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Induced Cuspidal Representations and Generalised Hecke Rings

R.B. Howlett [1] and G.I. Lehrer [2]*

[1] Mathematics Department, University of Western Australia, Nedlands, W.A.6009
[2] Mathematics Institute, University of Warwick, Coventry CV47AL, England

Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group defined over a finite field k. In the present work we are concerned with the complex representation theory of the finite group $\mathbf{G}(k)$ of rational points of $\mathbf{G}$ over k. Any parabolic k-subgroup $\mathbf{P}$ of $\mathbf{G}$ has a Levi k-decomposition $\mathbf{P}=\mathbf{MU}$ where $\mathbf{U}$ is its unipotent radical and $\mathbf{M}$ is a Levi k-subgroup of $\mathbf{P}$. A complex representation ρ of $\mathbf{G}(k)$ is called *cuspidal* (or "*discrete series*") if the intertwining number $(\rho, \mathrm{Ind}_{\mathbf{U}(k)}^{\mathbf{G}(k)}(1))=0$ for all proper parabolic k-subgroups $\mathbf{P}=\mathbf{MU}$ of $\mathbf{G}$.

The Harish-Chandra principle for $\mathbf{G}(k)$ (see [9] or [22]) indicates that in order to elucidate the representation theory of $\mathbf{G}(k)$, two problems must be solved:

I. Construct the irreducible cuspidal representations of all $\mathbf{G}(k)$.

II. Decompose representations of the form $\mathrm{Ind}_{\mathbf{P}(k)}^{\mathbf{G}(k)}(D^*)$, where D^* is an irreducible cuspidal representation of $\mathbf{M}(k)$, lifted to $\mathbf{P}(k)$ ($\mathbf{P}$ as above).

The methods of Deligne-Lusztig ([7]) have solved "most" of problem I, although some work remains to be done (c.f. also [10] and [20]). The present work is concerned with problem II.

Problem II has been the subject of an extensive literature in recent years (see, e.g. [1, 5, 10, 13] and their bibliographies) almost all of which deals with the case when $\mathbf{P}=\mathbf{B}$, a Borel k-subgroup of $\mathbf{G}$ (the "principal series" case). One of the main results of the present work is that the general case can "almost" be reduced to the case $\mathbf{P}=\mathbf{B}$. More specifically, we show (Theorem (4.14)) that the endomorphism algebra $E(D)=\mathrm{End}_{\mathbf{G}(k)}(\mathrm{Ind}_{\mathbf{P}(k)}^{\mathbf{G}(k)}(D^*))$ has generators and relations which are very similar to the ones which occur when $\mathbf{P}=\mathbf{B}$. This means in particular that known results on rationality (e.g. [1]) and generic degrees (e.g. [10, 12]) for "principal series" can be applied to the general situation. We intend to do this in a forthcoming paper.

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Present address: Department of Pure Mathematics, University of Sydney NSW 2006, Australia

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Let $\mathbf{P}=\mathbf{MU}$ be a parabolic k-subgroup of $\mathbf{G}$ and let $\mathbf{A}$ be the maximal k-split torus in the centre of $\mathbf{M}$. Let $W(\mathbf{A})$ be the set of bijections: $\mathbf{A}\to\mathbf{A}$ induced by conjugation by elements of G (and hence ${}_kW$). Let

$$W(D)=\{w\in W(\mathbf{A})\,|\,\chi_D\circ w=\chi_D\},$$

where w also denotes the automorphism of M induced by w, and χ_D is the character of D. This is the ramification group of D. Springer conjectured ([22], 4.14) that $E(D)\cong\mathbb{C}W(D)_\mu$, the group algebra of $W(D)$ twisted by a certain 2-cocycle μ. In the present work we prove Springer's conjecture (Corollary (5.4)) by a deformation argument due to Tits (c.f. [3] Chap. IV exx), using a generic algebra which arises from our presentation for $E(D)$. We also show that "generically" the cocycle μ is trivial.

There are, as mentioned above, cases for which Springer's conjecture has been known. In particular, the case $\mathbf{P}=\mathbf{B}$ and $D=1$ was the inspiration for the conjecture, and it has been proved in varying degrees of generality for $\mathbf{P}=\mathbf{B}$, and arbitrary D during the last fifteen years (c.f. [25, 27] and [12]). Apart from this, the result has been known implicitly for $G=GL(n)$ since the work of Green [8] on the characters of $GL(n,q)$, but in this case the theorem was derived from the classification of the characters, whereas its philosophy is that the opposite should occur. Similarly, it was also known for $\mathbf{G}=SL(n)$ (c.f. Lehrer [17]), post factum, but nevertheless significantly, since in the latter case the ramification groups $W(D)$ are not necessarily reflection groups. In addition, the work of Lusztig ([20]) on the characters of the 'conformal' classical groups also implies Springer's conjecture for the relevant cases post factum.

It might also be mentioned here that results of a similar nature exist for representations of semisimple Lie groups, where certain integrals depending on elements of W play an analogous role to our operators $B_{D,w}$ (see §3). For information on this, the reader is referred to Knapp ([14, 15]) or Knapp and Zuckermann [16].

An important step in the study of the structure of the endomorphism algebra $E(D)$ is the recognition of a rather large reflection subgroup of $W(D)$, together with its root system (which is the projection of a subset of that of W). This is treated in detail (including a classification for the simple groups) in [11], and the main features are summarised in §2 below.

The basic strategy is to start with a known basis $\{B_w\,|\,w\in W(D)\}$ of $E(D)$ (3.9), and to decompose the elements B_w as products, not of elements of $E(D)$, but of homomorphisms between various spaces of induced representations (c.f. (3.12)). These decompositions are in analogy with the expression of elements of $W(D)$ as products of certain distinguished elements $v(a,K)$ (see (2.17) below) of W, which are not necessarily in $W(D)$. Rules for the composition of these "elementary homomorphisms" are derived (3.16), and used to produce the presentation of $E(D)$ (Theorem(4.14)).

We remark finally that some of the ideas of which we make use already appear in Lusztig's work ([19], §5) in which he treats the case where D is unipotent.

§ 1. Notation and Preliminaries

For any affine group $\mathbf{H}$ defined over k, the group $\mathbf{H}(k)$ of its k-points will be denoted by H. Let $\mathbf{B}$ be a Borel k-subgroup of $\mathbf{G}$, and choose a maximal k-split torus $\mathbf{T}$ in $\mathbf{B}$. The finite group $G=\mathbf{G}(k)$ has a BN-pair (Tits system) (B, N) where $B=\mathbf{B}(k)$ and $N=N_{\mathbf{G}}(\mathbf{T})(k)$ with Weyl group $W={}_kW=N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$ (c.f. [2], § 5). We denote by Σ the (relative) root system ${}_k\Phi$ $(=\Phi(\mathbf{T},\mathbf{G})$ in the notation of [2]) of W. The Borel subgroup $\mathbf{B}$ determines a positive system $\Sigma^+\subset\Sigma$, and a corresponding set $\Pi\subset\Sigma^+$ of simple roots and simple reflections in W. Let $\ell(w)$ be the associated length function on W. The standard parabolic subgroups of G $(=\mathbf{G}(k))$ are those which contain B $(=\mathbf{B}(k))$. They are in bijective correspondence with the subsets J of Π, and each parabolic subgroup $P\supset B$ is of the form $P=\mathbf{P}(k)$ for a unique parabolic k-subgroup $\mathbf{P}$ of $\mathbf{G}$. Corresponding to a Levi k-decomposition $\mathbf{P}=\mathbf{MU}$ of $\mathbf{P}$, the finite group P also has a "Levi decomposition" $P=MU$. We define "root subgroups" of G as in Borel-Tits ([2], § 5.2), i.e. (c.f. also Richen [21]) as the root subgroups of the split BN-pair (B, N) of G.

The standard Levi decomposition of the standard parabolic subgroup P_J $(=BW_JB$, where W_J is the subgroup of W generated by the reflections corresponding to $J\subset\Pi)$ of G can then be expressed as follows:

$$(1.1)\qquad P_J=M_J\,U_J \quad \textit{where} \begin{cases} M_J=\langle T, U_a \mid a\in\Sigma_J\rangle \\ U_J=\langle U_a \mid a\in\Sigma^+-\Sigma_J\rangle\end{cases}$$

where Σ_J is the sub-root system of Σ spanned by J. In this decomposition we have

$$(1.2)\qquad U_J\trianglelefteq P_J \quad\text{and}\quad U_J\cap M_J=1.$$

The pair $(B\cap M_J, N\cap M_J)$ provides a split BN-pair for M_J, and the standard parabolic subgroups of M_J are of the form $M_{J,H}=M_H\cdot M_J\cap U_H$ for subsets $H\subset J$. The unipotent radical $M_J\cap U_H$ is generated by the U_a with $a\in\Sigma_J^+-\Sigma_H$. Thus $U_H=M_J\cap U_H\cdot U_J$, and one verifies easily that

(1.3) the complex representation ρ of M_J is cuspidal if and only if $(\rho^*, \mathrm{Ind}_{U_H}^{P_J}(1))=0$ for all $H\subsetneqq J$.

Here ρ^* denotes the lift of ρ from M_J to P_J.

For $w\in W$ we define $U_w^+=U\cap U^w$ and $U_w^-=U\cap U^{w_0w}$ (where w_0 is the longest element in W). Write $\mathrm{ind}(w)=[U\colon U_w^+]$ $(=|U_w^-|)$. The following facts are well known and easily proved (see, e.g. [21]).

(1.4) *Let $v, w\in W$.*

(i) $U=U_w^+\cdot U_w^-$ *and* $U_w^+\cap U_w^-=1$.

(ii) U_w^+ *is the product of the* U_a *with a satisfying* $a\in\Sigma^+$ *and* $wa\in\Sigma^+$; U_w^- *is the product of those* U_a *with* $a\in\Sigma^+$ *and* $wa\in\Sigma^-$.

(iii) *For* $a\in\Sigma$, $wU_aw^{-1}=U_{wa}$.

(iv) *If* $\ell(vw)=\ell(v)+\ell(w)$ *then* $\mathrm{ind}(vw)=\mathrm{ind}\,v\cdot\mathrm{ind}\,w$.

For any element $w\in W$, we define $N(w)=\{a\in\Sigma^+\mid wa\in\Sigma^-\}$. The following assertions are then standard.

(1.5) (i) *For any* $w \in W$, *we have* $|N(w)| = \ell(w)$.
(ii) *If* $w, w' \in W$ *are such that* $\ell(ww') = \ell(w) + \ell(w')$, *then we have* $N(ww') = N(w') \cup w'^{-1} N(w)$.

For each element $w \in W$, we take $\dot{w}$ to be a *fixed representative* for w ($\in W = N/B \cap N$) in N. The results of Tits ([26]) and Borel-Tits ([2], Théorème 7.2) show that the $\dot{w}$ may be chosen so that they satisfy

(1.6) (i) *If* $\ell(ww') = \ell(w) + \ell(w')$ *then* $(ww')^{\cdot} = \dot{w}\dot{w}'$.
(ii) *For any elements* $w, w' \in W$, *the element* $h = (ww')^{\cdot} \dot{w}'^{-1} \dot{w}^{-1}$ *of* $B \cap N'$ *has order at most* 2. *In other words, the group generated by the* $\dot{w}$ ($w \in W$) *is an elementary abelian* 2*-group, extended by* W.

We now fix a subset $J \subset \Pi$ and an irreducible cuspidal representation D of M_J, whose character is χ_D (or χ when there is no risk of confusion). The ramification group $W(D)$ is defined as above, i.e. as the group of automorphisms of M_J which fix χ_D and which are induced by automorphisms of $\mathbf{A}$ (the maximal k-split torus in $Z(\mathbf{M}_J)$) which come from conjugations in G. When there is no risk of confusion, we write $M = M_J$ and $P = P_J$.

§ 2. The Structure of $W(D)$

(2.1) **Proposition.** *With notation as above, we have* $W(\mathbf{A}) \cong N_W(W_J)/W_J$.

This is proved by Springer in ([23], Lemma 2.19).

(2.2) **Lemma** (Howlett [11]). *Let* V *be a finite dimensional Euclidean space, and let* $G \subset GL(V)$ *be a group satisfying* $G \trianglerighteq R$, *where* R *is a finite reflection group with root system* $\Phi \subset V$. *Then* R *has a complement* C *in* G, *given by*

$$C = \{g \in G \mid g\Phi^+ \subset V^+\}$$

where Φ^+ *is a positive system in* Φ.

(2.3) **Corollary.** *We have* $N(W_J) = W_J \rtimes S_J$ (*semi-direct product*), *where* $S_J = \{w \in W \mid wJ = J\}$. *Thus* $W(\underline{A}) \cong S_J$. *Moreover*

$$W(D) = \{w \in S_J \mid \chi_D(m^w) = \chi_D(m) \text{ for all } m \in M_J\}.$$

The group S_J has a large reflection subgroup defined as follows. Let $\hat{\Omega} = \{a \in \Sigma \mid w(J \cup \{a\}) \subset \Pi$ for some $w \in W\}$. For $a \in \hat{\Omega}$, if $L = J \cup \{a\}$, let $v(a, J) = w_L w_J$ where w_K is the longest element in W_K.

(2.4) **Lemma** (Howlett, op.cit.). *Let* $a \in \Pi - J$ *and write* $v = v(a, J)$, $L = J \cup \{a\}$. *Then*

(i) $N(v) = \Sigma_L^+ - \Sigma_J$,
(ii) $vJ = K \subset L (\subset \Pi)$,
(iii) *If* $\{b\} = L - vJ$, *then* $v(a, J)^{-1} = v(b, K) = v(b, vJ)$,
(iv) $v^2 = 1 \Leftrightarrow vJ = J \Leftrightarrow v \in S_J$.

Let $\Omega = \{a \in \hat{\Omega} \mid v(a, J)^2 = 1\}$.

(2.5) **Lemma** (Howlett, op.cit.). *Ω is the root system*[1] *of the group*

$$R_J = \langle v(a, J) \mid a \in \Omega \rangle.$$

This is best seen by projecting Ω to $\langle J \rangle^\perp$. The $v(a, J)$ then become reflections in $\langle J \rangle^\perp$, which generate a reflection group whose root system is the projection of Ω.

(2.6) **Corollary.** *R_J has a complement C_J in S_J, where*

$$C_J = \{w \in S_J \mid w\Omega^+ \subset \Omega^+\}.$$

Thus $S_J = R_J \rtimes C_J$ is a decomposition of S_J as a semi-direct product (here $\Omega^+ = \Omega \cap \Sigma^+$).

The purpose of the next Lemma is to show that $W(D)$ has a decomposition similar to the one of S_J given in (2.6) (c.f. (2.3)).

Let Γ be any subset of the root system Ω which satisfies

(i) If $a \in \Gamma$ then $v(a, J) \in W(D)$.
(ii) If $a \in \Gamma$ and $w \in W(D)$ then $wa \in \Gamma$.

(2.7) **Lemma.** *With Γ as above, we have*

(i) *$R_\Gamma(D) = \langle v(a, J) \mid a \in \Gamma \rangle$ is a normal reflection subgroup of $W(D)$ whose root system*[1] *is Γ, and $\Gamma \cap \Sigma^+ = \Gamma^+$ is a positive system in Γ. The set $\Delta = \{a \in \Gamma^+ \mid N(v(a, J)) \cap \Gamma = \{a\}\}$ is the corresponding fundamental system.*
(ii) *$W(D)$ is the semi-direct product $W(D) = R_\Gamma(D) \rtimes C_\Gamma(D)$ where*

$$C_\Gamma(D) = \{w \in W(D) \mid w\Gamma^+ \subset \Gamma^+\}.$$

This is a simple application of (2.2).

(2.8) **Lemma.** *For $a \in \Omega \cap \Pi$ write $L = J \cup \{a\}$. Then* $\operatorname{ind} v(a, J) = |U_J|/|U_L|$ *in the notation of* (1.1).

Proof. We have $v(a, J) = w_L w_J$ and $\ell(v(a, J)) = \ell(w_L) - \ell(w_J)$. By (1.4) (iv) it follows that $\operatorname{ind} v(a, J) = (\operatorname{ind} w_L)(\operatorname{ind} w_J)^{-1}$. But $U \cap U^{w_J} = U_J$ by (1.4) (ii), and the result follows. □

(2.9) For $w \in W$ such that $wJ \subset \Pi$, define the group

$$U^-_{w,J} = \langle U_a \mid a \in \Sigma^+ - \Sigma_{wJ}, \quad w^{-1}a < 0 \rangle \subset U_{wJ}.$$

(2.10) **Lemma.** *Suppose that $a \in \Pi$ is such that $v(a, J) \in S_J$. Then* $\operatorname{ind} v = |U^-_{v,J}|$ *where $v = v(a, J)$.*

Proof. We have $U^-_{v,J} = \langle U_a \mid a \in \Sigma^+ - \Sigma_J,\ va < 0 \rangle$ since $v^2 = 1$. But $N(v) = \Sigma^+_L - \Sigma^+_J$ where $L = J \cup \{a\}$, by (2.4)(i), and so $N(v) \cap \Sigma_J$ is empty. Thus $U_{v,J} = \langle U_a \mid a \in N(v) \rangle = U^-_v$, and the result follows. □

Let $\mathscr{J} = \{K \subset \Pi \mid K = wJ$ for some $w \in W\}$.

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[1] It should be noted that Ω and Γ are not root systems in V. They only become root systems in the usual sense when projected to $\langle J \rangle^\perp$

Then for any $K\in\mathscr{J}$ we may define the sets $\hat{\Omega}_K$ and Ω_K of roots corresponding to $\hat{\Omega}$ and Ω which were defined above for J. Of course $\hat{\Omega}_K\supset\Pi-K$.

(2.11) **Lemma** ([11], Lemma 5). (i) *Suppose $H\in\mathscr{J}$ and that $wH=K\subset\Pi$. Then for all $a\in\Pi-K$, we have*

$$\ell(v(a,K)w)=\begin{cases}\ell(w)+\ell(v(a,K)) & \text{if } w^{-1}a\in\Sigma^+\\ \ell(w)-\ell(v(a,K)) & \text{if } w^{-1}a\in\Sigma^-.\end{cases}$$

(ii) *If $w\in W$, $K\in\mathscr{J}$ and $wK\subset\Pi$ then there exist $K_i\in\mathscr{J}$ $(i=1,2,\dots,n+1)$ and $a_i\in\Pi-K_i$ $(i=1,2,\dots,n)$ satisfying*

(a) $K_1=K$,
(b) $v(a_i,K_i)K_i=K_{i+1}$ $(i=1,2,\dots,n)$,
(c) $w=v(a_n,K_n)\,v(a_{n-1},K_{n-1})\dots v(a_1,K_1)$,
(d) $\ell(w)=\sum_{i=1}^{n}\ell(v(a_i,K_i))$.

An expression as in (c) above shall be referred to as "a standard expression" for $w\in W$, and written

$$w=v(a_n,a_{n-1},\dots,a_1,K).$$

We conclude this section with two group-theoretic results concerning G, which depend on properties of root systems, and which we shall require later.

(2.12) **Lemma.** *Let $w\in W$ and suppose $H=wJ\in\mathscr{J}$ (i.e. $\subset\Pi$). Then $wU_Jw^{-1}\cap P_H\subset U_H$.*

Proof. If $v\in W_H$ then $\ell(vw)=\ell(v)+\ell(w)$ since $w^{-1}a\in\Sigma^+$ for all $a\in\Sigma_H^+$. Hence $BvBw\subset BvwB$ and so

$$\begin{aligned}wU_Jw^{-1}\cap BvB&=[wU_J\cap BvBw]\,w^{-1}\\&\subset[BwB\cap BvwB]\,w^{-1}\\&=\emptyset\quad\text{unless } v=1.\end{aligned}$$

Thus

$$\begin{aligned}wU_Jw^{-1}\cap P_H&=wU_Jw^{-1}\cap B\\&=w(U\cap w_JUw_J)\,w^{-1}\cap B\\&\subset U\cap ww_JUw_Jw^{-1}\end{aligned}$$

where w_J is the inversion element in W_J.

But $U\cap ww_JUw_Jw^{-1}=\langle U_a\,|\,a\in\Sigma^+ \text{ and } w_Jw^{-1}a\in\Sigma^+\rangle$

(by (1.4) (ii)) $\subset\langle U_a\,|\,a\in\Sigma^+-\Sigma_H\rangle$

$=U_H$. □

(2.13) **Proposition.** *Let $H\in\mathscr{J}$ and take $a\in\Pi-H$. Write $v=v(a,H)$, $L=H\cup\{a\}$ and $K=vH$ $(\subset L)$. Then we have*

(i) $vU_Hv^{-1}\cap P_K=U_L$
(ii) $W_L\cap S_K\subset\{1,v\}$, *where* $S_K=\{w\in W\,|\,wK=K\}$ (c.f. (2.3)).

Proof. (i) Since $K = vH$, we have from (2.12), with H replacing J, that $vU_H v^{-1} \cap P_K \subset U_K \subset U$. Moreover $N(v^{-1}) \subset \Sigma_L$ ((2.4) (i)) and $v\Sigma_H = \Sigma_K \subset \Sigma_L$. Hence if $a \in \Sigma^+ - \Sigma_L$, $v^{-1} a \in \Sigma^+ - \Sigma_H$. Thus $v^{-1} U_L v \subset U_H$, whence $U_L \subset vU_H v^{-1} \cap U$.

Conversely, we have $U_H = U \cap w_H U w_H$, so that $vU_H v^{-1} \cap U \subset vUv^{-1} \cap w_L U w_L \cap U \subset U_L$ (recall that $v = w_L w_H$). Thus $vU_H v^{-1} \cap U = U_L$, and the result follows.

(ii) Suppose $t \in W_L \cap S_K$. Since $tK = K$ it follows that $tb \in \Sigma^-$, where $\{b\} = L - K$, or else $t = 1$, since $N(t) \subset \Sigma_L$.

By (2.11), $\ell(tv(b,K)^{-1}) = \ell(v(b,K)t^{-1}) = \ell(t) - \ell(v(b,K))$. But since $t\Sigma_K^+ \subset \Sigma^+$ and $N(t) \subset \Sigma_L^+$, $\ell(t)$ is at most $|\Sigma_L^+ - \Sigma_K^+|$, which is $\ell(v(b,K))$ by (2.4). Hence $t = v(b,K)$.

But $v(b,K) = v(a,H)^{-1}$ by (2.4). Hence $t = v = v^{-1}$. □

§3. The Endomorphism Algebra $E(D)$-Basic Relations

We now fix a complex vector space V, and an irreducible cuspidal representation $D\colon M_J \to GL(V)$. Denote by $E(D)$ the endomorphism ("commuting") algebra

$$E(D) = \mathrm{End}_G(\mathrm{Ind}_{P_J}^G(D^*))$$

where D^* is the lift of D from M_J to P_J through the projection $P_J \to M_J$ with kernel U_J.

For any $w \in W$ such that $wJ \subset \Pi$, define a representation $\dot{w}D$ of M_{wJ} in V by transport of structure:

$$(3.1) \qquad (\dot{w}D)(x) = D(\dot{w}^{-1} x \dot{w}) \qquad (x \in M_{wJ} = \dot{w} M_J \dot{w}^{-1}).$$

When $w \in W(D)$, then by definition the representations D and $\dot{w}D$ of M_J are equivalent. Hence by Schur's Lemma there is a linear operator $\tilde{D}(\dot{w})$ on V, uniquely determined to within a scalar multiple, such that

$$(3.2) \qquad (\dot{w}D)(x) = \tilde{D}(\dot{w})^{-1} D(x) \tilde{D}(\dot{w}) \qquad (x \in M_J).$$

By uniqueness, we have, for any two elements $w_1, w_2 \in W(D)$,

$$(3.3) \qquad \tilde{D}(\dot{w}_1 \dot{w}_2) = \lambda(w_1, w_2) \tilde{D}(\dot{w}_1) \tilde{D}(\dot{w}_2)$$

where $\lambda\colon W(D) \times W(D) \to \mathbb{C}$ is a 2-cocycle.

Moreover by replacing λ with an equivalent cocycle if necessary (i.e. replacing the $\tilde{D}(\dot{w})$ by appropriate scalar multiples), we may assume that (c.f. [6], Theorem 63.7)

(3.4) *For any $u, v, w \in W(D)$, we have*

(a) $\lambda(u,v)\lambda(uv,w) = \lambda(u,vw)\lambda(v,w)$,

(b) $\lambda(u,v)$ *is a root of unity,*

(c) $\lambda(u^{-1},u) = \lambda(u,1) = \lambda(1,u) = 1$,

(d) $\lambda(v^{-1}w^{-1},w) = \lambda^{-1}(v^{-1},w^{-1}) = \lambda(w,v)$.

Let $\tilde{M}=\langle m, \dot{w} \mid m \in M_J, w \in W(D)\rangle$; then $\tilde{M}$ is an extension (possibly non-split) of M_J by $W(D)$. Define the projective representation $\tilde{D}$ of $\tilde{M}$ on V by

(3.5) $$\tilde{D}(m\dot{w})=D(m)\tilde{D}(\dot{w}) \qquad (m \in M_J,\ w \in W(D)).$$

It is then trivially verified that for $x \in M_J w$ and $y \in M_J v$ $(w, v \in W(D))$, we have

(3.6) $$\tilde{D}(xy)=\lambda(w,v)\tilde{D}(x)\tilde{D}(y).$$

The representation $R=\mathrm{Ind}_{P_J}^G(D^*)$ is realised on the space $F=F(M_J,D)$ of functions $f\colon G \to V$ satisfying

(3.7) $$f(xy)=D^*(x)f(y) \qquad (x \in P_J,\ y \in G).$$

The group G acts by right translation on $F(M_J,D)$: for $f \in F$, $g \in G$, $R(g)f(x)=f(xg)$.

Define the following operators on F: for $w \in W(D)$, $f \in F$

(3.8) $$(B_w f)(x)=|U_J|^{-1}\tilde{D}(\dot{w})\sum_{y \in U_J} f(\dot{w}^{-1}yx).$$

(3.9) **Proposition.** (i) *For each* $w \in W(D)$, $B_w \in E(D)$.

(ii) *The set* $\{B_w \mid w \in W(D)\}$ *forms a* $\mathbb{C}$*-linear basis of* $E(D)$.

This is proved in ([23], pp. 635–636).

In order to study how the B_w compose, we introduce the following maps. Suppose $w \in W$ is such that $wJ \subset \Pi$. The map $B_{D,\dot{w}}\colon F(M_J,D) \to F(M_{wJ},\dot{w}D)$ is defined by

(3.10) $$(B_{D,\dot{w}}f)(x)=|U_{wJ}|^{-1}\sum_{y \in U_{wJ}} f(\dot{w}^{-1}yx)$$

where $f \in F(M_J,D)$ and $x \in G$.

(3.11) **Lemma.** *Suppose* $w \subset W$ *is such that* $wJ \subset \Pi$.

(i) $B_{D,\dot{w}}$ *is a* G*-equivariant map from* $F(M_J,D)$ *to* $F(M_{wJ},\dot{w}D)$.

(ii) *We have* $(B_{D,\dot{w}}f)(x)=|U_{w,J}^-|^{-1}\sum_{y \in U_{w,J}^-} f(\dot{w}^{-1}yx)$ *for* $f \in F(M_J,D)$ *and* $x \in G$, *where* $U_{w,J}^-=U_{wJ}\cap U_{w^{-1}}^-$.

(iii) *If* $w \in W(D)$, *then* $B_w=\tilde{D}(\dot{w})B_{D,\dot{w}}$.

(iv) *For any* $h \in B \cap N$, *we have* $B_{D,\dot{w}h}=D(h^{-1})B_{D,\dot{w}}$.

Proof. (i) If $m \in M_{wJ}$ $(=\dot{w}M_J\dot{w}^{-1})$ and $f \in F(M_J,D)$, then

$$(B_{D,\dot{w}}f)(mx)=|U_{wJ}|^{-1}\sum_{y \in U_{wJ}} f(\dot{w}^{-1}ymx).$$

But

$$\dot{w}^{-1}ymx=\dot{w}^{-1}m(m^{-1}ym)x=(\dot{w}^{-1}m\dot{w})\dot{w}^{-1}(m^{-1}ym)x.$$

Hence

$$\begin{aligned}(B_{D,\dot{w}}f)(mx)&=|U_{wJ}|^{-1}D(\dot{w}^{-1}m\dot{w})\sum_{z \in U_{wJ}} f(\dot{w}^{-1}zx)\\&=(\dot{w}D)(m)(B_{D,\dot{w}}f)(x).\end{aligned}$$

Moreover if $u \in U_{wJ}$, clearly $(B_{D,\dot{w}} f)(ux) = (B_{D,\dot{w}} f)(x)$. Hence for any $a \in P_{wJ}$, we have $(B_{D,\dot{w}} f)(ax) = (\dot{w} D)^*(a)(B_{D,\dot{w}} f(x)$, and so $B_{D,\dot{w}} f \in F(M_{wJ}, \dot{w} D)$. It is trivial that $B_{D,\dot{w}}$ is G-equivariant.

(ii) This follows from the fact that each element y of U_{wJ} has a unique expression (c.f. (1.4)) $y = y_1 y_2$ where $y_1 \in U^+_{w,J} = U_{wJ} \cap U^+_{w^{-1}}$ and $y_2 \in U^-_{w,J}$, together with the remark that for $u \in U^+_{w,J}$, $D(\dot{w}^{-1} u \dot{w}) = \mathrm{id}_V$.

(iii) and (iv) are simple observations. □

Remarks 1. Note that from (iv) above, and (3.5), it follows that $B_w = \tilde{D}(\dot{w}) B_{D,\dot{w}}$ is independent of the representative $\dot{w}$ of w, given the choice of the projective representation $\tilde{D}$ of $\tilde{M}$. This justifies the notation.

2. For any $m \in \tilde{M}$ we could equally well define operators $B_{D,m}\colon F(M_J, D) \to F(M_J, mD)$ just as in (3.10). This will be implicit in some of the statements which follow. Similarly, for any $n \in N$, we have $B_{D,n}\colon F(M_J, D) \to F(M_{nJ}, nD)$, whenever $nJ \subset \Pi$.

We now derive the basic relations for composing the $B_{D,\dot{w}}$.

(3.12) **Proposition.** *Let w_1 and w_2 be elements of W such that $w_2 J \subset \Pi$, $w_1 w_2 J \subset \Pi$, and $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Then we have*

$$B_{D,(w_1 w_2)^{\cdot}} = B_{\dot{w}_2 D, \dot{w}_1} B_{D, \dot{w}_2}.$$

Proof. Take $f \in F(M_J, D)$. Then we have

$$\begin{aligned}(B_{\dot{w}_2 D, \dot{w}_1} B_{D,\dot{w}_2} f)(x) &= |U^-_{w_1 w_2, J}| \sum_{y \in U^-_{w_1 w_2, J}} (B_{D,\dot{w}_2} f)(\dot{w}_1^{-1} y x) \\ &= |U^-_{w_1 w_2, J}|^{-1} \sum_{y \in U^-_{w_1 w_2, J}} |U^-_{w_2, J}| \sum_{z \in U^-_{w_2, J}} f(\dot{w}_2^{-1} z \dot{w}_1^{-1} y x) \\ &= |U^-_{w_1 w_2, J}|^{-1} |U^-_{w_2, J}|^{-1} \sum_{\substack{y \in U^-_{w_1 w_2, J} \\ z \in U^-_{w_2, J}}} f(\dot{w}_2^{-1} \dot{w}_1^{-1} (\dot{w}_1 z \dot{w}_1^{-1}) y x).\end{aligned}$$

But $U^-_{w_2, J} = \langle U_a \mid a \in (\Sigma^+ - \Sigma_{w_2}) \cap N(w_2^{-1}) \rangle$. Hence

$$\dot{w}_1 U_{w_2, J} \dot{w}_1^{-1} = \langle U_a \mid a \in (w_1 \Sigma^+ - \Sigma_{w_1 w_2 J}) \cap w_1 N(w_2^{-1}) \rangle.$$

Moreover since $\ell(w_2^{-1} w_1^{-1}) = \ell(w_2^{-1}) + \ell(w_1^{-1})$ we have

$$N(w_2^{-1} w_1^{-1}) = N(w_1^{-1}) \cup w_1 N(w_2^{-1}) \text{ (1.5(ii))}.$$

Thus $\dot{w}_1 U^-_{w_2, J} \dot{w}_1^{-1} \subset \langle U_a \mid a \in (\Sigma^+ - \Sigma_{w_1 w_2}) \cap N(w_2^{-1} w_1^{-1}) \rangle = U_{w_1 w_2, J}$.
Hence the summand on the right hand side above is independent of z, and the whole expression may be written

$$|U^-_{w_1 w_2, J}| \sum_{y \in U^-_{w_1 w_2, J}} f(\dot{w}_2^{-1} \dot{w}_1^{-1} y x) = (B_{D, \dot{w}_1 \dot{w}_2} f)(x).$$

The result follows since $\dot{w}_1 \dot{w}_2 = (w_1 w_2)^{\cdot}$ by (1.6)(i). □

(3.13) **Lemma.** *Let L be a subset of Π which contains J. Denote by $F_L = F_L(M_J, D)$ the linear subspace of $F = F(M_J, D)$ consisting of $\{f \in F \mid \operatorname{supp} f \subset P_L\}$. Then we have*

(i) *Any element* $B \in E(D)$ *is determined by its action on* F_L.

(i) *The space* $E_L(D) = \mathrm{End}_{P_L}(F_L)$ *has linear basis* $\{\mathrm{Res}_{F_L}(B_w) \mid w \in W_L \cap W(D)\}$, *where* Res_{F_L} *denotes restriction to* F_L.

Proof. (i) is trivial, because F is a sum of G-translates of F_L, and the action of B on any translate is determined by its action on F_L.

(ii) The representation of P_L on F_L is $\mathrm{Ind}_{P_J}^{P_L}(D^*) = [\mathrm{Ind}_{P_J \cap M_L}^{M_L}(D^*)]^*$ where the *'s denote the lift through the appropriate unipotent radical. Thus

$$\mathrm{End}_{P_L}(F_L) = \mathrm{End}_{M_L}(F_L) = \mathrm{End}_{M_L}(\mathrm{Ind}_{P_J \cap M_L}^{M_L}(D^*)).$$

Since Res_{F_L} is injective ((i) above), the set $\{\mathrm{Res}_{F_L}(B_w) \mid w \in W_L \cap W(D)\}$ is linearly independent. But by (3.9)(ii) and the above remarks, $\dim_{\mathbb{C}} E_L(D) = |W_L \cap W(D)|$, whence the result follows. □

(3.14) **Proposition.** *Let* $a \in \Pi - J$, *and write* $v = v(a, J)$, $L = J \cup \{a\}$. *Then we have*

$$B_{\dot v D, \dot v^{-1}} B_{D, \dot v} = (\mathrm{ind}\, v)^{-1}\, \mathrm{id} + \beta B_v$$

where $\beta \in \mathbb{C}$, *and* $\beta = 0$ *unless* $v \in W(D)$. *Moreover when* $v \in W(D)$, $(\mathrm{ind}\, v)\,\beta$ *is an algebraic integer.*

Proof. By (3.11)(i), $B = B_{\dot v D, \dot v^{-1}} B_{D, \dot v} \in E(D)$. Moreover, since $v \in W_L$, B fixes the subspace F_L. Hence by (3.13), B is a linear combination of $\{B_w \mid w \in W_L \cap W(D)\}$. But from (2.13)(ii), we see that $W_L \cap W(D) \subset \{1, v\}$, whence

$$B = \alpha\, \mathrm{id} + \beta B_v, \quad \text{and} \quad \beta = 0 \quad \text{unless } v \in W(D).$$

To find α, β observe that

$$\begin{aligned}(Bf)(x) &= |U^-_{v^{-1}, vJ}|^{-1}\, |U^-_{v,J}|^{-1} \sum_{\substack{y \in U^-_{v^{-1}, vJ} \\ z \in U^-_{v,J}}} f(\dot v^{-1} z \dot v y x) \\ &= \alpha f(x) + \beta \tilde D(\dot v)\, |U^-_{v,J}|^{-1} \sum_{y \in U^-_{v,J}} f(\dot v^{-1} y x).\end{aligned} \tag{3.14.1}$$

Now take $e \in V$ and let $f = f_e$ be the unique element of $F_J \subset F$ such that $f_e(1) = e$. Then $f_e(t) = 0$ for $t \notin P_J$, and $\dot v^{-1} z \dot v y x \in P_J$ (above)$\Leftrightarrow \dot v^{-1} z \dot v \in P_J$. But $U^-_{v,J} = \langle U_a \mid a \in \Sigma^+ - \Sigma_{vJ},\ v^{-1} a < 0 \rangle$, so that $\dot v^{-1} U_{v,J} \dot v = \langle U_a \mid a \in (v^{-1}\Sigma^+ - \Sigma_J) \cap \Sigma^- \rangle$. Thus $\dot v^{-1} z \dot v \in P_J$ only when $z = 1$. Hence

$$\begin{aligned}(Bf_e)(1) &= |U^-_{v^{-1}, vJ}|^{-1}\, |U^-_{v,J}|^{-1} \sum_{y \in U^-_{v^{-1}, vJ}} f_e(y) \\ &= |U^-_{v,J}|^{-1} e \\ &= (\mathrm{ind}\, v)^{-1} e.\end{aligned}$$

But

$$\begin{aligned}((\alpha\, \mathrm{id}_F + \beta B_v) f_e)(1) &= \alpha e + \beta \tilde D(\dot v)\, |U^-_{v,J}|^{-1} \sum_{y \in U^-_{v,J}} f(\dot v^{-1} y) \\ &= \alpha e.\end{aligned}$$

Hence $\alpha = (\mathrm{ind}\, v)^{-1}$.

To determine β, take $f=f_e$ and $x=\dot{v}$ in 3.14.1. We obtain, using the same argument as above, together with the fact that $U^-_{v,vJ}=U^-_v$

$$\beta(\operatorname{ind} v)\, e=\tilde{D}(\dot{v})^{-1}\sum_{y,\,z\in U^-_v} f(\dot{v}^{-1}z\dot{v}y\dot{v})$$
$$=\tilde{D}(\dot{v})^{-1}\sum_{\substack{t\in P_J\\ \dot{v}t\dot{v}^{-1}\in U_v\dot{v}U_v^-}} f(t). \qquad (3.14.2)$$

Since $f(t)=D^*(t)\,e$, and (3.14.2) holds for each $e\in V$, we have

$$\beta(\operatorname{ind} v)\,\tilde{D}(\dot{v})=\sum_{\substack{t\in P_J,\, t=mU\\ \dot{v}t\dot{v}^{-1}\in U_v\dot{v}U_v}} D(m) \qquad (3.14.3)$$

or equivalently,

$$\beta(\operatorname{ind} v)\,\mathrm{id}_V=\sum_{\substack{t\in P_J,\, t=mU\\ \dot{v}t\dot{v}^{-1}\in U_v\dot{v}U_v^-}} \tilde{D}(\dot{v}^{-1}m).$$

From this last equation it follows that $\beta(\operatorname{ind} v)$ is an algebraic integer, since the (projective) representation $\tilde{D}$ of $\tilde{M}$ is equivalent to a representation over some ring of algebraic integers. □

Note that Eq. (3.14.2) provides, in principle at least, a practical method for determining β. We give an example of the computations in (4.15)

An easy calculation proves the following

(3.15) **Lemma.** (i) *Let $m,\, n\in\tilde{M}$. The* map $f\mapsto\tilde{D}(m)f$ *is an isomorphism from $F(M_J, nD)$ onto $F(M_J, nm^{-1}D)$.*

(ii) *Let $m\in\tilde{M}$ and $n\in N$. Then we have (assuming $nJ\subset\Pi$)*

$$B_{D,n}\,\tilde{D}(m)=\tilde{D}(m)\,B_{mD,n}.$$

(3.16) **Corollary.** *With notation as in* (3.14), *we have*

(i) *If $v\notin W(D)$, then*

$$B_{\dot{v}D,(v^{-1})}\cdot B_{D,\dot{v}}=\dot{v}D((v^{-1})^{\cdot -1}\dot{v}^{-1})(\operatorname{ind} v)^{-1}$$
$$=D(\dot{v}^{-1})(v^{-1})^{\cdot -1})(\operatorname{ind} v)^{-1}$$

(ii) *If $v\in W(D)$, then*

$$B_v^2=(\operatorname{ind} v)^{-1}\,\mathrm{id}+\beta B_v.$$

Proof. (i) Since $(v^{-1})^{\cdot}=\dot{v}^{-1}[\dot{v}(v^{-1})^{\cdot}]$, this follows from (3.11)(iv) and (3.14).

(ii) From (3.11)(iii) we have $B_v=\tilde{D}(\dot{v})\,B_{D,\dot{v}}$.

Hence $B_v^2=\tilde{D}(\dot{v})^2\,B_{\dot{v}D,\dot{v}}\,B_{D,\dot{v}}$ (using (3.15)). But $\dot{v}=\dot{v}^{-1}(\dot{v})^2$ (recall $v^2=1$) and so

$$B_v^2=\tilde{D}(\dot{v})^2 D((\dot{v})^2)((\operatorname{ind} v)^{-1}\mathrm{id}+\beta B_v) \qquad \text{(from (3.14))}.$$

Moreover $\tilde{D}(\dot{v})^2=\lambda(v,v)\ \tilde{D}(\dot{v}^2)=D(\dot{v}^2)$ since $\lambda(v,v)=1$ by (3.4)(c). The result follows. □

We remark that the statements (3.13), (3.14), (3.15) and (3.16) all hold with J replaced by any element K of $\mathscr{J}$, and D by an appropriate representation of M_K. We shall freely make use of the statements in their more general form.

(3.17) **Corollary.** *For any $w \in W$ such that $wJ \subset \Pi$, the map $B_{D,\dot{w}}$ is invertible.*

Proof. From (2.77)(ii), w has an expression

$$w = v(a_n, K_n)\, v(a_{n-1}, K_{n-1}) \dots v(a_1, K_1)$$

such that $K_1 = J$, $v_i K_i = K_{i+1}$ ($v_i = v(a_i, K_i)$), and $l(w) = \sum_{i=1}^{n} l(v_i)$.

Hence by (3.12), we have

$$B_{D,\dot{w}} = B_{\dot{v}_{n-1} \dots \dot{v}_1 D, \dot{v}_n} \dots B_{\dot{v}_1 D, \dot{v}_2}\, B_{D, \dot{v}_1}.$$

However, it follows from (3.16) that if $uJ = K \subset \Pi$, and $a \in \Pi - K$, then $B_{\dot{u}D, \dot{v}(a,K)}$ is invertible. Hence $B_{D,\dot{w}}$ is invertible, since each factor on the right hand side above is of this form and hence invertible. □

We now obtain more specific information concerning the β of (3.14).

(3.18) **Theorem.** *Let $a \in \Pi - J$, $L = J \cup \{a\}$ and assume that $v(a, J) \in W(D)$.*

(i) *The representation $\operatorname{Ind}_{P_J}^{P_L}(D^*)$ is the sum of two inequivalent irreducible components, which have degree d and $p^c d$, where p^c is an integral power of the characteristic of k.*

(ii) *The number β appearing in* (3.14) *and* (3.16) *satisfies*

$$\beta^2 = (p^c - 1)^2 / p^c \operatorname{ind} v.$$

Proof. (i) As we observed in (3.13)(ii), $\operatorname{Ind}_{P_J}^{P_L}(D^*) = (\operatorname{Ind}_{P_J \cap M_L}^{M_L}(D^*))^*$. Hence for the purposes of the theorem, we may as well take $L = \Pi$, so that $P_L = G$ (effectively we are replacing M_L by G).

By (3.9)(ii), $\operatorname{End}_G(\operatorname{Ind}_{P_J}^G(D^*))$ has dimension 2, and has basis id and B_v. It follows that $\operatorname{Ind}_{P_J}^G(D^*)$ has two inequivalent irreducible components, whose characters we shall denote by ξ, ξ', with $\xi'(1) \geq \xi(1)$. Writing R for the representation of G on $F(M_J, D) = F$, the projection ρ_ξ of F onto its ξ-isotypic component is given by

$$\rho_\xi = \frac{\xi(1)}{|G|} \sum_{y \in G} \xi(y^{-1})\, R(y).$$

It is a non-zero, non-identity element of $E(D)$, hence a linear combination of id and B_v. Thus we have

$$\rho_\xi = \lambda\, \mathrm{id}_F + \mu B_v \qquad (\lambda, \mu \in \mathbb{C}). \tag{3.18.1}$$

Applying both sides to $f \in F$, and evaluating at $x \in G$, we obtain

$$\frac{\xi(1)}{|G|} \sum_{y \in G} \xi(y^{-1}) f(xy) = \lambda f(x) + (\operatorname{ind} v)^{-1} \mu \tilde{D}(\dot{v}) \sum_{z \in U_{v,J}^-} f(\dot{v}^{-1} z x).$$

Now take $f=f_e$ and $x=1$ in the above equation:

$$\frac{\xi(1)}{|G|}\sum_{y\in P_J}\xi(y^{-1})\,D^*(y)\,e=\lambda e+0.$$

But $\frac{\xi(1)}{|G|}\sum_{y\in P_J}\xi(y^{-1})\,D^*(y)$ is a scalar multiplication in V by Schur's Lemma (V is an irreducible P_J-module), and the scalar is given by

$$\begin{aligned}\frac{1}{\dim V}\,\frac{\xi(1)}{|G|}\sum_{y\in P_J}\xi(y^{-1})\,\chi_{D^*}(y)&=\frac{\xi(1)}{\dim V|G|}\,|P_J|(\xi,\chi_{D^*})_{P_J}\\ &=\xi(1)\cdot[\dim V|G|/|P_J|]^{-1}\ \text{since}\ (\xi,\chi_{D^*})_{P_J}=(\xi,\chi_{D^*}^G)=1\\ &=\xi(1)/(\xi(1)+\xi'(1)).\end{aligned}$$

Hence $\lambda=\xi(1)/(\xi(1)+\xi'(1))$.

Now let η, ν be the eigenvalues of B_v, i.e. the roots of $X^2=\alpha+\beta X$, where $\alpha=(\operatorname{ind} v)^{-1}$ and β are as in (3.14).

From (3.18.1), the eigenvalues of ρ_ξ are then $\lambda+\mu\eta$ and $\lambda+\mu\nu$ respectively and since these are 1 and 0 (ρ_ξ is a projection on to a non-trivial subspace) we have

$$\lambda+\mu\eta=1,\qquad \lambda+\mu\nu=0. \tag{3.18.2}$$

Hence

$$\eta\nu^{-1}=-(1-\lambda)\,\lambda^{-1}=-\xi'(1)/\xi(1). \tag{3.18.3}$$

Now $(\operatorname{ind} v)\eta$ and $(\operatorname{ind} v)\nu$ are algebraic integers, since they satisfy $\left(\frac{X}{\operatorname{ind} v}\right)^2=\alpha+\beta\left(\frac{X}{\operatorname{ind} v}\right)$, i.e. $X^2=\alpha(\operatorname{ind} v)^2+\beta(\operatorname{ind} v)X$, and by (3.14) $\alpha(\operatorname{ind} v)$ and $\beta(\operatorname{ind} v)$ are integral.

Also $\eta\nu=-\alpha=-(\operatorname{ind} v)^{-1}$, whence $\eta\nu^{-1}=\eta^2(\eta\nu)^{-1}=-\eta^2(\operatorname{ind} v)$. Therefore $\eta\nu^{-1}(\operatorname{ind} v)$ is integral. But the same argument may be applied to $\nu\eta^{-1}$ to show that $\nu\eta^{-1}(\operatorname{ind} v)$ is integral. Since $(\operatorname{ind} v)$ is a power of p (the characteristic), this implies that $\eta\nu^{-1}=\pm p^c$, $c\in\mathbb{Z}$. Since $\xi'(1)\geqq\xi(1)$ we have

$$\xi'(1)=p^c\,\xi(1),\qquad c\geqq 0,\ \text{which proves (i).}$$

(ii) We deduce the value of β from the equation $X^2=\alpha+\beta X$, which has roots η, ν. We have $\eta\nu^{-1}=-p^c$ and $\eta\nu=-(\operatorname{ind} v)^{-1}$. Hence $\eta^2=p^c(\operatorname{ind} v)^{-1}$ and $\nu^2=(p^c\operatorname{ind} v)^{-1}$. Hence $\beta^2=\eta^2+\nu^2+2\eta\nu=(p^c-1)^2/p^c(\operatorname{ind} v)$. □

Suppose now that $a\in\Omega$, $v(a,J)\in W(D)$ and that $w\in W$ is such that $wJ=K$, $wa=b$ and $K\cup\{b\}=L\subset\Pi$. Let $H=J\cup\{a\}$ and let M_H be the image of M_L under the map ad $\dot w^{-1}$: $x\mapsto\dot w^{-1}x\dot w$. This map takes M_K to M_J, $v(b,K)$ to $v(a,J)$, and $M_L\cap P_K$ to a parabolic subgroup $P_{J,\dot w}$ of M_H. The group $P_{J,\dot w}$ has a Levi component M_J, and the representation $\operatorname{Ind}_{P_{J,\dot w}}^{M_H}(D^*)$ decomposes into irreducible components in the same way as $\operatorname{Ind}_{M_L\cap P_K}^{M_L}(\dot w D^*)$ (ad $\dot w$ "transports structure"), viz. into two irreducible components of degree d' and $p^{c'}d'$ ($c'\geqq 0$) by (3.18).

(3.19) *Definition.* For $a \in \Sigma$ such that $v(a, J) \in W(D)$, define $p_a = p^{c'}$ where c' is as in the above preamble.

To justify this definition we have

(3.20) **Lemma.** *The integer p_a defined above is independent of w (and hence of K).*

Proof. First, notice that $M_H = \langle B \cap N, U_a \mid a \in \Sigma_H \rangle$ is independent of w. Next, let Q be any parabolic subgroup of M_H, such that Q has a Levi component equal to M_J. Then it follows from Springer ([22], Theorem 4.7) that the equivalence class of the representation $\mathrm{Ind}_Q^{M_H}(D^*)$ is independent of Q; for if Q_1 and Q_2 are two such parabolic subgroups, and the corresponding characters of the induced representations are χ_1 and χ_2, then we have for the intertwining numbers that

$$|W(D) \cap W_H| = (\chi_1, \chi_1) = (\chi_1, \chi_2) = (\chi_2, \chi_2).$$

Hence $(\chi_1 - \chi_2, \chi_1 - \chi_2) = 0$, i.e. $\chi_1 = \chi_2$. The result follows. □

(3.21) *Definition.* Write $\Gamma' = \{a \in \Sigma \mid v(a, J)^2 = 1$ and $v(a, J) \in W(D)\}$ and write $\Gamma = \{a \in \Gamma' \mid p_a \neq 1\}$.

(3.22) **Lemma.** *Let $a \in \Gamma'$ and $w \in W(D)$. Then $p_{wa} = p_a$.*

Proof. Let $H = J \cup \{a\}$, $H' = w(J \cup \{a\}) = J \cup \{wa\}$, and let Q' be a parabolic subgroup of $M_{H'}$ which contains M_J as a Levi component. Then the map $\mathrm{ad}\,\dot{w}^{-1}: x \mapsto \dot{w}^{-1} x \dot{w}$ defines an isomorphism from $M_{H'}$ to M_H which takes M_J to M_J, and Q' to a parabolic subgroup Q of M_H, which also contains M_J as a Levi component. Moreover, composition with $\mathrm{ad}\,\dot{w}^{-1}$ takes the representation $\mathrm{Ind}_Q^{M_H}(D^*)$ of M_H to $\mathrm{Ind}_{Q'}^{M_{H'}}(\dot{w}D^*)$. But $\dot{w}D$ is equivalent to D, and hence $\mathrm{Ind}_{Q'}^{M_H}(\dot{w}D^*)$ is equivalent to $\mathrm{Ind}_{Q'}^{M_{H'}}(D^*)$. Thus $\mathrm{Ind}_Q^{M_H}(D^*)$ and $\mathrm{Ind}_{Q'}^{M_H^*}(D^*)$ have irreducible components of the same degree, and it follows from the definition that $p_{wa} = p_a$. □

(3.23) **Corollary.** *The set Γ ((3.21)) is invariant under $W(D)$.*

§4. A Presentation for $E(D)$

In this section we use the basic relations among the $B_w (w \in W(D))$ derived in §3 above to give a simple presentation for $E(D) = \mathrm{End}_G(\mathrm{Ind}_{P_J}^G(D^*))$, which makes the application of generic algebra methods possible.

(4.1) **Lemma.** *Suppose $w \in W$ is such that $wJ \in \Pi$ and take $a \in \Pi - wJ$. Write $v = v(a, wJ)$, and assume that either*

(i) *$w^{-1} a \in \Sigma^+$ or*

(ii) *$w^{-1} a \notin \Gamma$.*

Then we have

$$B_{\dot{w}D, \dot{v}} B_{D, \dot{w}} = |(\mathrm{ind}\, v)^{-1} (\mathrm{ind}\, w)^{-1} (\mathrm{ind}\, vw)|^{1/2} B_{D, \dot{v}\dot{w}}.$$

Proof. If $w^{-1}a\in\Sigma^+$ then $\ell(vw)=\ell(v)+\ell(w)$ (by (2.11)) and the result is clear from (3.12).

Thus we may assume that $b=w^{-1}a\in\Sigma^- -\Gamma$, by (ii). In this case $\ell(vw)=\ell(w)-\ell(v)$, and if $w_1=vw$, we have $w=v^{-1}w_1$, with $\ell(w)=\ell(v^{-1})+(w_1)$. Further, if $L=wJ\cup\{a\}$, and $\{c\}=L-vwJ=L-w_1J$, then $v^{-1}=v(c,w_1J)$ (by (2.4)(iii)). Hence from (3.12) we see that

$$B_{D,\dot{w}}=B_{\dot{w}_1D,(v^{-1})^{\cdot}}\cdot B_{D,\dot{w}_1},$$

whence

$$B_{\dot{w}D,\dot{v}}\,B_{D,\dot{w}}=B_{\dot{w}D,\dot{v}}\,B_{\dot{w}_1D,(v^{-1})^{\cdot}}\,B_{D,(vw)^{\cdot}} \tag{4.1.1}$$

To evaluate the product of the first two terms on the right above, we apply (3.16) with D replaced by $\dot{w}_1D$, and v replaced by v^{-1}.

First suppose that $v^{-1}\notin W(\dot{w}_1D)$. Then by (3.16)(i),

$$\begin{aligned}B_{\dot{w}D,\dot{v}}\,B_{\dot{w}_1D,(v^{-1})^{\cdot}}&=\dot{w}_1D((v^{-1})^{\cdot-1}\dot{v}^{-1})(\operatorname{ind}v^{-1})^{-1}\\&=D(\dot{w}_1^{-1}(v^{-1})^{\cdot-1}\dot{v}^{-1}\dot{w}_1)(\operatorname{ind}v)^{-1}\\&=D(\dot{w}^{-1}\dot{v}^{-1}(vw)^{\cdot})(\operatorname{ind}v)^{-1}\end{aligned} \tag{4.1.2}$$

since $\dot{w}=(v^{-1})^{\cdot}\dot{w}_1$.

Combining with (4.1.1) and using (3.11)(iv), we see that

$$\begin{aligned}B_{\dot{w}D,\dot{v}}\,B_{D,\dot{w}}&=(\operatorname{ind}v)^{-1}B_{D,\dot{v}\dot{w}}\\&=[(\operatorname{ind}v)^{-1}(\operatorname{ind}w)^{-1}(\operatorname{ind}vw)]^{1/2}B_{D,\dot{v}\dot{w}}\end{aligned}$$

since $\operatorname{ind}w=(\operatorname{ind}v^{-1})\operatorname{ind}w_1=\operatorname{ind}v\operatorname{ind}vw$.

Finally, if $v^{-1}\in W(\dot{w}_1D)$ then $v^2=1$, and $vwJ=wJ$, whence

$$w^{-1}vw=w^{-1}v(a,wJ)w=v(b,J)\in W(D).$$

But by hypothesis $b=w^{-1}a\notin\Gamma$, whence $p_b=1$. By the familiar "transport of structure" argument (c.f. the proof of (3.22)) this implies that in the computation of $B_{\dot{w}D,\dot{v}}B_{\dot{w}_1D,\dot{v}}$, the parameter β of (3.14) is zero, by (3.18)(ii). Hence we again obtain (4.1.2) and the proof proceeds as above. □

(4.2) **Main Lemma.** *Suppose $v\in W(D)$ and $w\in W$ is such that $wJ\subset\Pi$. If $v^{-1}a\in\Sigma^+$ for all $a\in\Gamma\cap N(w)$ (i.e. $N(v^{-1})\cap N(w)\cap\Gamma=\emptyset$) then we have*

$$B_{\dot{v}D,\dot{w}}\,B_{D,\dot{v}}=[(\operatorname{ind}w)^{-1}(\operatorname{ind}v)^{-1}(\operatorname{ind}wv)]^{1/2}B_{D,\dot{w}\dot{v}}.$$

Proof. This is by induction on $\ell(w)$. The case $\ell(w)=0$ is trivial. If $\ell(w)>0$, there is a set $K\subset\Pi$ and $a\in\Pi-K$ such that

$$w=v(a,K)w_1,\quad w_1J=K,\quad\text{and}\quad\ell(v(a,K)w_1)=\ell(v(a,K))+\ell(w_1).$$

Write $v_1=v(a,K)$. Then by (3.12) we have

$$B_{\dot{v}D,\dot{w}}=B_{\dot{w}_1\dot{v}D,\dot{v}_1}\,B_{\dot{v}D,\dot{w}_1}. \tag{4.2.1}$$

Moreover since $N(w)\supset N(w_1)$, we have $N(v^{-1})\cap N(w_1)\cap\Gamma=\emptyset$, so that the inductive hypothesis yields

$$B_{\dot{v}D,\dot{w}_1}\,B_{D,\dot{v}}=[(\operatorname{ind} w_1)^{-1}(\operatorname{ind} v)^{-1}(\operatorname{ind} w_1 v)]^{1/2}\,B_{D,\dot{w}_1\dot{v}}. \tag{4.2.2}$$

Combining (4.2.1) and (4.2.2), we are left with the problem of evaluating $B_{\dot{w}_1\dot{v}D,\dot{v}_1}\,B_{D,\dot{w}_1\dot{v}}$.

For this we use (4.1), and so we need to show that the hypotheses apply: if $v^{-1}w_1^{-1}a\notin\Sigma^+$, then since $N(v^{-1})\cap N(w)\cap\Gamma=\emptyset$, we have $w_1^{-1}a\notin\Gamma\cap N(w)$. But $w_1^{-1}a\in\Sigma^+$ and $ww_1^{-1}a=v(a,K)a\in\Sigma^-$. Hence $w_1^{-1}a\in N(w)$, from which it follows that $w_1^{-1}a\notin\Gamma$. But $v\in W(D)$, and Γ is $W(D)$-invariant ((3.23)), so that $v^{-1}w_1^{-1}a\notin\Gamma$. Thus we may apply (4.1) (with $\dot{w}_1\dot{v}$ in place of $\dot{w}$) to deduce that

$$B_{\dot{w}_1\dot{v}D,\dot{v}_1}\,B_{D,\dot{w}_1\dot{v}}=[(\operatorname{ind} v_1)^{-1}(\operatorname{ind}(w_1 v))^{-1}(\operatorname{ind} v_1 w_1 v)]^{1/2}\,B_{D,\dot{v}_1\dot{w}_1\dot{v}}.$$

Combining this with (4.2.1) and (4.2.2), using the fact that $w=v_1w_1$ with $\ell(w)=\ell(v_1)+\ell(w_1)$, we obtain the required relation. □

We are now in a position to give a complete set of relations for the basis $\{B_w\mid w\in W(D)\}$ of $E(D)$. Recall first (c.f. (2.7)) that with Γ as defined above ((3.21)) the ramification group $W(D)$ has a semi-direct decomposition $W(D)=R(D).C(D)$ ($R(D)\trianglelefteq W(D)$) where $R(D)=R_\Gamma(D)$ and $C(D)=C_\Gamma(D)$ in the notation of (2.7). In particular $R(D)$ is a reflection group with root system (the projection to $\langle J\rangle^\perp$ of) Γ and fundamental system (the projection to $\langle J\rangle^\perp$ of) $\Delta=\{a\in\Gamma^+\mid N(v(a,J))\cap\Gamma=\{a\}\}$.

(4.3) **Proposition.** *With notation as above, let $w\in W(D)$, $t\in C(D)$, $a\in\Delta$ and write $v=v(a,J)$. Then we have*

(i) $B_wB_t=[(\operatorname{ind} w)^{-1}(\operatorname{ind} t)^{-1}(\operatorname{ind} wt)]^{1/2}\,\lambda(w,t)\,B_{wt}$.
(ii) $B_tB_w=[(\operatorname{ind} t)^{-1}(\operatorname{ind} w)^{-1}(\operatorname{ind} tw)]^{1/2}\,\lambda(t,w)\,B_{tw}$.
(iii) *If $wa\in\Gamma^+$ then* (i) *holds, with v replacing t.*
(iv) *If $w^{-1}a\in\Gamma^+$ then* (ii) *holds, with v replacing t.*
(v) $B_v^2=(\operatorname{ind} v)^{-1}\,\mathrm{id}+\varepsilon_a(p_a-1)/(p_a\operatorname{ind} v)^{1/2}\,B_v$.

Proof. (i) We have

$$\begin{aligned}B_wB_t&=\tilde{D}(\dot{w})\,B_{D,\dot{w}}\,\tilde{D}(\dot{t})\,B_{D,\dot{t}}\\&=\tilde{D}(\dot{w})\,\tilde{D}(\dot{t})\,B_{\dot{t}D,\dot{w}}\,B_{D,\dot{t}}\qquad\text{by (3.15)(ii)}\\&=\lambda(w,t)\,\tilde{D}(\dot{w}\dot{t})\,|(\operatorname{ind} w)^{-1}(\operatorname{ind} t)^{-1}(\operatorname{ind} wt)|^{1/2}\,B_{D,\dot{w}\dot{t}}\end{aligned}$$

by (4.2), since $N(t^{-1})\cap\Gamma=\emptyset$.

But $\tilde{D}(\dot{w}\dot{t})\,B_{D,\dot{w}\dot{t}}=B_{wt}$ (c.f. Remark 1 following (3.11)), and the statement follows.

(ii) The proof is the same as (i), with (4.2) being applicable because $N(t)\cap\Gamma=\emptyset$.

(iii) The proof is again the same as (i), since

$$N(v^{-1})\cap N(w)\cap\Gamma\subset\{a\}\cap N(w)=\emptyset.$$

(iv) Here $N(w^{-1}) \cap N(v) \cap \Gamma \subset \{a\} \cap N(w^{-1}) = \emptyset$, so that again (4.2) applies, and the proof is the same as in (i).

(v) There is an element $u \in W$ such that $u(J \cup \{a\}) = K \cup \{ua\} = K \cup \{b\} \subset \Pi$, and writing $v_1 = v(b, K)$, we have $v = u^{-1} v_1 u$. Moreover by ([11], Theorem 8), u may be chosen so that $\ell(v) = 2\ell(u) + \ell(v_1)$.

Hence in particular $N(u) \subset N(v)$. So $N(u) \cap \Gamma \subset N(v) \cap \Gamma = \{a\}$. Moreover since $ua = b \in \Sigma^+$, we have $N(u) \cap \Gamma = \emptyset$. Hence from (4.2) we obtain that

$$B_{\dot{v}D, \dot{u}} B_{D, \dot{v}} = |(\operatorname{ind} u)^{-1} (\operatorname{ind} v)^{-1} (\operatorname{ind} uv)|^{1/2} B_{D, \dot{u}\dot{v}}.$$

On the other hand

$$B_{\dot{u}D, \dot{v}_1} B_{D, \dot{u}} = B_{D, \dot{v}_1 \dot{u}} = B_{D, (v_1 u)^{\cdot}}.$$

Combining these two equations, we see (using (3.11)(iv)) that

$$B_{\dot{v}D, \dot{u}} B_{D, \dot{v}} = \gamma D(h) B_{\dot{u}D, \dot{v}_1} B_{D, \dot{u}} \tag{4.3.1}$$

where $\gamma = [(\operatorname{ind} v_1)(\operatorname{ind} v)^{-1}]^{1/2}$ and $h = \dot{v}^{-1} \dot{u}^{-1} (uv)^{\cdot}$.

Composing the two sides of (4.3.1) on the left with $\tilde{D}(\dot{v})$ and using (3.15)(ii), we obtain

$$B_{D, \dot{u}} B_v = \gamma \tilde{D}(\dot{u}^{-1} (uv)^{\cdot}) B_{\dot{u}D, \dot{v}_1} B_{D, \dot{u}}. \tag{4.3.2}$$

But $(uv)^{\cdot} = \dot{v}_1 \dot{u}$, whence $\dot{u}^{-1} (uv)^{\cdot} = \dot{u}^{-1} \dot{v}_1 \dot{u}$. Hence

$$\tilde{D}(\dot{u}^{-1} (uv)^{\cdot}) = \tilde{D}(\dot{u}^{-1} \dot{v}_1 \dot{u}) = \widetilde{\dot{u}D}(\dot{v}_1),$$

where $\widetilde{\dot{u}D}$ has the obvious meaning (observe that $v_1 \in W(\dot{u}D)$). Thus

$$\tilde{D}(\dot{u}^{-1} (uv)^{\cdot}) B_{\dot{u}D, \dot{v}_1} = \widetilde{\dot{u}D}(\dot{v}_1) B_{\dot{u}D, \dot{v}_1} = B_{v_1} \in E(\dot{u}D).$$

Substituting into (4.3.2), we see that

$$B_{D, \dot{u}} B_v = \gamma B_{v_1} B_{D, \dot{u}}. \tag{4.3.3}$$

Hence

$$\begin{aligned} B_{D, \dot{u}} B_v^2 &= \gamma^2 B_{v_1}^2 B_{D, \dot{u}} \qquad \text{(applying (4.3.3) twice)} \\ &= \gamma^2 [(\operatorname{ind} v_1)^{-1} \operatorname{id}_{\dot{u}D} \pm (p_b - 1)/(p_b \operatorname{ind} v_1)^{1/2} B_{v_1}] B_{D, \dot{u}} \qquad \text{(by (3.18))} \\ &= B_{D, \dot{u}} [\gamma^2 (\operatorname{ind} v_1)^{-1} \operatorname{id}_D \pm \gamma (p_b - 1)/p_b (\operatorname{ind} v_1)^{1/2} B_v]. \end{aligned}$$

Finally, note that $p_a = p_b$ by definition and that $B_{D, \dot{u}}$ is invertible by (3.17). The result follows on substitution of the value of γ $(= [(\operatorname{ind} v_1)(\operatorname{ind} v)^{-1}]^{1/2})$. This completes the proof of (4.3). □

We shall now modify the basis $\{B_w \mid w \in W(D)\}$ of $E(D)$ to produce a "normalised basis" which has a particularly simply multiplication table.

(4.4) *Definition.* If $a \in \Delta$ and $v = v(a, J)$, define

$$T_v = \varepsilon_a (p_a \operatorname{ind} v)^{1/2} B_v, \qquad \text{where } \varepsilon_a \text{ is as in (4.3)(v).}$$

It is then a simple consequence of (4.3)(v) that

(4.5) $T_v^2 = p_a\,\mathrm{id} + (p_a - 1)\,T_v$.

(4.6) **Lemma.** *Let $w \in W(D)$ and let a and b be elements of Δ such that $wa = b$. Write $v = v(a, J)$, $u = v(b, J)$. Then $B_w T_v = T_u B_w$.*

Proof. Using (4.3)(ii), we have

$$B_w T_v = \varepsilon_a p_a [(\mathrm{ind}\, w)^{-1} (\mathrm{ind}\, wv)]^{1/2}\, \lambda(w, v)\, B_{wv} \tag{4.6.1}$$

while from (4.3)(iv) we have

$$T_u B_w = \varepsilon_b p_b [(\mathrm{ind}\, w)^{-1} (\mathrm{ind}\, uw)]^{1/2}\, \lambda(u, w)\, B_{uw}. \tag{4.6.2}$$

Moreover since a and b are in the same $W(D)$ - orbit, we have $p_a = p_b$.

Now apply formula (3.4)(a) with v and w interchanged, recalling that $uw = wv$, $u = u^{-1}$ and $v = v^{-1}$:

$$\lambda(u, w)\, \lambda(uw, v) = \lambda(u, wv)\, \lambda(w, v). \tag{4.6.3}$$

But

$$\begin{aligned} \lambda(uw, v) &= \lambda(vw^{-1}u, uw) \quad \text{(by (3.4)(d))} \\ &= \lambda(w^{-1}, uw) \\ &= \lambda(w^{-1}uw, w^{-1}) \quad \text{(again using (3.4)(d))} \\ &= \lambda(v, w^{-1}) \\ &= \lambda^{-1}(w, v) \quad \text{(again by (3.4)(d)).} \end{aligned}$$

Similarly, $\lambda(u, wv) = \lambda^{-1}(u, w)$.

Hence from (4.6.3) we deduce that

$$\lambda(u, w)^2 = \lambda(w, v)^2. \tag{4.6.4}$$

It follows from (4.6.1) and (4.6.2) that

$$B_w T_v = \varepsilon T_u B_w, \qquad \varepsilon = \pm 1. \tag{4.6.5}$$

Therefore

$$B_w T_v^2 = \varepsilon T_u B_w T_v = \varepsilon^2 T_u^2 B_w = T_u^2 B_w.$$

Using (4.5), this implies that

$$B_w(p_a\,\mathrm{id} + (p_a - 1)\,T_v) = (p_b\,\mathrm{id} + (p_b - 1)\,T_u)\,B_w,$$

and hence that

$$(p_a - 1)\,B_w T_v = (p_b - 1)\,\varepsilon B_w T_v.$$

Since $p_a = p_a \neq 1$ (since $a, b \in \Delta \subset \Gamma$) we have that $\varepsilon = 1$, and the result follows. □

(4.7) **Lemma.** *Let $w \in R(D)$ and suppose that $w = v_1 \dots v_n = u_1 \dots u_n$ are two reduced expressions for w in $R(D)$, where $v_i = v(a_i, J)$, $u_i = v(b_i, J)$ and $a_i, b_i \in \Delta (i = 1, 2, \dots, n)$. Then*

$$T_{v_1} \dots T_{v_n} = T_{u_1} \dots T_{u_n}.$$

Proof. This is by induction on n. The case $n=1$ is trivial. By the "exchange rule" applied to $R(D)$, if r is the greatest integer such that $u_1 v_1 \dots v_r$ is reduced, then

$$u_1 v_1 \dots v_r = v_1 \dots v_{r+1} \qquad (r \leqq n-1).$$

By (4.3)(iii), $T_{v_1} \dots T_{v_r}$ is a non-zero scalar multiple of $B_{v_1 \dots r_r}$. Taking $w = v_1 \dots v_r$ in (4.6), we have $u_i w = w v_{r+1}$, and so

$$(T_{v_1} \dots T_{v_r}) T_{v_{r+1}} = T_{u_1} (T_{v_1} \dots T_{v_r}).$$

Thus

$$T_{u_1} T_{v_1} \dots \hat{T}_{v_{r+1}} \dots T_{v_n} = T_{v_1} \dots T_{v_n} \tag{4.7.1}$$

where ^ denotes a term omitted.

However since $v_1 \dots \hat{v}_{r+1} \dots v_n = u_2 \dots u_n$ are both reduced expressions in $R(D)$, we have by induction that

$$T_{v_1} \dots \hat{T}_{v_{r+1}} \dots T_{v_n} = T_{u_2} \dots T_{u_n}. \tag{4.7.2}$$

Combining (4.7.1) and (4.7.2), the result follows. □

(4.8) *Definition.* (i) For $w \in R(D)$, define $T_w = T_{v_1} \dots T_{v_n}$, where $w = v_1 \dots v_n$ is any reduced expression for w in $R(D)$.

(ii) For $x \in C(D)$, define $T_x = (\operatorname{ind} x)^{1/2} B_x$.

(iii) If $x \in C(D)$ and $w \in R(D)$, define $T_{xw} = T_x T_w$.

The Definition (4.8)(i) is justified by (4.7).

(4.9) *Definition.* For any $w \in W(D)$ we define

$$p_w = \Pi p_a, \qquad \text{where the product is taken over } \{a \in N(w) \cap \Gamma\}.$$

One verifies trivially that

(4.10) If $w \in W(D)$ and $w_1 w_2$ with $w_1 \in C(D)$, $w_2 \in R(D)$, then $p_w = p_{w_2}$.

(4.11) **Proposition.** *For each $w \in W(D)$, we have*

$$T_w = \varepsilon_w [p_w (\operatorname{ind} w)]^{1/2} B_w$$

where ε_w is a root of unity.

Proof. This is by induction on $N = |N(w) \cap \Gamma|$. If $N=0$ then $w \in C(D)$, $p_w = 1$ and the result is trivial.

If $N>0$, there is an element $a \in \Delta$ such that $wa \in \Gamma^-$. Write $v = v(a, J)$, $u = wv$. For any element $t \in W$, write $N_\Gamma(t) = N(t) \cap \Gamma$. If $t = t_1 t_2$ with $t_1 \in C(D)$, $t_2 \in R(D)$ then $N_\Gamma(t) = N_\Gamma(t_2)$.

Now by using (1.5)(ii), applied to the reflection group $(R(D), \Gamma)$, we have, if $w = w_1 w_2$ with $w_1 \in C(D)$, $w_2 \in R(D)$ that

$$N_\Gamma(w_2) = N_\Gamma(v) \cup v N_\Gamma(w_2 v),$$

so that

$$N_\Gamma(w) = N_\Gamma(v) \cup v N_\Gamma(u).$$

But

$$N_\Gamma(v) = \{a\},$$

and so

$$N(w)\cap\Gamma=v(N(u)\cap\Gamma)\cup\{a\}. \tag{4.11.1}$$

By the induction hypothesis applied to u, we have

$$T_u=\varepsilon_u(p_u \operatorname{ind} u)^{1/2} B_u.$$

Now since $|N_\Gamma(w_2)|=|N_\Gamma(w_2 v)|+|N_\Gamma(v)|$, it follows from (4.7) that $T_{w_2}=T_{w_2 v} T_v$. Hence by (4.8)(iii) we have

$$T_w=T_{w_1} T_{w_2}=T_{w_1} T_{w_2 v} T_v=T_u T_v.$$

Hence

$$\begin{aligned}T_w&=\varepsilon_u(p_u \operatorname{ind} u)^{1/2}\, \varepsilon_a(p_a \operatorname{ind} v)^{1/2} B_u B_v\\ &=\varepsilon_u \varepsilon_a p_u p_a \lambda(u,v)(\operatorname{ind} uv)^{1/2} B_{uv}\\ &=\varepsilon_w p_w(\operatorname{ind} w)^{1/2} B_w\end{aligned}$$

since it follows from (4.11.1) that $p_w=p_u p_a$, and $\lambda(u,v)$ is a root of unity by (3.4)(b). □

(4.12) *Definition.* For $v, w\in W(D)$, define

$$\mu(v,w)=\varepsilon_v \varepsilon_w \varepsilon_{vw}^{-1} \lambda(v,w).$$

Clearly μ is a 2-cocycle which is cohomologous to λ.

(4.13) **Lemma.** *If $x, y\in C(D)$ and $v, w\in R(D)$, then*

$$\mu(xv, yw)=\mu(x,y)=\lambda(x,y).$$

Proof. An easy computation using (4.11), and the fact that $T_{xv}=T_x T_v$ shows that $\mu(x,v)=\mu(v,x)=1$ for any $v\in R(D)$, $x\in C(D)$. Similarly one shows that for $t=v(a,J)$ with $a\in\Delta$, if $\ell_\Gamma(tw)=\ell_\Gamma(w)+1$ (ℓ_Γ denoting length in $(R(D),\Gamma)$) then $\mu(t,w)=1$ (any $w\in R(D)$). Using (3.4)(a), it is easy to deduce that $\mu(t,w)=1$ holds without the condition on tw, and hence (by induction on $\ell(w)$) that $\mu(v,w)=1$ for all v, w in $R(D)$. From (3.4)(a) we now have

$$\mu(x,v)\,\mu(xv,w)=\mu(x,vw)\,\mu(v,w).$$

Hence $\mu(xv,w)=1$, i.e. $\mu(y,w)=1$ for all $y\in W(D)$, $w\in R(D)$. Applying (3.4)(a) again to the triple (x,y,v) we see that

$$\mu(x,yv)=\mu(x,y)=\mu(xw,y).$$

Finally, another application of (3.4)(a) to the triple (xw,y,v) proves the lemma. □

Note that (4.13) shows that μ is really a 2-cocycle of $C(D)$.

We are now able to prove the main theorem of this paper.

(4.14) **Theorem.** *Let P_J be a parabolic subgroup of G, with Levi component M_J. Suppose that D is an irreducible cuspidal representation of M_J, and write $E(D)$*

$=\mathrm{End}_G(\mathrm{Ind}_{P_J}^G(D^*))$. *Then* $E(D)$ *has a* $\mathbb{C}$*-basis* $\{T_w \mid w\in W(D)\}$ *whose multiplication table is given as follows. Let* $w\in W(D)$, $x\in C(D)$, $v=v(a,J)$ *for some* $a\in\Delta$. *Then*

(i) $T_w T_x = \mu(w,x)\, T_{wx}$,

(ii) $T_x T_w = \mu(x,w)\, T_{xw}$,

(iii) $T_v T_w = \begin{cases} T_{vw} & \text{if } w^{-1}a\in\Gamma^+ \\ p_a T_{vw} + (p_a-1)\, T_w & \text{if } w^{-1}a\in\Gamma^-, \end{cases}$

(iv) $T_w T_v = \begin{cases} T_{wv} & \text{if } wa\in\Gamma^+ \\ p_a T_{wv} + (p_a-1)\, T_w & \text{if } wa\in\Gamma^- \end{cases}$

where μ *is a 2-cocycle of* $C(D)$, *and the* p_a *are powers of the characteristic, defined in* (3.19).

Proof. These relations are really restatements of (4.3), using the results (4.1) and (4.11). We give proofs of (i) and (iii); the proofs of (ii) and (iv) are similar.

(i) We have

$$T_w T_x = \varepsilon_w \varepsilon_x (p_w \operatorname{ind} w)^{1/2} (\operatorname{ind} x)^{1/2} B_w B_x$$
$$= \varepsilon_w \varepsilon_x \varepsilon_{wx}^{-1} \lambda(w,x) (p_w \operatorname{ind} wx)^{1/2} \varepsilon_{wx} B_{wx}$$

(by (4.3)(ii)),

$$= \mu(w,x)\, T_{wx} \qquad \text{since } p_{wx}=p_w.$$

(iii) If $w^{-1}a\in\Gamma^+$, then $T_v T_w = \mu(v,w)\, T_{vw}$, the computation being the same as that in (i), since (4.3)(iv) applies. Since $\mu(v,w)=1$ by (4.13), we have $T_v T_w = T_{vw}$.

If $w^{-1}a\notin\Gamma^+$, then $w^{-1}va\in\Gamma^+$. Hence by the first case just considered, we have

$$T_v T_{vw} = T_{v^2 w} = T_w.$$

Hence

$$T_v T_w = T_v^2 T_{vw} = (p_a \operatorname{id} + (p_a-1) T_v)\, T_{vw}$$
$$= p_a T_{vw} + (p_a-1)\, T_w. \quad \square$$

We conclude this section with two examples.

(4.15) Let $G=GL(n,q)$, and let $\Pi=\{a_1,\dots,a_{n-1}\}$ be the set of simple roots corresponding to the split torus of diagonal elements. Assume that $n=dm$ (d, m rational integers) and take P_J to be the standard parabolic subgroup with $M_J = GL(d,q)\times GL(d,q)\times\dots\times GL(d,q)$ (m times). This corresponds to $J=\{a_i \mid d$ does not divide $i\}\subset\Pi$. Take the representation D to be $J^{\langle\psi\rangle}\otimes J^{\langle\psi\rangle}\otimes\dots\otimes J^{\langle\psi\rangle}$ (m times) where ψ is a sufficiently general character of $\mathbb{F}_{q^d}^*$ (see [17]).

The following facts are easily verified.

(i) $W(D)=\langle v(a,J) \mid a=a_{dj}, j=1,2,\dots,m-1\rangle$. Thus $W(D)$ is isomorphic to the symmetric group on m symbols, and the set $\{v(a_{dj},J) \mid j=1,2,\dots,m-1\}$ is the set of "Coxeter generators" for $W(D)$.

(ii) For $a=a_{dj}$ ($j\in\{1,\dots,m-1\}$) the parameter p_a is computed by decomposing

$$\mathrm{Ind}_{[GL(d,q)\times GL(d,q)]^*}^{GL(2d,q)} [(J^{\langle\psi\rangle}\otimes J^{\langle\psi\rangle})^*],$$

and comparing the degrees of the two components, i.e. by considering the special case $m=2$.

We actually compute p_a by using (3.14.3) to evaluate

$$\beta=(p_a-1)/(p_a \operatorname{ind} v)^{1/2}.$$

Note that in this case ($m=2$) if we write $v=v(a,J)$ ($a=a_d\in\Pi$) then $U_v^-=U_J$, which is the set of matrices in $GL(2d,q)$ of the form $\begin{pmatrix}1 & A\\ 0 & 1\end{pmatrix}$, where all symbols denote $d\times d$ matrices. Moreover, we may take $\dot{v}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Then $\begin{pmatrix}1 & A\\ 0 & 1\end{pmatrix}\dot{v}\begin{pmatrix}1 & B\\ 0 & 1\end{pmatrix}\in\dot{v}P_J\dot{v}^{-1}$ if and only if $B=A^{-1}$, and when this condition is satisfied, we have

$$\begin{pmatrix}1 & A\\ 0 & 1\end{pmatrix}\dot{v}\begin{pmatrix}1 & B\\ 0 & 1\end{pmatrix}=\dot{v}\begin{pmatrix}A^{-1} & 1\\ 0 & A\end{pmatrix}\dot{v}^{-1}.$$

Using (3.14.3) we therefore obtain

$$\sum_{A\in GL(d,q)} J^{\langle\psi\rangle}(A)\otimes J^{\langle\psi\rangle}(A^{-1})=(\beta \operatorname{ind} v)\,\tilde{D}(\dot{v}). \tag{4.15.1}$$

We now take traces of both sides, obtaining

$$\sum_{A\in GL(d,q)} |\chi_{J^{\langle\psi\rangle}}(A)|^2=|GL(d,q)|=\beta(\operatorname{ind} v)\operatorname{trace}\tilde{D}(\dot{v}).$$

But if V is the space of $J^{\langle\psi\rangle}$, then $\tilde{D}(\dot{v})$ is the map on $V\otimes V$ which takes $v_1\otimes v_2$ into $v_2\otimes v_1$. Hence

$$\operatorname{trace}(\tilde{D}(\dot{v}))=\dim V=(q^{d-1}-1)(q^{d-2}-1)\dots(q-1).$$

Hence $\beta=(q^d-1)/q^{\frac{1}{2}d(d+1)}=(q^d-1)/(q^d \operatorname{ind} v)^{1/2}$. It follows that $p_a=q^d$, and hence in the general case (any m) that $p_a=q^d$ for all $a\in\Gamma=\Pi-J$. In particular, $W(D)=R(D)$ in this case.

(iii) Using (4.14), we see that in this case $E(D)$ has the following presentation. $E(D)$ has generators $T_1,\dots,T_{m-1}$ ($T_j=T_{v(a_{dj},J)}$) and relations

$$T_w T_j=\begin{cases}T_{wv_j} & \text{if } \ell_{W(D)}(wv_j)=\ell_{W(D)}(w)+1\\ q^d T_{wv_j}+(q^d-1)\,T_w & \text{otherwise}\end{cases}$$

where $w\in W(D)(=R(D))$ and $v_j=v(a_{dj},J)$.

(4.16) *Example.* We include this example to show that it may happen that $p_a=1$ for some $a\in\Gamma'$.

Take $G=SL(2d,q)$ and take $J=\{a_1,\dots,\hat{a}_d,\dots,a_{2d-1}\}$ as in the above example, with M the corresponding Levi subgroup of P. Then

$$M'=SL(d,q)\times SL(d,q)\subset M.$$

For the background to the present example, we refer the reader to ([17], §4). Let D_0 be an irreducible component of the restriction of $J^{\langle\psi\rangle}\otimes J^{\langle\psi\rangle}$ to M, and write $J^{\langle\psi\rangle}|_{SL(d,q)}=J^0+\ldots+J^{e-1}$. Then (cf. [17], 4.14)

$$D_0|_{M'}=\sum_{i=0}^{e-1} J^i\otimes J^{i_0-i} \tag{4.16.1}$$

where the superscripts are taken modulo e and $i_0\in\{0,\ldots,e-1\}$.

The same computation as in example (4.15)(ii) shows that here (for $a=a_d$ and $v=v(a,J)$) we again have

$$\sum_{A\in GL(d,q)} D_0^*\begin{pmatrix}A & 1\\ 0 & A^{-1}\end{pmatrix}=\beta(\operatorname{ind} v)\,\tilde{D}(\dot{v}). \tag{4.16.2}$$

Suppose d and q are such that $D_0=\operatorname{Ind}_{M'}^{M}(J^0\otimes J^{i_0})$. This can occur when $q-1=d$, e.g. when $d=2$, $q=3$. In this case $e=q-1=d$.

Then taking traces in (4.16.2), and using (4.16.1) we have

$$\beta\operatorname{ind} v\operatorname{trace}(\tilde{D}(\dot{v}))=\sum_{i=0}^{e-1}\sum_{A\in SL(d,q)}\chi_i(A)\,\overline{\chi_{i_0-i}(A)} \tag{4.16.3}$$

where χ_i is the character of J^i.

But the inner sum is zero unless $i\equiv i_0-i \pmod e$, i.e. $i_0\equiv 2i \pmod e$. Hence if q is odd, it is possible to choose D_0 (e.g. $i_0=1$) such that $\beta=0$, i.e. $p_a=1$.

The reader is referred to [17], §6 for an explicit discussion of $W(D)$ and $E(D)$ in this case.

§5. The Generic Algebra - Proof of Springer's Conjecture

Let q be the largest power of p, such that for all $w\in W(D)$, we have

(5.1) (i) $\operatorname{ind} w=q^{n_w}$ (ii) $p_w=q^{m_w}$ for rational integers n_w, $m_w\geqq 0$.

Such a q always exists.

Let $\mathbb{C}[u]$ be the ring of polynomials in an indeterminate u over $\mathbb{C}$ and define an algebra $A(u)$ over $\mathbb{C}[u]$ as follows: for $w\in W(D)$ let $u_w=u^{m_w}$ (m_w as in (5.1)(ii)), and let μ be the 2-cocycle of (4.12).

(5.2) *Definition.* $A(u)$ is the associative algebra over $\mathbb{C}[u]$ which has basis $\{a_w\mid w\in W(D)\}$ and multiplication given by: for $w\in W(D)$, $x\in C(D)$, $v=v(a,J)$ for $a\in\Delta$.

(i) $a_w a_x=\mu(w,x)\,a_{wx}$,

(ii) $a_x a_w=\mu(x,w)\,a_{xw}$,

(iii) $a_v a_w=\begin{cases}a_{vw} & \text{if } w^{-1}a\in\Gamma^+\\ u_v a_{vw}+(u_v-1)\,a_w & \text{if } w^{-1}a\in\Gamma^-,\end{cases}$

(iv) $a_w a_v=\begin{cases}a_{wv} & \text{if } wa\in\Gamma^+\\ u_v a_{wv}+(u_v-1)\,a_w & \text{if } wa\in\Gamma^-.\end{cases}$

For any ring F such that $F\supset\mathbb{C}[u]$, write $A(u)^F=A(u)\otimes_{\mathbb{C}[u]}F$. If $f\colon\mathbb{C}[u]\to\mathbb{C}$ is an algebra homomorphism, and $f(u)=b\in\mathbb{C}$, we write $A(b)=A(u)\otimes_f C$. The $A(b)$ are "specializations" of $A(u)$.

(5.3) **Theorem.** *Let $F=\mathbb{C}(u)$ be the quotient field of $\mathbb{C}[u]$. Then $A(u)^F$ is a separable F-algebra and for each $b\in\mathbb{C}$ such that $A(b)$ is separable (and so semisimple), the algebras $A(u)^F$ and $A(b)$ have the same numerical invariants.*

Proof. Since $A(q)=E(D)$ (see (4.14)) is semisimple, it follows from Tits' theorem ([3], p. 56 ex. 26; see also [25], p. 249) that $A(u)^F$ is separable, and that for any $b\in\mathbb{C}$ such that $A(b)$ is semisimple, the algebras $A(u)^F$ and $A(b)$ have the same munerical invariants. □

(5.4) **Corollary** (Springer's Conjecture). *Let $\mathbb{C}W(D)_\mu$ be the group algebra of $W(D)$, twisted by the 2-cocycle μ, i.e. the associative $\mathbb{C}$-algebra with basis $\{[w]\mid w\in W(D)\}$ and multiplication table given by $[w][v]=\mu(w,v)[wv]$. Then we have*

$$E(D)=\mathrm{End}_G(\mathrm{Ind}_P^G(D^*))\cong\mathbb{C}W(D)_\mu.$$

Proof. The algebras $A(q)$ and $A(1)$ are respectively isomorphic to $E(D)$ and $\mathbb{C}W(D)_\mu$ (by (4.14)) and since they are both semisimple, they have the same numerical invariants (by (5.3)) and so are isomorphic. □

(5.5) **Corollary.** *The irreducible components of* $\mathrm{Ind}_P^G(D^*)$ *are in bijective correspondence with the irreducible representations of the algebra* $\mathbb{C}W(D)_\mu$.

The methods of Benson-Curtis ([1]) can also be applied to the algebra $A(u)$ to produce information concerning the degrees and rationality of these irreducible components. Roughly speaking, "all the irrationality" is introduced by the cuspidal respresentations - i.e. their field of definition suffices for all representations of G. The authors plan a sequel to this paper in which these questions, as well as explicit determination of the $W(D)$ and the parameters p_a will be addressed.

(5.6) *Example.* Applied to the example (4.15) introduced above, these results show that (in view of (4.13), which shows that μ is trivial in this case) the irreducible components of $\mathrm{Ind}_{P_J}^G(D^*)$ correspond bijectively to the irreducible representations of the symmetric groups on m symbols, and hence may be denoted $J^{\langle\psi\rangle}(\lambda)$, where λ is a partition of m (cf. [17]).

Note also that $E(D)$ is just the standard generic algebra $H(G,B)$ for $GL(m,q^d)$ in this case, so that "generic degrees" (cf. [5]) are available in the literature, and the degree of $J^{\langle\psi\rangle}(\lambda)$ can be written down explicitly.

§6. On the Nature of μ and Other Complements

(6.1) **Theorem.** *We have*

$$\mathrm{End}_{\tilde{M}}(\mathrm{Ind}_M^{\tilde{M}}(D))\cong\mathbb{C}W(D)_\lambda\cong\mathbb{C}W(D)_\mu$$

where $\tilde{M}$ and λ are the group and cocycle defined in (3.3) *and* (3.4).

Proof. Let X be the space of functions $f\colon\tilde{M}\to V$ (V being the space of D) satisfying

$$f(m\tilde{m})=D(m)f(\tilde{m})\qquad(m\in M,\tilde{m}\in\tilde{M}).$$

$\tilde{M}$ acts by right translation as $\mathrm{Ind}_M^{\tilde{M}}(D)$ on X. Now define $\sigma_w (w \in W(D))$ by

$$\sigma_w f(x) = \tilde{D}(w) f(\dot{w}^{-1} x). \tag{6.1.1}$$

As noted in Remark 1 following (3.11), σ_w is independent of $\dot{w}$. One verifies easily that σ_w is an $\tilde{M}$-equivariant linear transformation of X, i.e. is in $\mathrm{End}_{\tilde{M}}(X)$.

We show that σ_w are linearly independent as elements of $\mathrm{End}_{\tilde{M}}(X)$. If $\sum_{w \in W(D)} \alpha_w \sigma_w = 0$ $(\alpha_w \in \mathbb{C})$ then $\sum_w \alpha_w \sigma_w f = 0$ for each $f \in X$. Take f to be a function in X, whose support is $M u^{-1}$. Then

$$\sum_w \alpha_w \sigma_w f(1) = \sum_w \alpha_w \tilde{D}(w) f(w^{-1}) = \alpha_u \tilde{D}(u) f(u^{-1}).$$

Hence $\alpha_u = 0$ for each $u \in W(D)$, and the σ_w are linearly independent. Moreover using Mackey's formula one sees easily that $\dim_{\mathbb{C}}(\mathrm{End}_{\tilde{M}}(X) = |W(D)|$, so that $\{\sigma_w \mid w \in W(D)\}$ forms a basis of $\mathrm{End}_{\tilde{M}}(X)$.

Next, observe that

$$\begin{aligned}(\sigma_{w_1} \sigma_{w_2} f) x &= \tilde{D}(w_1) \tilde{D}(w_2) f(w_2^{-1} w_1^{-1} x) \\ &= (\lambda(w_1, w_2) \sigma_{w_1 w_2} f)(x).\end{aligned}$$

Thus

$$\sigma_{w_1} \sigma_{w_2} = \lambda(w_1, w_2) \sigma_{w_1 w_2}. \tag{6.1.2}$$

It follows from (6.1.2) and the definition, that $\mathrm{End}_{\tilde{M}}(X) = \mathbb{C} W(D)_\lambda$, and since μ is cohomologous to λ, the result follows. □

(6.2) **Corollary.** *The multiplicities of the irreducible components of* $\mathrm{Ind}_M^{\tilde{M}}(D)$ *are the same as those of* $\mathrm{Ind}_{P_J}^G(D^*)$.

This is a consequence of (5.4) and (6.1), which show that the corresponding endomorphism algebras are isomorphic.

To carry out computations in practice, it is important to determine when the cocycle μ is trivial.

(6.3) **Conjecture.** *The cocycle μ is always trivial.*

We are at present unable to prove this in complete generality, but in this section will give some sufficient conditions for the triviality of μ. We begin with the obvious remark that

(6.4) *If $W(D) = R(D)$, then μ is trivial.*

(6.5) **Lemma.** *If either of the representations* (i) $\mathrm{Ind}_M^{\tilde{M}}(D)$ *or* (ii) $\mathrm{Ind}_{P_J}^G(D^*)$ *has an irreducible constituent of multiplicity one, then μ is trivial.*

Proof. From (5.4) we have that $\mathrm{End}_G(\mathrm{Ind}_P^G(D^*)) \cong \mathbb{C} W(D)_\mu$. Thus if $\mathrm{Ind}_P^G(D^*)$ has an irreducible constituent of multiplicity one, the endomorphism algebra $\mathbb{C} W(D)$ has a representation of degree one, i.e. there is an algebra homomorphism $\zeta: \mathbb{C} W(D)_\mu \to \mathbb{C}$. This shows that μ is cohomologous to the trivial 2-cocycle, i.e. $\mu = 1$. The other case follows by the same argument, since by (6.1).

$$\mathrm{End}_{\tilde{M}}(\mathrm{Ind}_M^{\tilde{M}}(D)) = \mathbb{C} W(D)_\mu. \quad \square$$

(6.6) *Definition.* We say that the linear character α of U (the unipotent radial of B) is in general position if

(i) $\alpha|_{U_a} \neq 1$ for $a \in \Pi$, and (ii) $\alpha|_{U_a} = 1$ for $a \notin \Pi$.

Note that in most cases (e.g. if the characteristic is good for $\underline{G}$), condition (ii) is automatic for all linear characters α, since for $a \notin \Pi$, $U_a \subset U'$ (the derived group).

(6.7) **Proposition.** *Let α be a linear character of U in general position, and let χ be the character of D on M_J. Then the intertwining number*

$$(\mathrm{Ind}_P^G(\chi^*), \mathrm{Ind}_U^G(\alpha)) = (\xi, \mathrm{Ind}_{U \cap M_K}^{M_K}(\alpha)) = 0 \text{ or } 1$$

where $v = w_0 w_J$, and $(K, \xi) = v(J, \chi)$.

Proof. This is a formula of Rodier, and may be found in [24].

(6.8) **Corollary.** *If the restriction $\chi|_{M \cap U}$ contains a linear character of $M \cap U$ which is in general position, then μ is trivial.*

Proof. In this case we may choose α so that $\alpha|_{U \cap M_K}$ is an appropriate general position character which makes the multiplicity of (6.7) equal to 1. The corollary then follows from (6.5). □

(6.9) **Corollary.** *Suppose that* (a) *the degree of χ is prime to p and* (b) *M contains no component of type $B_\ell(2)$, $C_\ell(2)$, $F_4(2)$, $G_2(2)$, $G_2(3)$ or $F_4'(2)$. Then μ is trivial.*

Proof. For such M, all linear characters of $U \cap M$ are trivial on non-fundamental root subgroups, by a result of Howlett (Ph.D. Thesis, Adelaide University, 1974). Since the degree of χ is not divisible by p and $U \cap M$ is a p-group, the restriction $\chi|_{M \cap U}$ contains a linear character, which must (c.f. [18]) be in general position since χ is cuspidal. The result now follows from (6.8). □

(6.10) **Corollary.** *Suppose that all components of J are of type A_ℓ (for various ℓ). Then μ is trivial.*

Proof. For groups of type A_ℓ, all irreducible cuspidal characters have degree prime to p (c.f. [17]). Thus the result follows from (6.9). □

(6.11) **Corollary.** *Let $B = TU$ be a Levi decomposition of the Borel subgroup B of G, and let χ be a (linear) character of T, whose centralizer in $W = N(T)/Z(T)$ is $W(\chi)$. Then*

(i) $\mathrm{End}_G(\mathrm{Ind}_B^G(\chi^*)) \cong \mathbb{C}W(\chi)$,
(ii) *χ has an extension $\tilde{\chi}$ to $\tilde{T} = \langle T, \dot{w} \mid w \in W(\chi) \rangle$.*

Proof. The first statement follows from the fact that the cocycle μ is trivial in this case, which in turn follows from (6.8) since the condition on χ is vacuous. Alternatively, one may apply (6.7) directly to the present situation, obtaining (since J is empty)

$$(\mathrm{Ind}_B^G(\chi^*), \mathrm{Ind}_U^G(\alpha)) = (w_0 \chi, \mathrm{Ind}_{\{1\}}^T(\alpha)) = 1$$

since $\mathrm{Ind}_{\{1\}}^T(\alpha)$ is the regular representation of the abelian group T.

The second statement now follows from the remark that μ is trivial if and only if in the general case D has an extension $\tilde{D}$ from M to $\tilde{M}$; the implication here is that χ has an extension $\tilde{\chi}$ to $\tilde{T}$. □

The above special case of our result has been discussed by Steinberg and Yokunuma (see [25] and [27]; c.f. also [12]).

The following result is proved by Lusztig in ([19], §5).

(6.12) **Proposition** (Lusztig). *If D is a cuspidal unipotent representation, then μ is trivial.*

We note in closing that Lusztig informs us that he is able to prove that μ is trivial whenever G is adjoint, by using his classification of the characters of G. It would nevertheless be desirable to have a direct proof of (6.3).

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