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Characterization of the Unit Ball in $\mathbb{C}^n$ by Its Automorphism Group

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For a complex manifold M we denote by $\mathrm{Aut}(M)$ the group of biholomorphic automorphisms of M. The purpose of this paper is to prove the following

Theorem. *Let G be strongly pseudoconvex bounded domain with smooth boundary in $\mathbb{C}^n$. Then the following statements are equivalent:*

i) *G is biholomorphic to the unit ball $B_n \subset \mathbb{C}^n$.*

ii) $\mathrm{Aut}(G)$ *is non-compact.*

iii) *G is homogeneous.*

iv) *There is a subgroup $Z \subset \mathrm{Aut}(G)$ acting properly discontinuously on G such that G/Z is compact.*

Remarks. a) Notice, that the necessity of conditions ii), iii), iv) for i) to hold is trivial. Furthermore, the implication iii) $\Rightarrow$ ii) is obvious. Therefore, it will suffice to prove the implications ii) $\Rightarrow$ i) and iv) $\Rightarrow$ ii).

b) Some closely related results on C.R. transformations of strongly pseudoconvex hypersurfaces can be found in the Berkeley thesis (1975) of Webster and an article of Burns and Shnider ([1]). Furthermore, Burns told the author in a letter that he and Shnider also proved the theorem given here (with a slightly weaker version in the case $n=2$) by using Chern-Moser invariants and Feffermans hard theorem on biholomorphic mappings between strongly pseudoconvex domains. However, we want to point out, that in our proof Feffermans theorem and also Chern-Moser invariants are not used.

We shall give two proofs of our theorem. The first one involves the results of Diederich and Graham on the boundary behavior of the Bergman, Caratheodory and Kobayashi metrics, together with an observation on their holomorphic curvatures ([7]). The second one is an application of the boundary estimates for the corresponding intrinsic measures which were derived in [8].

§ 1. Definitions and Known Results

Let G be a bounded domain in $\mathbb{C}^n$. We denote by $B_1(G)$ (resp. $B_n(G)$) the family of holomorphic mappings from G into the unit disc B_1 (resp. the unit ball B_n in

$\mathbb{C}^n$) and by $G(B_1)$ (resp. $G(B_n)$) the family of holomorphic maps from B_1 (resp. B_n) into G. Furthermore, $T(G)$ means the holomorphic tangent bundle on G.

The *differential Carathéodory metric* on G is given by

$$C_G\colon T(G) \to \mathbb{R}^+ \cup \{0\},$$
$$C_G(z, v) = \sup\{|df(v)| : f \in B_1(G), f(z) = 0\}$$

where $df(v)$ is measured with respect to the Poincaré mezric on B_1. The *differential Kobayashi metric* on G is given by

$$K_G\colon T(G) \to \mathbb{R}^+ \cup \{0\},$$
$$K_G(z, v) = \inf\{|t| \colon t \text{ is a tangent vector to } B_1 \text{ at } 0,$$
$$\exists f \in G(B_1) \text{ with } f(0) = z \text{ and } df(t) = v\}$$

where t again is measured with respect to the Poincaré metric on B_1. The *Carathéodory measure* on G is defined by

$$M_G^C(z) = \sup\{f^*(M_n)(z) \colon f \in B_n(G), f(z) = 0\}$$

where M_n is the volume form of the Bergman metric on B_n. Finally, the *Eisenman-Koboyashi measure* on G (with respect to the unit ball) is defined as follows:

$$M_G^E(z) = \inf\{df(M_n)(z) \colon f \in G(B_n), f(0) = z\}.$$

We now summarize several known results on intrinsic metrics and measures which will be needed for the proof of the main theorem.

Theorem A (Schwartz lemma). *Let G_1, G_2 be bounded domains in $\mathbb{C}^n$ and I_1, I_2 either one of the intrinsic metrics or measures on G_1, G_2 respectively, as defined above. Suppose $f\colon G_1 \to G_2$ is a holomorphic map. Then one has $f^*(I_2) \leqq I_1$.*

(For the proof see Koboyashi [5].)

Theorem B (Diederich, Graham). *Let G be a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ with smooth boundary and denote by B_G the Bergman metric on G. Then for any $s>0$ there exists an $\eta>0$ such that for any $(z, v) \in T(G) = G \times \mathbb{C}^n$ with $v \neq 0$ and $d(z, \partial G) < \eta$. The following inequalities hold:*

$$\left|\frac{B_G(z, v)}{C_G(z, v)} - (n+1)^{\frac{1}{2}}\right| < s; \qquad \left|\frac{B_G(z, v)}{K_G(z, v)} - (n+1)^{\frac{1}{2}}\right| < s$$

(where d denotes the euclidean distance).

(This theorem is implicitly contained in Diederich ([2, 3]) and Graham ([4]).)

In Wong [7] the following characterization of the unit ball is given:

Theorem C. *Let G be a simply connected bounded domain in $\mathbb{C}^n$ (or even an arbitrary simply connected complex manifold), such that 1) G is complete hyper-*

bolic; 2) $K_G = C_G$; 3) K_G is a C^2 hermition metric. Then G is biholomorphic to the unit ball.

Finally, from Wong [8], one easily obtains:

Theorem D. *Let G be a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ with smooth boundary. Then for each $s>0$, there exists an $\eta>0$, such that for any $z \in G$ with $d(z, \partial G) < \eta$*

$$\left| \frac{M_G^E(z)}{M_G^C(z)} - 1 \right| < s.$$

§ 2. A Characterization of the Unit Ball by a Condition on Intrinsic Measures

We want to prove now:

Theorem E. *Let G be a complete hyperbolic bounded domain in $\mathbb{C}^n$. Suppose that for a certain point $z \in G$ one has*

$$M_G^E(z) = M_G^C(z).$$

Then G is biholomorphic to the unit ball.

Remark. If M^E and M^C are defined with respect to the unit polydisc $\Delta_n \subset \mathbb{C}^n$ and the condition of the theorem is satisfied with respect to these measures, then G is biholomorphic to Δ_n.

Proof. Since G is complete hyperbolic, the subset $\{f \in G(B_n)\colon f(0) = z\} \subset G(B_n)$ is compact in the CO-topology (see f.i. Theorem 3.2 in [5]). Therefore, there is a map $f_1 \in G(B_n)$, such that $M_G^E(z) = df_1(M_n)(z)$, $f_1(0) = z$. For the same reason, there is a map $f_2 \in B_n(G)$, such that $M_G^C(z) = f_2^*(M_B)(z)$, $f_2(z) = 0$. We consider the composite mapping $g = f_2 \circ f_1$, which maps B_n into B_n and the origin to the origin. From our assumption $M_G^E(z) = M_G^C(z)$ it now follows immediately that $|\det dg(0)| = 1$. As a consequence, g must be a biholomorphism from B_n to B_n according to Cartan's theorem (see f.i. Theorem 3.3 in [5]). In particular, $f_1\colon B_n \to G$ is injective and locally biholomorphic near the origin. It therefore suffices to prove that f_1 is also proper.

For this purpose we use the Koboyashi distance function d^K on G and B_n. If $\{x_i\}$ is any sequence in B_n tending to the boundary, the sequence $\{g(x_i) = f_2 \circ f_1(x_i)\} \subset B_n$ also tends to ∂B_n. Therefore, we get

$$d_G^K(z, f_1(x_i)) \geqq d_{B_n}^K(0, g(x_i)) \to \infty$$

because of the distance decreasing property of the Koboyashi distance. This shows that $\{f_1(x_i)\}$ tends to ∂G, which proves that f_1 is proper.

§ 3. Proof of the Main Theorem

We will need the following lemma, which was kindly communicated to us by Professor R. Greene:

Lemma. *Suppose G is a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ with smooth boundary. Suppose that there exists a sequence $\{g_i\} \subset \mathrm{Aut}(G)$, such that, for any point $z \in G$, the sequence $\{g_i(z)\}$ approaches ∂G. Then G must be simply connected.*

Proof. Suppose that G is not simply connected. Then there is a closed curve C in G not homotopic to a point. By the assumption of the lemma, the sequence of closed curves $\{g_i(C)\}$ approaches ∂G. Furthermore, all these curves have the same finite length with respect to the Bergman metric B_G and none of them is homotopic to a point in G. Since, on the other hand, by the result of Diederich ([2]), B_G is complete, there must be a subsequence of $\{g_i(C)\}$ tending to a point $q \in \partial G$. We can choose a neighborhood U of q, such that $U \cap G$ is simply connected, and just have proved that $g_i(C) \subset U \cap G$ for a certain i. This is a contradiction.

We now shall give two different proves for the implication

ii) $\Rightarrow$ *i):* 1) According to Graham [4], the domain G is complete hyperbolic. Furthermore, $\mathrm{Aut}(G)$ acts properly on G, i.e., for any two compact subsets K, L of G, the family $\{g \in \mathrm{Aut}(G)\colon g(K) \cap L \neq \emptyset\}$ is compact (see f.i. [6], p. 84, Prop. 8). Therefore, since $\mathrm{Aut}(G)$ is supposed to be non-compact, there is a sequence $\{g_i\}$ in $\mathrm{Aut}(G)$ such that $\{g_i(z)\}$ approaches ∂G for all $z \in G$. This shows at first by the above lemma, that G is simply connected. And secondly, it will enable us to prove that conditions 2) and 3) of Theorem C are satisfied by G, which then gives the claim. If namely $(z, v) \in T(G) = G \times \mathbb{C}^n$ is arbitrary and F_G denotes either B_G, C_G or K_G, then one has

$$F_G(z, v) = F_G(g_i(z), dg_i(v)) \qquad \text{for all } i.$$

Together with Theorem B and the fact that $g_i(z) \to \partial G$, we get the equality

$$K_G(z, v) = (n+1)^{\frac{1}{2}} B_G(z, v) = C_G(z, v)$$

This completes the first proof.

2) The second proof of ii) $\Rightarrow$ i) is now even shorter. Using the same sequence $\{g_i\} \subset \mathrm{Aut}(G)$ as in 1) together with Theorem D one obtains at once $M_G^E(z) = M_G^C(z)$ for arbitrary points $z = G$. Therefore, Theorem E can be applied.

It remains to prove the implication

iv) $\Rightarrow$ *ii):* From the compactness of the quotient G/Z and the properly discontinuous action of Z on G one obtains the existence of a compact subset $C \subset G$ such that $Z \cdot C = G$. Therefore, Z must be infinite. On theother hand, $Z \in \mathrm{Aut}(G)$ is discrete (see f.i. 6, p. 84, Prop. 9), showing that $\mathrm{Aut}(G)$ is non-compact, since any discrete subgroup of a compact group is finite.

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