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Note on Matrices with a Very Ill-Conditioned Eigenproblem

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Summary. Gives a bound for the distance of a matrix having an ill-conditioned eigenvalue problem from a matrix having a multiple eigenvalue which is generally sharper than that which has been published hitherto.

In the Algebraic Eigenvalue Problem [1] Wilkinson has discussed the sensitivity of an eigenvalue λ_i of a matrix A in terms of corresponding right-hand and left-hand eigenvectors x_i and y_i. A quantity s_i is defined by the relations

$$s_i = |y_i^H x_i| / \|y_i\|_2 \|x_i\|_2. \tag{1}$$

It will be convenient to assume that $\|y_i\|_2 = \|x_i\|_2 = 1$ so that $s_i = |y_i^H x_i|$ and clearly the vectors can be chosen so that $s_i = y_i^H x_i$.

A zero value of s_i necessarily implies that λ_i is a simple eigenvalue. From rather crude heuristic arguments Wilkinson has conjectured that when s_i is "small" there is a matrix $A+F$ having a multiple eigenvalue and such that $\|F\|_2/\|A\|_2$ is also "small" and recently Ruhe [2] has shown that this is indeed true. In this note we give a bound for the distance to the nearest matrix having a multiple eigenvalue which is generally much sharper than that of Ruhe. We observe that by means of a diagonal similarity a matrix $B = D^{-1}AD$ could be derived having arbitrarily small s_i. Our result would still be true for the matrix B, however 'unbalanced' the scaling, but of course when a small s_i is 'induced' in this way, $\|B\|_2$ would be artificially large. The result is at its most cogent when the given matrix A is 'balanced' in the sense that $\|A\|_2 \leqq \|D^{-1}AD\|_2$ for all non-singular diagonal D, (see eg Parlett and Reinsch [3, 4]) in which case we may say that a small s_i is 'genuine'.

The proof is motivated by considering the case when $s = y^H x = 0$, where

$$Ax = \lambda x, \quad y^H A = \lambda y^H. \tag{2}$$

Let P be a unitary matrix such that

$$Px = e_1 \tag{3}$$

where e_1 is the first column of I. Writing

$$B = PAP^H \tag{4}$$

we have

$$PAP^H Px = \lambda Px \qquad y^H P^H PAP^H = \lambda y^H P^H \tag{5}$$

giving

$$Be_1=\lambda e_1 \quad \text{and} \quad z^H B=\lambda z^H \tag{6}$$

where

$$z=Py. \tag{7}$$

The relation $y^H x=0$ implies that

$$0=y^H P^H P x=z^H e_1 \tag{8}$$

and hence

$$z_1=0. \tag{9}$$

From Eq. (6), B is of the form

$$B=\left[\begin{array}{c|c}\lambda & b^H \\ \hline 0 & B_1\end{array}\right] \tag{10}$$

while Eq. (9) implies that z^H is of the form

$$z^H=[0\,|\,w^H] \tag{11}$$

where w is a unit vector of order $n-1$. From Eq. (6)

$$[0\,|\,w^H]\left[\begin{array}{c|c}\lambda & b^H \\ \hline 0 & B_1\end{array}\right]=\lambda\,[0\,|\,w^H] \tag{12}$$

giving

$$w^H B_1=\lambda w^H. \tag{13}$$

Hence λ is an eigenvalue of B_1 showing that λ is an eigenvalue of B of multiplicity at least two. The same is therefore true of A.

Consider now the case when

$$y^H x=\varepsilon \tag{14}$$

where we are particularly interested in small values of ε. Again let $Px=e_1$, $Py=z$ and proceed as before. We now have

$$z^H e_1=\varepsilon \quad \text{and} \quad z^H=[\varepsilon\,|\,w^H] \tag{15}$$

where $\|w\|_2^2=1-\varepsilon^2$. Eq. (12) now becomes

$$[\varepsilon\,|\,w^H]\left[\begin{array}{c|c}\lambda & b^H \\ \hline 0 & B_1\end{array}\right]=\lambda\,[\varepsilon\,|\,w^H] \tag{16}$$

giving

$$\varepsilon b^H+w^H B_1=\lambda w^H \tag{17}$$

which may be written in the form

$$w^H[B_1+\varepsilon w b^H/(w^H w)]=\lambda w^H. \tag{18}$$

Hence λ is an eigenvalue of $B_1+\varepsilon w b^H/(w^H w)$ and

$$\begin{aligned}\|\varepsilon w b^H/w^H w\|&=\varepsilon\|w\|_2\|b\|_2/\|w\|_2^2\\ &\leq\varepsilon\|B\|_2/(1-\varepsilon^2)^{\frac{1}{2}}=\varepsilon\|A\|_2/(1-\varepsilon^2)^{\frac{1}{2}}.\end{aligned} \tag{19}$$

There is therefore a matrix $B+E$ having λ as an eigenvalue of multiplicity at least two, showing that $A+F$ with

$$F=P^H E P, \ \|F\|_2=\|E\|_2 \leqq \varepsilon\|A\|_2/(1-\varepsilon^2)^{\frac{1}{2}} \tag{20}$$

has the same property.

Notice that there will often be a much closer matrix having a multiple eigenvalue. In Eq. (19) we have used the inequality

$$\|b\|_2 \leqq \|B\|_2 \tag{21}$$

and this could be far from sharp. Indeed if A is normal b is null and E is therefore null assuming $\varepsilon \neq 1$. This is otherwise obvious since if λ is a simple root of a normal matrix then $y=x$ and $y^H x$ must be unity.

We have exhibited a neighbouring matrix having λ itself as the multiple eigenvalue. In general there will be a closer matrix having a multiple eigenvalue not exactly equal to λ. For example the matrix

$$\begin{bmatrix} 1 & \theta \\ 1 & 1 \end{bmatrix} \tag{22}$$

has an eigenvalue $1+\theta^{\frac{1}{2}}$. The corresponding left hand and right hand vectors are

$$[1,\ \theta^{\frac{1}{2}}] \quad \text{and} \quad [\theta^{\frac{1}{2}},\ 1] \tag{23}$$

giving $s=2\theta^{\frac{1}{2}}/(1+\theta) \equiv \varepsilon$ of the above result. If the $(1,2)$ element is changed from θ to 0 the matrix now has unity as a double eigenvalue; the multiplicity has been induced by a perturbation which is of order ε^2 not ε.

If we proceed as in the proof of the above theorem a matrix is derived having $1+\theta^{\frac{1}{2}}$ as a double root. Since the trace of this matrix is $2+2\theta^{\frac{1}{2}}$, changes of order $\theta^{\frac{1}{2}}$ (i.e. of order ε) are necessarily required. This illuminates a phenomenon of which many practical numerical analysts may be aware. When an s_i is small, A is often surprisingly near a matrix having a multiple eigenvalue.

References

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