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Sard Kernel Theorems on Triangular Domains with Application to Finite Element Error Bounds*

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Summary. Error bounds for interpolation remainders on triangles are derived by means of extensions of the Sard Kernel Theorems. These bounds are applied to the Galerkin method for elliptic boundary value problems. Certain kernels are shown to be identically zero under hypotheses which are, for example, fulfilled by tensor product interpolants on rectangles. This removes certain restrictions on how the sides of the triangles and/or rectangles tend to zero. Explicit error bounds are computed for piecewise linear interpolation over a triangulation and applied to a model problem.

1. Introduction

In this paper, a Kernel Theorem of Sard [8] is extended to construct error bounds for interpolation remainders defined on triangles. The Kernel Theorem provides an exact representation of linear functionals which are admissible on spaces of functions with a prescribed smoothness. The theory has application to the finite element analysis of elliptic boundary value problems, since the interpolation remainder is an upper bound on the finite element remainder in the energy norm (see Section 2).

The theory of Sard is well suited to the calculation of interpolation remainders defined on rectangles. Birkhoff, Schultz, and Varga [6] use the theory to derive bounds for tensor product Hermite interpolation. We show that the theory can be extended to treat triangular and other domains (see Section 3). This theory provides a constructive method of computing the constants in the error bounds, which the Bramble-Hilbert Lemma approach does not yield.

In Section 4, we prove a *Zero Kernel Theorem* which states sufficient conditions for certain of the Sard kernels to be identically zero. This theorem implies that finite element remainder functionals do not involve all possible derivatives of a certain order, and this permits avoidance of mesh restrictions. In particular, tensor product interpolants satisfy the hypotheses of the Zero Kernel Theorem (see Section 5) and thus the mesh restrictions in Birkhoff, Schultz, and Varga are not necessary. Lagrange interpolants for triangles also are covered by the Zero Kernel Theorem, in Section 6.

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We conclude with a computed example of the constants in the error bounds for piecewise linear interpolation in Section 6 and their application to a model elliptic problem in Section 7.

2. Interpolation Remainders and the Galerkin Method

In Sections 5 and 6 we consider bounds for interpolation remainders of a bivariate function $F(x, y)$ defined on the Sobolev space $W_r^n(\Omega)$, where Ω is a rectangle or a triangle. More generally, let Ω be a simply connected bounded region that satisfies a restricted cone condition in the xy-plane. The Sobolev space $W_r^n(\Omega)$, $1 \leqq r \leqq \infty$, integer $n \geqq 0$, is the space of functions such that all generalized derivatives of order $\leqq n$ are in $L_r(\Omega)$. A norm for $W_r^n(\Omega)$ is

$$\|F\|_{W_r^n(\Omega)} = \left\{ \sum_{0 \leqq |\alpha| \leqq n} \|D^\alpha F\|_{L_r(\Omega)}^r \right\}^{1/r}, \quad 1 \leqq r \leqq \infty \tag{2.1}$$

where $\alpha = (\alpha_i, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, and $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$.

Let P be an interpolation projector defined on $F \in W_r^n(\Omega)$. The remainders of interest in the Sobolev space are

$$R_{h,k}[F(x, y)] = \frac{\partial^{h+k}}{\partial s^h \partial t^k} R[F(x, y)] \qquad 0 \leqq h + k < n, \tag{2.2}$$

where

$$R[F(x, y)] \equiv F(s, t) - P[F(s, t)]. \tag{2.3}$$

For fixed (s, t) (2.2) and (2.3) define linear functionals on $F(x, y) \in W_r^n(\Omega)$.

If the interpolation function is piecewise defined over a subdivision of a polygonal region Ω into a union of disjoint triangular elements Ω_e, then each element can be considered separately since

$$\|F\|_{W_r^n(\Omega)} = \left\{ \sum_e \|F\|_{W_r^n(\Omega_e)}^r \right\}^{1/r}, \quad 1 \leqq r \leqq \infty. \tag{2.4}$$

Interpolation remainder theory has application to finite element remainder theory. Following Varga [9], we consider linear elliptic operators in divergence form:

$$L u(x, y) = \sum_{|\alpha| \leqq n} (-1)^{|\alpha|} D^\alpha [p_\alpha(x, y)\, D^\alpha u(x, y)] \tag{2.5}$$

where the p_α are in $L_\infty(\Omega)$. The nonhomogeneous boundary value problem corresponding to L is to find $u \in W_2^n(\Omega)$ such that:

$$L u(x, y) = g(x, y), \tag{2.6}$$

$$D^\beta u(x, y) = f_\beta(x, y), \qquad (x, y) \in \partial\Omega \qquad \text{for } 0 \leqq |\beta| \leqq n - 1. \tag{2.7}$$

The homogeneous problem is that all the f_β are identically zero, the relevant Sobolev space then being called $\mathring{W}_2^n(\Omega)$. A norm in $\mathring{W}_2^n(\Omega)$ is

$$\|v\|_{\mathring{W}_n^2(\Omega)} = \left\{ \sum_{|\alpha| = n} \|D^\alpha v\|_{L_2(\Omega)}^2 \right\}^{\frac{1}{2}}.$$

Let

$$a(u, v) = \sum_{|\alpha| \leqq n} \iint_\Omega p_\alpha(x, y)\, D^\alpha u(x, y)\, D^\alpha v(x, y)\, dx\, dy. \tag{2.8}$$

Then the *weak problem* corresponding to (2.6) and (2.7) is to find u satisfying (2.7) and such that

$$a(u, v) = (g, v) \quad \text{for all } v \text{ in } \mathring{W}_2^n(\Omega). \tag{2.9}$$

We consider interpolants $\tilde{u}$ to u, where the interpolation conditions are the following:

$$L_i(\tilde{u}) = L_i(u), \quad i = 1, \ldots, I,$$
$$M_j(\tilde{u}) = M_j(u), \quad j = 1, \ldots, J,$$

and the L_i and M_j are interpolation functionals such that the $L_i(u)$ are unknown and the $M_j(u)$ are known a priori from (2.7).

Let V^h be an $(I+J)$-dimensional subspace of $W_2^n(\Omega)$ such that the L_i and M_j are linearly independent over V^h. Then V^h has a basis of functions $\{B_i(x, y)\}_{i=1}^I$ and $\{C_j(x, y)\}_{j=1}^J$ that are biorthonormal with respect to the L_i and M_j [2]. Let S^h be the subset of $W_2^n(\Omega)$ which consists of functions U of the form

$$U(x, y) = \sum_{i=1}^{I} a_i B_i(x, y) + \sum_{j=1}^{J} M_j(u)\, C_j(x, y)$$

where the a_i are numbers. Let S_0^h be the m-dimensional subspace generated by the B_i. The *Galerkin method* is to find U in S^h such that

$$a(U, v) = (g, v) \quad \text{for all } v \text{ in } S_0^h. \tag{2.10}$$

Under the assumption that

$$S_0^h \subset \mathring{W}_2^n(\Omega),$$

the following lemma applies:

Lemma 2.1. The Galerkin approximation U is the *best approximation* from S^h to u in the energy norm induced by the inner product $a(u, v)$. That is,

$$a(u-U, u-U) \leqq a(u-\tilde{u}, u-\tilde{u}) \quad \text{for all } \tilde{u} \text{ in } S^h. \tag{2.11}$$

In fact,

$$a(u-U, u-U) + a(\tilde{u}-U, \tilde{u}-U) = a(u-\tilde{u}, u-\tilde{u}). \tag{2.12}$$

3. Sard Kernel Theorems and Interpolation Remainder Theory

The Kernel Theorems of Sard [8] involve Taylor series expansions of a function $F(x, y)$ about a point (a, b). These expansions have a rectangular domain of influence and Sard restricts the theory to function spaces defined on rectangles. However, the theory can be extended to bounded regions Ω which satisfy the following property:

Property 3.1. There is a rectangular coordinate system and a point $(a, b) \in \bar{\Omega}$ such that for all $(x, y) \in \bar{\Omega}$ the rectangle with opposite corners at (a, b) and (x, y) is contained in $\bar{\Omega}$, where the sides of the rectangle are parallel to the axes $x = 0$ and $y = 0$ of the rectangular coordinate system.

If Ω is a rectangle, then (a, b) can be an arbitrary point in the rectangle. If Ω is a triangle, then (a, b) can be the point on the longest side which is at the foot of the perpendicular to this side from the opposite vertex. Alternatively, for a right-angled triangle the vertex at the right angle can be taken as (a, b).

Sard Taylor expansion. Sard defines a function space "boldface $B_{p,q}(\Omega)$" of functions for which a certain type of Taylor expansion exists, where p and q are positive integers and $p+q=n$. The Taylor series expansion involves the triangular scheme of all derivatives of order less than or equal to n executed in a certain manner, see Fig. 3.1. The Taylor expansion is presented in the following theorem for $F\in C^n(\Omega)$ and the space boldface $B_{p,q}$ is then developed in Corollary 3.1.

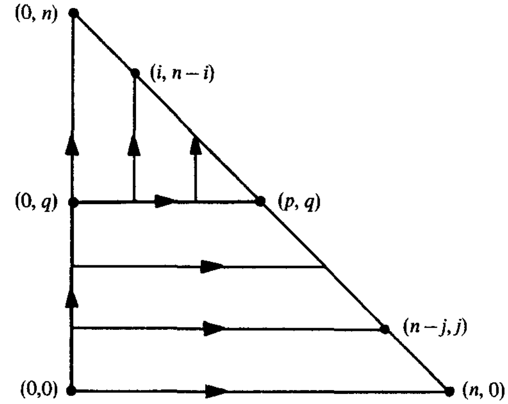

Fig. 3.1. Index triangle of partial derivatives

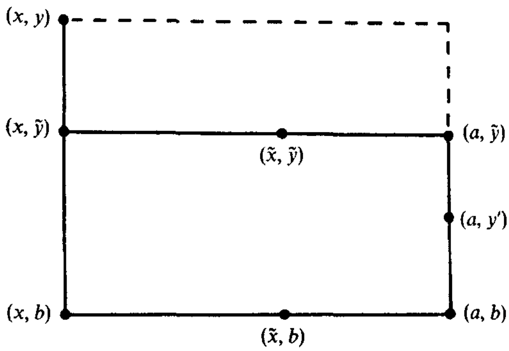

Fig. 3.2. Function arguments in Taylor expansion

Theorem 3.1. Let $F\in C^n(\Omega)$ where Ω is a region which satisfies Property 3.1. Then F has the following Taylor expansion at (x, y) about (a, b):

$$\begin{aligned} F(x,y) = & \sum_{i+j<n} (x-a)^{(i)}(y-b)^{(j)} F_{i,j}(a,b) \\ & + \sum_{j<q} (y-b)^{(j)} \int_a^x (x-\tilde{x})^{(n-j-1)} F_{n-j,j}(\tilde{x}, b)\, d\tilde{x} \\ & + \int_b^y (y-\tilde{y})^{(q-1)} \int_a^x (x-\tilde{x})^{(p-1)} F_{p,q}(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \\ & + \sum_{i<p} (x-a)^{(i)} \int_b^y (y-\tilde{y})^{(n-i-1)} F_{i,n-i}(a,\tilde{y})\, d\tilde{y}, \end{aligned} \tag{3.1}$$

where p and q are positive integers such that $p+q=n$ and $(x-a)^{(i)}\equiv(x-a)^i/i!$ etc.

Proof. For $F\in C^n(\Omega)$ the following single variable expansions can be obtained by integration by parts (cf. Figs. 3.1 and 3.2)

$$F(x,y)=\sum_{j=0}^{q-1}(y-b)^{(j)}F_{0,j}(x,b)+\int_b^y(y-\tilde{y})^{(q-1)}F_{0,q}(x,\tilde{y})\,d\tilde{y}, \tag{3.2}$$

$$F_{0,q}(x,\tilde{y})=\sum_{i=0}^{p-1}(x-a)^{(i)}F_{i,q}(a,\tilde{y})+\int_a^x(x-\tilde{x})^{(p-1)}F_{p,q}(\tilde{x},\tilde{y})\,d\tilde{x}, \tag{3.3}$$

$$F_{0,j}(x,b)=\sum_{j=0}^{n-j-1}(x-a)^{(i)}F_{i,j}(a,b)+\int_a^x(x-\tilde{x})^{(n-j-1)}F_{n-j,j}(\tilde{x},b)\,d\tilde{x}, \tag{3.4}$$

$$F_{i,q}(a,\tilde{y})=\sum_{j=q}^{n-i-1}(\tilde{y}-b)^{(j-q)}F_{i,j}(a,b)+\int_b^{\tilde{y}}(\tilde{y}-y')^{(p-i-1)}F_{i,n-i}(a,y')\,dy'. \tag{3.5}$$

The Taylor expansion (3.1) is derived by combining Eqs. (3.2)–(3.5), where, by integration by parts,

$$\int_b^y(y-\tilde{y})^{(q-1)}\int_b^{\tilde{y}}(\tilde{y}-y')^{(p-i-1)}F_{i,n-i}(a,y')\,dy'd\tilde{y}$$
$$=\int_b^y(y-\tilde{y})^{(n-i-1)}F_{i,n-i}(a,\tilde{y})\,d\tilde{y}.\quad \text{Q.E.D.}$$

Corollary 3.1. Let F satisfy the following properties on Ω:

(i) $F(x,y)$ is $q-1$ times continuously differentiable with respect to y and $F_{0,q-1}(x,y)$ is absolutely continuous with respect to y.

(ii) $F_{0,q}(x,\tilde{y})$ is $p-1$ times continuously differentiable with respect to x and $F_{p-1,q}(x,\tilde{y})$ is absolutely continuous with respect to x, almost everywhere $\tilde{y}$.

(iii) $F_{0,j}(x,b)$ is $n-j-1$ times continuously differentiable with respect to x and $F_{n-j-1,j}(x,b)$ is absolutely continuous with respect to x, $0\leqq j\leqq q-1$

(iv) $F_{i,q}(a,\tilde{y})$ is $n-i-1$ times continuously differentiable with respect to $\tilde{y}$ and $F_{i,n-i-1}(a,\tilde{y})$ is absolutely continuous with respect to $\tilde{y}$.
Then the Taylor expansion (3.1) exists.

Proof. The properties (i)–(iv) are required for the existence of expansions (3.2)–(3.5). The property (ii) need exist only a.e. $\tilde{y}$ since it is required under the Lebesgue integral in (3.2). Q.E.D.

Remark. In Corollary 3.1 the differentiations are performed according to a particular ordering rule consistent with the expansion (3.2)–(3.5), namely

$$F_{i,j}=D_y^{j-j_0}D_x^iD_y^{j_0}F,$$

where $D_x\equiv\partial/\partial x$, $D_y\equiv\partial/\partial y$ and $j_0=\min(j,q)$.

Properties (i)–(iv) are an equivalent definition of the function space boldface $B_{p,q}(\Omega)$ to that in Sard [8, p. 172], when Ω is a rectangle.

Corollary 3.2. Let $F \in W_r^n(\Omega)$, $1 \leqq r \leqq \infty$. Then the Taylor expansion (3.1) exists a.e. (a, b) where the derivatives are now generalized derivatives.

The Taylor expansion (3.1) can be amended to an expansion involving definite integrals by the following device: Let

$$\begin{aligned} \max_{(x,b)\in\bar{\Omega}} (x) &= \bar{\alpha}, & \min_{(x,b)\in\bar{\Omega}} (x) &= \underline{\alpha} \\ \max_{(a,y)\in\bar{\Omega}} (y) &= \bar{\beta}, & \min_{(a,y)\in\bar{\Omega}} (y) &= \underline{\beta} \end{aligned} \tag{3.6}$$

and let the function ψ be defined by

$$\psi(a, \tilde{x}, x) = \begin{cases} 1 & \text{if } a \leqq \tilde{x} < x, \\ -1 & \text{if } x \leqq \tilde{x} < a, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Then the Taylor expansion (3.1) can be written as

$$\begin{aligned} F(x, y) = {} & \sum_{i+j<n} (x-a)^{(i)} (y-b)^{(j)} F_{i,j}(a, b) \\ & + \sum_{j<q} \int_{\underline{\alpha}}^{\bar{\alpha}} (x-\tilde{x})^{(n-j-1)} \psi(a, \tilde{x}, x) (y-b)^{(j)} F_{n-j,j}(\tilde{x}, b)\, d\tilde{x} \\ & + \iint_{\Omega} (x-\tilde{x})^{(p-1)} \psi(a, \tilde{x}, x) (y-\tilde{y})^{(q-1)} \psi(b, \tilde{y}, y) F_{p,q}(\tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y} \\ & + \sum_{i<p} \int_{\underline{\beta}}^{\bar{\beta}} (x-a)^{(i)} (y-\tilde{y})^{(n-i-1)} \psi(b, \tilde{y}, y) F_{i,n-i}(a, \tilde{y})\, d\tilde{y}. \end{aligned} \tag{3.8}$$

Sard Kernel Theorem. The Sard Kernel Theorem applies to a class of admissible functionals defined on the space boldface $B_{p,q}$ as follows:

Definition 3.1. The admissible functionals R on boldface $B_{p,q}(\Omega)$, where Ω is a region satisfying Property 3.1, are of the following form:

$$\begin{aligned} R[F(x, y)] = {} & \sum_{\substack{i<p \\ j<q}} \iint_{\Omega} F_{i,j}(x, y)\, d\mu^{i,j}(x, y) \\ & + \sum_{\substack{i+j<n \\ i \geqq p}} \int_{\underline{\alpha}}^{\bar{\alpha}} F_{i,j}(x, b)\, d\mu^{i,j}(x) \\ & + \sum_{\substack{i+j<n \\ j \geqq q}} \int_{\underline{\beta}}^{\bar{\beta}} F_{i,j}(a, y)\, d\mu^{i,j}(y), \end{aligned} \tag{3.9}$$

where the $\mu^{i,j}$ are functions of bounded variation and $\underline{\alpha}$, $\bar{\alpha}$, $\underline{\beta}$, and $\bar{\beta}$ are defined by (3.6).

Example. Let $R[F] = F_{i,j}(c, d)$, $(c, d) \in \Omega$.
Then $R[F] = \iint_{\Omega} F_{i,j}(x, y)\, d\mu(x, y)$

where

$$\mu(x, y) = \begin{cases} 1 & c \leqq x, \quad d \leqq y \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2 (Kernel Theorem). Let Ω satisfy Property 3.1 and let R be an admissible functional on boldface $B_{p,q}(\Omega)$. If $F(x,y)\in$ boldface $B_{p,q}(\Omega)$, then

$$\begin{aligned} R[F] = & \sum_{i+j<n} c^{i,j} F_{i,j}(a,b) \\ & + \sum_{j<q} \int_{\underline{\alpha}}^{\bar{\alpha}} K^{n-j,j}(\tilde{x}) F_{n-j,j}(\tilde{x},b)\, d\tilde{x} \\ & + \iint_{\Omega} K^{p,q}(\tilde{x},\tilde{y}) F_{p,q}(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \\ & + \sum_{i<p} \int_{\underline{\beta}}^{\bar{\beta}} K^{i,n-i}(\tilde{y}) F_{i,n-i}(a,\tilde{y})\, d\tilde{y}, \end{aligned} \tag{3.10}$$

where

$$c^{i,j} = R_{(x,y)}[(x-a)^{(i)}(y-b)^{(j)}], \quad i+j<n, \tag{3.11}$$

$$K^{n-j,j}(\tilde{x}) = R_{(x,y)}[(x-\tilde{x})^{(n-j-1)}\psi(a,\tilde{x},x)(y-b)^{(j)}], \quad j<q,\ \tilde{x}\notin Jx, \tag{3.12}$$

$$K^{i,n-i}(\tilde{y}) = R_{(x,y)}[(x-a)^{(i)}(y-\tilde{y})^{(n-i-1)}\psi(b,\tilde{y},y)], \quad i<p,\ \tilde{y}\notin Jy, \tag{3.13}$$

$$K^{p,q}(\tilde{x},\tilde{y}) = R_{(x,y)}[(x-\tilde{x})^{(p-1)}\psi(a,\tilde{x},x)(y-\tilde{y})^{(q-1)}\psi(b,\tilde{y},y)], \quad \tilde{x}\notin \bar{J}x,\ \tilde{y}\notin \bar{J}y. \tag{3.14}$$

The notation $R_{(x,y)}$ means that R is applied to functions in the variables x and y. Jx is the jump set consisting of the points of discontinuity of the functions of total variation $|\mu^{n-j-1,j}|(x)$, $j<q$, and Jy is dual. $\bar{J}x$ is the jump set consisting of the points of discontinuity of $|\mu^{p-1,j'}|(x,y)$ evaluated at $y(x)=\max_{(x,y)\in\Omega}\{y\}$, $j'<q$, and $\bar{J}y$ is dual.

Remark. The jump sets are the points at which the kernels (3.12)–(3.14) are undefined, but these sets are of measure zero with respect to the Lebesgue-Stieltjes integrals in (3.10).

Proof of Theorem 3.2. The application of the functional R defined by (3.9) to the Taylor expansions (3.8), and the use of Fubini's Theorem to change the order of integration, give the desired result. Q.E.D.

Remark. If the functional is not admissible, then it can be applied to the Taylor expansion (3.1) directly.

Corollary 3.3. Let $F\in W_r^n(\Omega)$, $1\leq r\leq\infty$. Then the Kernel Theorem 3.2 is true a.e. (a,b) where the derivatives are now generalised derivatives.

Remark. For the interpolation remainder functionals (2.2) the corresponding kernels (3.12)–(3.14) depend on s and t and will be written:

$$K^{n-j,j}(s,t;\tilde{x}),\ K^{i,n-i}(s,t;\tilde{y}),\ K^{p,q}(s,t;\tilde{x},\tilde{y}). \tag{3.15}$$

Also, if $n=p+q$ is chosen such that $\mathscr{P}_{n-1}=\{x^i y^j/0\leq i+j\leq n-1\}$ is contained in the precision set of the interpolation operator P, then the $c_{i,j}$ in (3.11) are zero for the remainder functionals $R_{h,k}$.

4. Zero Kernels

Theorem 4.1 (Zero Kernel Theorem). Let R be a linear operator which maps $F(x,y)\in$ boldface $B_{p,q}(\Omega)$ to functions of (s,t) such that R is an admissible linear functional for fixed s and t. Also let

$$R_{h,k}[F(x,y)] = \frac{\partial^{h+k}}{\partial s^h \partial t^k} R[F(x,y)], \quad h, k > 0, \tag{4.1}$$

be an admissible functional for fixed s and t. Let $p_i(x)$ be a polynomial in x of degree $\leqq i$ and $q_j(y)$ be a polynomial in y of degree $\leqq j$. Then the Sard kernels for $R_{h,k}$ have the property that

$$K^{i,n-i}(s,t;\tilde{y}) = 0, \quad 0 \leqq i < \min\{h, p\}, \tag{4.2}$$

if

$$R[p_i(x) g(y)] = r(s,t) \tag{4.3}$$

is such that $r(s,t)$ considered as a function of s alone is a polynomial of degree $< h$. Dually

$$K^{n-j,j}(s,t;\tilde{x}) = 0, \quad 0 \leqq j < \min\{k, q\}, \tag{4.4}$$

if

$$R[f(x)\, q_j(y)] = r(s,t) \tag{4.5}$$

is such that $r(s,t)$ considered as a function of t alone is a polynomial of degree $< k$.

Proof. It is sufficient to prove (4.2) as (4.4) is a dual result. Let i be an integer such that $0 \leqq i < \min(h, p)$. Then the kernel $K_R^{i,n-i}(s,t;\tilde{y})$ corresponding to R is

$$K_R^{i,n-i}(s,t;\tilde{y}) = R_{(x,y)}[(x-a)^{(i)}(y-\tilde{y})^{(n-i-1)}\psi(b,\tilde{y},y)], \ \tilde{y} \notin J_y.$$

which, considered as a function of s, is a polynomial of degree $< h$, by the hypothesis (4.3). Thus the kernel $K^{i,n-i}(s,t;\tilde{y})$ corresponding to $R_{h,k}$ is

$$K^{i,n-i}(s,t;\tilde{y}) = \frac{\partial^{h+k}}{\partial s^h \partial t^k} K_R^{i,n-i}(s,t;\tilde{y}) = 0, \quad 0 \leqq i < \min\{h, p\}. \quad \text{Q.E.D.}$$

Schematically, the possibilities for the domain of influence of a functional $R_{h,k}$, which satisfies the hypotheses of the Zero Kernel Theorem in the Sard space boldface $B_{p,q}$, are shown in Fig. 4.1.

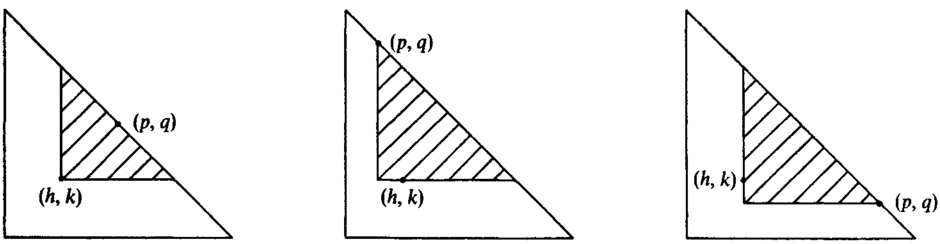

Fig. 4.1. Non-zero kernels in the Zero Kernel Theorem

Remark. Although not stated in the Theorem, the $c^{i,j}$, (3.11), are zero for $0 \leqq i < \min(h, p)$, $0 \leqq j < \min(k, q)$.

5. Error Bounds for Tensor Product Interpolation

We next show that the kernels (4.2) and (4.4) are always zero for tensor product schemes with sufficient polynomial precision.

Theorem 5.1. Tensor product interpolants of polynomial precision of at least $h-1$ in the variable x and at least $k-1$ in the variable y satisfy the hypotheses (4.3) and (4.5) of the Zero Kernel Theorem.

Proof. P a tensor product interpolation operator implies that P is of the form

$$P = P_x P_y = P_y P_x \tag{5.1}$$

where P_x is an operator in the variable x and P_y is dual in y. Therefore, if $F(x, y) = p_i(x)\, g(y)$ where $p_i(x)$ is a polynomial in x of degree $i < h$, then

$$\begin{aligned} P[p_i(x)\, g(y)] &= P_y P_x [p_i(x)\, g(y)] = P_y [p_i(x)\, g(y)] \\ &= p_i(x)\, P_y [g(y)] \equiv r(x, y). \end{aligned}$$

$r(x, y)$ satisfies (4.3), so that (4.2) follows. The argument for (4.5) is dual. Q.E.D.

Birkhoff, Schultz, and Varga derive error bounds for tensor product Hermite interpolation using the Sard Kernel Theorem. Here, P_x is defined by

$$P_x[f(x)] = \sum_{i=0}^{N} \phi_i(x)\, f^{(i)}(0) + \sum_{i=0}^{N} \psi_i(x)\, f^{(i)}(1), \tag{5.2}$$

where the $\phi_i(x)$ and $\psi_i(x) = (-1)^i \phi_i(1-x)$ are the cardinal basis functions for Hermite two point Taylor interpolation on $[0, 1]$. P_y is dual and the resulting tensor product interpolant is on $S = [0, 1] \times [0, 1]$. An important observation is that the point (a, b) of the Taylor expansion can be chosen as the point (s, t) of the remainder functionals $R_{h,k}$ (cf. Property 3.1) where $R_{h,k}$ is defined by (2.2). The remainder functional $R_{h,k}$ is then admissible in boldface $B_{m,m}(S)$ where $m = N + 1$ and $0 \leqq h + k \leqq 2N + 1$. ($R_{h,k}$ is precise for the set $\mathcal{P}_{2N+1}$.) An analysis similar to that in Birkhoff, Schultz, and Varga using the Sard Kernel Theorem, but with the application also of the Zero Kernel Theorem then gives that

$$\|R_{h,k}[F]\|_{L_p(S)} \leqq \sum_{j=\min(k,\, m)}^{2m-\min(h,\, m)} C_j \|F_{2m-j,j}\|_{L_r(S)}, \quad 0 \leqq h + k < 2m,\ p \leqq r \tag{5.3}$$

where the C_j are constants (see Gregory [7] for further details). The choice of $(a, b) = (s, t)$ enables the derivation of the above bound in $W_r^{2m}(S)$ which is not possible in application of the Sard Kernel Theorem on a triangle (cf. (6.2)).

The summation over the range $\min(k, m) \leqq j \leqq 2m - \min(h, m)$ in (5.3) is a consequence of the Zero Kernel Theorem. A change of variable leads to the bound

$$\begin{aligned} \|\tilde{R}_{h,k}[\tilde{F}]\|_{L_p(\tilde{S})} \leqq &\sum_{j=\min(k,\, m)}^{2m-\min(h,\, m)} C_j\, H^{2m-j-h+1/p-1/r}\, K^{j-k+1/p-1/r} \|\tilde{F}_{2m-j,j}\|_{L_r(\tilde{S})}, \\ &0 \leqq h + k < 2m,\ p \leqq r. \end{aligned} \tag{5.4}$$

for the Hermite interpolation remainder $\tilde{R}_{h,k}$ for the function $\tilde{F}$ defined on $\tilde{S} = [0, H] \times [0, K]$. Now, when $h, k \leqq m\,(h + k \neq 2m)$ the summation in (5.4) is over $k \leqq j \leqq 2m - h$ and the exponents of H and K are then greater than or equal to zero and not simultaneously zero. Thus negative exponents of H and K are not possible in this case, which removes the need of a "regular" mesh restriction.

6. Error Bounds for Interpolation on Triangles

Let P be an interpolation operator on the standard triangle T with vertices at $(1, 0)$, $(0, 1)$, and $(0, 0)$, which is precise for the set $\mathscr{P}_{n-1}$ but not all of $\mathscr{P}_n$. Consider the remainder functional

$$R[F(x, y)] = F(s, t) - P[F(s, t)].$$

A point (a, b) which satisfies Property 3.1 on the standard triangle T is $(a, b) = (0, 0)$. Consider the Sard space boldface $B_{n-m,m}(T)$, where $m = [n/2]$. Suppose R is an admissible functional on boldface $B_{n-m,m}(T)$. Then application of the Sard Kernel Theorem and the precision of the interpolation operator P give that

$$\begin{aligned} R[F] = \sum_{j<m} \int_0^1 K^{n-j,j}(s, t; \tilde{x}) F_{n-j,j}(\tilde{x}, 0)\, d\tilde{x} \\ + \iint_T K^{n-m,m}(s, t; \tilde{x}, \tilde{y}) F_{n-m,m}(\tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y} \\ + \sum_{i<n-m} \int_0^1 K^{i,n-i}(s, t; \tilde{y}) F_{i,n-i}(0, \tilde{y})\, d\tilde{y}. \end{aligned} \tag{6.1}$$

From (6.1) it follows by the triangle inequality and Hölder's inequality that

$$\begin{aligned} \|R[F]\|_{L_p(s,t)} \leqq \sum_{j<m} \big\| \|K^{n-j,j}(s, t; \tilde{x})\|_{L_{r_1'}(\tilde{x})} \big\|_{L_p(s,t)} \|F_{n-j,j}(\tilde{x}, 0)\|_{L_{r_1}(\tilde{x})} \\ + \big\| \|K^{n-m,m}(s, t; \tilde{x}, \tilde{y})\|_{L_{r_2'}(\tilde{x},\tilde{y})} \big\|_{L_p(s,t)} \|F_{n-m,m}(\tilde{x}, \tilde{y})\|_{L_{r_2}(\tilde{x},\tilde{y})} \\ + \sum_{i<n-m} \big\| \|K^{i,n-i}(s, t; \tilde{y})\|_{L_{r_1'}(\tilde{y})} \big\|_{L_p(s,t)} \|F_{i,n-i}(0, \tilde{y})\|_{L_{r_1}(\tilde{y})}, \end{aligned} \tag{6.2}$$

where $L_p(s, t)$ denotes the L_p norm over the triangle T with respect to (s, t), $L_{r_1}(\tilde{x})$ denotes the L_{r_1} norm over $[0, 1]$ with respect to $\tilde{x}$, etc., and $1/r_1 + 1/r_1' = 1$, $1/r_2 + 1/r_2' = 1$. The norms of the kernels in (6.2) are constants which can be estimated. However, (6.2) is not a bound in the Sobolev space $W_r^n(T)$ because of the presence of the univariate norm terms. The device of taking $(a, b) = (s, t)$ to obtain Sobolev space results is not possible here, as it is for the rectangle, since the rectangular domain of influence of the Sard Taylor expansion would then go outside the triangle.

Remark. For $x, \tilde{x} \in T$, $(a, b) = (0, 0)$, (3.7) becomes

$$\psi(0, \tilde{x}, x) = \begin{cases} 1, & \tilde{x} < x \\ 0 & \text{otherwise.} \end{cases}$$

Thus the functions which occur in the kernels can be expressed in terms of the $+$ function as follows:

$$(x - \tilde{x})^{(i)} \psi(0, \tilde{x}, x) = (x - \tilde{x})_+^{(i)} = \begin{cases} (x - \tilde{x})^{(i)} & \text{if} \quad \tilde{x} < x, \\ 0 & \text{otherwise,} \end{cases} \tag{6.3}$$

etc.

The treatment of the functional

$$R_{h,k}[F] = \frac{\partial^{h+k}}{\partial s^h \partial t^k} R[F]$$

is best considered in relation to the particular interpolation operator P. Firstly, one must consider whether the functional is admissible, and secondly whether or not the Zero Kernel Theorem, is applicable. In the particular example of linear interpolation considered below, the $R_{1,0}$ and $R_{0,1}$ functionals are not admissible; however, results are derived by direct application of the functional to the Sard Taylor expansion. One example of an interpolant for which the hypotheses of the Zero Kernel Theorem holds is given in the following theorem:

Theorem 6.1. Let P be the interpolation operator defined by the Lagrange polynomial of degree N which interpolates $F(x, y)$ at the nodes $(x_i, y_j) = \left(\frac{i}{N}, \frac{j}{N}\right)$, $0 \leqq i + j \leqq N$, i.e.,

$$P[F(x, y)] = \sum_{0 \leqq i+j \leqq N} p_{i,j}(x, y)\, F(x_i, y_j), \tag{6.4}$$

where

$$p_{i,j}(x, y) = \frac{1}{i!} \prod_{\nu=0}^{i-1} (n x - \nu) \frac{1}{j!} \prod_{\nu=0}^{j-1} (n y - \nu) \frac{1}{k!} \prod_{\nu=0}^{k-1} (n z - \nu) \tag{6.5}$$

and $i + j + k = N$. Then P satisfies the hypotheses (4.2) and (4.5) of the Zero Kernel Theorem, where the i and j of that theorem satisfy $i < h \leqq N$; $j < k \leqq N$.

Proof. We show that $P[x^i g(y)]$ is a polynomial in x of degree $\leqq i$ for the nontrivial case $0 \leqq i < N$, from which the conclusion follows. Now,

$$\begin{aligned} P[x^i g(y)] &= \sum_{j'=0}^{N} \sum_{i'=0}^{N-j'} p_{i',j'}(x, y)\, x_{i'}^i\, g(y_{j'}) \\ &= \sum_{j'=0}^{N} a_{j'}(x, y)\, g(y_{j'}), \end{aligned} \tag{6.6}$$

where

$$a_{j'}(x, y) = \sum_{i'=0}^{N-j'} p_{i',j'}(x, y)\, x_{i'}^i. \tag{6.7}$$

The cardinal function $p_{i',j'}(x, y)$ is a polynomial of degree $N - j'$ in x. Thus for $j' \geqq N - i$, $a_{j'}(x, y)$ is a polynomial in x of degree $\leqq i$. Now for $0 \leqq j \leqq N - i - 1$, (6.6) with $g(y) = g_j(y) \equiv \prod_{\substack{\nu=0 \\ \nu \neq j}}^{N-i} (y - y_\nu)$ gives

$$\begin{aligned} P[x^i g_j(y)] &= \sum_{j'=0}^{N} g_j(y_{j'})\, a_{j'}(x, y) \\ &= g_j(y_j)\, a_j(x, y) + \sum_{j'=N-i+1}^{N} g_j(y_{j'})\, a_{j'}(x, y) \\ &= x^i g_j(y), \quad \text{by the precision of } P. \end{aligned}$$

The last two steps of the above equation can be considered as an equation in $a_j(x, y)$, from which it follows that $a_j(x, y)$ is a polynomial in x of degree $\leqq i$, $0 \leqq j \leqq N - i - 1$. The proof for $P[f(x)\, y^j]$ is dual. Q.E.D.

Example of linear interpolation on the triangle T. The following results are generalizations of those given in Barnhill and Whiteman [5]. Further details including calculations for quadratic interpolation on T can be found in Barnhill and Gregory [2]. Consider the linear interpolation remainder functionals:

$$\begin{aligned} R[F] &= F(s,t) - [F(1,0)\,s + F(0,1)\,t + F(0,0)\,(1-s-t)] \\ R_{1,0}[F] &= F_{1,0}(s,t) + F(0,0) - F(1,0) \\ R_{0,1}[F] &= F_{0,1}(s,t) + F(0,0) - F(0,1) \end{aligned} \tag{6.8}$$

defined on the Sard space boldface $B_{1,1}(T)$. The functional R is admissible on boldface $B_{1,1}(T)$ and thus from (6.2) it follows that

$$\begin{aligned} \|R[F]\|_{L_p(s,t)} \leqq C_1 \|F_{2,0}(\tilde{x},0)\|_{L_{r_1}(\tilde{x})} + C_2 \|F_{1,1}(\tilde{x},\tilde{y})\|_{L_{r_2}(\tilde{x},\tilde{y})} \\ + C_1 \|F_{0,2}(0,\tilde{y})\|_{L_{r_1}(\tilde{y})}, \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} C_1 &= \| \, \|K^{2,0}(s,t;\tilde{x})\|_{L_{r_1'}(\tilde{x})} \|_{L_p(s,t)} \\ &= \| \, \|K^{0,2}(s,t;\tilde{y})\|_{L_{r_1'}(\tilde{y})} \|_{L_p(s,t)}, \\ C_2 &= \| \, \|K^{1,1}(s,t;\tilde{x},\tilde{y})\|_{L_{r_2'}(\tilde{x},\tilde{y})} \|_{L_p(s,t)}, \end{aligned} \tag{6.10}$$

$1/r_1 + 1/r_1' = 1$, $1/r_2 + 1/r_2' = 1$, and

$$\begin{aligned} K^{2,0}(s,t;\tilde{x}) &= K^{0,2}(t,s;\tilde{x}) = R[(x-\tilde{x})_+] = (s-\tilde{x})_+ - s(1-\tilde{x}), \\ K^{1,1}(s,t;\tilde{x},\tilde{y}) &= R[(x-\tilde{x})_+^0 \,(y-\tilde{y})_+^0] = (s-\tilde{x})_+^0 \,(t-\tilde{y})_+^0 . \end{aligned} \tag{6.11}$$

Careful evaluation of (6.10) yields

$$C_1 = \begin{cases} \left(\dfrac{1}{r_1'+1}\right)^{1/r_1'} \{B(p+1, p+2)\}^{1/p}, & r_1' \leqq \infty, \quad p < \infty, \\ \dfrac{1}{4}\left(\dfrac{1}{r_1'+1}\right)^{1/r_1'}, & r_1' \leqq \infty, \quad p = \infty, \end{cases} \tag{6.12}$$

$$C_2 = \begin{cases} \{B(p/r_2'+1, p/r_2'+2)\}^{1/p}, & r_2', p < \infty, \\ \left(\dfrac{1}{4}\right)^{1/r_2'}, & r_2' < \infty, \quad p = \infty, \\ \left(\dfrac{1}{2}\right)^{1/p}, & r_2' = \infty, \quad p \leqq \infty, \end{cases} \tag{6.13}$$

where B is the Beta function.

The functional $R_{1,0}$ and its dual $R_{0,1}$ are not admissible on boldface $B_{1,1}(T)$. However direct application of $R_{1,0}$ to the Sard Taylor expansion in boldface $B_{1,1}(T)$ gives

$$\begin{aligned} R_{1,0}[F] &= R_{1,0}\left[\int_0^x (x-\tilde{x})\,F_{2,0}(\tilde{x},0)\,d\tilde{x}\right] + R_{1,0}\left[\int_0^y \int_0^x F_{1,1}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y}\right] \\ &\quad + R_{1,0}\left[\int_0^y (y-\tilde{y})\,F_{0,2}(0,\tilde{y})\,d\tilde{y}\right] \\ &= \int_0^1 K^{2,0}(s,t;\tilde{x})\,F_{2,0}(\tilde{x},0)\,d\tilde{x} + \int_0^1 K^{0,2}(s,t;\tilde{y})\,F_{0,2}(0,\tilde{y})\,d\tilde{y} \\ &\quad + \frac{\partial}{\partial s}\int_0^s \int_0^t F_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x} \end{aligned} \tag{6.14}$$

where the first and last terms can be evaluated in Sard kernel form with

$$K^{2,0}(s,t;\tilde{x}) = R_{1,0}[(x-\tilde{x})_+] = (s-\tilde{x})_+^0 - 1 + \tilde{x}, \qquad \tilde{x} \neq s,$$
$$K^{0,2}(s,t;\tilde{y}) = R_{1,0}[(y-\tilde{y})_+] = 0, \tag{6.15}$$

Also

$$\frac{\partial}{\partial s}\int_0^s\int_0^t F_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x} = \int_0^t F_{1,1}(s,\tilde{y})\,d\tilde{y}. \tag{6.16}$$

Remark. For the first kernel in (6.15) $\tilde{x}=s$ is a jump set. The second kernel is an example of the Zero Kernel Theorem to which Theorem 6.1 is applicable.

The L_p norm of (6.14) and the triangle inequality give

$$\|R_{1,0}[F]\|_{L_p(s,t)} \leqq C_3\|F_{2,0}(\tilde{x},0)\|_{L_{r_1}(\tilde{x})} + \left\|\int_0^t F_{1,1}(s,\tilde{y})\,d\tilde{y}\right\|_{L_p(s,t)} \tag{6.17}$$

where

$$C_3 = \|\,\|K^{2,0}(s,t;\tilde{x})\|_{L_{r_1'}(\tilde{x})}\|_{L_p(s,t)} \tag{6.18}$$

cf. (6.2).

Now, for $p<\infty$, Hölder's inequality gives that

$$\begin{aligned}\left\|\int_0^t F_{1,1}(s,\tilde{y})\,d\tilde{y}\right\|_{L_p(s,t)}^p &\leqq \int_0^1\int_0^{1-t}\|1\|_{L_{r_2'}(\tilde{y})}^p\|F_{1,1}(s,\tilde{y})\|_{L_{r_2}(\tilde{y})}^p\,ds\,dt\\ &\leqq \int_0^1 \|\,\|t^{p/r_2'}\|_{L_{q'}(s)}\|\,\|F_{1,1}(s,\tilde{y})\|_{L_{r_2}(\tilde{y})}^p\|_{L_q(s)}\,dt\\ &\leqq \int_0^1 (1-t)^{1/q'}\,t^{p/r_2'}\,dt\,\|F_{1,1}(s,\tilde{y})\|_{L_{r_2}(s,\tilde{y})}^p,\end{aligned} \tag{6.19}$$

where $L_{r_2}(\tilde{y})$ is over $[0,t]$, $L_q(s)$ is over $[0,1-t]$, $q=r_2/p$ and hence $p\leqq r_2\leqq\infty$, $q'=r_2/(r_2-p)$. Thus

$$\|R_{1,0}[F]\|_{L_p(s,t)} \leqq C_3\|F_{2,0}(\tilde{x},0)\|_{L_{r_1}(\tilde{x})} + C_4\|F_{1,1}(s,\tilde{y})\|_{L_{r_2}(s,\tilde{y})} \tag{6.20}$$

where

$$C_4 = \{B(p/r_2'+1,\,2-p-p/r_2')\}^{1/p}, \tag{6.21}$$

and C_3 is defined by (6.18) and (6.15). Calculation of the constant C_3 for the case $p=2$ gives

$$C_3 = \begin{cases}\sqrt{7}/2\sqrt{30}, & r_1'=1, \quad r_1=\infty,\\ 1/2\sqrt{3}, & r_1'=r_1=2,\\ \sqrt{7}/\sqrt{24}, & r_1'=\infty, \quad r_1=1.\end{cases} \tag{6.22}$$

The bound $\|R_{0,1}[F]\|_{L_p(s,t)}$ is the dual of (6.20).

Bounds for linear interpolation over an arbitrary triangle can be obtained by affine transformation. For example, the transformation of (6.20) to the triangle $\tilde{T}$ say with sides $[0,H]$ and $[0,K]$ gives

$$\begin{aligned}\|\tilde{R}_{1,0}[\tilde{F}]\|_{L_p(\tilde{T})} \leqq\; & C_3\,H^{1+1/p-1/r_1}\,K^{1/p}\|\tilde{F}_{2,0}(\tilde{s},0)\|_{L_{r_1}(\tilde{s})}\\ &+ C_4\,H^{1/p-1/r_2}\,K^{1+1/p-1/r_2}\|\tilde{F}_{1,1}\|_{L_{r_2}(\tilde{T})},\end{aligned} \tag{6.23}$$

where $L_{r_1}(s)$ is on $0 \leqq \tilde{s} \leqq H$. Such a result (which allows the degenerate case of one angle of the triangle *not* being bounded away from zero) is also shown by Babuska [1].

Remark. The derivation of error bounds for more general interpolants on a general triangle by consideration of an affine transformation from the standard triangle is only applicable when the interpolant is invariant under such a transformation.

Application of the Sard Kernel Theorem to calculate the constants in the error bounds for smooth interpolation on triangles is considered in Barnhill and Mansfield [4].

7. Application to Differential Equations

We consider the model problem

$$\Delta u = 0 \quad \text{on} \quad \Omega = [0, \pi] \times [0, \pi], \tag{7.1}$$

$$u|_{\partial\Omega} = 0 \text{ except that } u(x, 0) = \sin x, \qquad 0 \leqq x \leqq \pi,$$

discussed in Barnhill, Gregory, and Whiteman [3]. For linear interpolation on a triangle T_h, we obtain the following bounds:

$$\|u - p_1\|_{L_2(T_h)} \leqq \frac{h^3}{\sqrt{40}}, \tag{7.2}$$

$$\|u - p_1\|_{C(T_h)} \leqq \frac{h^2}{2}, \tag{7.3}$$

$$\|\nabla(u - p_1)\|_{L_2(T_h)} \leqq \sqrt{\tfrac{2}{5}}\, h^2, \tag{7.4}$$

$$\begin{aligned} \|\nabla(u - p_1)\|_{L_2}^2 &= \|R_{1,0}\, u\|_{L_2}^2 + \|R_{0,1}\, u\|_{L_2}^2 \\ &= \sum_{(h)} [\|R_{1,0}\, u\|_{L_2(T_h)}^2 + \|R_{0,1}\, u\|_{L_2(T_h)}^2], \end{aligned} \tag{7.5}$$

where the summation is over all T_h and T_h^*, T_h^* being the upper right triangle analogous to the lower left right triangle T_h. Inequalities (7.2), (7.4) and (7.5) lead to the following bounds:

$$\|u - p_1\|_{L_2(\Omega)} \leqq .70\, h^2, \tag{7.6}$$

$$\|\nabla(u - p_1)\|_{L_2(\Omega)} \leqq \frac{2\pi}{\sqrt{5}}\, h. \tag{7.7}$$

The actual errors at the midpoints of the sides of one subtriangle are 0.0043, −0.0107, and 0.0127, which compare with (7.3) which is 0.078 if $h = \frac{\pi}{8}$.

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