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Examples of Eigenvalues of Hecke Operators on Siegel Cusp Forms of Degree Two

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Introduction

We present some examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two together with some generalizations of the Ramanujan conjecture. We obtain examples of eigenvalues for ten cusp forms which are denoted by χ_{10}, χ_{12}, χ_{14}, $\chi_{16}^{(+)}$, $\chi_{16}^{(-)}$, $\chi_{18}^{(+)}$, $\chi_{18}^{(-)}$, $\chi_{20}^{(1)}$, $\chi_{20}^{(2)}$, $\chi_{20}^{(3)}$. Our examples, which are listed in the last section, suggest that the above cusp forms are divided into two classes. The first class consists of the former nine cusp forms. It follows from our examples that an analogue of the Ramanujan conjecture formulated in this paper does not hold for the cusp forms in this class. Instead, it appears that for each such cusp form (of weight k) there exists an elliptic cusp form (of weight $2k-2$) with an explicit relation between the eigenvalues of Hecke operators for these two cusp forms. This relation is stated as Conjecture 1. The second class consists of the tenth cusp form $\chi_{20}^{(3)}$. It seems that the analogue of the Ramanujan conjecture holds for $\chi_{20}^{(3)}$ (Conjecture 3). Our examples for $\chi_{20}^{(3)}$ support this.

In [3], Langlands gives an interpretation of our Conjecture 1 (for χ_{10}) in a representation-theoretical formulation. This interpretation is deduced from his general philosophy containing the principle of functoriality. A general formulation of the "Ramanujan conjecture" is given in [3a]. (In [3], the "Ramanujan conjecture" is restricted to the case of $GL(n)$.) Our examples suggest that the automorphic representations attached to the former nine cusp forms are counter-examples to this "Ramanujan conjecture", and that these automorphic representations are "anomalous" in the sense of [3]. It seems that the automorphic representation attached to $\chi_{20}^{(3)}$ satisfies the "Ramanujan conjecture". We remark that Howe and Piatetski-Shapiro [2] have given counter-examples to the "generalized Ramanujan conjecture" (in a representation-theoretical formulation) by using Weil representations called oscillator representations.

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Notations. For Siegel modular forms, we follow the notations in Andrianov [1], Maass [4], or Resnikoff-Saldaña [5] in general. For each integer $n \geqq 1$, Γ_n denotes the Siegel modular group of degree n, and $M_k(\Gamma_n)$ (resp. $S_k(\Gamma_n)$) denotes the vector space over the complex number field $\mathbf{C}$ consisting of all Siegel modular (resp. cusp) forms of degree n and weight k for a positive integer k. For each integer $m \geqq 1$, $T(m)$: $M_k(\Gamma_n) \to M_k(\Gamma_n)$ denotes the Hecke operator with the usual normalization. If $f \in M_k(\Gamma_n)$ is an eigenfunction of $T(m)$ for an integer $m \geqq 1$, then we denote by $\lambda(m)=\lambda(m,f)$ the eigenvalue: $T(m)f=\lambda(m)f$. If $f \in M_k(\Gamma_n)$ is an eigenfunction of all Hecke operators on $M_k(\Gamma_n)$ (see [1] Chap. 1 for the precise meaning), then we call f an eigen modular form (or "eigen cusp form" if f is a cusp form) and we denote by $L(s,f)=\prod_p H_p(p^{-s},f)^{-1}$ the Euler product defined in [1] (Chap. 1), where p runs over all prime numbers and $H_p(T,f) \in 1+T\cdot\mathbf{C}[T]$ is a polynomial of degree 2^n in an indeterminate T for each p. For $n=1$ or 2, $H_p(T,f)=1-\lambda(p)T+p^{k-1}T^2$ or $H_p(T,f)=1-\lambda(p)T+(\lambda(p)^2-\lambda(p^2)-p^{2k-4})T^2 -p^{2k-3}\lambda(p)T^3+p^{4k-6}T^4$ respectively, where $\lambda(m)=\lambda(m,f)$. In this paper we say f satisfies *Ramanujan conjecture* if the absolute values of the zeros of $H_p(T,f)$ are equal to $p^{-n(2k-n-1)/4}$ for all p. We say a basis of $M_k(\Gamma_n)$ over $\mathbf{C}$ is an eigen basis if the basis is consisting of eigen modular forms. It is known that $M_k(\Gamma_n)$ has an eigen basis ([1] Theorem 1.3.4). If $f \in M_k(\Gamma_n)$, then f has the Fourier expansion: $f(Z)=\sum_T a(T)e^{2\pi i\sigma(TZ)}$ where Z is a variable on the Siegel upper half space of degree n, T runs over all $n \times n$ symmetric semi-integral positive semi-definite matrices, and $\sigma(TZ)$ is the trace of TZ. We call $a(T)=a(T,f)$ the Fourier coefficient of f at T. We denote by $a(m;T)=a(m;T,f)$ the Fourier coefficient of $T(m)f$ at T for each $f \in M_k(\Gamma_n)$ and $m \geqq 1$. For each matrix $T=\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}$ with t_1, t_2, t integers, we put $e(T)=\gcd(t_1,t_2,t)$ the greatest common divisor, $\Delta(T)=\det(2T)=4t_1t_2-t^2$, and $\langle T\rangle=\begin{pmatrix} 1 & t/2 \\ t/2 & t_1t_2 \end{pmatrix}$. For simplicity we denote by (t_1,t_2,t) the above matrix T, and we denote by $a(t_1,t_2,t)$ the Fourier coefficient of $f \in M_k(\Gamma_2)$ at T if T is positive semi-definite.

For special elements in $M_k(\Gamma_2)$ we use the following notation. For each even integer $k \geqq 4$, the Eisenstein series of weight k is denoted by φ_k. We denote by χ_k the normalized cusp form of weight $k=10$ or 12, and we put $\chi_{14}=\chi_{10}\cdot\varphi_4$. The Fourier coefficient of φ_k (resp. χ_k) at T is denoted by $a_k(T)$ (resp. $c_k(T)$).

For special elements in $M_k(\Gamma_1)$ we use the following notation. For each even integer $k \geqq 4$, the Eisenstein series of weight k is denoted by $E_k=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$ $(q=e^{2\pi iz})$ where B_k is the k-th Bernoulli number and $\sigma_{k-1}(n)=\sum_{d|n}d^{k-1}$. We denote by Δ_k the normalized cusp form of weight $k=12, 16, 18, 20, 22$ or 26. For simplicity, Δ_{12} is denoted by Δ also.

§ 1. Examples for Weights 10, 12 and 14

1.1. Eigenvalues. We obtain the following eigenvalues of Hecke operators for χ_{10}, χ_{12} and χ_{14}.

χ_{10}: $\lambda(2)=240$, $\lambda(3)=21960$, $\lambda(4)=135424$.

χ_{12}: $\lambda(2)=2784$.

χ_{14}: $\lambda(2)=12240$, $\lambda(3)=1929960$.

The calculation is as follows. Put $F=\chi_{10}$, $D=3$ and $N=(1,1,1)$ in Theorem 2.4.1 of Andrianov [1], then we have:

$$\zeta_{\mathbf{Q}(\sqrt{-3})}(s-8)\sum_{m=1}^{\infty} c_{10}(m,m,m)\,m^{-s}=c_{10}(1,1,1)\,\zeta(2s-16)\sum_{m=1}^{\infty}\lambda(m)\,m^{-s},$$

where $\zeta(s)$ (resp. $\zeta_{\mathbf{Q}(\sqrt{-3})}(s)$) is the zeta function of the rational number field $\mathbf{Q}$ (resp. the imaginary quadratic field $\mathbf{Q}(\sqrt{-3})$). Since $\left(\frac{-3}{2}\right)=-1$, we have the following identity by comparing the 2 (-power)-factors:

$$\sum_{\nu=0}^{\infty} c_{10}(2^\nu,2^\nu,2^\nu)\,T^\nu=c_{10}(1,1,1)\sum_{\nu=0}^{\infty}\lambda(2^\nu)\,T^\nu \quad \text{with } T=2^{-s}.$$

From Table IV in Resnikoff-Saldaña [5], we have: $c_{10}(1,1,1)=-1/4$, $c_{10}(2,2,2)=-240/4$, and $c_{10}(4,4,4)=-135424/4$. Hence we have: $\lambda(2)=240$ and $\lambda(4)=135424$. The other values are also calculated in the similar manner. For the calculation of $\lambda(m,\chi_{14})$, we use the following values: $c_{14}(1,1,1)=-1/4$, $c_{14}(2,2,2)=-12240/4$, and $c_{14}(3,3,3)=-1398519/4$. These values are obtained from Tables I–IV in [5] with using $\chi_{14}=\chi_{10}\cdot\varphi_4$. We remark the following general fact: if f_i ($i=1,\dots,r$; $r\geqq 1$) are Siegel modular forms of degree n and weights k_i ($k_i>0$ integers), then $f=f_1\dots f_r$ is a Siegel modular form of degree n and weight $k=k_1+\dots+k_r$, and the Fourier coefficient $a(T,f)$ of f at an $n\times n$ symmetric semi-integral positive semi-definite matrix T is given by $a(T,f)=\sum a(T_1,f_1)\dots a(T_r,f_r)$ where T_i runs over all $n\times n$ symmetric semi-integral positive semi-definite matrices such that $T_1+\dots+T_r=T$. Thus, $c_{14}(1,1,1)=c_{10}(1,1,1)\,a_4(0,0,0)$, $c_{14}(2,2,2)=c_{10}(1,1,0)\,a_4(1,1,2)+c_{10}(1,1,1)\,a_4(1,1,1)+2c_{10}(1,2,2)\,a_4(1,0,0)+c_{10}(2,2,2)\,a_4(0,0,0)$, and $c_{14}(3,3,3)=c_{10}(1,1,0)\,a_4(2,2,3)+c_{10}(1,1,-1)\,a_4(2,2,4)+c_{10}(1,1,1)\,a_4(2,2,2)+2c_{10}(1,2,1)\,a_4(2,1,2)+2c_{10}(1,2,2)\,a_4(2,1,1)+c_{10}(2,2,2)\,a_4(1,1,1)+c_{10}(2,2,3)\,a_4(1,1,0)+c_{10}(2,2,1)\,a_4(1,1,2)+2c_{10}(1,3,3)\,a_4(2,0,0)+2c_{10}(2,3,3)\,a_4(1,0,0)+c_{10}(3,3,3)\,a_4(0,0,0)$. (We note here that $4c_{10}(3,3,3)=-15399$ in Table IV in [5].)

1.2. An Euler Factor. We obtain the following Euler factor of $L(s,\chi_{10})$: $H_2(T,\chi_{10})=(1-2^8T)(1-2^9T)(1+528\,T+2^{17}T^2)$.

In fact $H_2(T,\chi_{10})=1-\lambda(2)\,T+(\lambda(2)^2-\lambda(4)-2^{16})\,T^2-2^{17}\cdot\lambda(2)\,T^3+2^{34}\,T^4$ with $\lambda(2)=240$ and $\lambda(4)=135424$. In particular, χ_{10} does not satisfy Ramanujan conjecture.

§ 2. A Conjecture and Remarks

2.1. A Conjecture. From the examples in § 1, we pose the following Conjecture 1.

Conjecture 1. $L(s,\chi_k)=\zeta(s-k+2)\,\zeta(s-k+1)\,L(s,\Delta_{2k-2})$ *for* $k=10$, 12 *and* 14.

We can check that the examples in §1 are compatible with Conjecture 1. In fact:

$$\Delta_{18}=\Delta\cdot E_6=q-528\cdot q^2-4284\cdot q^3+\cdots,$$
$$\Delta_{22}=\Delta\cdot E_4\cdot E_6=q-288\cdot q^2+\cdots,$$
$$\Delta_{26}=\Delta\cdot E_4^2\cdot E_6=q-48\cdot q^2-195804\cdot q^3+\cdots.$$

Hence it is easy to check $\lambda(p,\chi_k)=p^{k-2}+p^{k-1}+\lambda(p,\Delta_{2k-2})$ for the following cases: $(k,p)=(10,2)$, $(10,3)$, $(12,2)$, $(14,2)$ and $(14,3)$. Moreover, since $H_2(T,\Delta_{18})=1+528\,T+2^{17}T^2$, the following equality holds:

$$H_2(T,\chi_{10})=(1-2^8T)(1-2^9T)\,H_2(T,\Delta_{18}).$$

We extend the above Conjecture 1 for each even integer $k\geqq 10$ in the following form.

Conjecture 1. *Let $k\geqq 10$ be an even integer. Then there exists an injective* **C***-linear mapping $\sigma_k\colon S_{2k-2}(\Gamma_1)\to S_k(\Gamma_2)$ such that the following holds: if $f\in S_{2k-2}(\Gamma_1)$ is an eigen cusp form, then $\sigma_k(f)\in S_k(\Gamma_2)$ is an eigen cusp form and $L(s,\sigma_k(f))=\zeta(s-k+2)\,\zeta(s-k+1)\,L(s,f)$.*

Conjectural examples: $\sigma_k(\Delta_{2k-2})=\chi_k$ for $k=10$, 12 and 14.

Numerical examples for weights $k=16$, 18 and 20 with further examples for weights $k=10$, 12 and 14 will be given in §3, §5 and §7. These examples also support Conjecture 1.

2.2. Remarks. We remark the following facts on Conjecture 1. In the remarks (2), (3) and (4), let $k\geqq 10$ be an even integer.

(1) If k is an odd positive integer, then for each eigen cusp form $f\in S_k(\Gamma_2)$ $(k=35,\ldots)$, $L(s,f)$ is holomorphic on **C**; see Theorem 3.1.1 (IV) in [1].

(2) For each eigen cusp form $f\in S_{2k-2}(\Gamma_1)$, put $L(s,f,\sigma_k)=\zeta(s-k+2)\cdot\zeta(s-k+1)\,L(s,f)$ and $\Lambda(s,f,\sigma_k)=\Gamma_{\mathbf{C}}(s)\,\Gamma_{\mathbf{C}}(s-k+2)\,L(s,f,\sigma_k)$ with $\Gamma_{\mathbf{C}}(s)=2(2\pi)^{-s}\,\Gamma(s)$. Then $\Lambda(s,f,\sigma_k)$ is holomorphic on **C** except for two simple poles at $s=k-2$ and k, and $\Lambda(s,f,\sigma_k)$ satisfies the following functional equation: $\Lambda(s,f,\sigma_k)=\Lambda(2k-2-s,f,\sigma_k)$. This is compatible with Conjecture 1; see Theorem 3.1.1 (II), (IV) in [1]. In fact, $\Lambda(s,f)=\Gamma_{\mathbf{C}}(s)\,L(s,f)$ is holomorphic on **C** and $\Lambda(s,f)=-\Lambda(2k-2-s,f)$, so $\Lambda(k-1,f)=0$. Hence if we remark that $\Lambda(s,f,\sigma_k)=(2\pi)^{-1}\cdot(s-k+1)\cdot\Gamma_{\mathbf{R}}(s-k+1)\,\zeta(s-k+1)\cdot\Gamma_{\mathbf{R}}(s-k+2)\,\zeta(s-k+2)\cdot\Lambda(s,f)$ with $\Gamma_{\mathbf{R}}(s)=\pi^{-s/2}\,\Gamma(s/2)$, we have the desired results. (Remark that $\Gamma_{\mathbf{R}}(s)\,\Gamma_{\mathbf{R}}(s+1)=\Gamma_{\mathbf{C}}(s)$.) $L(s,f,\sigma_k)$ is holomorphic on **C** except for a simple pole at $s=k$ with the residue $\zeta(2)\,L(k,f)$ which is a positive real number. We note also that $L(k-1,f,\sigma_k)=-L'(k-1,f)/2$ where $L'(s,f)=\dfrac{d}{ds}L(s,f)$.

(3) Let $f\in S_k(\Gamma_2)$, then by using a generalization of Rankin's method ("Rankin convolution") we can prove that $a(T,f)=O(\Delta(T)^{(k-3/16)/2})$ as $\Delta(T)\to\infty$. In particular, if $f\in S_k(\Gamma_2)$ is an eigen cusp form, then $\lambda(m,f)=O(m^{k-3/16})$ as $m\to\infty$.

(4) It is easy to see from Igusa's dimension formula for $S_k(\Gamma_2)$ that: $\dim_{\mathbf{C}} S_k(\Gamma_2)\geqq\dim_{\mathbf{C}} S_{2k-2}(\Gamma_1)$. (For a proof, see §4.)

(5) For each even integer $k \geqq 4$, the following equalities hold:

$$L(s, \varphi_k) = \zeta(s-k+2)\,\zeta(s-k+1)\,L(s, E_{2k-2})$$
$$= \zeta(s)\,\zeta(s-k+2)\,\zeta(s-k+1)\,\zeta(s-2k+3).$$

See § 3.2 of [1]. Hence, the mapping $E_{2k-2} \mapsto \varphi_k$ may be considered as an analogue of σ_k for the spaces of Eisenstein series.

§ 3. Examples for Weights 16 and 18

For $k=16$ or 18, $\dim_{\mathbf{C}} S_k(\Gamma_2) = \dim_{\mathbf{C}} S_{2k-2}(\Gamma_1) = 2$ and an eigen basis of $S_k(\Gamma_2)$ (resp. $S_{2k-2}(\Gamma_1)$) over $\mathbf{C}$ is given by the following $\{\chi_k^{(+)}, \chi_k^{(-)}\}$ (resp. $\{\Delta_{2k-2}^{(+)}, \Delta_{2k-2}^{(-)}\}$).

k = 16 $\chi_{16}^{(\pm)} = 185 \cdot f + (-128 \pm \sqrt{51\,349}) \cdot g$
with $f = 4 \cdot \chi_{10} \cdot \varphi_6$ and $g = 12 \cdot \chi_{12} \cdot \varphi_4$,
$\lambda(2, \chi_{16}^{(\pm)}) = 53\,472 \pm 96\sqrt{51\,349}$,
$\Delta_{30}^{(\pm)} = A + (5\,856 \pm 96\sqrt{51\,349}) \cdot B$
with $A = \Delta \cdot E_6^3$ and $B = \Delta^2 \cdot E_6$,
$\lambda(2, \Delta_{30}^{(\pm)}) = 4\,320 \pm 96\sqrt{51\,349}$.

k = 18 $\chi_{18}^{(\pm)} = 2\,590 \cdot f + (1\,149 \pm \sqrt{2\,356\,201}) \cdot g$
with $f = 4 \cdot \chi_{10} \cdot \varphi_4^2$ and $g = 12 \cdot \chi_{12} \cdot \varphi_6$,
$\lambda(2, \chi_{18}^{(\pm)}) = 135\,768 \pm 72\sqrt{2\,356\,201}$,
$\Delta_{34}^{(\pm)} = A + (-59\,544 \pm 72\sqrt{2\,356\,201}) \cdot B$
with $A = \Delta \cdot E_6^3 \cdot E_4$ and $B = \Delta^2 \cdot E_6 \cdot E_4$,
$\lambda(2, \Delta_{34}^{(\pm)}) = -60\,840 \pm 72\sqrt{2\,356\,201}$.

From these values, it is easy to check the following equality for $k=16$ and 18: $\lambda(2, \chi_k^{(\pm)}) = \lambda(2, \Delta_{2k-2}^{(\pm)}) + 2^{k-2} + 2^{k-1}$. Hence, these examples support the Conjecture 1; for $k=16$ and 18, it takes the following form:

$$L(s, \chi_k^{(\pm)}) = \zeta(s-k+2)\,\zeta(s-k+1)\,L(s, \Delta_{2k-2}^{(\pm)}).$$

We note here that 51349 is a prime number and that 2356201 is the product of two prime numbers 479 and 4919.

For the calculation about $S_k(\Gamma_2)$, we use the following values of Fourier coefficients.

k		$a(1,1,0)$	$a(1,1,1)$	$a(2,2,0)$	$a(2,2,2)$	$a(1,4,0)$	$a(1,3,0)$
16	f	2	-1	280192	-47616	214656	-14848
	g	10	1	283520	23424	-44160	-9344
18	f	2	-1	263008	-24240	864	106832
	g	10	1	1902560	32016	591840	-99056

The values of $a(1,4,0)$ and $a(1,3,0)$ are given for later use in § 4. These values in the above table are calculated from Tables I–V in [5] and the following two values: $12c_{12}(1,4,0)=-2880$ and $12c_{12}(2,2,0)=17600$. The value of $12c_{12}(1,4,0)$ is calculated in Maass [4] (p. 174), and the value of $12c_{12}(2,2,0)$ is deduced from $12c_{12}(1,1,0)=10$ and $\lambda(2,\chi_{12})=2784$ with using Theorem 2.4.1 in [1] as in 1.1, or with using $c_{12}(2;(1,1,0))=c_{12}(2,2,0)+2^{10}c_{12}(1,1,0)=\lambda(2,\chi_{12})c_{12}(1,1,0)$, cf. [1] (the formula (2.1.11)). The method of calculation is similar to the calculation of $c_{14}(T)$ in 1.1.

To determine $\chi_k^{(\pm)}$, we use the method in Ribet [6] (§ 8). We obtain the following results concerning the action of $T(2)$ on $S_k(\Gamma_2)$.

$S_{16}(\Gamma_2)$: $T(2)f=65760\cdot f+18144\cdot g$,
$T(2)g=17760\cdot f+41184\cdot g$,
the characteristic polynomial of $T(2)$ is
$X^2-106944\,X+2386022400$.

$S_{18}(\Gamma_2)$: $T(2)f=53040\cdot f+28800\cdot g$,
$T(2)g=186480\cdot f+218496\cdot g$,
the characteristic polynomial of $T(2)$ is
$X^2-271536\,X+6218403840$.

In this calculation we use the following fact: $a(2;(1,1,1),f)=a((2,2,2),f)$ and $a(2;(1,1,0),f)=a((2,2,0),f)+2^{k-2}a((1,1,0),f)$ for each $f\in S_k(\Gamma_2)$; see (2.1.11) in [1]. The calculation is as follows. Let $k=16$, then we can write $T(2)f=\alpha\cdot f+\beta\cdot g$ with $\alpha,\beta\in\mathbf{C}$. By comparing the Fourier coefficients at $(1,1,0)$ in the both sides, we have $a(2;(1,1,0),f)=\alpha\cdot a((1,1,0),f)+\beta\cdot a((1,1,0),g)$. From the above table, we have

$$a(2;(1,1,0),f)=a((2,2,0),f)+2^{14}a((1,1,0)\,f)$$
$$=280192+2^{14}\cdot 2=312960,\quad a((1,1,0),f)=2,$$

and $a((1,1,0),g)=10$. Hence we have: $2\alpha+10\beta=312960$. Similarly, by comparing the Fourier coefficients at $(1,1,1)$, we have: $-\alpha+\beta=-47616$. Thus we have $\alpha=65760$ and $\beta=18144$. Other cases are similar.

For the determination of $\Delta_{2k-2}^{(\pm)}$, we use the following Fourier coefficients.

$S_{30}(\Gamma_1)$: $A=q-1536\cdot q^2+*q^3-95571968\cdot q^4+\cdots$,
$B=q^2+*q^3+8640\cdot q^4+\cdots$.

$S_{34}(\Gamma_1)$: $A=q-1296\cdot q^2+*q^3+80803072\cdot q^4+\cdots$,
$B=q^2+*q^3-121680\cdot q^4+\cdots$.

The action of $T(2)$ on $S_{2k-2}(\Gamma_1)$ is as follows.

$S_{30}(\Gamma_1)$: $T(2)A=-1536\cdot A+438939648\cdot B$,
$T(2)B=A+10176\cdot B$,

the characteristic polynomial of $T(2)$ is

$X^2-8640\,X-454569984.$

$S_{34}(\Gamma_1)$: $T(2)\,A=-1296\cdot A+8669058048\cdot B,$

$T(2)\,B=A-120384\cdot B,$

the characteristic polynomial of $T(2)$ is

$X^2+121680\,X-8513040384.$

The calculation is easier than the case of $S_k(\Gamma_2)$, and it is sufficient to note that: $a(2;1,f)=a(2,f)$ and $a(2;2,f)=a(4,f)+2^{2k-3}\,a(1,f)$ for each $f\in S_{2k-2}(\Gamma_1)$.

Remark. From the above procedure, it would be easy to see that we can obtain further examples with some efforts. Such examples will be given in § 5 and § 7.

§ 4. Supplementary Conjectures

In § 4 we assume Conjecture 1, and we pose two conjectures.

Let $k\geqq 10$ be an even integer. Put

$$S_k^I(\Gamma_2)=\left\{f\in S_k(\Gamma_2)\,\middle|\,(*)\ a(T,f)=\sum_{d\mid e(T)} d^{k-1}\,a\left(\left\langle\frac{1}{d}\,T\right\rangle,f\right)\ \text{for all } T.\right\}.$$

Here T runs over all 2×2 symmetric semi-integral positive semi-definite matrices. $S_k^I(\Gamma_2)$ is a sub vector space (over $\mathbf{C}$) of $S_k(\Gamma_2)$. The above equation $(*)$ appeared in Maass [4] (the Eq. (19)). We pose the following Conjecture 2.

Conjecture 2. *Let $k\geqq 10$ be an even integer. Then*

$$S_k^I(\Gamma_2)\supset\sigma_k(S_{2k-2}(\Gamma_1)).$$

There exist some theoretical supports to Conjecture 2, and actually we may conjecture moreover that $S_k^I(\Gamma_2)=\sigma_k(S_{2k-2}(\Gamma_1))$. At the same time, it is conjectured also that for each $f\in\sigma_k(S_{2k-2}(\Gamma_1))$ the Fourier coefficients $a(T,f)$ depend only on $e(T)$ and $\Delta(T)$.

If Conjecture 2 is valid, then $S_k^I(\Gamma_2)=S_k(\Gamma_2)$ for $k=10$, 12, 14, 16 and 18. For these k, it is easy to check that the above equation $(*)$ holds for all $f\in S_k(\Gamma_2)$ at $T=(2,2,0)$ and $(2,2,2)$ by using the values in [5] and the values in § 3. For example, let $k=16$ and $f=4\cdot\chi_{10}\cdot\varphi_6$. Then $a((2,2,0),f)=280192$, $a((1,1,0),f)=2$, and $a((1,4,0),f)=214656$ by the table in § 3. Hence the following equality holds: $a((2,2,0),f)=2^{15}\,a((1,1,0),f)+a((1,4,0),f)$, which is the equation $(*)$ at $T=(2,2,0)$. Similarly, $(*)$ holds for $f=12\cdot\chi_{12}\cdot\varphi_4$ at $T=(2,2,0)$. Thus $(*)$ holds for all $f\in S_{16}(\Gamma_2)$ at $T=(2,2,0)$. Other cases are similarly checked. For the checking at $T=(2,2,2)$, we remark that $(1,4,2)$ is unimodularly equivalent to $(1,3,0)$, in fact $\begin{pmatrix}1&1\\1&4\end{pmatrix}=\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&3\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}$. We remark that the Fourier coefficients at $T=(1,3,0)$ and $(1,4,0)$ are calculated also by the method of Maass [4] (Satz 3).

We note that $S_k^I(\Gamma_2) \subsetneqq S_k(\Gamma_2)$ for each even integer $k \geqq 20$ (cf. Maass [4] p. 167). In fact, for example, let $f = \chi_{10}^2 \cdot \varphi_4^{(k-20)/4}$ if $k \equiv 0 \bmod 4$, and let $f = \chi_{10} \cdot \chi_{12} \cdot \varphi_4^{(k-22)/4}$ if $k \equiv 2 \bmod 4$. Then it is easy to see that: $a((2,2,1), f) = -1/4$ (resp. $-1/6$) if $k \equiv 0 \bmod 4$ (resp. $k \equiv 2 \bmod 4$), but $a((1,4,1), f) = 0$ (obviously). We remark also that $e(2,2,1) = e(1,4,1) = 1$ and $\Delta(2,2,1) = \Delta(1,4,1) = 15$. Hence this example shows also that $a(T, f)$ is not necessarily determined only by $e(T)$ and $\Delta(T)$.

Let $k \geqq 10$ be an even integer, and put $r = \dim_{\mathbf{C}} S_k(\Gamma_2)$ and $d = \dim_{\mathbf{C}} S_{2k-2}(\Gamma_1)$. Then $r \geqq d$ as is remarked in § 2. This inequality is proved as follows. Put $I_k = \{(a,b,c,d) \mid a,b,c,d \geqq 0$ integers, $10a + 12b + 4c + 6d = k$ and $a + b \geqq 1\}$, $I_k^1 = \{(a,b,c,d) \in I_k \mid a + b = 1\}$ and $I_k^2 = \{(a,b,c,d) \in I_k \mid a + b \geqq 2\}$. Put $f(a,b,c,d) = \chi_{10}^a \cdot \chi_{12}^b \cdot \varphi_4^c \cdot \varphi_6^d$ and $S_k^i(\Gamma_2) = \langle\{f(a,b,c,d) \mid (a,b,c,d) \in I_k^i\}\rangle$ for $i = 1$ or 2, where $\langle\{\ \}\rangle$ denotes the vector space over $\mathbf{C}$ spanned by the elements in $\{\ \}$. Then, by Igusa's structure theorem for $S_k(\Gamma_2)$, we see that

$$S_k(\Gamma_2) = \langle\{f(a,b,c,d) \mid (a,b,c,d) \in I_k\}\rangle$$

and $\dim_{\mathbf{C}} S_k(\Gamma_2) = \dim_{\mathbf{C}} S_k^1(\Gamma_2) + \dim_{\mathbf{C}} S_k^2(\Gamma_2)$. On the other hand, an easy calculation shows that $\dim_{\mathbf{C}} S_k^1(\Gamma_2) = \dim_{\mathbf{C}} S_{2k-2}(\Gamma_1)$. Hence $r - d = \dim_{\mathbf{C}} S_k^2(\Gamma_2) \geqq 0$. Moreover, if $k \geqq 20$ is an even integer, then the above f which is used to show $S_k^I(\Gamma_2) \subsetneqq S_k(\Gamma_2)$ is a non-zero element in $S_k^2(\Gamma_2)$, hence it holds that $r - d \geqq 1$.

Let $\{\chi_k^{(j)} \mid j = 1, \dots, r\}$ (resp. $\{\Delta_{2k-2}^{(i)} \mid i = 1, \dots, d\}$) be an eigen basis of $S_k(\Gamma_2)$ (resp. $S_{2k-2}(\Gamma_1)$) over $\mathbf{C}$. Then, by Conjecture 1, we may assume that $\sigma_k(\Delta_{2k-2}^{(i)}) = \chi_k^{(i)}$ and $L(s, \chi_k^{(i)}) = \zeta(s-k+2)\,\zeta(s-k+1)\,L(s, \Delta_{2k-2}^{(i)})$ for $i = 1, \dots, d$. As a supplement to Conjecture 1, we pose the following Conjecture 3.

Conjecture 3. *Let $k \geqq 20$ be an even integer. Then $\chi_k^{(j)}$ satisfies Ramanujan conjecture for $j = d+1, \dots, r$.*

We note here that if $\chi_k^{(j)}$ satisfies Ramanujan conjecture, then the Euler product defining $L(s, \chi_k^{(j)})$ converges absolutely in $\mathrm{Re}(s) > k - \frac{1}{2}$, hence $L(s, \chi_k^{(j)})$ is holomorphic on $\mathbf{C}$. In fact, the possible simple pole of $L(s, \chi_k^{(j)})$ exists only at $s = k$ (see Theorem 3.1.1 (IV) in [1]), hence it does not occur for such $L(s, \chi_k^{(j)})$.

We note also the following point: since $r = k^3/8640 + O(k^2)$ and $d = k/6 + O(1)$ as $k \to \infty$, it follows from Conjecture 3 that the large part of Siegel eigen cusp forms of degree two will satisfy Ramanujan conjecture.

Since $r = 3$ and $d = 2$ for $k = 20$, the "first" example which should be checked is $\chi_{20}^{(3)}$. It will be interesting to determine the Euler factors of $L(s, \chi_{20}^{(3)})$. The Euler factors at primes 2 and 3 will be determined in § 5 and § 7. It turns out that these two Euler factors fit Conjecture 3.

§ 5. Examples for Weight 20

5.1. An Eigen Basis of $S_{20}(\Gamma_2)$. We give an eigen basis of $S_{20}(\Gamma_2)$ over $\mathbf{C}$. In this case, $\dim_{\mathbf{C}} S_{20}(\Gamma_2) = 3$ and an eigen basis of $S_{20}(\Gamma_2)$ over $\mathbf{C}$ is given by the following $\{\chi_{20}^{(1)}, \chi_{20}^{(2)}, \chi_{20}^{(3)}\}$. Put $f = 4 \cdot \chi_{10} \cdot \varphi_4 \cdot \varphi_6$, $g = 12 \cdot \chi_{12} \cdot \varphi_4^2$ and $h = 48 \cdot \chi_{10}^2$. For simplicity, we put $D = 63\,737\,521$ in § 5. We note that D is the product of the following three primes 181, 349 and 1009. Then:

$\chi_{20}^{(1)} = 460\,f - (7699 + \sqrt{D})\,g - 345600(8021 + \sqrt{D})\,h,$

$\lambda(2, \chi_{20}^{(1)}) = 689232 + 48\sqrt{D},$

$\chi_{20}^{(2)} = 460\,f - (7699 - \sqrt{D})\,g - 345600(8021 - \sqrt{D})\,h,$

$\lambda(2, \chi_{20}^{(2)}) = 689232 - 48\sqrt{D},$

$\chi_{20}^{(3)} = f - g + 595200\,h,$

$\lambda(2, \chi_{20}^{(3)}) = -840960.$

In the determination of $\chi_{20}^{(1)}$, $\chi_{20}^{(2)}$ and $\chi_{20}^{(3)}$, we use the following values of Fourier coefficients.

	$a(1,1,1)$	$a(1,1,0)$	$a(2,2,2)$	$a(2,2,0)$	$a(1,3,0)$	$a(4,4,4)$
f	-1	2	-59616	-187328	-261088	-309027859456
g	1	10	44064	1701440	556576	439982666752
h	0	0	3	18	0	424224

For the calculation of these Fourier coefficients, we refer to § 6.

For each $f \in S_k(\Gamma_2)$, k being a positive integer, the following equalities hold (see the equation (2.1.11) in [1]): $a(2;(1,1,1),f) = a((2,2,2),f)$, $a(2;(1,1,0),f) = a((2,2,0),f) + 2^{k-2} \cdot a((1,1,0),f)$, and $a(2;(2,2,2),f) = a((4,4,4),f) + 3 \cdot 2^{k-2} \cdot a((1,3,0),f) + 2^{2k-3} \cdot a((1,1,1),f)$. Hence, from the above table, we obtain the following values of $a(2;T)$ for $T = (1,1,1)$, $(1,1,0)$ and $(2,2,2)$.

	$a(2;(1,1,1))$	$a(2;(1,1,0))$	$a(2;(2,2,2))$
f	-59616	336960	-651794770944
g	44064	4322880	1015130797056
h	3	18	424224

Using these values of $a(2;T)$, we can determine the action of $T(2)$ on $S_{20}(\Gamma_2)$. The result is as follows:

$T(2)\,f = 77760\,f + 18144\,g - 215986176000\,h,$

$T(2)\,g = 323520\,f + 367584\,g + 339406848000\,h,$

$T(2)\,h = -f + 2g + 92160\,h.$

For example, we can write $T(2)\,f = \alpha \cdot f + \beta \cdot g + \gamma \cdot h$ with $\alpha, \beta, \gamma \in \mathbf{C}$. By comparing the Fourier coefficients at $T = (1,1,1)$, $(1,1,0)$ and $(2,2,2)$, we have:

$-\alpha + \beta = -59616,$

$2 \cdot \alpha + 10 \cdot \beta = 336960,$

$-59616 \cdot \alpha + 44064 \cdot \beta + 3 \cdot \gamma = -651794770944.$

Hence we have: $\alpha = 77760$, $\beta = 18144$, and $\gamma = -215986176000$. Other cases are similar.

Since the action of $T(2)$ on $S_{20}(\Gamma_2)$ is determined as above, we can calculate the characteristic polynomial of $T(2)$ on $S_{20}(\Gamma_2)$. It is given by:

$$X^3 - 537\,504\,X^2 - 831\,043\,584\,000\,X + 275\,994\,243\,130\,982\,400$$
$$= (X + 840\,960)(X^2 - 1\,378\,464\,X + 328\,189\,501\,440).$$

Hence we can determine eigenvalues and eigenfunctions of $T(2)$ on $S_{20}(\Gamma_2)$. Thus we obtain the eigen basis of $S_{20}(\Gamma_2)$ stated previously.

5.2. An Eigen Basis of $S_{38}(\Gamma_1)$. We give an eigen basis of $S_{38}(\Gamma_1)$ over $\mathbf{C}$. In this case, $\dim_{\mathbf{C}} S_{38}(\Gamma_1) = 2$ and an eigen basis of $S_{38}(\Gamma_1)$ over $\mathbf{C}$ is given by the following $\{\Delta_{38}^{(1)}, \Delta_{38}^{(2)}\}$. Put $A = \Delta \cdot E_6^3 \cdot E_4^2$ and $B = \Delta^2 \cdot E_6 \cdot E_4^2$. Then:

$$\Delta_{38}^{(1)} = A + (-96\,144 + 48\sqrt{D})\,B,$$
$$\lambda(2, \Delta_{38}^{(1)}) = -97\,200 + 48\sqrt{D},$$
$$\Delta_{38}^{(2)} = A + (-96\,144 - 48\sqrt{D})\,B,$$
$$\lambda(2, \Delta_{38}^{(2)}) = -97\,200 - 48\sqrt{D}.$$

In the determination of $\Delta_{38}^{(1)}$ and $\Delta_{38}^{(2)}$, we use the following Fourier coefficients:

$$A = q - 1056 \cdot q^2 + * q^3 + 169\,741\,312 \cdot q^4 + \cdots,$$
$$B = q^2 + * q^3 - 194\,400 \cdot q^4 + \cdots.$$

Then, the action of $T(2)$ on $S_{38}(\Gamma_1)$ is determined by the same method as in § 3. It is given by:

$$T(2)\,A = -1056\,A + 137\,607\,579\,648\,B,$$
$$T(2)\,B = A - 193\,344\,B.$$

The characteristic polynomial of $T(2)$ on $S_{38}(\Gamma_1)$ is: $X^2 + 194400\,X - 137403408384$. Thus we obtain the above eigen basis of $S_{38}(\Gamma_1)$.

5.3. Euler Factors at the Prime 2. We determine Euler factors at 2 of $L(s, \chi_{20}^{(j)})$ for $j = 1, 2$ and 3. Before going into this process, we remark the following point which gives a suggestion. From the results in 5.1 and 5.2, it is easy to check that the following equalities hold: $\lambda(2, \chi_{20}^{(i)}) = \lambda(2, \Delta_{38}^{(i)}) + 2^{18} + 2^{19}$ for $i = 1$ and 2. Hence, by Conjecture 1 in § 2, it would be that

$$\sigma_{20}(\Delta_{38}^{(i)}) = \chi_{20}^{(i)} \quad \text{and} \quad L(s, \chi_{20}^{(i)}) = \zeta(s-18)\,\zeta(s-19)\,L(s, \Delta_{38}^{(i)})$$

for $i = 1$ and 2. In particular, it will be that $H_2(T, \chi_{20}^{(i)}) = (1 - 2^{18}\,T)(1 - 2^{19}\,T) \cdot H_2(T, \Delta_{38}^{(i)})$ for $i = 1$ and 2. In fact, we prove below that the last equalities hold.

We first study the case of $\chi_{20}^{(1)}$. From the values of Fourier coefficients in 5.1, we have: $a((1,1,1), \chi_{20}^{(1)}) = -(8\,159 + \sqrt{D})$ and

$$a((4,4,4), \chi_{20}^{(1)}) = -(4\,705\,552\,729\,975\,808 + 586\,594\,481\,152\sqrt{D})$$
$$= -2^{10}(4\,595\,266\,337\,867 + 572\,846\,173\sqrt{D}).$$

Hence we have:

$$\lambda(4,\chi_{20}^{(1)})=a((4,4,4),\chi_{20}^{(1)})\cdot a((1,1,1),\chi_{20}^{(1)})^{-1}=2^9(692843369+55503\sqrt{D}).$$

Thus we have:

$$\begin{aligned}H_2(T,\chi_{20}^{(1)})&=1-2^4(43077+3\sqrt{D})\,T+2^{22}(47311+9\sqrt{D})\cdot T^2\\&-2^{41}(43077+3\sqrt{D})\,T^3+2^{74}T^4=(1-2^{18}T)(1-2^{19}T)\\&\cdot(1-2^4(-6075+3\sqrt{D})\,T+2^{37}T^2).\end{aligned}$$

Since $H_2(T,\Delta_{38}^{(1)})=1-2^4(-6075+3\sqrt{D})\,T+2^{37}\,T^2$, we have the required result: $H_2(T,\chi_{20}^{(1)})=(1-2^{18}\,T)(1-2^{19}\,T)\,H_2(T,\Delta_{38}^{(1)})$.

The calculation for $\chi_{20}^{(2)}$ is the same as for $\chi_{20}^{(1)}$, and the results for $\chi_{20}^{(2)}$ are obtained from the results for $\chi_{20}^{(1)}$ by changing the signs of $\sqrt{D}$. Hence $H_2(T,\chi_{20}^{(2)})=(1-2^{18}T)\cdot(1-2^{19}T)H_2(T,\Delta_{38}^{(2)})$ and $H_2(T,\Delta_{38}^{(2)})=1-2^4(-6075-3\sqrt{D})\,T+2^{37}\,T^2$.

Next, we study the case of $\chi_{20}^{(3)}$. In this case we have: $a((1,1,1),\chi_{20}^{(3)})=-2$ and $a((4,4,4),\chi_{20}^{(3)})=-496512401408$. Hence we have $\lambda(4,\chi_{20}^{(3)})=248256200704=2^{16}\cdot 3788089$. Thus we have $H_2(T,\chi_{20}^{(3)})=1+2^8\cdot 3285\cdot T+2^{26}\cdot 5815\cdot T^2+2^{45}\cdot 3285\cdot T^3+2^{74}T^4$. We prove that this Euler factor $H_2(T,\chi_{20}^{(3)})$ of $L(s,\chi_{20}^{(3)})$ fits Conjecture 3 in § 4. Put $T=2^{-37/2}\cdot t$, then $H_2(T,\chi_{20}^{(3)})=1+at+bt^2+at^3+t^4$ with $a=2^{-21/2}\cdot 3285=2.2684\ldots$ and $b=2^{-11}\cdot 5815=2.8393\ldots$. We denote by $P_2(t)$ the above polynomial in t. What we should prove is the absolute values of zeros of $P_2(t)$ are 1. Put $\alpha=(-a+\sqrt{a^2-4b+8})/4$ and $\beta=(-a-\sqrt{a^2-4b+8})/4$. Then, by a simple calculation, we see that the requirement is satisfied if (and only if) $-1\leqq\alpha\leqq 1$ and $-1\leqq\beta\leqq 1$, and that if these conditions are satisfied then the four zeros of $P_2(t)$ are given by $\exp(\pm i\theta_1)$ and $\exp(\pm i\theta_2)$ with $\theta_1=\arccos(\alpha)$ and $\theta_2=\arccos(\beta)$ $(0\leqq\theta_1\leqq\pi$ and $0\leqq\theta_2\leqq\pi)$ and $i=\sqrt{-1}$. In this case, the numerical values are $\alpha=-0.2327\ldots$ and $\beta=-0.9014\ldots$. Hence the requirements are satisfied, and the four zeros of $H_2(T,\chi_{20}^{(3)})$ are given by $2^{-37/2}\cdot\exp(\pm i\theta_1)$ and $2^{-37/2}\cdot\exp(\pm i\theta_2)$ with $\theta_1=1.805\ldots$ and $\theta_2=2.693\ldots$. Thus the Euler factor $H_2(T,\chi_{20}^{(3)})$ of $L(s,\chi_{20}^{(3)})$ fits Conjecture 3. In § 7, we obtain the Euler factor $H_3(T,\chi_{20}^{(3)})$, and we see that this also fits Conjecture 3.

§ 6. A Table of Fourier Coefficients

Let $f_{16}=4\cdot\chi_{10}\cdot\varphi_6$ and $g_{16}=12\cdot\chi_{12}\cdot\varphi_4$. Then we obtain the following table of Fourier coefficients of χ_{12}, f_{16} and g_{16}.

We remark here that Fourier coefficients of Siegel modular forms of degree two can be calculated in the following four steps (1), (2), (3) and (4) in principle. (1) The Fourier coefficients of Siegel modular forms of degree two can be calculated from the Fourier coefficients of φ_4, φ_6, χ_{10} and χ_{12} as in the calculation of $c_{14}(T)$ in § 1. (2) The Fourier coefficients of φ_4 and φ_6 can be calculated by the method of Maass [4] (see Satz 1 and the equation (19) in [4]). (3) The Fourier coefficients of χ_{10} can be calculated from the Fourier coefficients of φ_4 by the method of Resnikoff-Saldaña [5] (see the first equation in Theorem 3 in [5]).

T	$\Delta(T)$	$e(T)$	$12c_{12}(T)$	$a(T, f_{16})$	$a(T, g_{16})$
(1, 1, 1)	3	1	1	−1	1
(1, 1, 0)	4	1	10	2	10
(1, 2, 1)	7	1	−88	520	152
(1, 2, 0)	8	1	−132	−1044	2268
(1, 3, 1)	11	1	1275	8469	−17685
(1, 3, 0)	12	1	736	−14848	−9344
(2, 2, 2)	12	2	2784	−47616	23424
(1, 4, 1)	15	1	−8040	−93000	114600
(1, 4, 0)	16	1	−2880	214656	−44160
(2, 2, 0)	16	2	17600	280192	283520
(1, 5, 1)	19	1	24035	90973	274595
(1, 5, 0)	20	1	13080	−628200	199800
(1, 6, 1)	23	1	−14136	1104552	−4555656
(1, 6, 0)	24	1	−54120	−764616	−200040
(1, 7, 1)	27	1	−128844	−748692	14102676
(3, 3, 3)	27	3	48303	−15097599	28451583
(1, 7, 0)	28	1	115456	2874368	5533696
(2, 4, 2)	28	2	−64768	19913728	10514432
(1, 8, 1)	31	1	389520	−16100016	−7096080
(1, 8, 0)	32	1	38016	23811840	−36612864
(2, 4, 0)	32	2	−232320	−10397952	37704960
(1, 9, 1)	35	1	−256410	32117850	−32117850
(1, 9, 0)	36	1	−697950	−109122534	71214210
(3, 3, 0)	36	3	1073520	−80424720	214703280
(1, 10, 1)	39	1	−806520	54300264	−826440
(1, 10, 0)	40	1	963160	143559800	8656600
(1, 11, 1)	43	1	1892363	−157908251	140385563
(1, 11, 0)	44	1	938400	−41467392	1572480
(2, 6, 2)	44	2	3549600	236044800	−577929600
(1, 12, 1)	47	1	−1227600	−98103312	124508880
(1, 12, 0)	48	1	−2309120	120737792	−726118400
(2, 6, 0)	48	2	−801792	−365801472	−1032302592
(4, 4, 4)	48	4	3392512	−1439543296	41439232

(4) The Fourier coefficients of χ_{12} can be calculated from the Fourier coefficients of χ_{10} by the method of Resnikoff-Saldaña [5] (see the equation in § 5 of [5] which gives a relation between the Fourier coefficients of χ_{10} and χ_{12}).

In the above table, the values of $12c_{12}(T)$ for $e(T)=1$ were calculated by the method of Maass [4] (Satz 2), and the values of $12c_{12}(T)$ for $e(T)>1$ were calculated by the above method (4). The values of $a(T, f_{16})$ (resp. $a(T, g_{16})$) were calculated by the above method (1) from the Fourier coefficients of χ_{10} and φ_6 (resp. χ_{12} and φ_4). We note here that $4c_{10}(3, 3, 3) = -15399$ and $4c_{10}(2, 6, 0) = 126720$ in Table IV of [5]. The Fourier coefficients of f (resp. g) in § 5 were calculated by the above method (1) from the Fourier coefficients of f_{16} and φ_4 (resp. g_{16} and φ_4). The Fourier coefficients of h in § 5 were calculated by the method (1) from the Fourier coefficients of χ_{10}. Actually, in our calculation with using the method (1), we need Fourier coefficients at several T which are not unimodularly equivalent to any T listed in the above table. For example, we need Fourier coefficients at $T=(2, 2, 1)$. To calculate these Fourier coefficients, the methods (2), (3), (4) and (1) were used in this order. In the step (2), we can simplify our calculation by using Tables I–III and V in [5]. As a result, it turns out that for $\chi_{10}, \chi_{12}, f_{16}$ and g_{16}, the equation (*) in § 4 hold at these T. For example, the Fourier coefficients of $\chi_{10}, \chi_{12}, f_{16}$ and g_{16} at $T=(2,2,1)$ are equal to the Fourier coefficients at $T=(1,4,1)$, the latter values are listed in the above table or in Table IV in [5]. We note that, in the above table, the values of $a(T, f_{16})$ and $a(T, g_{16})$ at T with $e(T)=1$ are also calculated by the method of Maass [4] (Satz 3).

It may be noted that we can calculate some of the Fourier coefficients in the above table in a simpler way with using the method in the next section. For example, since $12c_{12}(1,1,1)=1$, $12c_{12}(2,2,2)=2784$ and $12c_{12}(1,3,0)=736$, we have by this method that $\lambda(2,\chi_{12})=2784$ and $\lambda(4,\chi_{12})=3392512$. Hence we have $12c_{12}(4,4,4)=12c_{12}(1,1,1)\cdot\lambda(4,\chi_{12})=3392512$. This method applies also for f_{16} and g_{16}, but in this case we must first apply this method to $\chi_{16}^{(+)}$ and $\chi_{16}^{(-)}$.

§ 7. Further Examples

We give here further examples. We formulate our method used in this section as follows. Let $k\geqq 10$ be an even integer. Let $f\in S_k(\Gamma_2)$ be an eigen cusp form, and suppose that we know the values of the following five Fourier coefficients of f: $a(1,1,1)$, $a(2,2,2)$, $a(1,3,0)$, $a(3,3,3)$ and $a(1,7,1)$. Suppose moreover that $a(1,1,1)\neq 0$. Then we can determine the four eigenvalues $\lambda(2)$, $\lambda(4)$, $\lambda(3)$ and $\lambda(9)$, and the two Euler factors $H_2(T,f)$ and $H_3(T,f)$. They are given by:

$$H_2(T,f)=1-\lambda(2)\,T+(\lambda(2)^2-\lambda(4)-2^{2k-4})\,T^2-2^{2k-3}\cdot\lambda(2)\,T^3+2^{4k-6}\,T^4$$

with

$$\lambda(2)=a(2,2,2)\cdot a(1,1,1)^{-1}$$

and

$$\lambda(4)=\lambda(2)^2-3\cdot 2^{k-2}\cdot a(1,3,0)\cdot a(1,1,1)^{-1}-2^{2k-3},$$

and

$$H_3(T,f)=1-\lambda(3)\,T+(\lambda(3)^2-\lambda(9)-3^{2k-4})\,T^2-3^{2k-3}\cdot\lambda(3)\,T^3+3^{4k-6}\,T^4$$

with

$$\lambda(3)=a(3,3,3)\cdot a(1,1,1)^{-1}+3^{k-2}$$

and

$$\lambda(9)=\lambda(3)^2-3^{k-2}\cdot\lambda(3)-3^{k-1}\cdot a(1,7,1)\cdot a(1,1,1)^{-1}-3^{2k-3}.$$

These formulas are obtained by simple calculations from the following equalities (see (2.1.11) in [1]):

$$\lambda(2)\,a(1,1,1)=a(2;(1,1,1))=a(2,2,2),$$

$$\lambda(4)\,a(1,1,1)=a(4;(1,1,1))=a(4,4,4),$$

$$\lambda(2)\,a(2,2,2)=a(2;(2,2,2))=a(4,4,4)+3\cdot 2^{k-2}\cdot a(1,3,0)+2^{2k-3}\cdot a(1,1,1),$$
$$\lambda(3)\,a(1,1,1)=a(3;(1,1,1))=a(3,3,3)+3^{k-2}\cdot a(1,1,1),$$
$$\lambda(9)\,a(1,1,1)=a(9;(1,1,1))=a(9,9,9)+3^{k-2}\cdot a(3,3,3),$$

and

$$\lambda(3)\,a(3,3,3)=a(3;(3,3,3))=a(9,9,9)+3^{k-2}\cdot a(3,3,3)+3^{k-1}\cdot a(1,7,1)$$
$$+3^{2k-3}\cdot a(1,1,1).$$

In our applications of this method, we use the following values of Fourier coefficients. We refer to § 6 for the calculation of these Fourier coefficients. In the following table, f and g for $k=16$ or 18 are as in § 3, and f, g and h for $k=20$ are as in § 5.

k		$a(1,1,1)$	$a(2,2,2)$	$a(1,3,0)$	$a(3,3,3)$	$a(1,7,1)$
10	$4\cdot\chi_{10}$	-1	-240	272	-15399	4284
12	$12\cdot\chi_{12}$	1	2784	736	48303	-128844
14	$4\cdot\chi_{14}$	-1	-12240	-4048	-1398519	195804
16	f	-1	-47616	-14848	-15097599	-748692
	g	1	23424	-9344	28451583	14102676
18	f	-1	-24240	106832	-182896839	-53756676
	g	1	32016	-99056	180470727	51330564
20	f	-1	-59616	-261088	-674624079	-73332
	g	1	44064	556576	444330063	-21201804
	h	0	3	0	2016	0

For reader's convenience, we list below the eigenvalues and Euler factors obtained by the above method with using the values of Fourier coefficients in the above table, although some of them have appeared in the previous sections. To simplify the expression, we give attention to the 2-power (resp. 3-power) factors of $\lambda(2)$ or $\lambda(4)$ (resp. $\lambda(3)$ or $\lambda(9)$), and we express Euler factors in simpler forms.

k = 10 (on χ_{10})

$$\lambda(2)=2^4\cdot 15,\qquad \lambda(4)=2^8\cdot 529,$$
$$H_2(T)=(1-2^8T)(1-2^9T)(1+2^4\cdot 33\cdot T+2^{17}T^2),$$
$$\lambda(3)=3^2\cdot 2440,\qquad \lambda(9)=3^4\cdot 3621529,$$
$$H_3(T)=(1-3^8T)(1-3^9T)(1+3^2\cdot 476\cdot T+3^{17}T^2).$$

k = 12 (on χ_{12})

$$\lambda(2)=2^5\cdot 87,\qquad \lambda(4)=2^{10}\cdot 3313,$$
$$H_2(T)=(1-2^{10}T)(1-2^{11}T)(1+2^5\cdot 9\cdot T+2^{21}T^2),$$
$$\lambda(3)=3^3\cdot 3976,\qquad \lambda(9)=3^6\cdot 24073249,$$
$$H_3(T)=(1-3^{10}T)(1-3^{11}T)(1+3^3\cdot 4772\cdot T+3^{21}T^2).$$

k = 14 (on χ_{14})

$\lambda(2)=2^4\cdot 765,\quad \lambda(4)=2^8\cdot 259849,$

$H_2(T)=(1-2^{12}T)(1-2^{13}T)(1+2^4\cdot 3\cdot T+2^{25}T^2),$

$\lambda(3)=3^3\cdot 71480,\quad \lambda(9)=3^6\cdot 2968411441,$

$H_3(T)=(1-3^{12}T)(1-3^{13}T)(1+3^3\cdot 7252\cdot T+3^{25}T^2).$

k = 16 We put $D=51349$ in this case, and we describe about $\chi_{16}^{(+)}$. The results for $\chi_{16}^{(-)}$ are obtained by changing the signs of $\sqrt{D}$.

$a(1,1,1)=-313+\sqrt{D},\ a(2,2,2)=2^7(-92244+183\sqrt{D})$
$=2^5(1671+3\sqrt{D})\,a(1,1,1),\ a(1,3,0)=2^7(-12116-73\sqrt{D})$
$=2^5(647+3\sqrt{D})\,a(1,1,1),\ a(3,3,3)=3^2(-714984271+3161287\sqrt{D})$
$=3^2(1318343-5888\sqrt{D})\,a(1,1,1),\ a(1,7,1)=3^2(-215961172$
$+1566964\sqrt{D})=3^2(-275980-5888\sqrt{D})\,a(1,1,1),\ \lambda(2)=2^5(1671+3\sqrt{D}),$
$\lambda(4)=2^{11}(868151+2709\sqrt{D}),\ H_2(T)=(1-2^{14}T)(1-2^{15}T)$
$\cdot(1-2^5(135+3\sqrt{D})T+2^{29}T^2),\ \lambda(3)=3^2(1849784-5888\sqrt{D}),$
$\lambda(9)=3^4(3811557505865-9266557952\sqrt{D}),$
$H_3(T)=(1-3^{14}T)(1-3^{15}T)(1+3^2(275980+5888\sqrt{D})T+3^{29}T^2).$

k = 18 We put $D=2356201$ in this case, and we describe about $\chi_{18}^{(+)}$. The results for $\chi_{18}^{(-)}$ are obtained by changing the signs of $\sqrt{D}$.

$a(1,1,1)=-1441+\sqrt{D},\ a(2,2,2)=2^4(-1624701+2001\sqrt{D})$
$=2^3(16971+9\sqrt{D})\,a(1,1,1),\ a(1,3,0)=2^4(10179971-6191\sqrt{D})$
$=2^3(587+9\sqrt{D})\,a(1,1,1),\ a(3,3,3)=3^3(-9864516581+6684101\sqrt{D})$
$=3^3(5485189-832\sqrt{D})\,a(1,1,1),\ a(1,7,1)=3^3(-2972258252+1901132\sqrt{D})$
$=3^3(702220-832\sqrt{D})\,a(1,1,1),\ \lambda(2)=2^3(16971+9\sqrt{D}),$
$\lambda(4)=2^7(165111641+42147\sqrt{D}),\ H_2(T)=(1-2^{16}T)(1-2^{17}T)$
$\cdot(1-2^3(-7605+9\sqrt{D})T+2^{33}T^2),\ \lambda(3)=3^3(7079512-832\sqrt{D}),$
$\lambda(9)=3^6(29479186252625-6474401024\sqrt{D}),$
$H_3(T)=(1-3^{16}T)(1-3^{17}T)(1-3^3(702220-832\sqrt{D})T+3^{33}T^2).$

k = 20 We put $D=63737521$ in this case, and we describe about $\chi_{20}^{(1)}$ and $\chi_{20}^{(3)}$. The results for $\chi_{20}^{(2)}$ are obtained from the results for $\chi_{20}^{(1)}$ by changing the signs of $\sqrt{D}$.

$\chi_{20}^{(1)}$: $a(1,1,1)=-8159-\sqrt{D},\ a(2,2,2)=2^5(-271338903-33777\sqrt{D})$
$=2^4(43077+3\sqrt{D})\,a(1,1,1),\ a(1,3,0)=2^5(-137661847-17393\sqrt{D})$
$=2^4(10309+3\sqrt{D})\,a(1,1,1),\ a(3,3,3)=3^3(-345173790851-42261469\sqrt{D})$

$=3^3(43305821-128\sqrt{D})\,a(1,1,1)$, $a(1,7,1)=3^3(6044405788+785252\sqrt{D})$
$=3^3(259100-128\sqrt{D})\,a(1,1,1)$, $\lambda(2)=2^4(43077+3\sqrt{D})$,
$\lambda(4)=2^9(692843369+55503\sqrt{D})$, $H_2(T)=(1-2^{18}T)(1-2^{19}T)$
$\cdot(1-2^4(-6075+3\sqrt{D})\,T+2^{37}T^2)$, $\lambda(3)=3^3(57654728-128\sqrt{D})$,
$\lambda(9)=3^6(1869002804420705-1881883486720\sqrt{D})$,
$H_3(T)=(1-3^{18}T)(1-3^{19}T)(1-3^3(259100-128\sqrt{D})\,T+3^{37}T^2)$.

$\chi_{20}^{(3)}$: $a(1,1,1)=-2$, $a(2,2,2)=2^9\cdot 3285=2^8(-3285)\,a(1,1,1)$,
$a(1,3,0)=2^9(-1597)=2^8\cdot 1597\cdot a(1,1,1)$, $a(3,3,3)=3^5\cdot 333206$
$=3^5(-166603)\,a(1,1,1)$, $a(1,7,1)=3^3\cdot 782536=3^3(-391268)\,a(1,1,1)$,
$\lambda(2)=2^8(-3285)$, $\lambda(4)=2^{16}\cdot 3788089$, $H_2(T)=1+2^8\cdot 3285\cdot T$
$+2^{26}\cdot 5815\cdot T^2+2^{45}\cdot 3285\cdot T^3+2^{74}T^4$, $\lambda(3)=3^5\cdot 1427720$,
$\lambda(9)=3^{10}(-7655524062959)$, $H_3(T)=1-3^5\cdot 1427720\cdot T$
$+3^{22}\cdot 13457830\cdot T^2-3^{42}\cdot 1427720\cdot T^3+3^{74}T^4$.

On the other hand, the eigenvalues for eigen cusp forms in $S_{2k-2}(\Gamma_1)$ are given as below. In these cases, Euler factors are given by $H_p(T)=1-\lambda(p)\,T+p^{2k-3}T^2$. We use the notations in the previous sections. In each of the cases for $k=16$, 18 or 20, the number D is as above, and we describe about the eigen cusp form which corresponds to the above eigen cusp form in $S_k(\Gamma_2)$. The results for another eigen cusp form are obtained by changing the signs of $\sqrt{D}$.

k = 10 $(2k-2=18)$

$\lambda(2)=-2^4\cdot 33$, $\quad\lambda(3)=-3^2\cdot 476$.

k = 12 $(2k-2=22)$

$\lambda(2)=-2^5\cdot 9$, $\quad\lambda(3)=-3^3\cdot 4772$.

k = 14 $(2k-2=26)$

$\lambda(2)=-2^4\cdot 3$, $\quad\lambda(3)=-3^3\cdot 7252$.

k = 16 $(2k-2=30)$

$\lambda(2)=2^5(135+3\sqrt{D})$, $\lambda(3)=3^2(-275980-5888\sqrt{D})$.

k = 18 $(2k-2=34)$

$\lambda(2)=2^3(-7605+9\sqrt{D})$, $\lambda(3)=3^3(702220-832\sqrt{D})$.

k = 20 $(2k-2=38)$

$\lambda(2)=2^4(-6075+3\sqrt{D})$, $\lambda(3)=3^3(259100-128\sqrt{D})$.

Thus, the above examples for χ_{10}, χ_{12}, χ_{14}, $\chi_{16}^{(+)}$, $\chi_{16}^{(-)}$, $\chi_{18}^{(+)}$, $\chi_{18}^{(-)}$, $\chi_{20}^{(1)}$ and $\chi_{20}^{(2)}$ fit Conjecture 1.

As is proved in § 5, the Euler factor $H_2(T, \chi_{20}^{(3)})$ fits Conjecture 3. The corresponding fact for $H_3(T, \chi_{20}^{(3)})$ is proved as follows. Put $T = 3^{-37/2} \cdot t$, then $H_3(T, \chi_{20}^{(3)}) = 1 - ct + dt^2 - ct^3 + t^4$ with $c = 3^{-27/2} \cdot 1427720 = 0.5170\ldots$ and

$$d = 3^{-15} \cdot 13457830 = 0.9378\ldots.$$

We denote by $P_3(t)$ the above polynomial in t. Put $\gamma = (c + \sqrt{c^2 - 4d + 8})/4$ and $\delta = (c - \sqrt{c^2 - 4d + 8})/4$. Then $\gamma = 0.6605\ldots$ and $\delta = -0.4020\ldots$, hence we see as in § 5 that the absolute values of the four zeros of $P_3(t)$ are 1, and these zeros are given by $\exp(\pm i\theta_3)$ and $\exp(\pm i\theta_4)$ with $\theta_3 = \arccos(\gamma)$ and $\theta_4 = \arccos(\delta)$ ($0 \leqq \theta_3 \leqq \pi$ and $0 \leqq \theta_4 \leqq \pi$). Hence, the absolute values of the four zeros of $H_3(T, \chi_{20}^{(3)})$ are $3^{-37/2}$, and these zeros are given by $3^{-37/2} \cdot \exp(\pm i\theta_3)$ and $3^{-37/2} \cdot \exp(\pm i\theta_4)$ with $\theta_3 = 0.849\ldots$ and $\theta_4 = 1.984\ldots$. Thus the Euler factor $H_3(T, \chi_{20}^{(3)})$ also fits Conjecture 3.

Lastly, we remark that the above examples fit Conjecture 2 in § 4. In fact, from the above values, it is easy to see that for $f = \chi_{10}, \chi_{12}, \chi_{14}, \chi_{16}^{(+)}, \chi_{16}^{(-)}, \chi_{18}^{(+)}, \chi_{18}^{(-)}, \chi_{20}^{(1)}$ and $\chi_{20}^{(2)}$ the following equalities hold: $a((2,2,2), f) = 2^{k-1} \cdot a((1,1,1), f) + a((1,3,0), f)$ and $a((3,3,3), f) = 3^{k-1} \cdot a((1,1,1), f) + a((1,7,1), f)$. (Note that $(1,9,3)$ is unimodularly equivalent to $(1,7,1)$.) It is remarked that these equalities do not hold for $f = \chi_{20}^{(3)}$.

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Note Added in Proof. (1) Our Ramanujan conjecture formulated in Notations says that $L(s + n(2k - n - 1)/4, f)$ is "unitary"; cf. Kurokawa, N.: On the meromorphy of Euler products. Proc. Japan Acad., **54**A, 163–166 (1978). (2) We have analogues for "Siegel wave forms" of results in Andrianov [1] and conjectures in this paper.