$U=X_-\cup Y_+, \qquad V=Y_-\cup X_+.^5$$

To fix smooth structures on U, V let us choose open neighbourhoods $X_\pm\subset\tilde{X}_\pm\subset X$, $Y_\pm\subset\tilde{Y}_\pm\subset Y$ such that $\tilde{X}_-\cap\tilde{X}_+$ and $\tilde{Y}_-\cap\tilde{Y}_+$ are diffeomorphic to $\tilde{Z}:=Z\times(0,1)$ and induced diffeomorphisms carry ∂X_-, ∂Y_- onto $Z\times\{\tfrac{1}{2}\}$. This uniquely determines embeddings:



[5] If X, Y are oriented, then all identifications and submanifolds are considered in the oriented category. In particular, then, U and V have natural orientations

which are agreed one with another, i.e. the following diagram of embeddings is commutative:



(8)

Solid arrows denote the canonical embeddings. Already, the diagram of this kind uniquely determines smooth structures on U and V. An analogous diagram of co-sphere bundles hangs over that.

3.5. Let us suppose, in addition, that there are *given* vector bundles $\mathscr{E}_X$, $\mathscr{E}_Y$ on the corresponding co-sphere bundles, which *coincide* on $S^*\tilde{Z}$.

3.6. *Remark.* For our purposes we are forced to make precise the notions "given" and "coincide". We will say below (in this and the following sections), that a vector bundle of rank r on a space W *is given* iff one exhibits a Čech 1-cocycle $\mathscr{E}\in Z^1(\{W^\alpha\}, GL_r(\mathbb{C}))$, where $\{W^\alpha\}$ is some locally finite covering of W.

By a restriction of $\mathscr{E}$ to a subspace $W_0\subset W$ we mean an element $\mathscr{E}|_{W_0}\in Z^1(\{W^\alpha\cap W_0\}, GL_r(\mathbb{C}))$.

By *coincidence* of two vector bundles $\mathscr{E}$ and $\mathscr{F}$ we mean that 1^{mo}: coincide the corresponding coverings, 2^{ndo}: coincide 1-cocycles themselves. Such coincidence will sometimes be called coincidence in *the narrow sense*.

3.7. Return to our bundles $\mathscr{E}_X\to S^*X$, $\mathscr{E}_Y\to S^*Y$. Thanks to their coincidence on $S^*\tilde{Z}$ they can be uniquely glued as new bundles $\mathscr{E}_U\to S^*U$, $\mathscr{E}_V\to S^*V$. In this situation the following analogue of the Lemma on the double holds.

3.8. **Lemma** *(on the cutting and gluing). Let $\mathscr{E}_X$, $\mathscr{E}_Y$ and $\mathscr{E}_U$, $\mathscr{E}_V$ be the vector bundles defined above. Then there exist operators A_X, A_Y and A_U, A_V acting on sections of trivial vector-bundles such that*

(a) $A_X\in Ell^m_{\mathscr{D}',\mathscr{D}''}(X)$, $A_Y\in Ell^m_{\mathscr{D}',\mathscr{D}''}(Y)$;

(a′) $A_U\in Ell^m_{\mathscr{D}',\mathscr{D}''}(U)$, $A_V\in Ell^m_{\mathscr{D}',\mathscr{D}''}(V)$;

(b) $\mathscr{E}'_{A_X}=\mathscr{E}_X$, $\mathscr{E}'_{A_Y}=\mathscr{E}_Y$;

(b′) $\mathscr{E}'_{A_U}=\mathscr{E}_U$, $\mathscr{E}'_{A_V}=\mathscr{E}_V$;

(c) $A_X|_{X_-}=A_U|_{X_-}$, $A_X|_{X_+}=A_V|_{X_+}$, $A_Y|_{Y_-}=A_V|_{Y_-}$, $A_Y|_{Y_+}=A_U|_{Y_+}$.

In particular, for any local invariant $\mathscr{R}$

$$\int_X\mathscr{R}(A_X)+\int_Y\mathscr{R}(A_Y)=\int_U\mathscr{R}(A_U)+\int_V\mathscr{R}(A_V).$$

3.9. *Remark.* As a matter of fact, out of arbitrary ψDOs A_X, A_Y agreeing on some neighbourhood $\mathscr{Z}\supset Z\times\{\frac{1}{2}\}$ one can always construct their "re-glues" A_U,

A_V satisfying the condition (c). This requires more subtle means than "coarse" partitions of unity, and exceeds the limits of this paper. For our purposes it is quite enough if there *exist* operators A_X, A_Y which could be glued anew into another pair of operators, and could have required sector splittings.

Proof. First of all, look for embeddings into the trivial bundles $\mathscr{E}_X \hookrightarrow \theta^k_{S^*X}$, $\mathscr{E}_Y \hookrightarrow \theta^k_{S^*Y}$ $(k \gg 0)$ such that the following diagram

$$\begin{array}{ccc} \mathscr{E}_X|_{S^*\tilde{Z}} & \hookrightarrow & \theta^k_{S^*\tilde{Z}} \\ \| & & \| \\ \mathscr{E}_Y|_{S^*\tilde{Z}} & \hookrightarrow & \theta^k_{S^*\tilde{Z}} \end{array} \tag{9}$$

commutes.

For a vector bundle on a compact space $\mathscr{E} \to W$ given as a 1-cocycle $\mathscr{E} \in Z^1(\{W^\alpha\}, GL_r(\mathbb{C}))$ there is a standard way how to construct an epimorphism $\theta^k_W \twoheadrightarrow \mathscr{E}$ $(k \gg 0)$. To do this it is sufficient to choose only a partition of unity subordinated to the covering $\{W^\alpha\}$ (for details we refer for example to [6, Theorem I.6.5]). Moreover, when there are given two vector bundles $\mathscr{E} \to W_1$, $\mathscr{F} \to W_2$, coinciding on some subspace $W_0 \subset W_i (i = 1, 2)$, then, by properly choosing partitions of unity, one can achieve the coincidence of the epimorphisms $\theta^k_{W_1} \twoheadrightarrow \mathscr{E}$, $\theta^k_{W_2} \twoheadrightarrow \mathscr{F}$ on W_0, i.e. commutativity of the diagram dual to the diagram of the type (9). Applying all this to $\mathscr{E} = \mathscr{E}^*_X$, $\mathscr{F} = \mathscr{E}^*_Y$ and dualizing the picture we obtain the required embeddings $\mathscr{E}_X \hookrightarrow \theta^k_{S^*X}$, $\mathscr{E}_Y \hookrightarrow \theta^k_{S^*Y}$. Furthermore, it costs nothing to provide splittings

$$\theta^k_{S^*X} = \mathscr{E}_X \oplus \mathscr{F}_X, \qquad \theta^k_{S^*Y} = \mathscr{E}_Y \oplus \mathscr{F}_Y, \tag{10}$$

which coincide on $S^*\tilde{Z}$ (this can be done, for example, using *canonical* metrics in $\theta^k_{S^*X}$, $\theta^k_{S^*Y}$).

The splittings (10) will serve as sector splittings of the operators A_X, A_Y, that we have to construct. The operators A_X, A_Y, A_U, A_V themselves one constructs exactly by the same scheme as in the proof of Lemma on the double (steps 1–4). Besides, requirements on the embeddings $S^*X \hookrightarrow T^*_0 X$, $S^*Y \hookrightarrow T^*_0 Y$ and on partitions of unity can be weakened (for example, "flatness" of $S^*X \hookrightarrow T^*_0 X$ or invariance of partitions of unity with respect to $t \mapsto 1 - t$ on $\tilde{Z} = Z \times (0, 1)$ are unnecessary).

Details are left to the reader as a routine exercise. □

4. The proof of Proposition Λ

Our aim in this section is to prove Proposition Λ_0 and the implication

$$\text{Theorem A and Theorem B}_0 \Rightarrow \text{Proposition } \Lambda.$$

Everywhere in this section N stands for a *closed* oriented manifold of *odd* dimension, all K-groups are tacitly considered to have rational coefficients, and $R \equiv \int \mathscr{R}$ is a (fixed) admissible local K-invariant (see 1.3). As we know (cf. 1.17) any such invariant gives rise to a $\mathbb{Q}$-linear mapping $K(S^*X)/\pi^* K(X) \to \mathbb{C}$, where X is a closed manifold. For another notation we refer to Sect. 2.

4.1. Assume N, for a moment, to admit of a diffeomorphism $f\colon N \to N$ reversing orientation. Let $\psi = S^*(\mathrm{id}_Q \times f)$ be the induced automorphism of $S^*(Q \times N)$. Obviously $\psi^*(\varepsilon \otimes \mathbb{1}_N \otimes \omega_S) = \varepsilon \otimes \mathbb{1}_N \otimes (-\omega_S) = -\varepsilon \otimes \mathbb{1}_N \otimes \omega_S$. In view of the Diff$X$-equivariance of the isomorphism $\delta \circ ch_{\mathbb{Q}}$ (cf. 2.3) this implies that

$$\psi^* \lambda(N; Q, \varepsilon) = -\lambda(N; Q, \varepsilon). \tag{1}$$

Put $\lambda \equiv \lambda(N; Q, \varepsilon)$. Due to (1) $R(\psi^* \lambda) = -R(\lambda)$, on the other hand, in view of Corollary 1.21, $R(\psi^* \lambda) = R(\lambda)$. Hence, $R(\lambda)$ must be zero.

So, we have proved the following assertion (making no use of the localness of R).

4.2. **Lemma.** *If N admits an orientation reversing diffeomorphism, then for any admissible invariant $R\colon K(S^*(Q \times N)) \to \mathbb{C}$*

$$R(\lambda(N; Q, \varepsilon)) = 0. \quad \square$$

4.3. *Remark.* We used the same argument with an orientation reversing diffeomorphism in the proof of the Lemma on the spheres (see 2.13). This will be used many times in this and the following sections.

4.4. In the general case, undoubtedly, N may lack such an orientation reversing diffeomorphism. Still, when dealing with local invariants it is permitted to apply surgery.

Thus, let $N = N_- \cup N_+$, $M = M_- \cup M_+$ be two closed manifolds each represented as a sum of its proper submanifolds glued together along a common boundary.

Assume that:

a) an oriented identification of normal neighbourhoods $\nu(\partial N_-) \subset N$ and $\nu(\partial M_-) \subset M$ is fixed such that the diagram:

$$\begin{array}{ccccc} \nu(\partial N_-) \cap N_- & \hookrightarrow & \nu(\partial N_-) & \hookleftarrow & \nu(\partial N_-) \cap N_+ \\ \wr\| & & \wr\| & & \wr\| \\ \nu(\partial M_-) \cap M_- & \hookrightarrow & \nu(\partial M_-) & \hookleftarrow & \nu(\partial M_-) \cap M_+ \end{array}$$

is commutative;

b) there are *given* vector bundles $\mathscr{E}_N$, $\mathscr{E}_M$ on $S^*(Q \times N)$, $S^*(Q \times M)$, and their restrictions to $S^*(Q \times N)|_{\nu(\partial N_-) \times Q}$ and resp. to $S^*(Q \times M)|_{\nu(\partial M_-) \times Q}$ *coincide* (in the sense of 3.6, and with respect to identifications fixed at point a));

c) $[\mathscr{E}_N] = s\lambda(N; Q, \varepsilon)$ in $\tilde{K}(S^*(Q \times N))$, and $[\mathscr{E}_M] = s\lambda(M; Q, \varepsilon)$ in $\tilde{K}(S^*(Q \times N))$, where s is some positive integer.

Then, N and M can be glued anew into a pair of oriented manifolds

$$N' := N_- \cup M_+, \qquad M' := M_- \cup N_+.$$

Moreover, the bundles $\mathscr{E}_N, \mathscr{E}_M$ glue anew into the bundles $\mathscr{E}_{N'}, \mathscr{E}_{M'}$:

$$\begin{aligned} \mathscr{E}_{N'} &:= \mathscr{E}_{N_-} \cup \mathscr{E}_{M_+} \quad \text{on} \quad S^*(Q \times N'), \\ \mathscr{E}_{M'} &:= \mathscr{E}_{M_-} \cup \mathscr{E}_{N_+} \quad \text{on} \quad S^*(Q \times M'). \end{aligned}$$

4.5. *Remark*. To ensure a fulfilment of the conditions b) and c) above one may proceed in the following way.

Denote for brevity $Q\times\nu(\partial N_-)\simeq Q\times\nu(\partial M_-)$ by $\mathscr{X}_0$. We can assume, that a bit larger normal neighbourhoods have been identified:

$$\begin{array}{ccc} \nu_1(\partial N_-) & \simeq & \nu_1(\partial M_-) \\ \cup & & \cup \\ \nu(\partial N_-) & \simeq & \nu(\partial M_-) \end{array}$$

and by $\mathscr{X}_1$ we denote $Q\times\nu_1(\partial N_-)\simeq Q\times\nu_1(\partial M_-)$. $\mathscr{X}_0,\mathscr{X}_1$ are closed subsets of $Q\times M$.

Let $\mathscr{E}_N^1$ be any vector bundle such that $[\mathscr{E}_N^1]=\lambda(N;Q,\varepsilon)$ in $\tilde{K}(S^*(Q\times N))$ and $\mathscr{F}$ - be its restriction to $S^*\mathscr{X}_1$. It would be sufficient to know that the *given* bundle $\mathscr{F}$ on $S^*\mathscr{X}_1$ could be extended on the whole $S^*(Q\times M)$ to some bundle $\mathscr{E}_M$ such that $[\mathscr{E}_M]=\lambda(M;Q,\varepsilon)$ in $\tilde{K}(S^*(Q\times M))$. One may attempt to provide such an extension in the following way. Let $\mathscr{E}_M^0$ be any vector bundle representing the class $\lambda(M;Q,\varepsilon)$ and with the same rank as $\mathscr{E}_N^1$. Put

$$\mathscr{X}_\pm=Q\times[M_\pm\cap\overline{(\nu_1(\partial M_-)\setminus\nu(\partial M_-))}]=Q\times M_\pm\cap\overline{(\mathscr{X}_1\setminus\mathscr{X}_0)}$$

(see Fig. 1). In view of the functoriality of the $(Q,\mathscr{E})$-class (cf. 2.9) the *rational* classes $[\mathscr{E}_M^0|_{S^*\mathscr{X}_\pm}]$ and $[\mathscr{F}|_{S^*\mathscr{X}_\pm}]$ are equal. Hereupon, for some $s_1\in\mathbb{N}$ $(s_1\neq0)$, we have an equality of the *integral* classes $s_1[\mathscr{E}_M^0|_{S^*\mathscr{X}_\pm}]$ and $s_1[\mathscr{F}|_{S^*\mathscr{X}_\pm}]$. And so, adding, if necessary, from the very beginning to each $\mathscr{E}_M^0$ and $\mathscr{E}_N^1$ the trivial vector bundle we may assume the existence od isomorphisms

$$\varphi_\pm:\mathscr{E}_M^1|_{S^*\mathscr{X}_\pm}\xrightarrow{\sim}\mathscr{G}|_{S^*\mathscr{X}_\pm}.$$

Here we have put $\mathscr{E}_M^1=(\mathscr{E}_M^0)^{\oplus s_1}$, $\mathscr{E}_N^2=(\mathscr{E}_N^1)^{\oplus s_1}$, $\mathscr{G}=\mathscr{F}^{\oplus s_1}=\mathscr{E}_N^2|_{S^*\mathscr{X}_1}$. Pasting $\mathscr{E}_M^1|_{S^*(Q\times M_\pm\setminus\mathscr{X}_0)}$ and $\mathscr{G}$ with the aid of the isomorphisms $\varphi_\pm$ we obtain some new bundle $\mathscr{E}_M^2$. In view of the Mayer-Vietoris exact sequence in K-theory with rational coefficients:

$$\begin{aligned}\dots\to K^{-1}(S^*\mathscr{X}_-)\oplus K^{-1}(S^*\mathscr{X}_+)\xrightarrow{\Delta}K(S^*(Q\times M))\to\\ \to K(S^*(\overline{Q\times M_-\setminus\mathscr{X}_0}))\oplus K(S^*(\overline{Q\times M_+\setminus\mathscr{X}_0}))\oplus K(S^*\mathscr{X}_0)\to\dots\end{aligned}$$

the difference $[\mathscr{E}_M^2]-s_1\lambda(M;Q,\varepsilon)$ must lie in the range of Δ, i.e. there exist

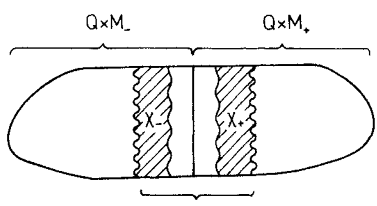


Fig. 1.

1) mappings $\gamma_\pm: S^*\mathscr{X}_\pm \to GL_k(\mathbb{C})$, where k is a sufficiently large integer,
2) a positive integer s_2, such that

$$s_2([\mathscr{E}_M^2] - s_1\lambda(M; Q, \varepsilon)) = \Delta[\gamma_-] + \Delta[\gamma_+],$$

where $[\gamma_\pm]$ are the corresponding classes in $K^{-1}(S^*\mathscr{X}_\pm)$.

Put $\mathscr{E}_N^3 = (\mathscr{E}_N^2)^{\oplus s_2}$, $\mathscr{E}_M^3 = (\mathscr{E}_M^2)^{\oplus s_2}$. Add to $\mathscr{E}_M^3$ a bundle obtained by pasting three trivial bundles: $\theta^k_{S^*(Q\times M_\pm \setminus \mathscr{X}_0)}$, $\theta^k_{S^*\mathscr{X}_1}$ with the aid of the mappings $\gamma_\pm^{-1}$. Denote the obtained bundle by $\mathscr{E}_M$. Of course

$$[\mathscr{E}_M] = [\mathscr{E}_M^3] - \Delta[\gamma_-] - \Delta[\gamma_+] = s_1 s_2 \lambda(M; Q, \varepsilon).$$

Add to $\mathscr{E}_N^3$ the trivial bundle $\theta^k_{S^*(Q\times N)}$ and denote the obtained bundle by $\mathscr{E}_N$. In view of all of the above $\mathscr{E}_N|_{S^*\mathscr{X}_0}$ and $\mathscr{E}_M|_{S^*\mathscr{X}_0}$ *coincide* (in the narrow sense). Moreover, the condition 4.4c is satisfied with $s := s_1 s_2$.

4.6. Return to the bundles $\mathscr{E}_{N'} = \mathscr{E}_{N_-} \cup \mathscr{E}_{M_+}$, $\mathscr{E}_{M'} = \mathscr{E}_{M_-} \cup \mathscr{E}_{N_+}$. Using again the functoriality of the (Q, ε)-class and the Mayer-Vietoris argument we conclude that

$$t([\mathscr{E}_{N'}] - s\lambda(N'; Q, \varepsilon)) = \Delta[\delta_1], \qquad t([\mathscr{E}_{M'}] - s\lambda(M'; Q, \varepsilon)) = \Delta[\delta_2],$$

where δ_1, δ_2 are some mappings $\mathscr{Z} \to GL_l(\mathbb{C})$, $\mathscr{Z}$ stands for $S^*(Q \times N)|_{Q\times\partial N_-} \simeq S^*(Q\times M)|_{Q\times\partial M_-}$, and t is some positive integer.

Embed $\mathscr{Z} \times [0,1]$ into $S^*(Q\times N)$ so that

a) $\mathscr{Z} \times \{\frac{1}{2}\}$ goes into $\mathscr{Z} \subset S^*(Q\times N)$;
b) $\mathscr{Z} \times \{0\} \subset S^*(Q\times[N_- \setminus \nu(\partial N_-)])$, $\mathscr{Z}\times\{1\} \subset S^*(Q\times[N_+\setminus\nu(\partial N_-)])$.

Let us consider a bundle $\mathscr{H}$ on $S^*(Q\times N)$ obtained by pasting three trivial bundles with the aid of the mappings:

$$\delta_1^{-1}: \mathscr{Z}\times\{0\} \to GL_l(\mathbb{C}), \qquad \delta_1: \mathscr{Z}\times\{1\}\to GL_l(\mathbb{C}).$$

Of course, $\mathscr{H}$ is trivializable. Replace the initial $\mathscr{E}_N, \mathscr{E}_M$ by

$$\bar{\mathscr{E}}_N := (\mathscr{E}_N)^{\oplus t} \oplus \mathscr{H} \quad \text{and} \quad \bar{\mathscr{E}}_M := (\mathscr{E}_M)^{\oplus t} \oplus \theta^l_{S^*(Q\times M)},$$

respectively. Surely, our "new" $\mathscr{E}_N, \mathscr{E}_M$ also satisfy the conditions 4.4b–c, but with another $\bar{s} := st$. Furthermore,

$$\begin{aligned}
[\mathscr{E}_{N'}] &= \bar{s}\lambda(N'; Q, \varepsilon) \quad \text{in } \tilde{K}(S^*(Q\times N')),\\
[\mathscr{E}_{M'}] &= \bar{s}\lambda(M'; Q, \varepsilon) + \Delta[\delta_2] + \Delta[\delta_1^{-1}]\\
&= \bar{s}\lambda(M'; Q, \varepsilon) + \Delta[\delta] \quad \text{in } \tilde{K}(S^*(Q\times M')),
\end{aligned} \tag{2}$$

where $\delta = \delta_2\delta_1^{-1}: \mathscr{Z} \to GL_l(\mathbb{C})$.

4.7. If we put $X_\pm = Q\times N_\pm$, $Y_\pm = Q\times M_\pm$, $\mathscr{E}_X = \bar{\mathscr{E}}_N$, $\mathscr{E}_Y = \bar{\mathscr{E}}_M$, then we land in a situation covered by Lemma on the cutting and gluing (cf. 3.8), and therefore

$$R(\bar{\mathscr{E}}_N) + R(\bar{\mathscr{E}}_M) = R(\bar{\mathscr{E}}_{N'}) + R(\bar{\mathscr{E}}_{M'}),$$

i.e. for some $\delta\colon \mathscr{X} \to GL_l(\mathbb{C})$ and integer $\bar{s} \neq 0$:

$$R(\lambda(N)) + R(\lambda(M)) = R(\lambda(N')) + R(\lambda(M') + \bar{s}^{-1} \Delta[\delta]). \tag{3}$$

Here, for brevity, we omit letters Q and ε.

4.8. Let us specialize the procedure of cutting and gluing anew, described above, to an important particular case of the surgery $N \rightsquigarrow N'$ done on a sphere $S^j \hookrightarrow N$ $(0 \leqq j \leqq n)$ embedded with a trivial normal bundle. The corresponding decompositions are:

$$N = N_- \bigcup_{S^j \times S^{n-j-1}} S^j \times D^{n-j}, \qquad M = D^{j+1} \times S^{n-j-1} \bigcup_{S^j \times S^{n-j-1}} D^{j+1} \times S^{n-j-1},$$

and

$$N' = N_- \bigcup_{S^j \times S^{n-j-1}} D^{j+1} \times S^{n-j-1}, \qquad M' = D^{j+1} \times S^{n-j-1} \bigcup_{S^j \times S^{n-j-1}} S^j \times D^{n-j}.$$

Observe that M is diffeomorphic to $S^{j+1} \times S^{n-j-1}$, and M' to S^n. In particular, M admits of an orientation reversing diffeomorphism, so we have $R(\lambda(M; Q, \varepsilon)) = 0$, in view of Lemma 4.2. Thus (3) transforms into

$$R(\lambda(N; Q, \varepsilon)) = R(\lambda(N'; Q, \varepsilon)) + R(e), \tag{4}$$

for some $e \in \tilde{K}(S^*(Q \times S^n))$. As a matter of fact $R(e)$ must be zero.

4.9. **Lemma.** *Let $N \rightsquigarrow N'$ denote the surgery, as above. Then*

$$R(\lambda(N; Q, \varepsilon)) = R(\lambda(N'; Q, \varepsilon)). \tag{5}$$

Proof. If Q is a product of spheres one can apply the Lemma on the spheres (cf. 2.13) to ensure that $R(e) = 0$.

In the general case one must appeal to part of the statements of Theorems A and B.

If $\dim Q = \text{odd}$, then according to Theorem A

$$R(\lambda(N; Q, \varepsilon)) = R(\lambda(N'; Q, \varepsilon)) = 0.$$

If $\dim Q = \text{even}$, then according to Theorem B_0 $R \equiv 0$ for manifolds of the form $Q^{\text{ev}} \times S^{\text{odd}}$, so, in particular, $R(e) = 0$ and the equality (5) holds. □

4.10. The last step consists in modifying N so as to obtain a manifold admitting of an orientation reversing diffeomorphism, and afterwards, of applying Lemma 4.2.

As is commonly known, the group of odd-dimensional oriented bordisms Ω_{odd} consists entirely of torsion (cf. [10, p. 42]). Hence, some multiplicity of our manifold is bordant to the standard sphere $kN \sim S^n$. By the Morse theory such a bordism can be factored into a sequence of traces of surgeries

$$kN = N_0 \rightsquigarrow N_1 \rightsquigarrow \ldots \rightsquigarrow N_r = S^n,$$

such as in 4.8. In view of Lemma 4.9 $R(\lambda(kN; Q, \varepsilon)) = R(\lambda(S^n; Q, \varepsilon))$. But S^n admits of an orientation reversing diffeomorphism, hence $R(\lambda(S^n; Q, \varepsilon)) = 0$, according to Lemma 4.2. On the other hand

$$R(\lambda(kN; Q, \varepsilon)) = kR(\lambda(N; Q, \varepsilon))$$

due to localness of R. As a result $R(\lambda(N; Q, \varepsilon)) = 0$.

Summarizing, we have proved Proposition Λ_0, and the implication

$$\text{Theorem A} \quad \text{and} \quad \text{Theorem B}_0 \Rightarrow \text{Proposition } \Lambda.$$

5. The proof of Theorem A

In this section we prove Theorem A making use of Proposition Λ_0 (cf. 2.12). The proof consists of a series of reductions, which reduce the question of local K-invariants of spectral asymmetry to the case of (Q, ω_Q)-classes, where $Q = S^k$ (standard odd-dimensional sphere). Everywhere in this section X^d stands for a closed even-dimensional orientable manifold.

5.1. If $N \subset X$ is a closed submanifold, then its normal neighbourhood will be denoted by $\mathcal{N}$. It can be considered as a submanifold with a boundary. We say that a vector bundle $\mathscr{E}$ on S^*X is *twisted along* N, if the restriction of $\mathscr{E}$ to $S^*(X \setminus \mathcal{N})$ is trivializable.

The following proposition holds.

5.2. **Proposition.** *$K(S^*X)/\pi^*K(X) \otimes \mathbb{Q}$ is spanned by vector bundles twisted along odd-dimensional submanifolds $N \subset X$ with trivial normal bundles.*

Proof. Step. 1. The cohomological Gysin sequence of the fibre bundle $\pi\colon S^*X \xrightarrow{S} X$, as we already observed in 2.2, determines the isomorphism

$$\delta\colon H^{\mathrm{ev}}(S^*X)/\pi^*H^{\mathrm{ev}}(X) \xrightarrow{\sim} H^{\mathrm{odd}}(X) \otimes H^{d-1}(S) \tag{1}$$

($d = \dim X$). Examine the group $H^{\mathrm{odd}}(X)$ in more detail.

Let $[\alpha] \in H^k(X)$ be represented by a map $\alpha\colon X \to K(\mathbb{Q}, k)$ ($k = $ odd). The embedding $S^k \hookrightarrow K(\mathbb{Z}, k)$ onto a cell of lowest dimension, for $k = $ odd, is a rational homotopy equivalence, i.e. induces a homotopy equivalence $S^k \otimes \mathbb{Q} \sim K(\mathbb{Q}, k)$. A standard model of the rational sphere has the form of an infinite telescope of a sequence of mappings

$$S^k \xrightarrow{l_1} S^k \xrightarrow{l_2} S^k \longrightarrow \ldots,$$

where $l_i\colon S^k \to S^k$ is a mapping of degree l_i. Since X is compact, any continuous map $X \to S^k \otimes \mathbb{Q}$ must factor through a telescope of some finite sequence

$$S^k \xrightarrow{l_1} S^k \xrightarrow{l_2} \ldots \xrightarrow{l_\nu} S^k.$$

Denote this telescope by T_l $\left(l := \prod_{i=1}^{\nu} l_i\right)$. The natural retraction onto the "last" sphere $T_l \to S^k$ is a homotopy equivalence, and $S^k \sim T_l \hookrightarrow S^k \otimes \mathbb{Q}$ induces a homomorphism $\tilde{H}_*(S^k, \mathbb{Z}) \to \tilde{H}_*(S^k, \mathbb{Z})$ of $\mathbb{Z}$-modules:

$$\mathbb{Z} \to \mathbb{Z}\langle 1/l \rangle \subset \mathbb{Q}$$
$$1 \mapsto 1/l.$$

Applying the above to the map $\alpha\colon X\to K(\mathbb{Q},k)\sim S^k\otimes\mathbb{Q}$ we get that α factors through some finite telescope T_l, and therefore, there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \alpha\ } & K(\mathbb{Q},k) \\ & \searrow^{\beta} & \uparrow{\scriptstyle 1/l} \\ & & S^k \end{array}$$

Of course $[\beta]:=\beta^*\omega_{S^k}=l\alpha^*\iota_k=l[\alpha]$, where $\iota_k\in H^k(K(\mathbb{Q},k))$ is a fundamental class. So, we have showed, that for any $[\alpha]\in H^{\mathrm{odd}}(X)$ its certain integral multiplicity is induced by a map $\beta\colon X\to S^{\mathrm{odd}}$. In establishing this we have required of X only that it be a compact CW-complex.

Step. 2. Performing smooth approximation, if necessary, we may assume $\beta\colon X\to S^k$ to be smooth. Put $N_\beta=\beta^{-1}(s)$ – preimage of some regular value of β. Orientation of X (and the standard one of S^k) uniquely determines orientation of the general fibre N_β. Besides $[N_\beta]\in H_{d-k}(X)$ and $[\beta]\in H^k(X)$ are mutually Poincaré-dual. Indeed, $[N_\beta]=\beta_![1_{S^k}]$, where $\beta_!$ is a homological transfer. Now, the assertion results from the relation between the transfer and the Poincaré-duality.

Step 3. The normal neighbourhood can be chosen in the form $\beta^{-1}(D)$, where $D\subset S^k$ is a sufficiently small (closed) disc containing the point s. In particular then, $\mathcal{N}$ is diffeomorphic to $N_\beta\times D$. Restrictions of $[\beta]$ to $\mathcal{N}$ or to $\overline{X\setminus\mathcal{N}}$ are easily seen to be zero. In fact, $[\beta]\in H^k(X)$ comes from $H^{k-1}(\partial\mathcal{N})$.

Moreover,

$$\delta^{-1}([\beta]\otimes\omega_S)|_{S*\overline{(X\setminus\mathcal{N})}}\subset\pi^*H^{k+d-1}(X)|_{S*\overline{(X\setminus\mathcal{N})}}. \tag{2}$$

Indeed, we have the commutative diagram (by $\mathcal{U}$ we denote $\overline{X\setminus\mathcal{N}}$):

$$\begin{array}{ccccccc} & & H^{k+d-1}(X) & \xrightarrow{\pi^*} & H^{k+d-1}(S^k X) & \xrightarrow{\delta} & H^k(X)\otimes H^{d-1}(S)\to 0 \\ & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle j^*\otimes\mathrm{id}} \\ & & H^{k+d-1}(\mathcal{U}) & \xrightarrow{\pi^*} & H^{k+d-1}(S^*\mathcal{U}) & \xrightarrow{\delta} & H^k(\mathcal{U})\otimes H^{d-1}(S) \\ & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\ H^k(X,\mathcal{U})\otimes H^{d-1}(S) & \xrightarrow{0} & H^{k+d}(X,\mathcal{U}) & \xrightarrow{\pi^*} & H^{k+d}(S^*X,S^*\mathcal{U}) & & \end{array}$$

whose columns are exact cohomological sequences of the pairs $(X,\mathcal{U})$, $(S^*X,S^*\mathcal{U})$ and rows are the corresponding absolute or relative Gysin sequences (for the relative Gysin sequence see e.g. [9, Theorem 9.5.2, p. 499]). The arrows j^*, ι^* are induced by the embeddings $j\colon\mathcal{U}\hookrightarrow X$, $\iota\colon S^*\mathcal{U}\hookrightarrow S^*X$. The commutativity of the top right square follows from the functoriality properties of the Gysin sequence (ibid., p. 498). The arrow $H^k(X,\mathcal{U})\otimes H^{d-1}(S)\to H^{k+d}(X,\mathcal{U})$ must be zero for all k by the dimensional argument and the observation that $H^0(X,\mathcal{U})=0$.

Now, let $\varepsilon \in \delta^{-1}([\beta]\otimes\omega_S)\subset H^{k+d-1}(S^*X)$. Since we have $(\delta\iota^*)\varepsilon = j^*[\beta]\otimes\omega_S=0$, there exists $\gamma_0\in H^{k+d-1}(\mathcal{U})$ such that $\iota^*\varepsilon=\pi^*\gamma_0$. In fact, γ_0 must belong to $\operatorname{Im} j^*$. Indeed, $(\pi^*\partial)\gamma_0=(\partial\pi^*)\gamma_0=(\partial\iota^*)\varepsilon=0$, but the lowest arrow $\pi^*: H^{k+d}(X,\mathcal{U})\to H^{k+d}(S^*X, S^*\mathcal{U})$ is mono, hence $\partial\gamma_0=0$ and therefore $\gamma_0=j^*\gamma$ for some $\gamma\in H^{k+d-1}(X)$. By observing that $(\iota^*\pi^*)\gamma=\pi^*\gamma_0=\iota^*\varepsilon$ we obtain the required inclusion (2). Therefore, let $\eta\in H^{k+d-1}(S^*X)$ be a such element that: 1° $\eta\in\delta^{-1}([\beta]\otimes\omega_S)$, 2° $\iota^*\eta=0$; and put $b = ch_{\mathbb{Q}}^{-1}(\eta)\in K(S^*X)\otimes\mathbb{Q}$. In view of the above $b|_{S^*\mathcal{U}}=0$ and we know that such classes generate $K(S^*X)/\pi^*K(X)\otimes\mathbb{Q}$ $\mathbb{Q}$-linearly. Because some integral multiplicity of b can be represented in the form $[\mathscr{E}]-[\theta^{\vee}_{S^*X}]$, where $\mathscr{E}$ is some vector-bundle *trivial* on $S^*\mathcal{U}$ (even on its neighbourhood in S^*X) the proposition holds. □

5.3. Let $\mathscr{E}$ be any vector bundle on S^*X which is trivial on a neighbourhood of $S^*\mathcal{U}$. We already know that such vector bundles generate $K(S^*X)/\pi^*K(X)\otimes\mathbb{Q}$. Cut the manifolds $X=\mathcal{N}\cup\mathcal{U}$, $-X=-\mathcal{U}\cup-\mathcal{N}$ along $\partial\mathcal{N}$, $-\partial\mathcal{N}$ and form two doubles $D\mathcal{N}=\mathcal{N}\cup-\mathcal{N}$ and $D\mathcal{U}=-\mathcal{U}\cup\mathcal{U}$. Triviality of $\mathscr{E}$ on $S^*X|_{\partial\mathcal{N}}$ guarantees that $\mathscr{E}|_{S^*\mathcal{N}}$ and $\mathscr{E}|_{S^*(-\mathcal{N})}$ glue together to give the bundle on $S^*(D\mathcal{N})$. The same holds for $\mathscr{E}|_{S^*\mathcal{U}}$ and $\mathscr{E}|_{S^*(-\mathcal{U})}$. Denote these new vector bundles by $\mathscr{E}_{\mathcal{N}}$ and $\mathscr{E}_{\mathcal{U}}$. Clearly, we can apply Lemma on the double (see 3.4) to obtain $R([\mathscr{E}])=\frac{1}{2}R(\mathscr{E}_{\mathcal{N}})$.

Since $\mathcal{N}$ is diffeomorphic to $S^k\times N^n$ ($d=k+n$; k,n – odd) we reduce the question of local K-invariants of spectral asymmetry on even-dimensional manifolds to the case of product-manifolds:

$$X=S^k\times N^n \qquad (k,n=\text{odd}).$$

Below, we can assume N to be connected.

5.4. Recall once more that

$$\delta\colon H^{\text{ev}}(S^*X)/\pi^*H^{\text{ev}}(X)\xrightarrow{\sim} H^{\text{odd}}(X)\otimes H^{d-1}(S),$$

if $d=$ even. In case $X=S^k\times N^n$, as above, $H^{\text{odd}}(X)$ breaks into two pieces

$$H^{\text{odd}}(S^k\times N)=1_{S^k}\otimes H^{\text{odd}}(N)\oplus\omega_{S^k}\otimes H^{\text{ev}}(N).$$

Any reversing orientation automorphism of S^k acts on $1_{S^k}\otimes H^{\text{odd}}(N)\otimes H^{d-1}(S)$ as multiplication by -1. Repeating argumentation from 4.1 and applying Corollary 1.21 we derive that any *admissible* invariant $R\colon K(S^*X)/\pi^*K(X)\to\mathbb{C}$ vanishes on that piece of $K(S^*X)/\pi^*K(X)$, which corresponds to $1_{S^k}\otimes H^{\text{odd}}(N)\otimes H^{d-1}(S)$.

5.5. A proof of the fact:

if R is, in addition, local, then R vanishes also on that piece of $K(S^*X)/\pi^*K(X)$ which *corresponds to* $\omega_{S^k}\otimes H^{\text{ev}}(N)\otimes H^{n+k-1}(S)$,

we are going to carry out by induction on $n=\dim N$ with fixed $k+n=d=\dim X$, and n – odd.

$n=1$. In this case N obviously admits of an orientation reversing diffeomorphism. Its action on $\omega_{S^k}\otimes H^{\text{ev}}(N)\otimes H^{d-1}(S)=\omega_{S^k}\otimes H^0(N)\otimes H^{d-1}(S)$ is, clearly, multiplication by -1. The standard argument (cf. the proof of the Lemma 2.13) shows that R must kill $ch_{\mathbb{Q}}^{-1} \circ \delta^{-1}(\omega_{S^k} \otimes H^{\mathrm{ev}}(N) \otimes H^{d-1}(S))$.

Suppose we have proved the assertion for $n \leqq v$ (v odd). Let $\dim N = v+2$, e is an arbitrary element of $K(S^*X)/\pi^* K(X)$.

It is sufficient to consider only two cases.

5.6. $\delta \circ ch_{\mathbb{Q}}(e) \in \omega_{S^k} \otimes H^0(N) \otimes H^{d-1}(S)$.

In this case e is a multiple of $\lambda(N; S^k, \omega_{S^k})$ (we assume N to be connected!). In view of Proposition Λ_0 (cf. 2.11) we conclude $R(e) = 0$.

5.7. $\delta \circ ch_{\mathbb{Q}}(e) \in \omega_{S^k} \otimes H^{k'}(N) \otimes H^{d-1}(S)$ (k' even and >0).

In this case some multiple of $\delta \circ ch_{\mathbb{Q}}(e)$ is induced by a map into a sphere $S^k \times N \to S^{k+k'}$. Acting as in the proof of Proposition 5.2 we can find some new manifold $N' \subset S^k \times N$ and an element $e' \in K(S^*(S^{k+k'} \times N'))$ such that

$$R(e') = 2R(e).$$

Besides $\dim N' = \dim N - k' \leqq v$, and therefore $R(e) = \frac{1}{2} R(e') = 0$ by the inductive assertion.

Theorem A has been proven for orientable manifolds. Because R is local, the non-orientable case reduces to the orientable one by passage to a two-fold covering.

5.8. It would be demonstrative to envision a run of reductions in the form of the following flow-chart:

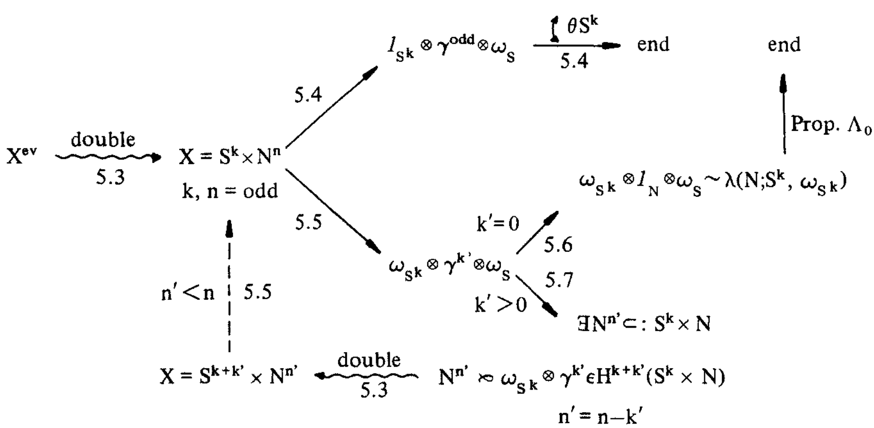


Fig. 2. **a** Solid arrows correspond to possible images in even-dimensional cohomology of generators of $K(S^*X)/\pi^* K(X) \otimes \mathbb{Q}$. **b** The arrow with the symbol $\zeta\theta\ S^k$ indicates the argument with an orientation reversing diffeomorphism (cf. the proof of Lemma 2.13 or 4.1). **c** The symbol $N^{n'} \rightarrowtail \omega_{S^k} \otimes \gamma^{k'}$ means that some *multiple* of the odd-dimensional class $\omega_{S^k} \otimes \gamma^{k'}$ is represented by a vector bundle on S^*X *twisted along* a submanifold $N^{n'}$; the symbol $N' \subset : S^k \times N$ means the normal neighbourhood of $N^{n'}$ is trivial. In this situation the argument of 5.3 applies. **d** The symbol $\omega_{S^k} \otimes 1_N \otimes \omega_S \sim \lambda(N; S^k, \omega_{S^k})$ means that the cohomology class $\omega_{S^k} \otimes 1_N \otimes \omega_S$ corresponds to the K-theory class λ via the construction of Sect. 2. **e** Waved arrows with the word "double" indicate passage to the double - the procedure described in 5.3. **f** The step of induction is indicated by the dashed arrow. The convenience of representing the proof in the form of flow-chart will be apparent in the next section

6. The proof of Theorem B

6.1. First, we are going to prove the implication

$$\text{Proposition } \Lambda_0 \Rightarrow \text{Theorem } \mathrm{B}_0.$$

Recall that in the case B_0 $X=\Sigma^l \times Y$, where Σ^l is a *standard odd-dimensional* sphere, and Y can be assumed to be orientable even-dimensional. The proof uses the same technique as the proof of Theorem A in the preceding section, and can be envisioned in the form of the following flow-chart:

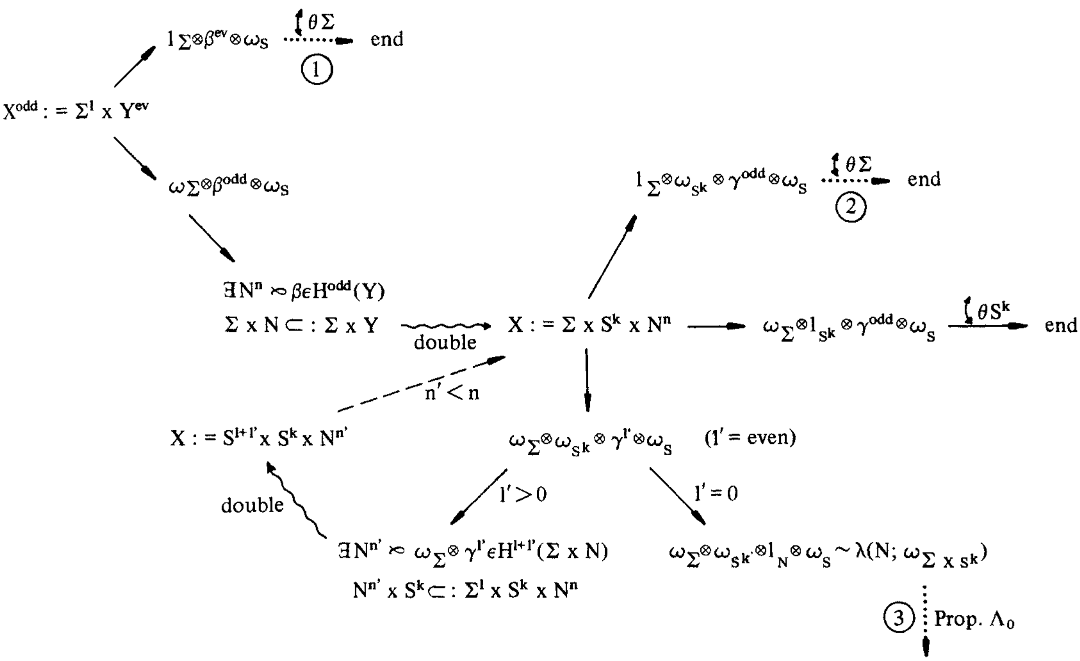


Fig. 3.

For explanations see 5.8. Recommendation to the reader: first to examine in parallel both proofs of Theorem A: "verbal", and in the form of flow-chart. Then the reconstruction of the "verbal" proof of Theorem B_0 will offer no difficulties.

6.2. The proof of the implication

$$\text{Proposition } \Lambda \Rightarrow \text{Theorem B}$$

is obtained, if at three places marked by the dotted arrows one applies Proposition Λ to the following classes:

① $\lambda(\Sigma; Y, \beta^{ev})$,

② $\lambda(\Sigma; S^k \times N, \omega_{S^k} \otimes \gamma^{odd})$,

③ $\lambda(N; \Sigma \times S^k, \omega_{\Sigma \times S^k})$

(recall that Σ, N are odd-dimensional).

7. Spectral asymmetry and pseudodifferential projectors

This section is independent of the rest of the paper and is devoted to the invariant setting of the question of coincidence of $\zeta_\theta(0)$ for various cuttings θ.

7.1. $\zeta_\theta(0)$ *for operators of order zero.* If Q is an elliptic ψDO of order zero whose principal symbol has no eigenvalues on a ray $\{\operatorname{Arg}\lambda=\theta\}$ then as usual, one can define the complex powers Q_θ^{-s}. Because the order of Q_θ^{-s} is zero for every complex s, $\operatorname{Tr} Q_\theta^{-s}$ does not exist for any s, and therefore the standard way of defining the zeta-function does not work.

For an operator of order $m_A>0$ we put

$$Z_0^\theta(A)=m_A\cdot\zeta_\theta(0;A). \tag{1}$$

It turns out, after all, that $Z_0^\theta(A)$ can be defined correctly also when $m_A=0$.

7.2. The important point in the following is to consider sections of the infinite-dimensional fibre bundle "of the order":

$$\text{ord}\colon\ CL(X,E)^\times\to\mathbb{C}, \tag{2}$$

where $CL(X,E)^\times$ denotes the group of all invertible (classical) ψDOs acting on sections of a fixed vector bundle $E\to X$. In fact, (2) can be seen as a principal $CL^0(X,E)^\times$-bundle. If Δ_E is the Laplace operator with respect to some riemannian metric on X and a connection on E then $(1+\Delta_E)^{z/2}$ provides a global section of (2). There is a convenient topology on $CL(X,E)^\times$,with respect to which one can define what sections of (2) are holomorphic. For our purposes this is superfluous; thus, in order not to overburden the paper, we pass these questions over in silence. Despite that, the following weak version of the holomorphicity of sections proves to be useful.

7.3. **Definition.** We say that a local section $\Phi\colon \mathscr{U}\to CL(X,E)^\times$ ($\mathscr{U}\subset\mathbb{C}$ open) is holomorphic (in a weak sense) if for every coordinate chart $X_0\subset\mathbb{R}^n$ and for every multiindices α, β the mappings

$$X_0\times\{\mathbb{R}^n\backslash 0\}\times\mathscr{U}\ni(x,\xi,z)\mapsto\partial_\xi^\alpha D_x^\beta\phi_{z-j}(x,\xi;z)\qquad (j\in\mathbb{N})$$

are smooth, and with respect to z also holomorphic. Here ϕ_{z-j} is a $(z-j)$-th homogeneous component of a complete symbol of $\Phi(z)$ in a given coordinate chart X_0. For brevity, the sentence "in a weak sense" we shall usually omit.

7.4. Let Q be an operator of order zero as in 7.1. One can assume it to be invertible. Suppose that over some neighbourhood of zero $\mathscr{U}_0\subset\mathbb{C}$ there is given a holomorphic section Φ of (2) with the properties:

a) $\Phi(0)=Q$,

b) there exists a conical neighbourhood of the ray $\{\operatorname{Arg}\lambda=\theta\}$ containing no eigenvalues of operators $\Phi(z)$ $(z\in\mathscr{U}_0)$. An example of such a section is provided by $z\mapsto Q(1+\Delta_E)^{z/2}$.

7.5. Let $\Phi_\theta^{-s}(z)$ be the complex powers defined for $z\in\mathcal{U}_0\cap(0,\infty)$. For any (fixed) $z\in\mathcal{U}_0\cap(0,\infty)$ the operators $\Phi_\theta^{-s}(z)$ are nuclear in the half-space $\operatorname{Re}s>\frac{\dim X}{z}$. Let $\Phi_\theta^{-s}(x,y;z)|dy|$ be the Schwartz kernel of $\Phi_\theta^{-s}(z)$. For $\operatorname{Re}s>\frac{\dim X}{z}$, this is a continuous section of the vector bundle $\mathcal{H}om(p_2^* E, p_1^* E)\otimes p_2^*|\Lambda|X$, where $p_i\colon X\times X\to X$ $(i=1,2)$ are natural projections. Recall that for every $x\neq y$ $s\mapsto\Phi_\theta^{-s}(x,y;z)|dy|$ is an entire function. When $x=y$ one has a meromorphic continuation of $s\mapsto\Phi_\theta^{-s}(x,x;z)|dx|$ from $\operatorname{Re}s>\frac{\dim X}{z}$ to the whole complex plane. This continued function takes its values in smooth sections of $\mathcal{E}nd\,E\otimes|\Lambda|X\simeq\mathcal{H}om(p_2^* E, p_1^* E)\otimes p_2^*|\Lambda|X_{|\Delta}$, and will be denoted by $\Phi_\theta^{(-s)}(x;z)|dx|$. In particular it is regular at $s=0$.

7.6. **Lemma.** *Let $\mathcal{U}_0\ni z\mapsto\Phi(z)$ be a local holomorphic section of* (2) (*as in* 7.4). *Then*

a) *the correspondence*

$$\mathcal{U}_0\cap(0,\infty)\ni z\mapsto\Phi_\theta^{(0)}(x;z)|dx| \tag{3}$$

defines a real-analytic function with values in $C^\infty(X,\mathcal{E}nd\,E\otimes|\Lambda|)$,

b) *the function* (3) *continues analytically to a meromorphic function in $\mathcal{U}_0$,*

c) *its only singularity is a simple pole at $z=0$ with the residue:*

$$Q_0^\theta(x)|dx|=-\int_{|\xi|=1}\int_0^{\infty e^{i\theta}} r_{-\dim X}(x,\xi,\lambda)\,d\lambda\,d\xi'\,dx, \tag{4}$$

where $r(x,\xi,\lambda)\sim\sum_{j=0}^{\infty}r_{-j}(x,\xi,\lambda)$ is the complete symbol of $(Q-\lambda)^{-1}$ in any coordinate chart containing a point $x\in X$, $d\xi'$ is the volume element of the unit sphere in $\mathbb{R}_\xi^{\dim X}$, divided by $(2\pi)^{\dim X}$.

In particular, the right hand side of (4) is a 1-density with values in $\mathcal{E}nd\,E$.

7.7. *Remark.* It would be more correct to write $|d\xi'||dx|$ instead of $d\xi'\,dx$. But for ξ-coordinates associated with those on the base this is the same, and we are dealing only with such coordinate systems on T^*X.

7.8. *Remark.* The expression (4) indicates, in particular, that the residue depends only on $\Phi(0)=Q$, and not on the choice of a local section $\Phi(z)$.

Proof. Let $\psi(x,\xi,\lambda;z)\sim\sum_{j=0}^{\infty}\psi_{-z-j}(x,\xi,\lambda;z)$ be the standard complete symbol *with parameter* of $(\Phi(z)-\lambda)^{-1}$ in any coordinate chart X_0 $(z\in\mathcal{U}_0)$. Recall that the components ψ_{-z-j} are determined by the following recurrent formula:

$$\psi_{-z}(x,\xi,\lambda;z)=(\phi_z(x,\xi;z)-\lambda)^{-1},$$

$$\psi_{-z-j}(x,\xi,\lambda;z)=-\psi_{-z}(x,\xi,\lambda;z)\sum_{\substack{|\alpha|+k+l=j\\ l<j}}\partial_\xi^\alpha\phi_{z-k}(x,\xi;z)\cdot D_x^\alpha\psi_{-z-l}(x,\xi,\lambda;z)/\alpha!, \tag{5}$$

where $\phi(x,\xi;z)\sim\sum_{k=0}^{\infty}\phi_{z-k}(x,\xi;z)$ is the complete symbol of $\Phi(z)$ in the same chart. In particular, any component ψ_{-z-j} expresses elementarily in terms of $(\phi_z(x,\xi;z)-\lambda)^{-1}$ and $\partial_\xi^\alpha D_x^\beta\phi_{z-k}(x,\xi;z)$ $(|\alpha|+|\beta|+k\leqq j)$.

By the assumption on $\Phi(z)$ the expression

$$-\int_{|\xi|=1}\int_0^{\infty e^{i\theta}}\psi_{-z-\dim X}(x,\xi,\lambda;z)\,d\lambda\,d\xi' \tag{6}$$

depends smoothly on $(x,z)\in X_0\times\mathcal{U}_0$, and holomorphically on $z\in\mathcal{U}_0$. On the other hand, for $z\in\mathcal{U}_0\cap(0,\infty)$, (6) divided by z equals to the value at $s=0$ of the analytic continuation (with resp. to s) of the diagonal part of kernal of $\Phi_\theta^{-s}(z)$. In another chart X_1 we can write an analogous expression $(6)_1$ using its own complete symbol of $\Phi(z)$. This will depend holomorphically on z too. By the remark just made, for $z\in\mathcal{U}_0\cap(0,\infty)$ (6) transforms as a 1-density with respect to change of coordinates. Due to an analytic continuation (with resp. to z) this must hold for every $z\in\mathcal{U}_0$. Thus, we have shown that the correspondence

$$\mathcal{U}_0\ni z\mapsto -\frac{1}{z}\left\{\int_{|\xi|=1}\int_0^{\infty e^{i\theta}}\psi_{-z-\dim X}(x,\xi,\lambda;z)\,d\lambda\,d\xi'\right\}dx$$

is a meromorphic function with values in $\mathscr{C}^\infty(X,\mathscr{End}\,E\otimes|\Lambda|)$ providing an analytic continuation of (3) to the whole $\mathcal{U}_0$. Clearly, its only singularity is a simple pole at $z=0$ with residue given by

$$Q_0^\theta(x)\,|dx|=-\int_{|\xi|=1}\int_0^{\infty e^{i\theta}}\psi_{-z-\dim X}(x,\xi,\lambda;0)\,d\lambda\,d\xi'\,dx. \tag{7}$$

Since the operator of multiplication by λ has order zero, then, for $z=0$, the symbol *with parameter* $\psi(x,\xi,\lambda;0)$ coincides with the *ordinary* classical symbol of $(\Phi(0)-\lambda)^{-1}=(Q-\lambda)^{-1}$. Therefore (7) and (4) are the same (up to change of notation). The Lemma has been proven. □

Put $Z_0^\theta(x;Q)\,|dx|=\operatorname{tr}Q_0^\theta(x)\,|dx|$. By the Lemma we know that this is a (scalar) 1-density. Its integral

$$Z_0^\theta(Q):=\int_X Z_0^\theta(x;Q)\,|dx|$$

is equal to $\operatorname{ord}Q\cdot\zeta_\theta(0;Q)$, when $\operatorname{ord}Q\in(0,\infty)$.

7.9. *Pseudodifferential projectors.* Let $P\equiv P^+$ be a ψD projector $\mathscr{C}^\infty(X,E)\to\mathscr{C}^\infty(X,E)$. Its order must be equal to 0 or $-\infty$. We would be interested only in the first case. Let $P^-=I-P^+$ be the complementary projector. Put $Q=P^+-P^-$, and let $Q(z)$ be any local section of the type 7.4. The complex powers can be defined by cuttings in the upper or lower half-planes. The corresponding Z_0^θ's will be denoted by $Z_0^\uparrow$ and $Z_0^\downarrow$ respectively. Let $R_0(Q(z))=Z_0^\uparrow(Q(z))-Z_0^\downarrow(Q(z))$. We know that $R_0(Q(z))$ is a holomorphic function on a neighbourhood of the origin. On the other hand, its restriction to the positive half-axis is constant (see the first argument of [11, Sect. 5.2]). Hence $R_0(Q(z)) \equiv \text{const}$. In particular $R_0(Q(z)) \equiv R_0(Q)$, where $Q = Q(0)$.

Besides

$$
\begin{aligned}
(Q-\lambda)^{-1} &= (P^+ - P^- - \lambda)^{-1} = \{(1-\lambda)P^+ - (1+\lambda)P^-\}^{-1} \\
&= (1-\lambda)^{-1}P^+ - (1+\lambda)^{-1}P^-, \qquad (\lambda \neq \pm 1),
\end{aligned}
$$

due to mutual orthogonality of P^+ and P^-. In particular, the complete symbol of $(Q-\lambda)^{-1}$ looks like

$$r(x, \xi, \lambda) = \frac{p^+(x,\xi)}{1-\lambda} - \frac{p^-(x,\xi)}{1+\lambda}.$$

Hence, applying Lemma 7.6, we get

$$
\begin{aligned}
R_0(Q(z)) &\equiv -\int_X \int_{|\xi|=1} \left\{ \int_0^{\infty e^{\pi i/2}} - \int_0^{\infty e^{-\pi i/2}} \right\} \operatorname{tr} r_{-\dim X}(x, \xi, \lambda)\, d\lambda\, d\!\!{}^-\xi'\, dx \\
&= -2\pi i \int_X \int_{|\xi|=1} \operatorname{tr} \operatorname{Res}_{\lambda=-1} r_{-\dim X}(x, \xi, \lambda)\, d\!\!{}^-\xi'\, dx \\
&= 2\pi i \int_X \int_{|\xi|=1} \operatorname{tr} p^-_{-\dim X}(x, \xi)\, d\!\!{}^-\xi'\, dx. \qquad (8)
\end{aligned}
$$

7.10. Conversely, let A be any elliptic ψDO of positive order admitting of two cuttings θ', θ'' in the spectral plane. Because $\operatorname{ord} A \cdot \{\zeta_{\theta'}(0; A) - \zeta_{\theta''}(0; A)\}$ depends only on the class $[\mathscr{E}'_A] \in K(S^*X)/\pi^* K(X)$ (ibid.), where $\mathscr{E}'_A \oplus \mathscr{E}''_A = \pi^* E$ is the sectorial splitting for A, we can find an invertible operator $B = \beta B_0$ ($|\beta| = 1$) with the properties:

a) B admits of the same cuttings θ', θ'', and $[\mathscr{E}'_B] = [\mathscr{E}'_A]$ in $K(S^*X)/\pi^* K(X)$;

b) B_0 is self-adjoint with respect to some volume density on X and hermitian metric on E, and $\operatorname{ord} B_0 = m > 0$.

Obviously, then, $\operatorname{ord} A \cdot \{\zeta_{\theta'}(0; A) - \zeta_{\theta''}(0; A)\}$ coincides, up to sign, with $R_0(B)$. Denoting by P^+, P^--orthogonal projectors onto positive (resp. negative) eigenvalues of B_0 we have $B_0 = (P^+ - P^-)|B_0|$, and $\Phi(z) = (P^+ - P^-)|B_0|^{z/m}$ is a section of (2) holomorphic in the sense of 7.4. Therefore the argument of 7.9 applies and we obtain

$$\operatorname{ord} A \cdot \{\zeta_{\theta'}(0; A) - \zeta_{\theta''}(0; A)\} = \pm 2\pi i \int_X \int_{|\xi|=1} \operatorname{tr} p^-_{-\dim X}(x, \xi)\, d\!\!{}^-\xi'\, d\!\!{}^-x. \qquad (9)$$

We have shown, hereby, the following proposition.

7.11. **Proposition.** 1) *For any pseudodifferential projector P on a closed manifold X the expression*

$$\int_{|\xi|=1} \operatorname{tr} p_{-\dim X}(x, \xi)\, d\!\!{}^-\xi'\, d\!\!{}^-x \qquad (10)$$

defines a 1-density.

2) *The following assertions are equivalent:*

a) *the value of the zeta-function of an elliptic ψDO at zero does not depend on a choice of cuttings in the spectral plane;*

b) *for any pseudodifferential projector the density* (10) *is a complete differential, i.e.*

$$\int_X \int_{|\xi|=1} \operatorname{tr} p_{-\dim X}(x,\xi)\, đ\xi'\, dx = 0. \quad \square$$

Because the assertion 2a) has been proven earlier (cf. 1.24) we obtain:

7.12. **Corollary.** *For an arbitrary pseudodifferential projector P on a closed manifold X*

$$\int_X \int_{|\xi|=1} \operatorname{tr} p_{-\dim X}(x,\xi)\, đ\xi\, dx = 0. \quad \square$$

7.13. *Final remarks.* In fact, the first assertion of 7.11 holds for *any* (classical) ψDO B: $\mathscr{C}^\infty(X,E)\to\mathscr{C}^\infty(X,E)$. This could be proven by a constant use of holomorphic (in a stronger sense) sections of the fibre bundle $\coprod_{z\in\mathbb{C}} CL^z(X,E) \xrightarrow{\text{ord}} \mathbb{C}$. The theory of completely integrable systems (the Adler-Lebiediev-Manin scheme, see [1, 7, 12]) prompts to introduce the following notation:

$$\operatorname{res} B := \int_X \int_{|\xi|=1} \operatorname{tr} b_{-\dim X}(x,\xi)\, đ\xi'\, dx, \tag{11}$$

and to call it the (higher) *non-commutative residue* of B[6]. If $\operatorname{ord} B \notin \mathbb{Z}$ it is understood that $b_{-\dim X} \equiv 0$. The non-commutative residue has some remarkable properties analogous to those of the ordinary finite-dimensional trace:

(1) $\operatorname{res}(\lambda B + \mu C) = \lambda \operatorname{res} B + \mu \operatorname{res} C$ $(\operatorname{ord} B - \operatorname{ord} C \in \mathbb{Z})$;

(2) $\operatorname{res}([B, C]) = 0$;

(3) $\operatorname{res} B = 0$ $(\dim X > 1) \Rightarrow \exists\, C_i, D_i$ $(i = 1, \ldots, k = k(X,E))$:

$$B - \sum_{i=1}^{k} [C_i, D_i] \in L^{-\infty};$$

(4) the "Killing form" $\langle B, C\rangle := \operatorname{res}(BC)$ is $\mathbb{C}$-linear, symmetric, ad-invariant and non-degenerated (on $CL(X,E) \bmod L^{-\infty}(X,E)$);

(5) if B admits of complex powers B_θ^{-s}, then $\operatorname{Res}_{s=s_0} \zeta_\theta(s;B) = \dfrac{\operatorname{res}(B_\theta^{-s_0})}{\operatorname{ord} B}$. In particular this could be non-zero only when $s_0 \cdot \operatorname{ord} B \in \mathbb{Z} \cap (-\infty, \dim X]$, and

[6] When $\dim X = 1$ this slightly differs from the original *ALM* non-commutative residue. In fact, the algebra of $\mathbb{Z}$-graduated 1-dimensional formal ψDOs $CL^*(X,E)/L^{-\infty}$ has a 2-dimensional center $\mathbb{C}\langle I, S\rangle$, where S is a formal ψDO with its complete symbol $s(x,\xi) = \operatorname{sign}\xi$. Then, one can define

$$\operatorname{res}_{ALM}(B) := \operatorname{res}\{\tfrac{1}{2}(I+S)B\}.$$

Note that $\operatorname{res}(\cdot)$ and $\operatorname{res}(S\cdot)$ are two different functionals $CL(X,E)\to\mathbb{C}$ satisfying conditions (1), (2), (4) below. The condition (3) for $\dim X = 1$ can be formulated as:

$$\operatorname{res} B = 0, \quad \operatorname{res}(SB) = 0 \Rightarrow \exists\, C_i, D_i \quad (i = 1, \ldots, k = k(E)):\ B - \sum_{i=1}^{k} [C_i, D_i] \in L^{-\infty}. \tag{3$_1$}$$

At last, any functional γ: $CL(X,E)/L^{-\infty}\to\mathbb{C}$, in the 1-dimensional case, must be equal to $c_1 \operatorname{res}(\cdot) + c_2 \operatorname{res}(B\cdot)$ for some constants c_1, c_2 depending on E

for $s_0 = -l \in \mathbb{Z}$ we have $\operatorname{Res}_{s=-l} \zeta(s;B) = \frac{\operatorname{res}(B^l)}{\operatorname{ord} B}$ not depending on θ. Using (3) it is easily seen that any map $\gamma\colon CL(X,E)/L^{-\infty} \to \mathbb{C}$ satisfying (1) and (2) must coincide with $\operatorname{res}(\cdot)$ up to some multiplicative constant $c = c(X,E)$. If, moreover, γ is local then c depends only on $\dim X$.

Proofs will be published in a subsequent paper (to appear in Funct. Anal. and its Appl.). Corollary 7.12 and 7.13(3) imply that *any* pseudodifferential projector on a closed manifold decomposes into a finite sum of commutators plus a smoothing operator. Proving this independently would give another proof of coincidence of the zero value of zeta-functions defined by different cuttings in the spectral plane.

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Note added in proof

Since $H^j(X^{2n+1};\mathbb{Q})$, for $j>n$, is generated by spherical classes, the argument of Sect. 5 and Prop. Λ_0 (for $Q=pt$ or S^4) apply, and it is easily seen that Theorem 1.10 remains valid for general 3- and 7-manifolds. In this last case one must in addition assume that $\int \mathcal{R}$ is invariant with respect to the natural $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on $Ell^m_{\mathcal{D}',\mathcal{D}''}(X)$ (here we are allowed to suppose that $\tau(\mathcal{D}')=\mathcal{D}'$, $\tau(\mathcal{D}'')=\mathcal{D}''$; τ complex conjugation). This condition is satisfied *e.g.* by any *spectral* K-invariant. In proof the induced action on $K(S^*X)/\pi^*K(X)$ is used.