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A Particle Method for First-order Symmetric Systems*

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Summary. We present and study a conservative particle method of approximation of linear hyperbolic and parabolic systems. This method is based on an extensive use of cut-off functions. We prove its convergence in L^2 at the order $\varepsilon^2 + \frac{h^m}{\varepsilon^{m+1}}$ as soon as the cut-off function belongs to $W^{m+1,1}$.

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Introduction

Most of the partial differential equations arising in Sciences and Engineering are conveniently solved numerically by using classical discretization methods such as finite-difference, finite element or spectral methods. However, due to the growing complexity of problems which need numerical solutions, an increasing number of them are not efficiently solved by these conventional methods and require specially fitted numerical techniques. This is the case in particular of convection dominated complex problems for which the particle method is able to provide effective numerical solutions. In fact, the particle method is commonly used in some specific domains in Physics and in Fluid Mechanics. In Physics, kinetic equations of Boltzmann and Fokker-Planck types are currently solved by the particle method which is frequently associated with Monte-Carlo techniques. In that direction, see for instance Duderstadt and Martin [11, Chap. 9]. In plasma Physics and more specifically in the study of inertial confinement fusion problems, the particle method is used in the numerical solution of the coupled Vlasov-Poisson or Vlasov-Maxwell equations: see the recent books of Hockney and Eastwood [15] and Birdsall and Langdon [5]. For applications to the computer simulation of semi-conductor devices, see again [15].

* Dedicated to Professor Joachim Nitsche on the occasion of his 60th birthday

The particle method is also used in Fluid Mechanics for both compressible and incompressible fluid flow simulations. On the one hand, vortex methods of solutions of the two- and three-dimensional incompressible Euler and Navier-Stokes equations are of growing practical importance: see for instance the survey papers of Leonard [16, 17] and the references theirein. On the other hand, for compressible multifluid flows, a particle-in-cell (P.I.C.) method has been introduced by Harlow [14] using a particle treatment of the convective terms coupled with a finite-difference treatment of the pressure terms. Recently, Gingold and Monaghan [12] have modified the P.I.C. method by deriving a pure particle treatment of the pressure terms.

The numerical analysis of the particle method has received a great deal of attention in the last few years. Since the pioneering work of Hald [13] on the convergence of the two-dimensional vortex method, many results have been obtained in that direction, see the papers of Anderson and Greengard [1], Beale [2], Beale and Majda [3, 4], Cottet [6, 7, 8], Raviart [19]. By using related mathematical techniques, Cottet and Raviart [9, 10] have studied the particle approximation of the one-dimensional Vlasov-Poisson equations.

Now, the purpose of this paper is to provide a mathematical analysis of a particle method of approximation of linear first-order systems closely related to the method of Gingold and Monaghan [12]. The analysis presented here extends previous works concerning the particle approximation of linear hyperbolic equations [19, 20]. An outline of the paper is as follows. Section 1 is devoted to the derivation of the particle method. We state in Sect. 2 the main result of convergence. The consistency analysis of the method is based on a theory of approximation of smooth functions by linear combinations of Dirac measures. This theory is given in Sect. 3. We derive in Sect. 4 a L^2 stability analysis of the method and we prove the main theorem. Finally, in Sect. 5, we extend the method and the analysis to the numerical approximation of parabolic systems.

In a subsequent paper [18], we shall present a somewhat different but related particle method of approximation of linear convection-diffusion problems.

1. Description of the Particle Method

Let $x=(x^1,\dots,x^n)\in\mathbb{R}^n$ be the space variables and t the time variable. Given $T>0$, we set:

$$Q_T=\mathbb{R}^n\times\,]0,T[$$

We denote by $\mathscr{L}(\mathbb{R}^p)$ the space of $p\times p$ matrices with real coefficients.

We introduce $(n+1)$ mappings A^i: $(x,t)\in Q_T\to A^i(x,t)\in\mathscr{L}(\mathbb{R}^p)$, $0\leqq i\leqq n$, with the following properties:

$$\begin{aligned}
&\text{(i)}\ A^i\in L^\infty(Q_T;\mathscr{L}(\mathbb{R}^p)), \qquad 0\leqq i\leqq n;\\
&\text{(ii)}\ \frac{\partial A^i}{\partial x^j}\in L^\infty(Q_T;\mathscr{L}(\mathbb{R}^p)), \qquad 1\leqq i,j\leqq n;\\
&\text{(iii)}\ A^i(x,t)=A^i(x,t)^T, \qquad 1\leqq i\leqq n,\ (x,t)\in Q_T.
\end{aligned} \tag{1.1}$$

Then, given two functions $u_0: x\in\mathbb{R}^n \to u_0(x)\in\mathbb{R}^p$ and $f: (x, t)\in Q_T \to f(x, t)\in\mathbb{R}^p$, we want to find a function $u: (x, t)\in Q_T \to u(x, t)\in\mathbb{R}^p$ solution of the first-order symmetric system written in *conservation form*

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n}\frac{\partial}{\partial x^i}(A^i u)+A^0 u=f \quad \text{in } Q_T \tag{1.2}$$

with the initial condition

$$u(x, 0)=u_0(x), \qquad x\in\mathbb{R}^n. \tag{1.3}$$

Now, setting

$$\mathbb{L}^2(\mathbb{R}^n)=L^2(\mathbb{R}^n)^p,$$

it is well known that Problem (1.2), (1.3) is well posed in $\mathbb{L}^2(\mathbb{R}^n)$: if $u_0\in\mathbb{L}^2(\mathbb{R}^n)$ and $f\in L^1(0, T; \mathbb{L}^2(\mathbb{R}^n))$. Problem (1.2), (1.3) has a unique weak solution $u\in C^0(0, T; \mathbb{L}^2(\mathbb{R}^n))$, i.e. u is continuous from $[0, T]$ into $\mathbb{L}^2(\mathbb{R}^n)$. We shall assume in all the sequel that the data A^i, $0\leqq i\leqq n$, u_0 and f are smooth enough so that the solution u satisfies the regularity properties that we shall require later on.

In order to approximate the solution u of (1.2), (1.3) by a particle method, we begin by introducing a system of moving coordinates. We write

$$A^i=a^i I+B^i, \qquad 1\leqq i\leqq n, \tag{1.4}$$

where I is the identity matrix of $\mathscr{L}(\mathbb{R}^n)$ and the functions a^i are realvalued functions defined on Q_T which satisfy

$$a^i\in L^\infty(Q_T), \qquad \frac{\partial a^i}{\partial x^j}\in L^\infty(Q_T), \qquad 1\leqq i,\ j\leqq n. \tag{1.5}$$

Then, we define the characteristic curves associated with the first order differential operator

$$\frac{\partial}{\partial t}+\sum_{j=1}^{n}a^i\frac{\partial}{\partial x^i}.$$

Consider the differential system

$$\frac{dx}{dt}=a(x, t), \qquad a=(a^1, \ldots, a^n). \tag{1.6}$$

We denote by $t\to x(\xi, t)$ the unique solution of (1.6) which satisfies the unitial condition

$$x(0)=\xi, \qquad \xi\in\mathbb{R}^n \tag{1.7}$$

and we set

$$J(\xi, t)=\det\left(\frac{\partial x^i}{\partial \xi^j}(\xi, t)\right). \tag{1.8}$$

Then, it is a simple and classical matter to check that

$$\frac{\partial J}{\partial t}(\xi, t)=J(\xi, t)(\operatorname{diva})(x(\xi, t), t), \qquad \operatorname{diva}=\sum_{i=1}^{n}\frac{\partial a^i}{\partial x^i} \tag{1.9}$$

Moreover, since $J(\xi, 0)=1$, we have

$$J(\xi, t)>0 \quad \forall \xi \in \mathbb{R}^n, \quad \forall t \in [0, T].$$

Note that (ξ, t) may be viewed as a system of Lagrangean coordinates associated with the "velocity" vector field $a=(a^1, \ldots, a^n)$. Hence, $t \to x(\xi, t)$ is the trajectory in the velocity field a of a material particle whose initial position is ξ.

The next step consists in deriving a general approximation of a continuous function by a linear combination of Dirac measures. Let $g \in C^0(\mathbb{R}^n)$ and let $\varphi \in C_0^0(\mathbb{R}^n)$, i.e., φ is a continuous function with compact support. By using the change of variables $x=x(\xi, t)$, we have

$$\int_{\mathbb{R}^n} g\varphi\, dx = \int_{\mathbb{R}^n} g(x(\xi, t))\, \varphi(x(\xi, t))\, J(\xi, t)\, d\xi.$$

Now, if we approximate the integral

$$\int_{\mathbb{R}^n} \psi(\xi)\, d\xi \quad \text{by} \quad \sum_{k \in K} \omega_k\, \psi(\xi_k)$$

for some set $(\xi_k, \omega_k)_{k \in K}$ of points $\xi_k \in \mathbb{R}^n$ and weights $\omega_k \in \mathbb{R}$, we obtain

$$\int_{\mathbb{R}^n} g\varphi\, dx \simeq \sum_{k \in K} w_k(t)\, g(x_k(t))\, \varphi(x_k(t))$$

where

$$x_k(t)=x(\xi_k, t), \qquad w_k(t)=\omega_k\, J(\xi_k, t), \qquad k \in K. \tag{1.10}$$

This amounts to approximate the function $g \in C^0(\mathbb{R}^n)$ by the following measure

$$\Pi^h(t)\, g = \sum_{k \in K} w_k(t)\, g(x_k(t))\, \delta(x - x_k(t)) \tag{1.11}$$

where $\delta(x-x_0)$ means the Dirac measure located at the point $x_0 \in \mathbb{R}^n$ and the subscript h refers to a discretization parameter to be specified.

Later on, it will be useful to associate with the measure $\Pi^h(t)g$ a continuous functions $\Pi_\varepsilon^h(t)g$ which approximates the function g in a more classical sense. In order to construct $\Pi_\varepsilon^h(t)g$, we first introduce a cut-off function $\zeta \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \zeta\, dx = 1.$$

For the sake of simplicity, we shall assume in all the sequel that the function ζ has a *compact support*. Next, we set for all $\varepsilon > 0$

$$\zeta_\varepsilon(x) = \frac{1}{\varepsilon^n}\, \zeta\left(\frac{x}{\varepsilon}\right) \tag{1.12}$$

and

$$\Pi_\varepsilon^h(t)\, g = \Pi^h(t)\, g * \zeta_\varepsilon$$

or equivalently

$$(\Pi_\varepsilon^h(t) g)(x) = \sum_{k \in K} w_k(t)\, g(x_k(t))\, \zeta_\varepsilon(x - x_k(t)). \tag{1.13}$$

We are now looking for a *particle approximation* u^h of the solution of Problem (1.2), (1.3) of the form

$$u^h(x,t)=\sum_{k\in K} w_k(t)\,u_k(t)\,\delta(x-x_k(t)) \tag{1.14}$$

where $u_k(t)$ stands for an approximation of $u(x_k(t),t)$. We consider also the regularized form u^h_ε of u^h given by

$$u^h_\varepsilon(.,t)=u^h(.,t)*\zeta_\varepsilon$$

i.e.,

$$u^h_\varepsilon(x,t)=\sum_{k\in K} w_k(t)\,u_k(t)\,\zeta_\varepsilon(x-x_k(t)). \tag{1.15}$$

In other words (1.15) consists in approximating the solution u by point particles which move along the characteristic curves passing through the points ξ_k, $k\in K$. Similarly, (1.15) amounts to approximate u by finite size particles, the motion of each particle being identical to that of its centroid $x_j(t)$ and its size being characterized by the function ζ_ε.

It remains to derive a discretized form of Problem (1.2), (1.3) in order to define the unknown functions $t\to u_k(t)$, $k\in K$, and therefore the approximate solutions u^h and u^h_ε. Using, (1.4), Equation (1.2) becomes

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n}\frac{\partial}{\partial x^i}(a^i u)+\sum_{i=1}^{n}\frac{\partial}{\partial x^i}(B^i u)+A^0 u=f \quad \text{in } Q_T.$$

We first notice the simple but crucial following result

Lemma 1. *If u^h is given by* (1.14), *we have in the sense of distributions on Q_T*

$$\frac{\partial}{\partial t}u^h+\sum_{i=1}^{n}\frac{\partial}{\partial x^i}(a^i u^h)=\sum_{k\in K}\frac{d}{dt}(w_k(t)\,u_k(t))\,\delta(x-x_k(t)). \tag{1.16}$$

Proof. Denote by $\langle .,. \rangle$ the duality pairing between the space $C_0^\infty(Q_T)$ of all C^∞ functions with compact support in Q_T and the space $\mathscr{D}'(Q_T)$ of all distributions on Q_T. If we set

$$Lu^h=\frac{\partial}{\partial t}u^h+\sum_{i=1}^{n}\frac{\partial}{\partial x^i}(a^i u^h),$$

we have for all $\varphi\in C_0^\infty(Q_T)$

$$\begin{aligned}\langle Lu^h,\varphi\rangle &= -\left\langle u^h,\frac{\partial\varphi}{\partial t}+\sum_{i=1}^{n}a^i\frac{\partial\varphi}{\partial x^i}\right\rangle\\ &= -\int_0^T\left\{\sum_{k\in K} w_k(t)\,u_k(t)\left(\frac{\partial\varphi}{\partial t}+\sum_{i=1}^{n}a^i\frac{\partial\varphi}{\partial x^i}\right)(x_k(t),t)\right\}dt.\end{aligned}$$

But it follows from (1.6) that

$$\left(\frac{\partial\varphi}{\partial t}+\sum_{i=1}^{n}a^i\frac{\partial\varphi}{\partial x^i}\right)(x(\xi,t),t)=\frac{d}{dt}\varphi(x(\xi,t),t).$$

Hence, we obtain

$$\langle Lu^h, \varphi\rangle = -\sum_{k\in K}\int_0^T w_k(t)\,u_k(t)\,\frac{d}{dt}\,\varphi(x_k(t), t)\,dt$$
$$=\left\langle \sum_{k\in K}\frac{d}{dt}(w_k(t)\,u_k(t))\,\delta(x-x_k(t)), \varphi\right\rangle$$

and the conclusion follows. □

Let us next derive a particle approximation of $\frac{\partial}{\partial x^i}(B^i u^h)$. We write

$$\frac{\partial}{\partial x^i}(B^i u^h) = \frac{\partial B^i}{\partial x^i}u^h + B^i\frac{\partial}{\partial x^i}u^h \cong \frac{\partial B^i}{\partial x^i}u^h + B^i\frac{\partial}{\partial x^i}u^h_\varepsilon$$

and we use the following approximations for B^i and $\frac{\partial B^i}{\partial x^i}$

$$B^i \simeq \sum_{k\in K} w_k(t)\,B^i_k(t)\,\delta(x-x_k(t)),$$
$$\frac{\partial B^i}{\partial x^i} \simeq \sum_{k\in K} w_k(t)\,B^i_k(t)\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x-x_k(t))$$

where

$$B^i_k(t) = B^i(x_k(t), t). \tag{1.17}$$

We find

$$\frac{\partial}{\partial x^i}(B^i u^h) \cong \left(\sum_{l\in K} w_l(t)\,B^i_l(t)\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x-x_l(t))\right)$$
$$\cdot\left(\sum_{k\in K} w_k(t)\,u_k(t)\,\delta(x-x_k(t))\right) + \left(\sum_{k\in K} w_k(t)\,B^i_k(t)\,\delta(x-x_k(t))\right)$$
$$\cdot\left(\sum_{l\in K} w_l(t)\,u_l(t)\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x-x_l(t))\right)$$

i.e.,

$$\frac{\partial}{\partial x^i}(B^i u^h) \cong \sum_{k,l\in K} w_k(t)\,w_l(t)(B^i_k(t)\,u_l(t) + B^i_l(t)\,u_k(t))$$
$$\cdot\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\,\delta(x-x_k(t)). \tag{1.18}$$

Finally, we have

$$A^0 u^h = \sum_{k\in K} w_k(t)\,A^0_k(t)\,u_k(t)\,\delta(x-x_k(t)) \tag{1.19}$$

where

$$A^0_k(t) = A^0(x_k(t), t) \tag{1.20}$$

and we consider the following approximation of f

$$f=\sum_{k\in K} w_k(t)\, f_k(t)\, \delta(x-x_k(t)) \tag{1.21}$$

where

$$f_k(t)=f(x_k(t), t). \tag{1.22}$$

Combining (1.16), (1.18), (1.19) and (1.21), we find that a (semi-)discretized form of Problem (1.2), (1.3) consists in finding functions $t\in[0, T]\to u_k(t)\in\mathbb{R}^p$, $k\in K$, solutions of the differential system

$$\frac{d}{dt}(w_k(t)\, u_k(t))+w_k(t)\left\{\sum_{l\in K} w_l(t) \sum_{i=1}^{n} (B_k^i(t)\, u_l(t) + B_l^i(t)\, u_k(t))\, \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right\}+w_k(t)\, A_k^0(t)\, u_k(t)=w_k(t)\, f_k(t), \tag{1.23}$$

$$u_k(0)=u_0(\xi_k), \qquad k\in K. \tag{1.24}$$

On the other hand, using (1.6), (1.7), (1.9) and (1.10), we note that the functions $t\to x_k(t)$ and $t\to w_k(t)$ can be characterized as the solutions of the differential equations

$$\frac{d x_k}{dt}(t)=a(x_k(t), t), \qquad x_k(0)=\xi_k, \tag{1.25}$$

and

$$\frac{d w_k}{dt}(t)=w_k(t)(\operatorname{div} a)(x_k(t), t), \qquad w_k(0)=w_k. \tag{1.26}$$

The numerical method is thus defined by the Eqs. (1.23)–1.26). It remains however to perform a suitable time – discretization in order to obtain a practically implementable numerical scheme.

Remark that any solution $u\in C^0(0, T; \mathbb{L}^1(\mathbb{R}^n))$ of (1.2) satisfies the following conservation property.

$$\int_{\mathbb{R}^n} u(x, t)\, dx+\int_0^t \int_{\mathbb{R}^n} A^0(x, s)\, u(x, s)\, dx\, ds$$

$$\int_{\mathbb{R}^n} u(x, 0)\, dx+\int_0^t \int_{\mathbb{R}^n} f(x, s)\, dx\, ds.$$

It is often required in practice that a similar property holds for any approximate solution (1.2). In fact, we shall say that the numerical scheme (1.23) satisfies the conservation property if

$$\begin{aligned}&\sum_{k\in K} w_k(t)\, u_k(t)+\int_0^t \sum_{k\in K} w_k(s)\, A_k^0(s)\, u_k(s)\, ds\\ &\quad=\sum_{k\in K} w_k(0)\, u_k(0)+\int_0^t \sum_{k\in K} w_k(s)\, f(s)\, ds\end{aligned} \tag{1.27}$$

holds for any solution $t\to(u_k(t))_{k\in K}\in C^0(0, T; l^1(K)^p)$ of (1.23).

Lemma 2. *Assume that the function* ζ *is even, i.e.*

$$\zeta(-x)=\zeta(x) \qquad \forall x\in\mathbb{R}^n \tag{1.28}$$

Then, the numerical scheme (1.23) *satisfies the conservation property.*

Proof. It follows from (1.28) that

$$\frac{\partial\zeta_\varepsilon}{\partial x^i}(-x)=-\frac{\partial\zeta_\varepsilon}{\partial x^i}(x). \tag{1.29}$$

Hence, by interchanging the roles of k and l, we obtain

$$\sum_{k,\,l\in K} w_k(t)\,w_l(t)(B_k^i(t)\,u_l(t)+B_l^i(t)\,u_k(t))\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))=0.$$

The conclusion follows at once by summing the Eq. (1.23) with respect to $k\in K$ and integrating from 0 to t. □

Remark 1. It would seem more natural at first glance to use in (1.23) the following discretization of $\frac{\partial}{\partial x^i}(B^i u)(x_k(t),t)$:

$$\sum_{l\in K} w_l(t)\,B_l^i(t)\,u_l(t)\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t)).$$

However the corresponding scheme does not satisfy the conservation property.

On the other hand, there exist other discretizations of $\frac{\partial}{\partial x^i}(B^i u)(x_k(t),t)$ which lead to a conservative scheme when the condition (1.28) holds, for instance

$$\sum_{l\in K} w_l(t)(B_l^i(t)\,u_l(t)+B_k^i(t)\,u_k(t))\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t)).$$

All these conservative or non conservative schemes can be studied in exactly the same way than the scheme (1.23). □

2. The Main Result

In order to present a *simple* analysis of the convergence of the particle method, we shall restrict ourselves in all the sequel to the following *model* situation. Given a discretization parameter $h>0$, we set:

$$K=\mathbb{Z}^n; \qquad \xi_k=(k_i h)_{1\leq i\leq n}, \qquad \omega_k=h^n\,\forall k=(k_1,\dots k_n)\in\mathbb{Z}^n.$$

Hence, the initial positions ξ_k of the particles are uniformly distributed in the space $\mathbb{R}^n$. Note that, in many applications, it may appear more adequate to choose the set $(\xi_k,\omega_k)_{k\in K}$ in a more sophisticated way depending on the initial condition u_0 but this leads to non essential technicalities in the proofs.

It will be also convenient to assume that the cut-off function ζ is sufficiently smooth. The case of a non-smooth cut-off will be discussed in the appendix.

In order to prove that Problem (1.23), (1.24) has a unique solution, we first introduce the space $\ell^2(\mathbb{Z}^n)$ of sequences $v=(v_k)_{k\in\mathbb{Z}^n}$ with values in $\mathbb{R}^p$ such that

$$\|v\|=\Big(\sum_{k\in\mathbb{Z}^n}|v_k|^2\Big)^{1/2}<+\infty, \tag{2.2}$$

where $|.|$ denotes the Euclidean norm of $\mathbb{R}^p$. We provide also $\ell^2(\mathbb{Z}^n)$ with the following time-dependent norm

$$\|v\|_{h,t}=\Big(\sum_{k\in\mathbb{Z}^n}w_k(t)|v_k|^2\Big)^{1/2}=\Big(h^n\sum_{k\in\mathbb{Z}^n}J(\xi_k,t)|v_k|^2\Big)^{1/2}. \tag{2.3}$$

Next, we state some simple preliminary results. In all the sequel of the paper, we shall denote by $C, c_1, \ldots, c_i, \ldots$ various positive constants independent of the parameters h and ε.

Let us begin with a standard result.

Lemma 3. *Assume that the hypothesis* (1.5) *holds. Then, there exists a constant* $C=C(T)>0$ *such that*

$$\exp(-Ct)\leqq J(\xi,t)\leqq\exp(Ct), \tag{2.4}$$

$$C^{-1}|\xi-\eta|\leqq|x(\xi,t)-x(\eta,t)|\leqq C|\xi-\eta| \tag{2.5}$$

for all $\xi,\eta\in\mathbb{R}^n$ *and all* $t\in[0,T]$.

As a consequence of (2.4), we obtain that $v\to\|v\|_{h,t}$ is indeed a norm on $\ell^2(\mathbb{Z}^n)$ which is equivalent (but not uniformly equivalent with respect to h) to the usual ℓ^2 norm (2.2).

Lemma 4. *Under the condition* (1.5), *there exists a constant* $C=C(T)>0$ *such that for all* $k\in\mathbb{Z}^n$

$$\sum_{l\in\mathbb{Z}^n}w_l(t)\left|\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right|\leqq\frac{C}{\varepsilon}. \tag{2.6}$$

Proof. Since the cut-off function ζ has a compact support, we observe that

$$\operatorname{meas}(\operatorname{supp}(\zeta_\varepsilon))\leqq c_1\,\varepsilon^n.$$

Now, let $x\in\mathbb{R}^n$ be fixed; using (2.5), we find that the number of indices $l\in\mathbb{Z}^n$ such that $x-x_l(t)$ belongs to $\operatorname{supp}(\zeta_\varepsilon)$ is bounded by $c_2\left(\frac{\varepsilon}{h}\right)^n$.

$$\left|\frac{\partial\zeta_\varepsilon}{\partial x^i}(x)\right|\leqq\frac{c_3}{\varepsilon^{n+1}}$$

that

$$\sum_{l\in\mathbb{Z}^n}w_l(t)\left|\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right|$$
$$=h^n\sum_{l\in\mathbb{Z}^n}J(\xi_l,t)\left|\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right|\leqq c_4\,h^n\left(\frac{\varepsilon}{h}\right)^n\frac{1}{\varepsilon^{n+1}}=\frac{c_4}{\varepsilon}.$$

Setting

$$\bar u_0=(u_0(\xi_k))_{k\in\mathbb{Z}^n},\ \bar f(t)=(f_k(t))_{k\in\mathbb{Z}^n}$$

we can now prove.

Theorem 1. *Assume that* (1.5) *holds and the cut-off function* ζ *belongs to the space* $C_0^1(\mathbb{R}^n)$. *Then, under the conditions*

$$\|\bar{u}_0\|_{h,0} < +\infty, \tag{2.7}$$

$$\int_0^T \|\bar{f}(t)\|_{h,t}\, dt < +\infty \tag{2.8}$$

Problem (1.23), (1.24) *has a unique solution* $t \to \bar{u}(t) = (u_k(t))_{k \in \mathbb{Z}^n}$ *which is continuous from* $[0, T]$ *into* $\ell^2(\mathbb{Z}^n)$.

Proof. Using (1.26), Eq. (1.23) can be equivalently written in the form

$$\frac{d}{dt} u_k(t) + \sum_{l \in K} w_l(t) \sum_{i=1}^n (B_k^i(t)\, u_l(t) + B_l^i(t)\, u_k(t)) \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t) - x_l(t))$$
$$+ \tilde{A}_k^0(t)\, u_k(t) = f_k(t)$$

where

$$\tilde{A}_k^0(t) = A_k^0(t) + \frac{1}{w_k(t)} \frac{d w_k(t)}{dt} = A_k^0(t) + (\operatorname{div} a)(x_k(t), t).$$

For all $t \in [0, T]$, we introduce the linear operator $\phi(t)$: $v \in \ell^2(\mathbb{Z}^n) \to \phi(t) v \in \ell^2(\mathbb{Z}^n)$ defined by

$$(\phi(t) v)_k = \sum_{l \in K} w_l(t) \sum_{i=1}^n (B_k^i(t)\, v_l + B_l^i(t)\, v_k) \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t) - x_l(t)) + \tilde{A}_k^0(t)\, v_k.$$

Let us check that $\phi(t)$ is indeed a linear continuous mapping. Using (1.1) (i), (1.5) and Lemma 4, we have

$$|(\phi(t) v)_k| \leqq c_1 \sum_{i=1}^n \sum_{l \in \mathbb{Z}^n} w_l(t) |v_l| \left| \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t) - x_l(t)) \right| + \left(\frac{c_2}{\varepsilon} + c_3 \right) |v_k|$$

and therefore

$$\|\phi(t) v\|_{h,t} \leqq c_1 \sum_{i=1}^n$$
$$\cdot \left\{ \sum_{k \in \mathbb{Z}^n} w_k(t) \left(\sum_{l \in \mathbb{Z}^n} w_l(t) |v_l| \left| \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t) - x_l(t)) \right| \right)^2 \right\}^{1/2} + \left(\frac{c_2}{\varepsilon} + c_3 \right) \|v\|_{h,t}.$$

Next, applying Cauchy-Schwarz' inequality gives

$$\left(\sum_{l \in \mathbb{Z}^n} w_l(t) |v_l| \left| \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t) - x_l(t)) \right| \right)^2$$
$$\leqq \left(\sum_{l \in \mathbb{Z}^n} w_l(t) \left| \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t) - x_l(t)) \right| \right)$$
$$\cdot \left(\sum_{l \in \mathbb{Z}^n} w_l(t) |v_l|^2 \left| \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t) - x_l(t)) \right| \right)$$

so that by Lemma 4 again

$$\sum_{k\in\mathbb{Z}^n} w_k(t)\left(\sum_{l\in\mathbb{Z}^n} w_l(t)|v_l|\left|\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right|\right)^2$$

$$\leqq\frac{c_4}{\varepsilon}\sum_{k,l\in\mathbb{Z}^n} w_k(t)\,w_l(t)|v_l|^2\left|\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right|$$

$$\leqq\frac{c_4^2}{\varepsilon^2}\sum_{l\in\mathbb{Z}^n} w_l(t)|v_l(t)|^2=\frac{c_4^2}{\varepsilon^2}\|v\|_{h,t}^2.$$

Hence, we obtain

$$\|\phi(t)v\|_{h,t}\leqq c_5\left(\frac{1}{\varepsilon}+1\right)\|v\|_{h,t}$$

Now, Problem (1.23), (1.24) can be equivalently written in the form

$$\frac{d\bar{u}}{dt}(t)+\phi(t)\,\bar{u}(t)=\bar{f}(t),\qquad 0\leqq t\leqq T\qquad \bar{u}(0)=\bar{u}_0 \tag{2.9}$$

Using (2.7), (2.8) and standard results in differential equations theory, we obtain that the linear differential problem (2.9) in $\ell^2(\mathbb{Z}^n)$ has a unique solution $t\to\bar{u}(t)\in C^0(0,T;\ell^2(\mathbb{Z}^n))$.

After having obtained a particle approximation of the original problem (1.2), (1.3), we then construct the function u^h defined by (1.15). Let us check that the function $t\to u_\varepsilon^h(.,t)$ belongs to $C^0(0,T;\mathbb{L}^2(\mathbb{R}^n))$. This will be an immediate consequence of

Lemma 5. *The mapping*

$$v=(v_k)_{k\in K}\to v_\varepsilon^h(x,t)=\sum_{k\in\mathbb{Z}^n} w_k(t)\,v_k\,\zeta_\varepsilon(x-x_k(t))$$

is continuous from $\ell^2(\mathbb{Z}^n)$ into $\mathbb{L}^2(\mathbb{R}^n)$ and there exists a constant $C=C(T)>0$ such that

$$\|v_\varepsilon^h(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)}\leqq C\|v\|_{h,t},\qquad 0\leqq t\leqq T. \tag{2.10}$$

Proof. Applying Cauchy-Schwarz' inequality gives

$$\Big|\sum_{k\in\mathbb{Z}^n} w_k(t)\,v_k\,\zeta_\varepsilon(x-x_k(t))\Big|^2$$

$$\leqq\Big(\sum_{k\in\mathbb{Z}^n} w_k(t)|\zeta_\varepsilon(x-x_k(t))|^2\Big)\Big(\sum_{k\in\mathbb{Z}^n} w_k(t)|v_k|^2\Big)$$

so that

$$\|v_\varepsilon^h(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)}^2\leqq\Big(\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^n} w_k(t)|\zeta_\varepsilon(x-x_k(t))|^2\,dx\Big)\|v\|_{h,t}^2.$$

Now, in the sum

$$\sum_{k\in\mathbb{Z}^n} w_k(t)|\zeta_\varepsilon(x-x_k(t))|^2,$$

we have to take into account only the indices $k \in \mathbb{Z}^n$ such that $x - x_k(t)$ belongs to $\mathrm{supp}(\zeta_\varepsilon)$. Argueing as in the proof of Lemma 4, we find that the number of such indices is bounded by $c_1 \left(\frac{\varepsilon}{h}\right)^n$. Hence, we obtain

$$\| v_\varepsilon^h(.,t)\|^2_{\mathbb{L}^2(\mathbb{R}^n)} \leq c_1 \left(\frac{\varepsilon}{h}\right)^n \sup_{k\in\mathbb{Z}^n} w_k(t) \left(\int_{\mathbb{R}^n} |\zeta_\varepsilon(x)|^2\, dx\right) \| v \|^2_{h,t}$$

and the result follows from the bounds

$$w_k(t) = h^n J(\xi_k, t) \leq c_2 h^n,$$

$$\int_{\mathbb{R}^n} |\zeta_\varepsilon(x)|^2\, dx = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} |\zeta(y)|^2\, dy \leq \frac{c_3}{\varepsilon^n}.$$

Now, we want to compare $u_\varepsilon^h(.,t)$ and $u(.,t)$ in $\mathbb{L}^2(\mathbb{R}^n)$. This is done in the next theorem which is the main result of this paper. Before stating the theorem, we need to introduce the standard Sobolev spaces

$$W^{k,p}(\Omega) = \left\{ \varphi \in L^p(\Omega);\ \partial^\alpha \varphi = \frac{\partial^{|\alpha|}\varphi}{(\partial x^1)^{\alpha_1} \dots (\partial x^n)^{\alpha_n}} \in L^p(\Omega),\ |\alpha| \leq k \right\}$$

where Ω is an open subset of $\mathbb{R}^n$. We provide $W^{k,p}(\Omega)$ with the norm

$$\| \varphi \|_{k,p,\Omega} = \left(\sum_{|\alpha| \leq k} \| \partial^\alpha \varphi \|^p_{L^p(\Omega)} \right)^{1/p}$$

and the semi-norm

$$| \varphi |_{k,p,\Omega} = \left(\sum_{|\alpha| = k} \| \partial^\alpha \varphi \|^p_{L^p(\Omega)} \right)^{1/p}$$

for $1 \leq p < \infty$ and their usual modification for $p = \infty$.

Theorem 2. *Assume that the cut-off function ζ satisfies the following hypotheses*

(i) *$\zeta \in C_0^1(\mathbb{R}^n)$ belongs to the space $W^{m+1,1}(\mathbb{R}^n)$ for some integer $m > n$;*

(ii) *there exists an integer $r \geq 1$ such that*

$$\begin{aligned} &\int_{\mathbb{R}^n} \zeta\, dx = 1, \\ &\int_{\mathbb{R}^n} x^\alpha \zeta\, dx = 0 \quad \forall \alpha \in \mathbb{N}^n \quad \textit{with } 1 \leq |\alpha| \leq r - 1\,[1]; \end{aligned} \tag{2.11}$$

(iii) *the condition* (1.28) *holds.*

Assume in addition that the parameters h and ε satisfy.

$$\frac{h^m}{\varepsilon^{m+1}} \leq c. \tag{2.12}$$

[1] $x^\alpha = (x^1)^{\alpha_1} \dots (x^n)^{\alpha_n}$ *if* $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, $\alpha = (a_1, \dots, a_n) \in \mathbb{N}^n$.

Suppose finally that the exact solution u belongs to the space $C^0(0, T; W^{\mu,\infty}(\mathbb{R}^n)^p)$ *where* $\mu = \max(r+1, m)$ *and satisfies for some* $\gamma > \frac{n}{2}$ *and for all* $\beta \in \mathbb{N}^n$ *with* $|\beta| \leqq \mu$

$$|\partial^\beta u(x, t)| \leqq c(1+|x|)^{-\gamma} \; x \in \mathbb{R}^n, \qquad t \in [0, T]. \tag{2.13}$$

Then, there exists a constant $C = C(u, T) > 0$ *such that*

$$\|u(.,t) - u_\varepsilon^h(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)} \leqq C\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+1}}\right), \qquad 0 \leqq t \leqq T. \tag{2.14}$$

Let us sketch the main steps in the proof. We write

$$u(.,t) - u_\varepsilon^h(.,t) = u(.,t) - \Pi_\varepsilon^h(t)\, u(.,t) + \Pi_\varepsilon^h(t)\, u(.,t) - u_\varepsilon^h(.,t).$$

Setting

$$e(t) = (e_k(t))_{k \in \mathbb{Z}^n}, \qquad e_k(t) = u(x_k(t), t) - u_k(t), \tag{2.15}$$

we have

$$(\Pi_\varepsilon^h(t)\, u(.,t) - u_\varepsilon^h(.,t))(x) = \sum_{k \in \mathbb{Z}^n} w_k(t)\, e_k(t)\, \zeta_\varepsilon(x - x_k(t))$$

and by Lemma 5

$$\|u(.,t) - u_\varepsilon^h(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)} \leqq \|u(.,t) - \Pi_\varepsilon^h(t)\, u(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)} + c_1 \|e(t)\|_{h,t}. \tag{2.16}$$

The key point is to estimate $\|e(t)\|_{h,t}$. We note that the function $t \to (u(x_k(t), t))_{k \in \mathbb{Z}^n}$ satisfies approximatively the Eq. (1.23). First, using (1.6) and (1.9), we observe that

$$\begin{aligned}
&\frac{d}{dt}(u(x(\xi, t), t)\, J(\xi, t)) \\
&= \left(\frac{\partial u}{\partial t} + \sum_{i=1}^n a^i \frac{\partial u}{\partial x^i}\right)(x(\xi, t), t)\, J(\xi, t) + u(x(\xi, t), t)\, \frac{\partial J}{\partial t}(\xi, t) \\
&= \left(\frac{\partial u}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial x^i}(a^i u)\right)(x(\xi, t), t)\, J(\xi, t).
\end{aligned}$$

Hence the solution u of Problem (1.2), (1.3) satisfies

$$\begin{aligned}
&\frac{d}{dt}(u(x_k(t), t)\, J(\xi_k, t)) + J(\xi_k, t)\left\{\sum_{i=1}^n \frac{\partial}{\partial x^i}(B^i u) + A^0 u\right\}(x_k(t), t) \\
&\quad = J(\xi_k, t)\, f_k(t), \qquad k \in \mathbb{Z}^n
\end{aligned}$$

or equivalently

$$\frac{d}{dt}(w_k(t)\,u(x_k(t),t))+w_k(t)\Big\{\sum_{l\in\mathbb{Z}^n} w_l(t)\sum_{i=1}^{n}(B_k^i(t)\,u(x_l(t),t)$$

$$+B_l^i(t)\,u(x_k(t),t))\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\Big\}$$

$$+w_k(t)\,A_k^0(t)\,u(x_k(t),t)=w_k(t)(f_k(t)+\varphi_k(t)) \tag{2.17}$$

where $\varphi(t)=(\varphi_k(t))_{k\in\mathbb{Z}^n}$ is defined by

$$\varphi_k(t)=-\sum_{i=1}^{n}\Big\{\frac{\partial}{\partial x^i}(B^i u)(x_k(t),t)$$

$$-\sum_{l\in\mathbb{Z}^n} w_l(t)(B_k^i(t)\,u(x_l(t),t)+B_l^i(t)\,u(x_k(t),t))\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\Big\}$$

Remark that

$$\varphi_k(t)=-\sum_{i=1}^{n}\Big\{B_k^i(t)\Big(\frac{\partial u}{\partial x^i}(x_k(t),t)-\sum_{l\in\mathbb{Z}^n} w_l(t)\,u(x_l(t),t)$$

$$\cdot\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\Big)+\Big(\frac{\partial B^i}{\partial x^i}(x_k(t),t)-\sum_{l\in\mathbb{Z}^n} w_l(t)\,B_l^i(t)$$

$$\cdot\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\Big)u(x_k(t),t)\Big\}$$

i.e.,

$$\varphi_k(t)=-\sum_{i=1}^{n}\Big\{B_k^i(t)\frac{\partial}{\partial x^i}(u(.,t)-\Pi_\varepsilon^h(t)\,u(.,t))(x_k(t))$$

$$+\frac{\partial}{\partial x^i}(B^i(.,t)-\Pi_\varepsilon^h(t)\,B^i(.,t))(x_k(t))\,u(x_k(t),t)\Big\}. \tag{2.18}$$

Now, substracting (1.23) from (2.16) gives

$$\frac{d}{dt}(w_k(t)\,e_k(t))+w_k(t)\Big\{\sum_{l\in\mathbb{Z}^n} w_l(t)\sum_{i=1}^{n}(B_k^i(t)\,e_l(t)+B_l^i(t)\,e_k(t))$$

$$\cdot\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\Big\}\,w_k(t)\,A_k^0(t)\,e_k(t)=w_k(t)\,\varphi_k(t). \tag{2.19}$$

Moreover, we have by (1.24)

$$e_k(0)=0. \tag{2.20}$$

Using (2.16), a proof of Theorem 2 will follow from

(i) estimates of the approximation errors

$$\|u(.,t)-\Pi_\varepsilon^h(t)\,u(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)},\qquad \|\varphi(t)\|_{h,t};$$

(ii) a stability inequality of the form

$$\| e(t)\|_{h,t} \leqq C \int_0^t \| \varphi(s) \|_{h,s}\, ds$$

for the solution $t \to e(t)$ of the equations (2.19), (2.20). We shall derive in Sect. 3 the *consistency estimates* (i). In Sect. 4, we shall establish the *stability property* (ii) and give the proof of Theorem 2.

Remark 2. Consider the case where

$$B^i = 0, \qquad 1 \leqq i \leqq n,$$

i.e., Eq. (1.2) reduces to

$$\frac{\partial u}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial x^i}(a^i u) + A^0 u = f \quad \text{in } Q_T.$$

Then, (2.19) becomes

$$\frac{d}{dt}(w_k(t)\, e_k(t)) + w_k(t)\, A_k^0(t)\, e_k(t) = 0.$$

Since $e_k(0) = 0$, we obtain $e_k(t) = 0$ and therefore

$$u^h(.,t) = \Pi^h(t)\, u(.,t), \qquad u_\varepsilon^h(.,t) = \Pi_\varepsilon^h(t)\, u(.,t).$$

Hence, in that case, finding a bound for the error

$$u(.,t) - u_\varepsilon^h(.,t)$$

exactly reduces in estimating the approximation error

$$u(.,t) - \Pi_\varepsilon^h(t)\, u(.,t). \quad \square$$

3. Some Results in Approximation Theory

Given a function $v \in C^0(\mathbb{R}^n)$, we want to derive bounds for the error $v - \Pi_\varepsilon^h(t) v$ between v and its generalized interpolate $\Pi_\varepsilon^h(t) v$ defined by (1.11). We begin by recalling two simple but essential results of [19].

Lemma 6. *Assume that the cut-off function ζ satisfies the conditions* (2.11) *for some integer $r \geqq 1$. Then, there exists a constant $C > 0$ such that for all function $g \in W^{r,p}(\mathbb{R}^n)$, $1 \leqq p \leqq +\infty$*

$$\| g - g * \zeta_\varepsilon \|_{\mathbb{L}^p(\mathbb{R}^n)} \leqq C \varepsilon^r |g|_{r,p,\mathbb{R}^n} \tag{3.1}$$

Next, we set for all $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$

$$D_k = \{x \in \mathbb{R}^n;\ (k_i - \tfrac{1}{2})\, h \leqq x^i \leqq (k_i + \tfrac{1}{2})\, h,\ 1 \leqq i \leqq n\}$$

and

$$E_k(g)=\int_{D_k} g(\xi)\,d\xi-h^n g(\xi_k), \qquad g\in C^0(D_k). \tag{3.2}$$

Then, we have

Lemma 7. *Let $m\geqq 1$ be an integer and $p>\frac{n}{m}$, $q=\frac{p}{p-1}$. There exists a constant $C>0$ such that, for all function $g\in W^{m,p}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ if $m\leqq 2$ or for all function $g\in W^{m,p}(\mathbb{R}^n)\cap W^{m-1,1}(\mathbb{R}^n)$ if $m\geqq 3$, we have*

$$\Big|\sum_{k\in\mathbb{Z}^n} E_k(g)\Big|\leqq C h^{m+\frac{n}{q}} \sum_{k\in\mathbb{Z}^n} |g|_{m,p,D_k} \tag{3.3}$$

Now, in order to estimate $v-\Pi_\varepsilon^h(t)v$, we begin by deriving a bound for $v-\Pi^h(t)v$ in the negative Sobolev space $W^{-m,p}(\mathbb{R}^n)$.

Lemma 8. *Let $m>n$ be an integer. Assume that*

$$a^i\in L^\infty(0,T;W^{m+1,\infty}(\mathbb{R}^n)), \qquad 1\leqq i\leqq n. \tag{3.4}$$

Then, there exists a constant $C=C(T)>0$ such that we have for all function $v\in W^{m,p}(\mathbb{R}^n)$, $1\leqq p\leqq+\infty$

$$\|v-\Pi^h(t)v\|_{-m,p,\mathbb{R}^n}\leqq C h^m \|v\|_{m,p,\mathbb{R}^n}, \qquad 0\leqq t\leqq T. \tag{3.5}$$

Proof. Since $m>n$, we have by the Sobolev's imbedding theorem

$$W^{m,p}(\mathbb{R}^n)\subset C^0(\mathbb{R}^n) \qquad \text{for all } 1\leqq p\leqq+\infty$$

Hence, we can associate with any function $v\in W^{m,p}(\mathbb{R}^n)$ the measure

$$\Pi^h(t)v=\sum_{k\in\mathbb{Z}^n} w_k(t)\,v(x_k(t))\,\delta(x-x_k(t)).$$

Consider first the case $1<p\leqq+\infty$. Let $\varphi\in\mathscr{D}(\mathbb{R}^n)$; we have

$$\langle v-\Pi^h(t)\,v,\varphi\rangle=\int_{\mathbb{R}^n} v\varphi\,dx-\sum_{k\in\mathbb{Z}^n} w_k(t)\,v(x_k(t))\,\varphi(x_k(t))$$

Using the change of variable $x=x(\xi,t)$, we can write

$$\langle v-\Pi^h(t)\,v,\varphi\rangle=\int_{\mathbb{R}^n} J(\xi,t)\,v(x(\xi,t))\,\varphi(x(\xi,t))\,d\xi$$
$$-h^n\sum_{k\in\mathbb{Z}^n} J(\xi_k,t))\,v(x(\xi_k,t))\,\varphi(x(\xi_k,t))$$

so that

$$\langle v-\Pi^h(t)\,v,\varphi\rangle=\sum_{k\in\mathbb{Z}^n} E_k(g(.,t))$$

where

$$g(\xi,t)=J(\xi,t)\,v(x(\xi,t))\,\varphi(x(\xi,t)).$$

Now, it is a simple matter to check that the hypothesis (3.5) implies that

$$t\to\partial_\xi^\alpha x(.,t)\in C^0(0,T;L^\infty(\mathbb{R}^n)), \qquad 1\leqq|\alpha|\leqq m+1$$

and

$$t \to J(.,t) \in C^0(0, T; W^{m,\infty}(\mathbb{R}^n)).$$

Hence, the function $t \to g(t)$ belongs to the space $C^0(0, T; W^{m,1}(\mathbb{R}^n))$. Moreover, applying Lemma 7 (with $p=1$) to the function $g(.,t)$ gives

$$\begin{aligned}|\langle v - \Pi^h(t)\, v, \varphi\rangle| &\leqq c_1\, h^m |g(.,t)|_{m,1,\mathbb{R}^n} \\ &\leqq c_2\, h^m \| v \|_{m,p,\mathbb{R}^n} \| \varphi \|_{m,q,\mathbb{R}^n}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Since $\mathscr{D}(\mathbb{R}^n)$ is dense in $W^{m,q}(\mathbb{R}^n)$, $1 \leqq q < \infty$, the above inequality holds for all function $\varphi \in W^{m,q}(\mathbb{R}^n)$ and we have

$$\| v - \Pi^h(t) v \|_{-m,p,\mathbb{R}^n} = \sup_{\varphi \in W^{m,q}(\mathbb{R}^n)} \frac{|\langle v - \Pi^h(t)\, v, \varphi\rangle|}{\| \varphi \|_{m,q,\mathbb{R}^n}} \leqq c_2\, h^m \| v \|_{m,p,\mathbb{R}^n}$$

In the case $p=1$, we take $\varphi \in W^{m,\infty}(\mathbb{R}^n)$ and the only additional difficulty is to show that $\langle v - \Pi_h(t)\, v, \varphi\rangle$ makes sense. This is left to the reader. □

As a corollary of the above result, we obtain

Lemma 9. *Assume the hypotheses of Lemma* 8. *Assume in addition that the cut-off function* ζ *belongs to the space* $W^{m+s,1}(\mathbb{R}^n)$ *for some integer* $s \geqq 0$. *Then, there exists a constant* $C = C(T) > 0$ *such that we have for all function* $v \in W^{m,p}(\mathbb{R}^n)$, $1 \leqq p \leqq \infty$

$$|v * \zeta_\varepsilon - \Pi_\varepsilon^h(t) v|_{s,p,\mathbb{R}^n} \leqq C \frac{h^m}{\varepsilon^{m+s}} \| v \|_{m,p,\mathbb{R}^n}, \qquad 0 \leqq t \leqq T \tag{3.6}$$

Proof. First, we note that

$$v * \zeta_\varepsilon - \Pi_\varepsilon^h(t)\, v = (v - \Pi^h(t) v) * \zeta_\varepsilon.$$

Next, we observe that, if $f \in W^{-m,p}(\mathbb{R}^n)$ and $g \in W^{m,1}(\mathbb{R}^n)$, we have $f * g \in L^p(\mathbb{R}^n)$ with

$$\| f * g \|_{L^p(\mathbb{R}^n)} \leqq c_1 \| f \|_{-m,p,\mathbb{R}^n} \| g \|_{m,1,\mathbb{R}^n} \tag{3.7}$$

In fact, assuming $1 \leqq p < +\infty$ (for specificity), any distribution $f \in W^{-m,p}(\mathbb{R}^n)$ may be put (in a nonunique way) in the form

$$f = \sum_{|\alpha| \leqq m} \partial^\alpha f_\alpha, \qquad f_\alpha \in L^p(\mathbb{R}^n), \qquad |\alpha| \leqq m. \tag{3.8}$$

Moreover, we have

$$\inf\Big(\sum_{|\alpha| \leqq m} \| f_\alpha \|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p} \leqq c_2 \| f \|_{-m,p,\mathbb{R}^n}$$

where the infimum is taken over all the decompositions (3.8) of f. Hence, using (3.8), we can write

$$f * g = \sum_{|\alpha| \leqq m} \partial^\alpha f_\alpha * g = \sum_{|\alpha| \leqq m} f_\alpha * \partial^\alpha g$$

so that

$$\|f*g\|_{L^p(\mathbb{R}^n)} \leqq \sum_{|\alpha|\leqq m} \|f_\alpha * \partial^\alpha g\|_{L^p(\mathbb{R}^n)} \leqq \sum_{|\alpha|\leqq m} \|f_\alpha\|_{L^p(\mathbb{R}^n)} \|\partial^\alpha g\|_{L^1(\mathbb{R}^n)}$$
$$\leqq c_3 \Big(\sum_{|\alpha|\leqq m} \|f_\alpha\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p} \|g\|_{m,1,\mathbb{R}^n}$$

and (3.7) follows at once.

Now, if $\zeta \in W^{m+s,1}(\mathbb{R}^n)$ with $m>n$ and $s \geqq 0$, we obtain from (3.7) and Lemma 8 that the function

$$\partial^\alpha(v * \zeta_\varepsilon - \Pi^h_\varepsilon(t)v) = (v - \Pi^h(t)v) * \partial^\alpha \zeta_\varepsilon$$

belongs to $L^p(\mathbb{R}^n)$ for $|\alpha| \leqq s$ and

$$\|\partial^\alpha(v * \zeta_\varepsilon - \Pi^h_\varepsilon(t)v)\|_{L^p(\mathbb{R}^n)} \leqq c_4 \|v - \Pi^h(t)v\|_{-m,p,\mathbb{R}^n} \|\partial^\alpha \zeta_\varepsilon\|_{m,1,\mathbb{R}^n}$$
$$\leqq c_5\, h^m \|v\|_{m,p,\mathbb{R}^n} \|\partial^\alpha \zeta_\varepsilon\|_{m,1,\mathbb{R}^n}$$

Together with

$$|\zeta_\varepsilon|_{i,1,\mathbb{R}^n} = \frac{1}{\varepsilon^i} |\zeta|_{i,1,\mathbb{R}^n} \leqq \frac{c_6}{\varepsilon^i},$$

this implies the desired inequality (3.6).

We are now able to state the following general approximation result.

Theorem 3. *Let $m>n$ be an integer. We assume that the hypothesis* (3.4) *holds. We assume in addition that the cut-off function $\zeta \in C^1_0(\mathbb{R}^n)$ satisfies the conditions* (2.11) *for some integer $r \geqq 1$ and belongs to the space $W^{m+s,1}(\mathbb{R}^n)$ for some other integer $s \geqq 0$. Then, there exists a constant $C = C(T) > 0$ such that we have for all function $v \in W^{\mu,p}(\mathbb{R}^n)$, $\mu = \max(r+s, m)$, $1 \leqq p \leqq +\infty$*

$$|v - \Pi^h_\varepsilon(t)v|_{s,p,\mathbb{R}^n} \leqq C\left(\varepsilon^r |v|_{r+s,p,\mathbb{R}^n} + \frac{h^m}{\varepsilon^{m+s}} \|v\|_{m,p,\mathbb{R}^n}\right), 0 \leqq t \leqq T. \tag{3.9}$$

Proof. For all $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$, we write

$$\partial^\alpha(v - \Pi^h_\varepsilon(t)v) = \partial^\alpha v - \partial^\alpha v * \zeta_\varepsilon + \partial^\alpha(v * \zeta_\varepsilon - \Pi^h_\varepsilon(t)v).$$

Applying Lemma 6 to the function $\partial^\alpha v$ gives

$$\|\partial^\alpha v - \partial^\alpha v * \zeta_\varepsilon\|_{L^p(\mathbb{R}^n)} \leqq c_1\, \varepsilon^r |v|_{r+s,p,\mathbb{R}^n}.$$

On the other hand, we have by Lemma 9

$$\|\partial^\alpha(v * \zeta_\varepsilon - \Pi^h_\varepsilon(t)v\|_{L^p(R^n)} \leqq c_2 \frac{h^m}{\varepsilon^{m+s}} \|v\|_{m,p,\mathbb{R}^n}$$

and (3.10) follows.

Corollary. *Assume the hypotheses of Theorem* 3. *Let v be a function of $W^{\mu,\infty}(\mathbb{R}^n)$, $\mu = \max(r+s, m)$, which satisfies for some $\gamma > 0$ and for all $\beta \in \mathbb{N}^n$ with $|\beta| \leqq \mu$*

$$|\partial^\beta v(x)| \leqq c(1+|x|)^{-\gamma}, \qquad x \in \mathbb{R}^n \tag{3.10}$$

Then, there exists a constant $C = C(v, T) > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$.

$$|\partial^\alpha (v - \Pi_\varepsilon^h(t)v)(x)| \leqq C\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+s}}\right)(1 + |x|)^{-\gamma}, \qquad x \in \mathbb{R}^n,\ t \in [0, T]. \tag{3.11}$$

Proof. Setting $O_\varepsilon(x) = x + \operatorname{supp}(\zeta_\varepsilon)$, we note that $(v - \Pi_\varepsilon^h(t)v)(x)$ depends only on the restriction of the function v to $O_\varepsilon(x)$. Hence, under the hypotheses of Theorem 3, we have if v belongs to $W^{\mu,\infty}(O_\varepsilon(x))$

$$|\partial^\alpha (v - \Pi_\varepsilon^h(t)v)(x)| \leqq c_1\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+s}}\right) \| v \|_{\mu,\infty,O_\varepsilon(x)}.$$

Now, if we assume that the function $v \in W^{\mu,\infty}(\mathbb{R}^n)$ satisfies the conditions (3.10) we have

$$\| v \|_{\mu,\infty,O_\varepsilon(x)} \leqq c_2 (1 + |x|)^{-\gamma}$$

and (3.1) follows.

Remark 3. Since we have always

$$\| g * \zeta_\varepsilon - g \|_{L^p(\mathbb{R}^n)} \leqq c \| g \|_{L^p(\mathbb{R}^n)}$$

we may apply Theorem 3 with $r = 0$.

4. Stability Analysis and Convergence Theorems

Before proving Theorem 2, we need to derive a stability inequality for the solution $e(t) = (e_k(t))_{k \in \mathbb{Z}^n}$ of (2.19), (2.20). We begin by recalling a classical simple result. Given real numbers a_{kl}, $k, l \in \mathbb{Z}^n$, we want to give sufficient conditions for the formula

$$(Ay)_k = \sum_{l \in \mathbb{Z}^n} a_{kl}\, y_l, \qquad k \in \mathbb{Z}^n \tag{4.1}$$

to define a linear continuous operator $A \in \mathscr{L}(l^2(\mathbb{Z}^n))$.

Lemma 11. *Assume that the coefficients a_{kl} satisfy*

$$\text{(i)}\ \sup_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} |a_{kl}| \leqq C.$$

$$\text{(ii)}\ \sup_{l \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} |a_{kl}| \leqq C \tag{4.2}$$

for some constant $C > 0$. Then (4.1) *defines a linear operator $A \in \mathscr{L}(l^2(\mathbb{Z}^n))$ with*

$$\| A \| \leqq C. \tag{4.3}$$

In (4.3), $\| A \|$ denotes the usual norm subordinate to the norm (2.2) of $l^2(\mathbb{Z}^n)$.

Proof. We give the proof for reader's convenience. Let $y \in l^2(\mathbb{Z}^n)$; we set for all $k \in \mathbb{Z}^n$

$$z_k = \sum_{l \in \mathbb{Z}^n} a_{kl}\, y_l$$

Using Cauchy-Schwarz' inequality and (4.2) (i), we have,

$$|z_k| \leqq \sum_{l \in \mathbb{Z}^n} |a_{kl}|^{1/2} |a_{kl}|^{1/2} |y_l| \leqq \Big(\sum_{l \in \mathbb{Z}^n} |a_{kl}|\Big)^{1/2} \Big(\sum_{l \in \mathbb{Z}^n} |a_{kl}| |y_l|^2\Big)^{1/2}$$
$$\leqq C^{1/2} \Big(\sum_{l \in \mathbb{Z}^n} |a_{kl}| |y_l|^2\Big)^{1/2}$$

so that z_k makes sense. Moreover

$$\sum_{k \in \mathbb{Z}^n} |z_k|^2 \leqq C \sum_{k,\, l \in \mathbb{Z}^n} |a_{kl}| |y_l|^2 \leqq C \sum_{l \in \mathbb{Z}^n} \Big(\sum_{k \in \mathbb{Z}^n} |a_{kl}|\Big) |y_l|^2.$$

Hence, by (4.2) (ii)

$$\sum_{k \in \mathbb{Z}^n} |z_k|^2 \leqq C^2 \sum_{l \in \mathbb{Z}^n} |y_l|^2$$

and the conclusion follows.

Next, denote by $(.,.)$ the Euclidean inner product in $\mathbb{R}^p$. We have

Lemma 12. *Assume that the cut-off function* $\zeta \in C_0^1(\mathbb{R}^n)$ *satisfies the condition (1.28). Then, there exists a constant* $C = C(T) > 0$ *such that for all* $v \in \ell^2(\mathbb{Z}^n)$

$$\left| \sum_{k,\, l \in \mathbb{Z}^n} w_k(t)\, w_l(t) (B_k^i(t)\, v_l, v_k) \frac{\partial \zeta_\varepsilon}{\partial x^i} (x_k(t) - x_l(t)) \right| \leqq C \| v \|_{h,t}^2, \quad 0 \leqq t \leqq T. \tag{4.4}$$

Proof. It follows from (1.29) and the symmetry of the matrix $B_k^i(t)$ that (by interchanging the roles of k and l)

$$\sum_{k,\, l \in \mathbb{Z}^n} w_k\, w_l ((B_k^i + B_l^i)\, v_l, v_k) \frac{\partial \zeta_\varepsilon}{\partial x^i} (x_k - x_l) = 0$$

where, for the sake of conciseness, we have dropped the explicit dependence in the variable t. Hence, we obtain

$$\sum_{k,\, l \in \mathbb{Z}^n} w_k\, w_l (B_k^i\, v_l, v_k) \frac{\partial \zeta_\varepsilon}{\partial x^i} (x_k - x_l) = \frac{1}{2} \sum_{k,\, l \in \mathbb{Z}^n} w_k\, w_l ((B_k^i - B_l^i)\, v_l, v_k) \frac{\partial \zeta_\varepsilon}{\partial x^i} (x_k - x_l)$$

Thus, we can write

$$\left| \sum_{k,\, l \in \mathbb{Z}^n} w_k\, w_l (B_k^i\, v_l, v_k) \frac{\partial \zeta_\varepsilon}{\partial x^i} (x_k - x_l) \right| \leqq \sum_{k,\, l \in \mathbb{Z}^n} a_{kl}\, y_l\, y_k$$

where

$$a_{kl} = \sqrt{w_k\, w_l}\, |B_k^i - B_l^i| \left| \frac{\partial \zeta_\varepsilon}{\partial x^i} (x_k - x_l) \right|, \qquad y_k = \sqrt{w_k}\, |v_k|$$

and $|B_k^i - B_l^i|$ is the spectral norm of the matrix $B_k^i - B_l^i$. The desired estimate (4.4) will follow from (4.3) if we check that the coefficients a_{kl} satisfy the hypotheses of Lemma 11 with a constant C independent of h and ε. Since $a_{kl} = a_{lk}$, it suffices to check the condition (4.2) (i).

First, given $k \in \mathbb{Z}^n$, we know that the number of indices $l \in \mathbb{Z}^n$ such that $x_k - x_l$ belongs to $\operatorname{supp}(\zeta_\varepsilon)$ and therefore the number of non zero elements a_{kl} are bounded by $c_1 \left(\frac{\varepsilon}{h}\right)^n$. Moreover, since

$$\frac{\partial B^i}{\partial x^i} \in L^\infty(Q_T; \mathscr{L}(\mathbb{R}^p)),$$

it follows from (2.5) that we have if $x_k - x_l \in \operatorname{supp}(\zeta_\varepsilon)$

$$|B_k^i - B_l^i| = |B^i(x_k, t) - B^i(x_l, t)| \leqq c_2 |x_k - x_l| \leqq c_3 \varepsilon.$$

Next, we obtain

$$w_k \leqq c_4 h^n, \left|\frac{\partial \zeta_\varepsilon}{\partial x^i}\right| \leqq \frac{c_5}{\varepsilon^{n+1}}$$

so that

$$|a_{kl}| \leqq c_6 \left(\frac{h}{\varepsilon}\right)^n.$$

Thus, we find

$$\sum_{l \in \mathbb{Z}^n} |a_{kl}| \leqq c_1 \left(\frac{\varepsilon}{h}\right)^n c_6 \left(\frac{h}{\varepsilon}\right)^n = c_1 c_6.$$

Using, Lemma 11, we obtain

$$\Big| \sum_{k, l \in \mathbb{Z}^n} a_{kl} y_k y_l \Big| \leqq c_1 c_6 \sum_{k \in \mathbb{Z}^n} y_k^2 = c_1 c_6 \| v \|_{h,t}^2$$

which proves (4.4).

We are now able to prove the stability result.

Theorem 4. *Assume that the cut-off function $\zeta \in C_0^1(\mathbb{R}^n)$ belongs to the space $W^{m+1,1}(\mathbb{R}^n)$ for some integer $m > n$ and the conditions* (1.28) *and* (2.12) *hold. Assume in addition that the functions a^i, B^i are smooth enough, i.e.,*

$$a^i \in L^\infty(0, T; W^{m+1,\infty}(\mathbb{R}^n)), \qquad B^i \in L^\infty(0, T; W^{m+1,\infty}(\mathbb{R}^n)^{p^2}), \qquad 1 \leqq i \leqq n.$$

Then, the solution $t \to e(t)$ of (2.19), (2.20) *satisfies*

$$\| e(t) \|_{h,t} \leqq C \int_0^t \| \varphi(s) \|_{h,s} \, ds, \qquad 0 \leqq t \leqq T$$

for some constant $C = C(T) > 0$.

Proof. We start from (2.19). First we observe that

$$\sum_{k \in \mathbb{Z}^n} \left(\frac{d}{dt}(w_k(t)\, e_k(t)), e_k(t)\right) = \frac{1}{2} \frac{d}{dt} \| e(t) \|_{h,t}^2 + \frac{1}{2} \sum_{k \in \mathbb{Z}^n} \frac{d w_k}{dt}(t) |e_k(t)|^2$$

Hence, using (1.26), we obtain

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|_{h,t}^2+\sum_{k,l\in\mathbb{Z}^n} w_k(t)\,w_l(t)$$

$$\cdot\left\{\sum_{i=1}^{n}(B_k^i(t)\,e_l(t)+B_l^i(t)\,e_k(t),\,e_k(t))\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right\}$$

$$+\sum_{k\in\mathbb{Z}^n} w_k(t)(\bar{A}_k^0(t)\,e_k(t),\,e_k(t))=\sum_{k\in\mathbb{Z}^n} w_k(t)(\varphi_k(t),\,e_k(t)),$$

where

$$\bar{A}^0=A^0+\tfrac{1}{2}(\operatorname{div} a)I,\qquad \bar{A}_k^0(t)=\bar{A}^0(x_k(t),\,t).$$

On the one hand, using Lemma 12 gives

$$\left|\sum_{k,l\in\mathbb{Z}^n} w_k(t)\,w_l(t)\left\{\sum_{i=1}^{n}(B_k^i(t)\,e_l(t),\,e_k(t))\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right\}\right|\leqq c_1\,\|e(t)\|_{h,t}^2$$

On the other hand, we have

$$\left|\sum_{k\in\mathbb{Z}^n} w_k(t)(\bar{A}_k^0(t)\,e_k(t),\,e_k(t))\right|\leqq c_2\,\|e(t)\|_{h,t}^2$$

and

$$\left|\sum_{k\in\mathbb{Z}^n} w_k(t)(\varphi_k(t),\,e_k(t))\right|\leqq\|\varphi(t)\|_{h,t}\,\|e(t)\|_{h,t}.$$

It remains to estimate

$$\sum_{k,l\in\mathbb{Z}^n} w_k(t)\,w_l(t)\left\{\sum_{i=1}^{n}(B_l^i(t)\,e_k(t),\,e_k(t))\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right\}$$

$$=\sum_{k\in\mathbb{Z}^n} w_k(t)\left\{\sum_{i=1}^{n}\left(\left(\frac{\partial}{\partial x^i}\Pi_\varepsilon^h(t)\,B^i(.,\,t)\right)(x_k(t))\,e_k(t),\,e_k(t)\right)\right\}.$$

Using the smoothness of the functions $a^i(.,t)$ and $B^i(.,t)$ and Theorem 3 with $r=0$, $s=1$, $p=\infty$ (see Remark 3), we obtain

$$\left\|\frac{\partial}{\partial x^i}\Pi_\varepsilon^h(t)\,B^i(.,\,t)\right\|_{L^\infty(\mathbb{R}^n)}\leqq c_3\left(1+\frac{h^m}{\varepsilon^{m+1}}\right)$$

and therefore

$$\left|\sum_{k,l\in\mathbb{Z}^n} w_k(t)\,w_l(t)\left\{\sum_{i=1}^{n}(B_l^i(t)\,e_k(t),\,e_k(t))\,\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right\}\right|$$

$$\leqq c_3\left(1+\frac{h^m}{\varepsilon^{m+1}}\right)\|e(t)\|_{h,t}^2.$$

By combining the above estimates, we find that, under the condition (2.12), we have the bound

$$\frac{d}{dt}\|e(t)\|_{h,t}^2 \leqq c_4 \|e(t)\|_{h,t}^2 + 2\|\varphi(t)\|_{h,t}\|e(t)\|_{h,t}$$

which implies

$$\|e(t)\|_{h,t} \leqq \|e(0)\|_{h,0}\exp\left(\frac{c_4}{2}t\right) + \int_0^t \|\varphi(s)\|_{h,s}\exp\left(\frac{c_4}{2}(t-s)\right)ds$$

Theorem 4 follows since $e(0)=0$. □

Proof of Theorem 2. Let us derive a bound for $\|\varphi(t)\|_{h,t}$. We start from (2.18). Using the hypothesis (2.13) and the corollary of Theorem 3, we obtain

$$\left|\frac{\partial}{\partial x^i}(u(.,t) - \Pi_\varepsilon^h(t)\,u(.,t))(x_k(t))\right| \leqq c_1\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+1}}\right)(1+|x_k(t)|)^{-\gamma}.$$

Similarly, we have by the smoothness of the functions B^i:

$$\left|\frac{\partial}{\partial x^i}(B^i(.,t) - \Pi_\varepsilon^h(t)\,B^i(.,t))(x_k(t))\,u(x_k(t),t)\right| \leqq c_2\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+1}}\right)(1+|x_k(t)|)^{-\gamma}$$

Hence, we find

$$|\varphi_k(t)| \leqq c_3\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+1}}\right)(1+|x_k(t)|)^{-\gamma}$$

so that

$$\|\varphi(t)\|_{h,t} \leqq c_3\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+1}}\right)\Big(\sum_{k\in\mathbb{Z}^n} w_k(t)(1+|x_k(t)|)^{-2\gamma}\Big)^{1/2}.$$

Now, since

$$x_k(t) = \xi_k + \int_0^t a(x_k(s),s)\,ds,$$

we have by (1.5)

$$|x_k(t)| \geqq |\xi_k| - c_4 t.$$

Therefore, we get for $|\xi_k| > R$ large enough

$$(1+|x_k(t)|)^{-2\gamma} \leqq c_5(R)(1+|\xi_k|)^{-2\gamma}, \qquad 0 \leqq t \leqq T.$$

Since $\gamma > \frac{n}{2}$, we obtain

$$\sum_{k\in\mathbb{Z}^n} w_k(t)(1+|x_k(t)|)^{-2\gamma} \leqq c_6 + c_7 h^n \sum_{|\xi_k|>R}(1+|\xi_k|)^{-2\gamma} \leqq c_8$$

and

$$\|\varphi(t)\|_{h,t} \leqq c_8\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+1}}\right), \qquad 0 \leqq t \leqq T. \tag{4.6}$$

Next, it follows from (2.13) that $u \in C^0(0,T;W^{\mu,2}(\mathbb{R}^n)^p)$. Thus, Theorem 3 gives

$$\|u(.,t) - \Pi_\varepsilon^h(t)\,u(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)} \leqq c_9\left(\varepsilon^r + \frac{h^m}{\varepsilon^m}\right), 0 \leqq t \leqq T \tag{4.7}$$

Combining the stability inequality (4.5) and the estimates (4.6) and (4.7) gives the desired error bound (2.14) of Theorem 2. □

Remark 4. One can easily generalize Theorem 2 to first order symmetrizable systems. Instead of the hypothesis (1.1) (iii), we require that there exists a continuous function $P: (x, t) \in Q_T \to P(x, t) \in \mathscr{L}(^p)$ with the following properties:

(i) $P, \dfrac{\partial P}{\partial t} \in L^\infty(Q_T, \mathscr{L}(\mathbb{R}^p))$;

(ii) the matrix $P(x, t)$ is symmetric and positive definite uniformly in Q_T, i.e., there exists a constant $\alpha > 0$ such that

$$(P(x, t)\eta, \eta) \geqq \alpha |\eta|^2 \quad \forall \eta \in \mathbb{R}^p, \qquad (x, t) \in Q_T;$$

(iii) the matrices $(PA^i)(x, t)$ are symmetric, $1 \leqq i \leqq n$.

Then, it is a simple matter to check that Theorem 4 and therefore Theorem 2 still hold in that case.

5. Particle Approximation of Parabolic Systems

Let us generalize the particle method to the numerical approximation of parabolic systems. In particular, we want to deal with convection-diffusion problems.

In addition to the $p \times p$ matrix-valued functions A^i, $0 \leqq i \leqq n$, we are given n^2 continuous mappings $A^{ij}: (x, t) \in Q_T \to A^{ij}(x, t) \in \mathscr{L}(\mathbb{R}^p)$, $1 \leqq i, j \leqq n$, with the following properties:

(i) $A^{ij} \in L^\infty(Q_T; \mathscr{L}(\mathbb{R}^p))$, $1 \leqq i, j \leqq n$,

(ii) there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} (A^{ij}(x, t)\eta^j, \eta^i) \geqq \alpha \sum_{i=1}^{n} |\eta^i|^2 \quad \forall \eta^i \in \mathbb{R}^p, 1 \leqq i \leqq n, \qquad (x, t) \in Q_T. \tag{5.1}$$

Then, we consider the parabolic system

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x^i}(A^i u) + A^0 u - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(A^{ij} \frac{\partial u}{\partial x^j}\right) = f \quad \text{in } Q_T \tag{5.2}$$

with the initial condition

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^n \tag{5.3}$$

Then, if $u_0 \in \mathbb{L}^2(\mathbb{R}^n)$ and $f \in L^2(0, T; \mathbb{L}^2(\mathbb{R}^n))$, it is a standard result that Problem (5.2), (5.3) has a unique solution $u \in C^0(0, T; \mathbb{L}^2(\mathbb{R}^n)) \cap L^2(0, T; W^{1,2}(\mathbb{R}^n)^p)$.

Let us put the system (5.2) into the form of a first-order system. Setting

$$p^i = -\sum_{j=1}^{n} A^{ij} \frac{\partial u}{\partial x^j}$$

and using (1.4), Eq. (5.2) becomes

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n}\frac{\partial}{\partial x^i}(a^i u+B^i u+p^i)+A^0 u=f,$$

$$p^i+\sum_{j=1}^{n}A^{ij}\frac{\partial u}{\partial x^j}=0,\qquad 1\leqq i\leqq n. \tag{5.4}$$

Now, we look for a particle approximation u_h of Problem (5.2), (5.3) based on the formulation (5.4). Using the ideas and the notations of Sect. 1, we first set:

$$u^h(x,t)=\sum_{k\in K}w_k(t)\,u_k(t)\,\delta(x-x_k(t)),$$

$$p^{h,i}(x,t)=\sum_{k\in K}w_k(t)\,p_k^i(t)\,\delta(x-x_k(t)),\qquad 1\leqq i\leqq n. \tag{5.5}$$

Next, assuming that the data u_0 and f are continuous functions, a (semi-) discretized form of Problem (5.2), (5.3) consists in finding functions $t\to u_k(t)$ and $t\to p_k^i(t)$, $1\leqq i\leqq n$, $k\in K$, from $[0,T]$ into $\mathbb{R}^p$ solutions of the equations

$$\frac{d}{dt}(w_k(t)\,u_k(t))+w_k(t)\left\{\sum_{l\in K}w_l(t)\sum_{i=1}^{n}(B_k^i(t)u_l(t)+B_l^i(t)\,u_k(t)+p_l^i(t)+p_k^i(t))\cdot\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\right\}$$
$$+w_k(t)\,A_k^0(t)\,u_k(t)=w_k(t)\,f_k(t), \tag{5.6}$$

$$p_k^i(t)+\sum_{j=1}^{n}A_k^{ij}(t)\sum_{l\in K}w_l(t)(u_l(t)-u_k(t))\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))=0,\qquad 1\leqq i\leqq n \tag{5.7}$$

and

$$u_k(0)=u_0(\xi_k),\qquad k\in K, \tag{5.8}$$

where

$$A_k^{ij}(t)=A^{ij}(x_k(t),t). \tag{5.9}$$

Remark 5. Note that the analogue of Lemma 2 still holds: Under the condition (1.28), the conservation property (1.27) is satisfied. Moreover, this property does not depend on the discretization of the 2nd equations (5.4). In fact, one could as well take instead of (5.7)

$$p_k^i(t)+\sum_{j=1}^{n}A_k^{ij}(t)\sum_{l\in K}w_l(t)\,u_l(t)\,\frac{\partial\zeta_\varepsilon}{\partial x^j}(x_k(t)-x_l(t))=0,$$

but the Eq. (5.7) lead to a simpler stability analysis (see the proof of Lemm 13).

In all the sequel, we shall assume that the sets K and $(\xi_k, w_k)_{k\in K}$ are chosen as in (2.1). Then, one can easily show that the exact analogue of Theorem 1 is valid.

Let us next study the convergence of the regularized particle approximation $(u_\varepsilon^h, p_\varepsilon^{h,i})$ defined by

$$u_\varepsilon^h(.,t)=u^h(.,t)*\zeta_\varepsilon, \qquad p_\varepsilon^{h,i}(.,t)=p^{h,i}(.,t)*\zeta_\varepsilon, \qquad 1\leqq i\leqq n \tag{5.10}$$

as the two parameters h and ε tend to zero. Again we assume that the data and therefore the solution u of Problem (5.2), (5.3) are smooth enough and we set:

$$\begin{aligned} e_k(t)&=u(x_k(t),t)-u_k(t),\\ \eta_k^i(t)&=p^i(x_k(t),t)-p_k^i(t), \qquad 1\leqq i\leqq n. \end{aligned} \tag{5.11}$$

Argueing as in Sect. 2, we find that $e(t)=(e_k(t))_{k\in\mathbb{Z}^n}$ and $\eta^i(t)=(\eta_k^i(t))_{k\in\mathbb{Z}^n}$, $1\leqq i\leqq n$, satisfy the equation

$$\frac{d}{dt}(w_k e_k)+w_k\sum_{l\in\mathbb{Z}^n} w_l\sum_{i=1}^n (B_k^i e_l+B_l^i e_k+\eta_k^i+\eta_l^i)\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k-x_l)=w_k\varphi_k, \tag{5.12}$$

where

$$\begin{aligned}\varphi_k(t)=-\sum_{i=1}^n\cdot\Big\{&\frac{\partial}{\partial x^i}(B^i u+p^i)(x_k(t),t)-\sum_{l\in\mathbb{Z}^n} w_l(B_k^i(t)\,u(x_l(t),t)\\ &+B_l^i(t)\,u(x_k(t),t)+p^i(x_l(t),t)+p^i(x_k(t),t))\frac{\partial\zeta_\varepsilon}{\partial x^i}(x_k(t)-x_l(t))\Big\}\end{aligned}$$

or equivalently

$$\begin{aligned}\varphi_k(t)=-\sum_{i=1}^n\Big\{&B_k^i(t)\frac{\partial}{\partial x^i}(u(.,t)-\Pi_\varepsilon^h(t)\,u(.,t))(x_k(t))\\ &+\frac{\partial}{\partial x^i}(p^i(.,t)-\Pi_\varepsilon^h(t)\,p^i(.,t))(x_k(t))\\ &+\frac{\partial}{\partial x^i}(B^i(.,t)-\Pi_\varepsilon^h(t)\,B^i(.,t))(x_k(t))\,u(x_k(t),t)\\ &-\frac{\partial}{\partial x^i}(\Pi_\varepsilon^h(t)\,1)(x_k(t))\,p^i(x_k(t),t)\Big\}.\end{aligned} \tag{5.13}$$

Similarly, (5.7) yields

$$\eta_k^i+\sum_{j=1}^n A_k^{ij}\sum_{l\in\mathbb{Z}^n} w_l(e_l-e_k)\frac{\partial\zeta_\varepsilon}{\partial x^j}(x_k-x_l)=\sigma_k^i, \qquad 1\leqq i\leqq n \tag{5.14}$$

where

$$\sigma_k^i(t)=-\sum_{j=1}^n A_k^{ij}(t)\Big\{\frac{\partial u}{\partial x^j}(x_k(t),t)-\sum_{l\in\mathbb{Z}^n}(u(x_l(t),t)-u(x_k(t),t))\frac{\partial\zeta_\varepsilon}{\partial x^j}(x_k(t)-x_l(t))\Big\}$$

i.e.,

$$\begin{aligned}\sigma_k^i(t)=-\sum_{j=1}^n A_k^{ij}(t)\Big\{&\frac{\partial}{\partial x^j}(u(.,t)-\Pi_\varepsilon^h(t)\,u(.,t))(x_k(t),t)\\ &+\frac{\partial}{\partial x^j}(\Pi_\varepsilon^h(t)\,1)(x_k(t))\,u(x_k(t),t)\Big\}.\end{aligned} \tag{5.15}$$

We now prove a stability result.

Lemma 13. *Assume the hypotheses of Theorem 4. Then, the functions $t \to e(t)$ and $t \to \eta^i(t)$, $1 \leqq i \leqq n$, satisfy the energy inequality*

$$\|e(t)\|_{h,t}^2 + \int_0^t \sum_{i=1}^n \|\eta^i(s)\|_{h,s}^2 \, ds \leqq C \int_0^t \left\{ \|\varphi(s)\|_{h,s}^2 + \sum_{i=1}^n \|\sigma^i(s)\|_{h,s}^2 \right\} ds, \quad 0 \leqq t \leqq T \tag{5.16}$$

for some constant $C = C(T) > 0$.

Proof. We start from (5.12). By using the same arguments as in the proof of Theorem 4, we obtain under the conditions (1.28) and (2.12)

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|_{h,t}^2 + \sum_{k,l \in \mathbb{Z}^n} w_k(t)\, w_l(t) \sum_{i=1}^n (\eta_l^i(t) + \eta_k^i(t), e_k(t)) \cdot \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k(t) - x_l(t)) \leqq c_1 \|e(t)\|_{h,t}^2 + (\varphi(t), e(t))_{h,t}. \tag{5.17}$$

Consider next (5.14). It follows from the hypothesis (5.1) that the $np \times np$ matrix $(A^{ij}(x,t))_{1 \leqq i,j \leqq n}$ is invertible and its inverse matrix $(X^{ij}(x,t))_{1 \leqq i,j \leqq n}$ satisfies the inequalities

$$c_2 \sum_{i=1}^n |\xi^i|^2 \leqq \sum_{i,j=1}^n (X^{ij}(x,t)\,\xi^j, \xi^i) \leqq c_3 \sum_{i=1}^n |\xi^i|^2 \quad \forall \xi^i \in \mathbb{R}^n, \quad 1 \leqq i \leqq n. \tag{5.18}$$

Since

$$\sum_{j=1}^n X^{ij} A^{jh} = \delta^{ih} I$$

where δ^{ih} is the Kronecker symbol and I is the $p \times p$ identity matrix, we infer from (5.14) that

$$\sum_{j=1}^n X_k^{ij} \eta_k^j + \sum_{l \in \mathbb{Z}^n} w_l (e_l - e_k) \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k - x_l) = \sum_{j=1}^n X_k^{ij} \sigma_k^j$$

where $X_k^{ij} = X_k^{ij}(t) = X^{ij}(x_k(t), t)$. Therefore, we have

$$\sum_{k \in \mathbb{Z}^n} w_k \sum_{i,j=1}^n (X_k^{ij} \eta_k^j, \eta_k^i) + \sum_{k,l \in \mathbb{Z}^n} w_k w_l \sum_{i=1}^n (e_l - e_k, \eta_k^i) \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k - x_l) = \sum_{k \in \mathbb{Z}^n} w_k \sum_{i,j=1}^n (X_k^{ij} \sigma_k^j, \eta_k^i). \tag{5.19}$$

Now, it follows from (1.28) that

$$\sum_{k,l\in\mathbb{Z}^n} w_k w_l\{(\eta_l^i+\eta_k^i, e_k)+(e_l-e_k, \eta_k^i)\} \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k-x_l)$$

$$= \sum_{k,l\in\mathbb{Z}^n} w_k w_l\{(\eta_l^i, e_k)+(e_l, \eta_k^i)\} \frac{\partial \zeta_\varepsilon}{\partial x^i}(x_k-x_l)=0.$$

Hence, combining (5.18) and (5.20) gives

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|_{h,t}^2+\sum_{k\in\mathbb{Z}^n} w_k(t)\sum_{i,j=1}^n (X_k^{ij}(t)\,\eta_k^j(t), \eta_k^i(t))$$

$$\leqq c_1\|e(t)\|_{h,t}^2+(\varphi(t), e(t))_{h,t}\sum_{k\in\mathbb{Z}^n} w_k(t)\sum_{i,j=1}^n (X_k^{ij}(t)\,\sigma_k^j(t), \eta_k^i(t))$$

and the inequality (5.17) follows easily.

Let us then state the convergence result.

Theorem 5. *Assume the hypotheses* (i), (ii), (iii) *of Theorem* 2 *together with the condition* (2.12). *Assume in addition that the solution u of Problem* (5.2), (5.3) *belongs to the space $C^0(0, T; W^{\mu+1,\infty}(\mathbb{R}^n)^p)$ where $\mu=\max(r+1, m)$ and satisfies for some $\gamma>\frac{n}{2}$ and for all $\beta\in\mathbb{N}^n$ with $|\beta|\leqq\mu+1$*

$$|\partial^\beta u(x,t)|\leqq c(1+|x|)^{-\gamma}, \qquad x\in\mathbb{R}^n,\ t\in[0,T]. \tag{5.20}$$

Then, there exists a constant $C=C(u,T)>0$ such that

$$\max_{0\leqq t\leqq T}\|u(.,t)-u_\varepsilon^h(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)} + \left(\int_0^T\sum_{i=1}^n\|p^i(.,t)-p_\varepsilon^{h,i}(.,t)\|_{\mathbb{L}^2(\mathbb{R}^n)}^2\,dt\right)^{1/2}\leqq C\left(\varepsilon^r+\frac{h^m}{\varepsilon^{m+1}}\right) \tag{5.21}$$

Proof. The proof mimics that of Theorem 2. Using (5.13), (5.20) for $|\beta|\leqq\mu$ and Theorem 3 together with its corollary, we obtain

$$\|\varphi(t)\|_{h,t}\leqq c_1\left(\varepsilon^r+\frac{h^m}{\varepsilon^{m+1}}\right)\Big(\sum_{k\in\mathbb{Z}^n} w_k(t)(1+|x_k(t)|)^{-2\gamma}\Big)^{1/2}\leqq c_2\left(\varepsilon^r+\frac{h^m}{\varepsilon^{m+1}}\right).$$

Next, using again (5.20) and the smoothness of the functions $x\to A^{ij}(x,t)$, we have

$$|\partial^\beta p^i(x,t)|\leqq c_3(1+|x|)^{-\gamma}, \qquad |\beta|\leqq\mu.$$

Hence, (5.15) gives as above

$$\|\sigma^i(t)\|_{h,t}\leqq c_4\left(\varepsilon^r+\frac{h^m}{\varepsilon^{m+1}}\right), \qquad 1\leqq i\leqq n.$$

By applying Lemma 13, we obtain

$$\|e(t)\|_{h,t}^2+\int_0^t\sum_{i=1}^n\|\eta^i(s)\|_{h,s}^2\,ds\leqq c_5\left(\varepsilon^r+\frac{h^m}{\varepsilon^{m+1}}\right)^2. \tag{5.22}$$

The desired error estimate (5.21) then follows from (5.22), Theorem 3 and Lemma 5.

6. Appendix

In Theorem 2, one can slightly relax the smoothness properties of the cut-off function ζ in order to consider the case $m<n$. We begin by stating an extension of Theorem 3.

Theorem 3 Bis. *Let $m \geqq 1$ be an integer. We assume that the hypothesis* (3.4) *holds. We assume in addition that the cut-off function ζ has a compact support, satisfies the conditions* (2.11) *for some integer $r \geqq 1$ and belongs to the space $W^{m+s,\infty}(\mathbb{R}^n)$ for some other integer $s \geqq 0$. Then, there exists a constant $C=C(T)>0$ such that we have for all function $v \in W^{\mu,p}(\mathbb{R}^n)$, $\mu=\max(r+s, m)$, $\frac{n}{m}<p \leqq +\infty$*

$$|v-\Pi_\varepsilon^h(t)v|_{s,p,\mathbb{R}^n} \leqq C\left(\varepsilon^r |v|_{r,s,p,\mathbb{R}^n} + \left(1+\frac{h}{\varepsilon}\right)^{n/q} \frac{h^m}{\varepsilon^{m+s}} \|v\|_{m,p,\mathbb{R}^n}\right. \tag{6.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In fact, using the techniques of the proof of [19, Theorem 5.1], it is an easy matter to check that, if $v \in W^{m,p}(\mathbb{R}^n)$ with $p>\frac{n}{m}$, we have

$$|(v-\Pi^h(t)v)*\zeta_\varepsilon|_{s,p,\mathbb{R}^n} \leqq c_1 \left(1+\frac{h}{\varepsilon}\right)^{n/q} \frac{h^m}{\varepsilon^{m+s}} \|v\|_{m,p,\mathbb{R}^n} \tag{6.2}$$

so that (6.1) holds.

If moreover v belongs to $W^{\mu,\infty}(\mathbb{R}^n)$ and satisfies the conditions (3.10) for $|s| \leqq \mu$, we obtain for all $\alpha \in \mathbb{N}^n$ with $|\alpha|=s$

$$|\partial^\alpha(v-\Pi_\varepsilon^h(t)v)(x)| \leqq c_2 \left(1+\frac{h}{\varepsilon}\right)^n \frac{h^m}{\varepsilon^{m+s}} (1+|x|)^{-\gamma} \tag{6.3}$$

Now, argueing as in the proof of Theorem 2, we can show that the error bound (2.14) still holds for $m \geqq 1$ provided that in the hypotheses of Theorem 2 we replace the assumption (i) by the following one:

(i)' the cut-off function ζ belongs to $W^{m+1,\infty}(\mathbb{R}^n)$ and has a compact support.

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