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Implicit Runge-Kutta Methods for Second Kind Volterra Integral Equations

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Summary. Implicit Runge-Kutta methods for ordinary differential equations which arise from interpolatory quadrature formulae are generalized to Volterra integral equations of the second kind. Two classes of methods are considered and shown to be convergent and numerically stable. In addition, for various choices of quadrature formulae the methods are A-stable and stiffly A-stable.

1. Introduction

The extension of explicit Runge-Kutta methods for ordinary differential equations to Volterra integral equations of the second kind

$$y(t) = g(t) + \int_0^t K(t, s, y(s))\,ds \tag{1.1}$$

has been considered by Laudet and Oules [9], Pouzet [10] and Beltjukov [2]. Implicit Runge-Kutta methods for the solution of ordinary differential equations have been studied by Butcher [3] and Axelsson [1] and have been shown to possess desirable stability properties combined with high order convergence. Butcher shows that each Runge-Kutta process can be generated by a numerical quadrature formula.

In this paper we extend the idea of constructing implicit Runge-Kutta methods based on quadrature formulae to (1.1). The basis of the quadrature formulae under consideration for the interval $[0, 1]$ is Lagrangian interpolation with respect to a set of points $\{u_1, u_2, \ldots, u_n\}$ with $0 \leqq u_1 < u_2 < \cdots < u_n = 1$. We show that for each choice of $\{u_1, u_2, \ldots, u_n\}$ the method is convergent of at least order n and that higher order convergence up to order $2n-1$ is possible for suitably chosen points. In addition the methods are shown to be numerically stable in the sense of Noble [11] and for special choices of $\{u_1, u_2, \ldots, u_n\}$ have the stronger property of being A-stable in the sense of Dahlquist [4].

From the theory of Volterra integral equations of the second kind it is well known (see for example [5; Ch. 13, p. 415]) that, if

(i) $g(t)$ is continuous on $0 \leqq t \leqq T$,

(ii) $K(t, s, y)$ is uniformly continuous in t and s for all finite y on $0 \leqq s \leqq t \leqq T$, and

(iii) $K(t, s, y)$ satisfies the Lipschitz condition

$$|K(t, s, y_1) - K(t, s, y_2)| \leqq L|y_1 - y_2|, \quad 0 \leqq s \leqq T \tag{1.2}$$

where L is a constant independent of s and t, then equation (1.1) has a unique continuous solution on $0 \leqq t \leqq T$. However, additional smoothness conditions will be imposed in the subsequent analysis.

It should be noted that although the analysis is presented only for the scalar equation (1.1), the generalization to a system of Volterra integral equations of the second kind follows immediately.

2. Preliminaries

In this section we introduce some notation and present three lemmas and two corollaries which will be required in subsequent analysis.

Let $0 \leqq u_1 < u_2 < \cdots < u_n = 1$. For convenience we shall consider only the case $u_1 > 0$. However, with slight notational modifications, the analysis is valid for $u_1 = 0$. Put

$$w(t) = (t-u_1)(t-u_2)\ldots(t-u_n) = \sum_{j=0}^{n} \alpha_j t^{n-j}.$$

Let

$$L_k(t) = \frac{w(t)}{(t-u_k)\,w'(u_k)}, \qquad k = 1, \ldots, n, \tag{2.1}$$

$$\begin{aligned} a_{jk} &= \int_0^{u_j} L_k(s)\,ds, \qquad j = 1, \ldots, n; \qquad k = 1, \ldots, n, \\ a_k &= a_{nk}. \end{aligned} \tag{2.2}$$

We denote the relation

$$\int_0^1 w(s)\,ds \neq 0$$

by $w(t) \in p_0$, and the relations

$$\int_0^1 s^r w(s)\,ds = 0, \qquad r = 0, \ldots, \nu-1, \nu > 0; \qquad \int_0^1 s^\nu w(s)\,ds \neq 0 \tag{2.3}$$

by $w(t) \in p_\nu$. Let

$$R_i(f) = \int_0^{u_i} f(s)\,ds - \sum_{k=1}^{n} a_{ik} f(u_k), \qquad i = 1, \ldots, n. \tag{2.4}$$

The following lemmas and corollaries are simple generalizations of results due to Axelsson [1]. Proofs are given in Weiss [15, Ch. II].

Lemma 2.1. *If $w(t) \in p_\nu$, then*

$$R_i(u^{n+q}) = \int_0^{u_i} s^q w(s)\,ds - \sum_{r=1}^{q} \alpha_r R_i(u^{n+q-r}), \qquad q = 0, \ldots, n,$$

and

$$R_n(u^{n+q}) = 0, \qquad q \leqq \nu - 1. \tag{2.5}$$

Lemma 2.2. *If $f_q(t)$ is a polynomial of degree less than or equal to q, and $r + q \leqq n + \nu - 2$, then*

$$\sum_{k=1}^{n} a_k u_k^r \int_0^{u_k} f_q(s)\,ds = \int_0^1 \frac{1-s^{r+1}}{r+1} f_q(s)\,ds.$$

Corollary 2.1.

$$\sum_{k=1}^{n} a_r u_k^r R_k(u^{n+p}) = 0, \qquad r+p=0, \ldots, \nu-2.$$

Corollary 2.2.

$$\sum_{k=1}^{n} a_k u_k^r R_k\big((u-u_k)^{n+p}\big) = 0, \qquad r+p=0, \ldots, \nu-2.$$

The following lemma, which provides an estimation of the growth of solutions of nonhomogeneous systems of difference equations is given by Henrici [7, Ch. 6, p. 313].

Lemma 2.3. *If* $|\varepsilon_i| \leqq A \sum_{k=0}^{i-1} |\varepsilon_k| + B, \quad i=1, 2, \ldots, A, B>0$ *and* $|\varepsilon_0| \leqq \eta$, *then*

$$|\varepsilon_i| \leqq (B+A\eta)(1+A)^{i-1}, \qquad i=1, 2, \ldots.$$

3. Numerical Schemes

In this section we define two classes of implicit Runge-Kutta methods corresponding to a fixed set $\{u_1, u_2, \ldots, u_n\}$.

We are interested in an approximation to the solution $y(t)$ on the grid

$$t_i = ih, \qquad i=0, \ldots, I; \qquad h=\frac{T}{I}.$$

Let

$$t_{ij} = t_i + u_j h, \qquad j=1, \ldots, n; \qquad i=0, \ldots, I-1.$$

Discretizing (1.1) we obtain

$$\begin{aligned} y(t_{ij}) &= g(t_{ij}) + \int_0^{t_i} K\big(t_{ij}, s, y(s)\big)ds + \int_{t_i}^{t_{ij}} K\big(t_{ij}, s, y(s)\big)ds, \\ j &= 1, \ldots, n; \qquad i=0, \ldots, I-1. \end{aligned} \tag{3.1}$$

To approximate the integrals in (3.1), we use the quadrature formulae

$$\int_{t_l}^{t_{l+1}} f(s)\,ds \approx h \sum_{k=1}^{n} a_k f(t_{lk}), \qquad l=0, \ldots, i-1$$

and

$$\int_{t_i}^{t_{ij}} f(s)\,ds \simeq \sum_{k=1}^{n} h a_{jk} f(t_{ik}).$$

Hence we obtain the numerical scheme

$$\begin{aligned} Y_{ij} &= g(t_{ij}) + \sum_{l=0}^{i-1} \sum_{k=1}^{n} h a_k K(t_{ij}, t_{lk}, Y_{lk}) + \sum_{k=1}^{n} h a_{jk} K(t_{ij}, t_{ik}, Y_{ik}), \\ j &= 1, \ldots, n; \qquad i=0, \ldots, I-1, \end{aligned} \tag{3.2}$$

where Y_{lk} denotes the approximation to $y(t_{lk})$.

Thus, for each i, (3.2) represents a system of n simultaneous equations in Y_{ij}, $j=1, \ldots, n$. Also it may be seen from (3.2) that values of $K(t, s, y(s))$ are required outside the region $0 \leqq s \leqq t \leqq T$ and this could cause difficulties in practice if the kernel is badly behaved outside the region (see for instance [6]).

This problem can be overcome by using a different approximation for $\int_{t_i}^{t_{ij}} K(t_{ij}, s, y(s))\,ds$. First approximate

$$\int_{t_i}^{t_{ij}} K(t_{ij}, s, y(s))\,ds \quad \text{by} \quad \int_{t_i}^{t_{ij}} K\left(t_{ij}, s, \sum_{k=1}^{n} L_k\left(\frac{s-t_i}{h}\right) Y_{ik}\right) ds$$

and then apply the quadrature formula

$$\int_{t_i}^{t_{ij}} f(s)\,ds \simeq \sum_{k=1}^{n} h u_j a_k f(t_i + u_j u_k h).$$

This yields the numerical scheme

$$\begin{aligned} Y_{ij} = g(t_{ij}) &+ \sum_{l=0}^{i-1} \sum_{k=1}^{n} h a_k K(t_{ij}, t_{lk}, Y_{lk}) \\ &+ \sum_{k=1}^{n} h u_j a_k K\left(t_{ij}, t_i + u_j u_k h, \sum_{r=1}^{n} L_r(u_j u_k) Y_{ir}\right), \qquad (3.3) \\ &\qquad j=1, \ldots, n; \quad i=0, \ldots, I-1. \end{aligned}$$

Clearly $K(t, s, y(s))$ is not required outside the region $0 \leqq s \leqq t \leqq T$. For the special case $K(t, s, y(s)) = \lambda y(s)$, $\lambda =$ constant, the schemes (3.2) and (3.3) are the same.

It follows from a simple contraction mapping argument that (3.2) and (3.3) are uniquely solvable if h is sufficiently small.

4. Convergence of the Numerical Schemes

In this and the following section we shall only present the analysis for the schemes (3.2). Analogous results for the schemes (3.3) can be obtained and the reader is referred to Weiss [15, Ch. II] for details.

Let $w(t) \in \not{p}_\nu$. In addition to (1.2) let $y(t) \in C^{n+\nu+1}[0, T]$ and $K(t, s, v)$ be $n+\nu+1$ times continuously differentiable for $0 \leqq s \leqq t+\delta$, $0 \leqq t \leqq T$, where δ is a fixed positive number, and for all v contained in an open neighborhood of $y(t)$. Define

$$y_{ij} = y(t_{ij}), \quad \varepsilon_{ij} = y_{ij} - Y_{ij}, \quad j=1, \ldots, n; \quad i=0, \ldots, I-1.$$

Let $h < \delta$. Subtracting (3.2) from (3.1) we obtain

$$\begin{aligned} \varepsilon_{ij} = &\sum_{l=0}^{i-1} \sum_{k=1}^{n} h a_k \{K(t_{ij}, t_{lk}, y_{lk}) - K(t_{ij}, t_{lk}, Y_{lk})\} \\ &+ \sum_{k=1}^{n} h a_{jk} \{K(t_{ij}, t_{ik}, y_{ik}) - K(t_{ij}, t_{ik}, Y_{ik})\} + R_{ij}, \qquad (4.1) \\ &\qquad j=1, \ldots, n; \quad i=0, \ldots, I-1, \end{aligned}$$

where

$$R_{ij} = P_{ij} + Q_{ij},$$

$$P_{ij} = \sum_{l=0}^{i-1} \left\{ \int_{t_l}^{t_{l+1}} K\big(t_{ij}, s, y(s)\big)\,ds - \sum_{k=1}^{n} h a_k K(t_{ij}, t_{lk}, y_{lk}) \right\}$$

and

$$Q_{ij} = \int_{t_i}^{t_{ij}} K\big(t_{ij}, s, y(s)\big)\,ds - \sum_{k=1}^{n} h a_{jk} K(t_{ij}, t_{ik}, y_{ik}).$$

We shall now obtain an asymptotic expansion for R_{ij}.

Lemma 4.1. *Let*

$$K(t, s) = K\big(t, s, y(s)\big), \quad K^{(m)}(t, s) = \frac{\partial^m}{\partial \eta^m} K(t, \eta)\big|_{\eta = s}.$$

Then

$$R_{ij} = \sum_{p=0}^{\nu} h^{n+p} \varphi_{pj}(t_{ij}) + O(h^{n+\nu+1}), \quad j = 1, \ldots, n; \quad i = 0, \ldots, I-1, \tag{4.2}$$

where

$$\varphi_{pj}(t) = K^{(n+p-1)}(t, t) \frac{R_j\big((u-u_j)^{n+p-1}\big)}{(n+p-1)!}, \quad p = 0, \ldots, \nu-1; \quad j = 1, \ldots, n,$$

$$\varphi_{\nu j}(t) = \frac{R_n(u^{n+\nu})}{(n+\nu)!} \int_0^t K^{(n+\nu)}(t, s)\,ds + \frac{K^{(n+\nu-1)}(t, t)}{(n+\nu-1)!} R_j\big((u-u_j)^{n+\nu-1}\big),$$

$$j = 1, \ldots, n.$$

Proof. The lemma follows from Taylor series expansion of $K(t_{ij}, s)$. #

Corollary 4.1. *If $\nu > 1$, then*

$$\sum_{j=1}^{n} a_j u_j^r \varphi_{pj}(t) = 0, \quad r = 0, \ldots, \nu - p - 1; \quad p = 0, \ldots, \nu - 1.$$

Proof. The result follows from Corollary 2.1. #

Lemma 4.2. *There exist positive constants K and h_0 such that for $h \leqq h_0$,*

$$e_i = \max_{1 \leqq j \leqq n} |\varepsilon_{ij}| \leqq K h^n, \quad i = 0, \ldots, I-1, \quad \text{if} \quad \nu = 0,$$

or

$$e_i \leqq K h^{n+1}, \quad i = 0, \ldots, I-1, \quad \text{if} \quad \nu > 0.$$

Proof. Taking absolute values in (4.1) and applying the Lipschitz condition (1.2), we obtain

$$|\varepsilon_{ij}| \leqq \sum_{l=0}^{i-1} \sum_{k=1}^{n} hL|a_k|\,|\varepsilon_{lk}| + \sum_{k=1}^{n} hL|a_{jk}|\,|\varepsilon_{ik}| + |R_{ij}|,$$

$$j = 1, \ldots, n; \quad i = 0, \ldots, I-1.$$

From Lemmas 2.1 and 4.1, there exists a constant C such that $|R_{ij}| \leqq C h^n$ if $\nu = 0$ or $|R_{ij}| \leqq C h^{n+1}$ if $\nu > 0$. Hence the result follows from Lemma 2.3. #

The above lemma gives a convergence result for the scheme (3.2). Generally however, this result is not the best possible and in particular we can obtain a more accurate estimate for ε_{in}, $i=0,\ldots,I-1$. For this analysis, four additional lemmas are required.

Lemma 4.3. *Let the functions* $\theta_j(t)$, $j=1,\ldots,n$, *satisfy*

$$\sum_{j=1}^{n} a_j u_j^r \theta_j(t)=0, \qquad r=0,\ldots,q, \qquad \textit{where} \quad 0<q<\nu-1.$$

Then

$$\sum_{j=1}^{n} a_j u_j^p \sum_{k=1}^{n} a_{jk} u_k^l \theta_k(t)=0, \qquad p+l=0,\ldots,q-1.$$

Proof. Using Lemma 2.2,

$$\begin{aligned}\sum_{j=1}^{n} a_j u_j^p \sum_{k=1}^{n} a_{jk} u_k^l \theta_k(t) &= \sum_{k=1}^{n} u_k^l \theta_k(t) \sum_{j=1}^{n} a_j u_j^p \int_0^{u_j} L_k(s)\,ds \\ &= \sum_{k=1}^{n} u_k^l \theta_k(t) \int_0^1 \frac{(1-s^{p+1})}{p+1} L_k(s)\,ds, \\ & \qquad p=0,\ldots,\nu-1.\end{aligned} \tag{4.3}$$

Using

$$s^r=\left((s-u_k)+u_k\right)^r=\sum_{q=0}^{r}\binom{r}{q}(s-u_k)^q u_k^{r-q},$$

we obtain

$$s^r L_k(s)=u_k^r L_k(s)+\sum_{q=1}^{r}\binom{r}{q}(s-u_k)^{q-1}u_k^{r-q}w(s)/w'(u_k).$$

Thus, from (2.3),

$$\int_0^1 s^r L_k(s)\,ds=u_k^r a_k, \qquad r=0,\ldots,\nu. \tag{4.4}$$

On substitution of (4.4) into (4.3) it follows that

$$\sum_{j=1}^{n} a_j u_j^p \sum_{k=1}^{n} a_{jk} u_k^l \theta(t)=\frac{1}{p+1}\sum_{k=1}^{n} a_k u_k^l (1-u_k^{p+1})\,\theta_k(t)=0,$$
$$l+p=0,\ldots,q-1. \quad \#$$

Lemma 4.4. *Let* $f(t,s)$ *be* $M+1$ *times continuously differentiable in the region* $0\leqq s\leqq t\leqq T$ *and denote*

$$f^{(r)}(t,s)=\frac{\partial^r}{\partial\eta^r}f(t,\eta)\big|_{\eta=s}.$$

Then

$$h\sum_{l=0}^{i-1} f(t_{ij},t_{lk})=\int_0^{t_{ij}} f(t_{ij},s)\,ds+\sum_{m=0}^{M-1}h^{m+1}\psi_{jkm}(t_{ij})+O(h^{M+1}), \qquad M\geqq 0,$$

where

$$\psi_{jkm}(t)=f^{(m)}(t,0)\sum_{r=0}^{m+1}C_{mr}u_k^r+f^{(m)}(t,t)\sum_{r=0}^{m+1}D_{mr}(u_k-u_j-1)^r$$

and C_{mr}, D_{mr}, $r=0,\ldots,m+1$; $m=0,\ldots,M-1$ *are constants.*

Proof. By the Euler-MacLaurin sum formula (see for example Ralston [13, p. 133]),

$$\begin{aligned} h\sum_{l=0}^{i-1} f(t_{ij}, t_{lk}) = \int_{u_k h}^{t_{i-1}+u_k h} f(t_{ij}, s)\,ds + \frac{h}{2}\,\big(f(t_{ij}, u_k h) \\ + f(t_{ij}, t_{i-1}+u_k h)\big) + \sum_{r=1}^{[M/2]} \frac{h^{2r} B_{2r}}{(2r!)} \\ \cdot \{f^{(2r-1)}(t_{ij}, t_{i-1}+u_k h) - f^{(2r-1)}(t_{ij}, u_k h)\} + O(h^{M+1}), \end{aligned} \tag{4.5}$$

where B_{2r}, $r=1, \dots, [M/2]$, are the Bernoulli numbers. By Taylor series expansion,

$$\int_0^{u_k h} f(t_{ij}, s)\,ds = \sum_{m=0}^{M-1} h^{m+1} \frac{f^{(m)}(t_{ij}, 0)}{(m+1)!}\, u_k^{m+1} + O(h^{M+1}), \tag{4.6}$$

$$\int_{t_{i-1}+u_k h}^{t_{ij}} f(t_{ij}, s)\,ds = -\sum_{m=0}^{M-1} h^{m+1} \frac{f^{(m)}(t_{ij}, t_{ij})}{(m+1)!}\,(u_k - u_j - 1)^{m+1} + O(h^{M+1}), \tag{4.7}$$

$$f^{(r)}(t_{ij}, u_k h) = \sum_{m=0}^{M-2r} h^m \frac{f^{(r+m)}(t_{ij}, 0)}{m!}\, u_k^m + O(h^{M-2r+1}), \tag{4.8}$$

and

$$f^{(r)}(t_{ij}, t_{i-1}+u_k h) = \sum_{m=0}^{M-2r} h^m \frac{f^{(r+m)}(t_{ij}, t_{ij})}{m!}\,(u_k - u_j - 1)^m + O(h^{M-2r+1}). \tag{4.9}$$

The result follows by substitution of (4.6), (4.7), (4.8) and (4.9) into (4.5). #

Lemma 4.5. *The scheme* (3.2) *is* 0*-stable in the sense of Stetter* [14].

Proof. The result follows in the same way as Lemma 4.2. #

Since from Lemma 4.1, R_{ij} has an asymptotic expansion in integral powers of h, it follows from the 0-stability of the scheme (3 2) and Stetter [14, Theorem 1, p. 21] that ε_{ij} also possesses such an expansion, viz. there exists a unique set of functions

$$\{e_{pj}(t) \in C^{\nu-p}[0, T],\ j=1, \dots, n;\ p=0, \dots, \nu\}$$

such that

$$\varepsilon_{ij} = \sum_{p=0}^{\nu} h^{n+p} e_{pj}(t_{ij}) + O(h^{n+\nu+1}), \quad j=1, \dots, n; \quad i=0, \dots, I-1. \tag{4.10}$$

In the following lemma, a recurrence relation for the functions $e_{pj}(t)$ is derived. Using this relation, it will be possible to obtain estimates for ε_{ij}, which are sharper than the bound given in Lemma 4.2.

Lemma 4.6. *If* $\nu > 1$, *then the functions* $e_{pj}(t)$ *satisfy*

$$\begin{aligned} &e_{0j}(t) = 0, \quad j=1, \dots, n, \\ &e_{pj}(t) = \varphi_{pj}(t) \\ &\quad + \sum_{m=0}^{p-1} \sum_{q=0}^{p-m-1} \binom{p-m-1}{q} \frac{k^{(p-m-q-1)}(t, t)}{(p-m-1)!} \sum_{k=1}^{n} a_{jk} e_{mk}^{(q)}(t)\,(u_k - u_j)^{p-m-1}, \\ &\qquad\qquad j=1, \dots, n; \quad p=1, \dots, \nu-1, \end{aligned} \tag{4.11}$$

and

$$\sum_{j=1}^{n} a_j u_j^r e_{pj}(t) = 0, \quad r = 0, \ldots, \nu - p - 1; \quad p = 0, \ldots, \nu - 1, \tag{4.12}$$

where

$$k(t, s) = \frac{\partial}{\partial \eta} K(t, s, \eta)\big|_{\eta = y(s)}, \qquad k^{(r)}(t, s) = \frac{\partial^r}{\partial \eta^r} k(t, \eta)\big|_{\eta = s}$$

and

$$e_{mk}^{(r)}(t) = \frac{d^r}{dt^r} e_{mk}(t).$$

Proof. From Taylor's theorem (4.1) becomes

$$\varepsilon_{ij} = \sum_{l=0}^{i-1} \sum_{k=1}^{n} h a_k k(t_{ij}, t_{lk}) \varepsilon_{lk} + \sum_{k=1}^{n} h a_{jk} k(t_{ij}, t_{ik}) \varepsilon_{ik} + R_{ij} + O(h^{2n}).$$

Substitution of (4.2) and (4.10) and division by h^n yields

$$\begin{aligned} \sum_{p=0}^{\nu} h^p e_{pj}(t_{ij}) = \sum_{p=0}^{\nu} h^p \sum_{k=1}^{n} \Big\{ a_k h \sum_{l=0}^{i-1} k(t_{ij}, t_{lk}) e_{pk}(t_{lk}) \\ + a_{jk} h k(t_{ij}, t_{ik}) e_{pk}(t_{ik}) \Big\} + \sum_{p=0}^{\nu} h^p \varphi_{pj}(t_{ij}) + O(h^{\nu+1}), \\ j = 1, \ldots, n; \quad i = 0, \ldots, I-1. \end{aligned} \tag{4.13}$$

From Lemma 4.4, with $f(t, s) = k(t, s) e_{pk}(s)$, there exists a set of functions $\{\Phi_{jkpm}(t),\ m = 0, \ldots, \nu - p\}$ such that

$$\begin{aligned} h \sum_{l=0}^{i-1} k(t_{ij}, t_{lk}) e_{pk}(t_{lk}) = \int_0^{t_{ij}} k(t_{ij}, s) e_{pk}(s)\, ds + \sum_{m=0}^{\nu-p-1} h^{m+1} \Phi_{jkpm}(t_{ij}) \\ + O(h^{\nu-p+1}), \quad j = 1, \ldots, n; \quad k = 1, \ldots, n; \quad p = 0, \ldots, \nu. \end{aligned} \tag{4.14}$$

Also, from Taylor series expansion,

$$\begin{aligned} k(t_{ij}, t_{ik}) e_{pk}(t_{ik}) = \sum_{m=0}^{\nu-p-1} h^m \frac{(u_k - u_j)^m}{m!} \frac{\partial^m}{\partial \eta^m} \big(k(t_{ij}, \eta) e_{pk}(\eta)\big)\big|_{\eta = t_{ij}} + O(h^{\nu-p}), \\ k = 1, \ldots, n; \quad p = 0, \ldots, \nu. \end{aligned} \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.13) and applying Leibnitz's rule gives

$$\begin{aligned} \sum_{p=0}^{\nu} h^p e_{pj}(t_{ij}) = \sum_{p=0}^{\nu} h^p \Big\{ \sum_{k=1}^{n} a_k \int_0^{t_{ij}} k(t_{ij}, s) e_{pk}(s)\, ds + \varphi_{pj}(t_{ij}) \Big\} \\ + \sum_{p=1}^{\nu} h^p \sum_{r=0}^{p-1} \sum_{k=1}^{n} \Big\{ a_k \varphi_{jkr,p-r-1}(t_{ij}) \\ + a_{jk} \sum_{q=0}^{p-r-1} \binom{p-r-1}{q} k^{(p-r-q-1)}(t_{ij}, t_{ij}) e_{rk}^{(q)}(t_{ij}) \frac{(u_k - u_j)^{p-r-1}}{(p-r-1)!} \Big\} + O(h^{\nu+1}), \\ j = 1, \ldots, n; \quad i = 0, \ldots, I-1. \end{aligned} \tag{4.16}$$

Clearly, from Lemma 4.2,

$$e_{0j}(t) = 0, \quad j = 1, \ldots, n.$$

We now consider the case $p = 1$. From Lemma 4.4,

$$\Phi_{jk00}(t) = 0, \quad j = 1, \ldots, n; \quad k = 1, \ldots, n.$$

Hence, equating coefficients of h in (4.16), we obtain

$$e_{1j}(t) = \int_0^t k(t, s) \sum_{k=1}^n a_k e_{1k}(s)\, ds + \varphi_{1j}(t), \quad j = 1, \ldots, n, \tag{4.17}$$

and since $\sum_{j=1}^n a_j = 1$, it follows that

$$\sum_{j=1}^n a_j e_{1j}(t) = \int_0^t k(t, s) \sum_{j=1}^n a_j e_{1j}(s)\, ds + \sum_{j=1}^n a_j \varphi_{1j}(t). \tag{4.18}$$

By Corollary 4.1, (4.18) is a homogeneous Volterra integral equation of the second kind and so

$$\sum_{j=1}^n a_j e_{1j}(t) = 0.$$

Hence, from (4.17)

$$e_{1j}(t) = \varphi_{1j}(t), \quad j = 1, \ldots, n,$$

and so Corollary 4.1 yields (4.12) for $p = 1$.

We now proceed by induction. Assume the lemma is true for $p = 1, \ldots, l-1 < \nu - 1$. Then from (4.12), (4.14) and Lemma 4.4,

$$\sum_{k=1}^n a_k \Phi_{jkr,\,l-r-1}(t) = 0, \quad j = 1, \ldots, n, \quad r = 0, \ldots, l-1.$$

Hence, equating coefficients of h^l in (4.16), we obtain

$$\begin{aligned} e_{lj}(t) = {} & \int_0^t k(t, s) \sum_{k=1}^n a_k e_{lk}(s)\, ds + \varphi_{lj}(t) \\ & + \sum_{r=0}^{l-1} \sum_{q=0}^{l-r-1} \binom{l-r-1}{q} \frac{k^{(l-r-q-1)}(t, t)}{(l-r-1)!} \sum_{k=1}^n a_{jk} e_{rk}^{(q)}(t)\, (u_k - u_j)^{l-r-1} \\ & \qquad j = 1, \ldots, n, \end{aligned} \tag{4.19}$$

and so,

$$\begin{aligned} \sum_{j=1}^n a_j e_{lj}(t) = {} & \int_0^t k(t, s) \sum_{j=1}^n a_j e_{lj}(s)\, ds + \sum_{j=1}^n a_j \varphi_{lj}(t) \\ & + \sum_{r=0}^{l-1} \sum_{q=0}^{l-r-1} \binom{l-r-1}{q} \frac{k^{(l-r-q-1)}(t, t)}{(l-r-1)!} \sum_{j=1}^n a_j \sum_{k=1}^n a_{jk} e_{rk}^{(q)}(t)\, (u_k - u_j)^{l-r-1}. \end{aligned} \tag{4.20}$$

From Corollary 4.1, (4.12) and Lemma 4.3, (4.20) is homogeneous and therefore

$$\sum_{j=1}^n a_j e_{lj}(t) = 0.$$

Thus, (4.19) yields (4.11) for $p=l$. From (4.12) with $p \leqq l-1$ and Lemma 4.3 we obtain (4.12) with $p=l$. Hence the lemma follows by induction. #

We are now in a position to prove the main convergence result.

Theorem 4.1. *If $w(t) \in \wp_\nu$, then*

$$\varepsilon_{in} = h^{n+\nu} e_{\nu n}(t_{i+1}) + O(h^{n+\nu+1}), \qquad i=0, \ldots, I-1, \tag{4.21}$$

where $e_{\nu n}(t)$ satisfies an equation of the form

$$e_{\nu n}(t) = \zeta_{\nu n}(t) + \int_0^t k(t,s)\, e_{\nu n}(s)\, ds. \tag{4.22}$$

Proof. For $\nu=0$ and $\nu=1$, (4.21) follows trivially from (4.10) and Lemma 4.2. If $\nu > 1$, then from Lemmas 4.1 and 4.6,

$$e_{pn}(t) = 0, \qquad p=0, \ldots, \nu-1,$$

and (4.21) again follows from (4.10).

Equating powers of h^ν in (4.16), we obtain equations of the form

$$e_{\nu j}(t) = \xi_{\nu j}(t) + \int_0^t k(t,s) \sum_{k=1}^n a_k e_{\nu k}(s)\, ds, \qquad j=1, \ldots, n, \tag{4.23}$$

and hence

$$\sum_{j=1}^n a_j e_{\nu j}(t) = \sum_{j=1}^n a_j \xi_{\nu j}(t) + \int_0^t k(t,s) \sum_{j=1}^n a_j e_{\nu j}(s)\, ds$$
$$= \sum_{j=1}^n a_j \xi_{\nu j}(t) - \xi_{\nu n}(t) + e_{\nu n}(t).$$

It follows that

$$e_{\nu n}(t) = \xi_{\nu n}(t) + \int_0^t k(t,s) \sum_{j=1}^n a_j \left(\xi_{\nu j}(s) - \xi_{\nu n}(s)\right) ds + \int_0^t k(t,s)\, e_{\nu n}(s)\, ds$$
$$= \zeta_{\nu n}(t) + \int_0^t k(t,s)\, e_{\nu n}(s)\, ds. \quad \#$$

From this theorem it is clear that the choice of the points $\{u_1, \ldots, u_n\}$ is important. A natural choice would appear to be the equally spaces points $u_i = (i-1)/(n-1)$, $i=1, \ldots, n$, $n \geqq 2$. For these points, $w(t) \in \wp_0$ for $n=2r$ and $w(t) \in \wp_1$ for $n=2r+1$ and so the methods with $n=2r+1$ and $n=2r+2$ have order $2r+1$ convergence.

More suitable sets of points however are those considered by Axelsson [1], namely the Radau points for $u_1 > 0$ and the Lobatto points for $u_1 = 0$. The corresponding orders of convergence are $2n-1$ and $2n-2$ respectively. This is optimal.

5. Numerical Stability

According to Noble [11] a finite difference method for (1.1) is *numerically stable* if the leading term in the asymptotic expansion of the error satisfies an equation whose kernel is the same as that of the linearised form of (1.1). For additional discussions of this concept see also Kobayasi [8] and Linz [10].

From Theorem 4.1 it is clear that the pure discretization error of (3.2) grows in a stable way. However, to investigate the numerical stability fully, it is necessary to consider the propagation of rounding errors which can be characterized by the propagation of perturbations in Y_{0j}, $j=1, \ldots, n$.

Suppose that in the first step approximations $\overline{Y}_{0j}$, $j=1, \ldots, n$, which satisfy

$$\overline{Y}_{0j}=Y_{0j}-\delta_j, \quad j=1, \ldots, n,$$

have been calculated instead of Y_{0j}, $j=1, \ldots, n$. Denote

$$\delta=\max_{j=1,\ldots,n} |\delta_j|.$$

Using the values $\overline{Y}_{0j}$, $j=1, \ldots, n$, (3.2) will generate a new sequence of approximations $\overline{Y}_{ij}$, $j=1, \ldots, n$, $i=1, \ldots, I-1$, given by

$$\overline{Y}_{ij}=\sum_{l=0}^{i-1}\sum_{k=1}^{n} h a_k K(t_{ij}, t_{lk}, \overline{Y}_{lk})+\sum_{k=1}^{n} h a_{jk} K(t_{ij}, t_{ik}, \overline{Y}_{ik}), \qquad (5.1)$$
$$j=1, \ldots, n; \quad i=1, \ldots, I-1.$$

Let

$$\bar{\varepsilon}_{ij}=y_{ij}-\overline{Y}_{ij}, \quad j=1, \ldots, n; \quad i=0, \ldots, I-1.$$

Then,

$$\bar{\varepsilon}_{0j}=\varepsilon_{0j}+\delta_j, \quad j=1, \ldots, n, \qquad (5.2)$$

where ε_{0j}, $j=1, \ldots, n$, are given by (4.1). By an argument similar to lemma 4.2 it follows that

$$\bar{\varepsilon}_{ij}=O(h^n)+O(h\delta), \quad j=1, \ldots, n; \quad i=1, \ldots, I-1. \qquad (5.3)$$

Subtraction of (5.1) from (3.1) and the use of Taylor's theorem, (5.2) and (5.3) yield

$$\bar{\varepsilon}_{ij}=\sum_{l=1}^{i-1}\sum_{k=1}^{n} h a_k k(t_{ij}, t_{lk})\,\bar{\varepsilon}_{lk}+\sum_{k=1}^{n} h a_{jk} k(t_{ij}, t_{ik})\,\bar{\varepsilon}_{ik}$$
$$+\sum_{k=1}^{n} h a_k k(t_{ij}, t_{0k})(\varepsilon_{0k}+\delta_k)+R_{ij}+O(h^{2n})+O(h^2\delta),$$
$$j=1, \ldots, n; \quad i=1, \ldots, I-1.$$

Hence, by superposition,

$$\bar{\varepsilon}_{ij}=\varepsilon_{ij}+\hat{\varepsilon}_{ij}, \quad j=1, \ldots, n; \quad i=0, \ldots, I-1,$$

where

$$\hat{\varepsilon}_{0j}=\delta_j, \quad j=1, \ldots, n,$$
$$\hat{\varepsilon}_{ij}=\sum_{l=1}^{i-1}\sum_{k=1}^{n} h a_k k(t_{ij}, t_{lk})\,\hat{\varepsilon}_{lk}+\sum_{k=1}^{n} h a_{jk} k(t_{ij}, t_{ik})\,\hat{\varepsilon}_{ik}$$
$$+\sum_{k=1}^{n} h a_k k(t_{ij}, t_{0k})\,\delta_k+O(h^{2n})+O(h^2\delta), \qquad (5.4)$$
$$j=1, \ldots, n; \quad i=1, \ldots, I-1,$$

and ε_{ij}, $j=1,\ldots,n$; $i=0,\ldots,I-1$, is the pure discretization error given by (4.1). By Lemma 2.3,

$$\hat{\varepsilon}_{ij}=O(h\delta)+O(h^{2n}), \qquad j=1,\ldots,n; \qquad i=1,\ldots,I-1.$$

Let $e_{ij}=\hat{\varepsilon}_{ij}/h$, $j=1,\ldots,n$; $i=1,\ldots,I-1$. Then, from (5.4), e_{ij} satisfy an equation which can be interpreted as a finite difference method applied to the system of integral equations

$$e_j(t)=\int_0^t k(t,s)\sum_{k=1}^{n} a_k e_k(s)\,ds+k(t,0)\sum_{k=1}^{n} a_k\delta_k, \qquad j=1,\ldots,n. \tag{5.5}$$

Using Lemma 2.3, this method can easily be shown to be convergent of order one, and hence

$$e_{ij}=e_j(t_{ij})+O(h^{2n-1})+O(h\delta), \qquad j=1,\ldots,n; \qquad i=1,\ldots,I-1.$$

From (5.5),

$$e_j(t)=e(t), \qquad j=1,\ldots,n,$$

where

$$e(t)=\int_0^t k(t,s)e(s)\,ds+k(t,0)\sum_{k=1}^{n} a_k\delta_k$$

and it follows that

$$\hat{\varepsilon}_{ij}=he(t_{ij})+O(h^{2n})+O(h^2\delta), \qquad j=1,\ldots,n; \qquad i=1,\ldots,I-1,$$

which implies that the scheme (3.2) is numerically stable.

6. A-Stability and Stiff A-Stability

The definitions of A-stability and stiff A-stability for methods for ordinary differential equations given by Dahlquist [4] and Axelsson [1], respectively, can be extended to methods for second kind Volterra integral equations in the following way.

A numerical method is called *A-stable* if, when applied to the problem

$$y(t)=1+\lambda\int_0^t y(s)\,ds, \qquad \mathrm{Re}(\lambda)<0, \tag{6.1}$$

with an arbitrary step size h, then

$$\lim_{\substack{i\to\infty\\ h\text{ fixed}}} Y_i=0,$$

where Y_i denotes the numerical approximation to $y(t_i)$. If, in addition,

$$\lim_{\substack{h\to\infty\\ i\text{ fixed}}} Y_i=0 \quad \text{for all} \quad i,$$

then the method is called *stiffly A-stable.*

To examine A-stability and stiff A-stability for our schemes, we use the observation that the schemes applied to (6.1) yield the same numerical approximations as the corresponding implicit Runge-Kutta methods for ordinary differential equations (see Axelsson [1] and Wright [16]) applied to

$$y' = \lambda y, \qquad y(0) = 1.$$

Hence we know from Wright [16] that the schemes based on equally spaced points are A-stable if $n \leqq 9$, and from Axelsson [1] that the schemes using Lobatto and Radau quadrature are A-stable and stiffly A-stable, respectively, for all n.

7. Numerical Results

In this section we report some calculations with the schemes (3.2) and (3.3) based on Radau points with degree of precision four ($\nu = 2$), i.e.

$$u_1 = \left(1 - \tfrac{1}{5}\left(1 + \sqrt{6}\right)\right)/2, \qquad u_2 = \left(1 + \tfrac{1}{5}\left(\sqrt{6} - 1\right)\right)/2, \qquad u_3 = 1 \tag{7.1}$$

The fifth order convergence is illustrated by the application of the methods to

$$y(t) = 1 + t - \cos(t) - \int_0^t \cos(t-s)\, y(s)\, ds, \qquad 0 \leqq t \leqq 2,$$

which has the solution $y(t) = t$. The errors for (3.2) and (3.3) are tabulated in Tables 7.1 and 7.2, respectively. It should be noted that the errors for (3.3) are appreciably smaller than those for (3.2). Numerical computations show that this is frequently the case.

The advantage of stiff A-stability is illustrated by the application of the methods to

$$y(t) = \left((1+t)\exp(-10t) + 1\right)^{\frac{1}{2}} + (1+t)\left((1 - \exp(-10)\right) + 10\log(1+t)$$
$$-10 \int_0^t \frac{1+t}{1+s}\, y(s)^2\, ds, \qquad 0 \leqq t \leqq 19, \tag{7.2}$$

Table 7.1

t	$h = 0.4$	$h = 0.2$	$h = 0.1$
0.4	$-2.396\ E-6$	$-7.446\ E-8$	$-2.316\ E-9$
0.8	$-4.608\ E-6$	$-1.446\ E-7$	$-4.519\ E-9$
1.2	$-6.391\ E-6$	$-2.022\ E-7$	$-6.349\ E-9$
1.6	$-7.638\ E-6$	$-2.437\ E-7$	$-7.682\ E-9$
2.0	$-8.347\ E-6$	$-2.685\ E-7$	$-8.498\ E-9$

Table 7.2

t	$h = 0.4$	$h = 0.2$	$h = 0.1$
0.4	$2.647\ E-7$	$7.704\ E-9$	$2.323\ E-10$
0.8	$4.165\ E-7$	$1.212\ E-8$	$3.655\ E-10$
1.2	$4.803\ E-7$	$1.398\ E-8$	$4.220\ E-10$
1.6	$4.843\ E-7$	$1.413\ E-8$	$4.271\ E-10$
2.0	$4.550\ E-7$	$1.334\ E-8$	$4.043\ E-10$

which has the solution

$$y(t) = \left((1+t)\exp(-10t)+1\right)^{\frac{1}{2}}.$$

The schemes were applied with $h=0.1$ on the interval $[0, 1]$ and then the step size was increased on $(1, 19)$. The errors for (3.2) and (3.3) are given in Tables (7.2) and (7.4) respectively.

Table 7.3

t	$h=1.5$	$h=3.0$
4.0	$-1.780\ E-3$	$-2.843\ E-2$
7.0	$-3.058\ E-4$	$-5.376\ E-3$
10.0	$-1.015\ E-4$	$-1.392\ E-3$
13.0	$-4.577\ E-5$	$-5.430\ E-4$
16.0	$-2.446\ E-5$	$-2.716\ E-4$
19.0	$-1.458\ E-5$	$-1.558\ E-4$

Table 7.4

t	$h=1.5$	$h=3.0$
4.0	$-3.271\ E-5$	$-2.302\ E-3$
7.0	$-1.717\ E-6$	$-1.834\ E-4$
10.0	$-2.699\ E-7$	$-1.952\ E-5$
13.0	$-7.191\ E-8$	$-3.757\ E-6$
16.0	$-2.452\ E-8$	$-1.150\ E-6$
19.0	$-1.077\ E-8$	$-4.554\ E-7$

Since the Lipschitz constant in (7.2) is effectively 20, a conventional multistep method will not work well for a large stepsize. To illustrate this, (7.2) was solved by the two-step scheme Simpson #1, which is convergent of order four and numerically stable, (see Linz [10] or Noble [11]). On $[0, 1]$, $h=1/30$ in Simpson #1 yields comparable accuracy to $h=0.1$ for (3.3). On $(1, 19)$ the largest gridspacing for which the error in Simpson #1 does not exhibit unstable growth is approximately $h=0.09$. The accuracy with this h is about the same as for (3.3) with $h=1.5$.

To compare the efficiency of (3.2) (or (3.3)) with Simpson #1, note that when solving Volterra integral equations the bulk of the computations comes from approximating integrals of the form $\int_0^{t_j} K(t_j, s, y(s))\,ds$ by sums. Thus the number of necessary evaluations of $K(t, s, y)$, $N(h, T)$, can serve as measure for the amount of work required. For (3.2) (and (3.3)) with points (7.1), $N(h, T) \approx (9(T/h)^2/2)$, while for Simpson #1, $N(h, T) \approx (T/h)^2/2$. Computing these figures for our situation ($T=18$, $h=1.5$ for (3.2) and $h=0.9$ for Simpson #1) we see that the number of evaluations in Simpson #1 is about 30 times that for (3.2). This illustrates well the superiority of stiffly A-stable schemes over linear multistep methods when solving "stiff" equations.

To solve the nonlinear systems arising from (3.2) and (3.3), Newton iteration was used. Although convergence of the iteration can only be established if $hL < 1$, when solving stiff equations, convergence is usually observed even if $hL \gg 1$. This is the same situation as in differential equations.

The nonlinear equations arising in Simpson #1 were also solved by Newton iteration.

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