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Direct and Inverse Error Estimates for Finite Elements With Mesh Refinements

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Abstract. The finite element method is used to solve a second order elliptic boundary value problem on a polygonal domain. Mesh refinements and weighted Besov spaces are used to obtain optimal error estimates and inverse theorems.

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1. Introduction

The problem of the behavior of the finite element method in the presence of singularities caused by corners of the domain or abrupt changes in boundary condition has been the focus of intense interest in recent years, see [see 1, 3, 6, 10–12, 14, 15]. This paper studies in detail mesh refinements near the singular points. Using weighted Besov spaces, we obtain bounds for the error in the energy norm, and we provide inverse theorems to show that our results are optimal. This approach removes the necessity to distinguish between uniform and non-uniform estimates when the usual Sobolev spaces are used, as in [2]. Measured in terms of the number of unknowns (number of degrees of freedom) our analysis shows that a proper mesh refinement gives the same rate of convergence of the error as in the case of smooth solutions and a quasiuniform mesh. In addition we use a duality argument to obtain a weighted L_2 error estimate. Our results are related to [16], which deals with approximation properties of piecewise polynomial functions on quasiuniform meshes in R^n. [16] contains direct and inverse approximation theorems in the framework of Sobolev and Besov spaces, and shows the analogies with well known results, e.g., for approximation by trigonometric polynomials.

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In Sect. 2 we define certain weighted Besov spaces that will play a role in our analysis. In Sect. 3 we give the singular properties of the solution to our problem at the vertices of Ω. We specify the refined triangulations in Sect. 4, and we give some properties of the associated spaces of piecewise linear functions defined on these triangulations. Sect. 5 contains our error estimates, and Sect. 6 contains the inverse theorem. We have put in an appendix some results required by us, dealing with weighted Besov spaces, and with interpolation between Sobolev spaces satisfying boundary conditions.

2. Some Weighted Spaces

Our analysis will be carried out on a polygonal domain $\Omega \subset R^2$ with boundary Γ. Let x_i, $1 \leqq i \leqq M$ denote the vertices of Ω with θ_i being the interior angle of Ω at x_i. Let $\beta = (\beta_1, \dots, \beta_M)$, $0 \leqq \beta_i < 1$ be an M-tuple and let

$$\phi_\beta(x) = \prod_{i=1}^{M} |x - x_i|^{\beta_i} \tag{2.1}$$

where $|x|$ means the Euclidian norm.

Further let $\Gamma = \Gamma_D \cup \Gamma_N$ where Γ_D is the union of some (closed) sides of Ω and $\Gamma_N = \Gamma \setminus \Gamma_D$. For each vertex x_i, we let $\kappa_i = 1$ if both sides of Γ having endpoint x_i belong to Γ_D or Γ_N, and we let $\kappa_i = \frac{1}{2}$ otherwise. We set $\alpha_i = \min\{1, \kappa_i \pi / \theta_i\}$. We let $\mathcal{M}$ denote the indices i such that $\alpha_i < 1$. (Note that by allowing a "vertex" x_i of Ω to have interior angle $\theta_i = \pi$, we include the later possibility of an abrupt change of boundary condition within a line segment of Γ.) We remark that many of the results do not utilize the specific form (2.1) of ϕ_β, but are also true for any positive weight function. This will be utilized in Sect. 6, where we shall consider polygonal subdomains of Ω but will use the function ϕ_β defined with respect to the domain Ω.

We let $H^m(\Omega)$, $m = 1, 2, \dots$ denote the usual Sobolev space of functions with square integrable derivatives of order $\leqq m$ and with the norm $\|u\|_m$ and scalar product $(.,.)_m$. We let $H_D^1(\Omega)$ denote the set of $u \in H^1(\Omega)$ with $u = 0$ on Γ_D (in the sense of traces) and we let $H_D^{-1}(\Omega) = (H_D^1(\Omega))'$ be the linear functionals on $H_D^1(\Omega)$. Further we let $H^{m,\beta}(\Omega)$ $m \geqq 0$ be the closure of smooth functions on Ω with the norm $\|u\|_{m,\beta}$ defined by

$$\|u\|_{m,\beta}^2 = \|u\|_{m-1}^2 + \int_\Omega \phi_\beta^2 |D^m u|^2 \, dx, \qquad m \geqq 1,$$

$$\|u\|_{0,\beta}^2 = \int \phi_\beta^2 u^2 \, dx,$$

where we denote

$$|D^m u|^2 = \sum_{m_1 + m_2 = m} \left| \frac{\partial^m u}{\partial x_1^{m_1} \partial x_2^{m_2}} \right|^2, \qquad m_1 \geqq 0, \ m_2 \geqq 0,$$

and $dx = dx_1 \, dx_2$.

We will deal with various domains Ω. When we wish to emphasize the domain, we will write $\|u\|_{H^m(\Omega)}$, $\|u\|_{H^{m,\beta}(\Omega)}$ or $(.,.)_{H^m(\Omega)}$. In particular if $\beta=(0,\dots,0)$ then we have $H^{m,\beta}(\Omega)=H^m(\Omega)$.

We remark that

$$H^{m,\beta}(\Omega)\subset C^{m-2}(\bar{\Omega}),\qquad m\geqq 2, \tag{2.2}$$

with continuous imbedding, where $C^{m-2}(\bar{\Omega})$ is the class of $m-2$ times continuously differentiable functions on Ω with continuous extensions of its $m-2$ derivatives on $\bar{\Omega}$, furnished with the norm $\|u\|_{C^k(\Omega)}=\sup_{x\in\Omega, i\leqq k}|D^i u|$. For, if $m=2$, the continuity of u at all the points except x_i, $i=1,\dots,M$ follows from Sobolev's imbedding theorem. To verify the continuity of u at $x_1=0$, say, it suffices to show the inequality

$$\sup|u(x)|^2\leqq C\int_\Omega[|u|^2+|D^1u|^2+r^{2\beta_1}|D^2u|^2]\,dx,\qquad 0<\beta_1<1, \tag{2.3}$$

with $r^2=|x|^2$ for all smooth functions u. By use of Theorem A.1 we extend u to a full neighborhood S of 0 preserving the norm of the right side of (2.3). Let ρ be a smooth (C^∞) function with its support contained in S and with $\rho\equiv 1$ in some neighborhood $S'\subset S$ of 0. Setting $v=\rho u$, $f=\Delta v$ we understand v to be defined on R^2 by zero extension and define

$$V(x)=\frac{1}{2\pi}\int_S f(y)\ln|x-y|\,dy. \tag{2.4}$$

Because $\beta_1<1$ we get by Schwartz's inequality

$$|V(x)|\leqq C\|u\|_{2,\beta}. \tag{2.5}$$

We also obviously have $\Delta V=\Delta v$. Integrating in (2.4) by parts and realizing that f has support in S, we see easily that $V(x)\to 0$ as $|x|\to\infty$. Because v has compact support, applying the standard (uniqueness) theorem we get $V=v$. Now (2.5) yields immediately (2.3) and hence (2.2) in the case $m=2$. The proof of (2.2) for $m>2$ follows by induction.

Our weighted Besov spaces will be defined in terms of interpolation between the above spaces. We shall use the interpolation spaces of Peetre which are developed for example in [4]. However, we shall require only the definition and a few properties of these spaces and we state these explicitly. If $0\leqq k<m$ we define

$$X(k,m,\beta,\theta,p)=[H^{m,\beta}(\Omega),H^k(\Omega)]_{\theta,p}.$$

If $1\leqq k<m$, we define

$$X_D(k,m,\beta,\theta,p)=[H^{m,\beta}(\Omega)\cap H_D^1(\Omega),\ H^k(\Omega)\cap H_D^1(\Omega)]_{\theta,p}.$$

We shall only require these spaces in the case $p=\infty$, and we now define them explicitly in this case. For $u\in H^k\cap H_D^1$, we set

$$K(u,t)=\inf\{\|v\|_k+t\|w\|_{m,\beta}\};$$
$$u=v+w,\ v\in H^k(\Omega)\cap H^1_D(\Omega),\ w\in H^{m,\beta}(\Omega)\cap H^1_D(\Omega);$$

and we define the norm

$$\|u\|_{X(k,m,\beta,\theta,\infty)}=\sup\{t^{-\theta}K(u,t);\ t>0\}.$$

Then $X_D(k,m,\beta,\theta,\infty)$ is defined to be the set of all $u\in H^k(\Omega)\cap H^1_D(\Omega)$ with finite norm. The space $X(k,m,\beta,\theta,\infty)$ is defined in a similar manner. In Sect. 6 we shall need the reiteration formula [4, Theorem 3.5.3]

$$X_D(1,2,\beta,\theta,\infty)=[X_D(1,2,\beta,\theta_1,\infty),\ H^1_D(\Omega)]_{\theta_2,\infty},\qquad \theta=\theta_1\theta_2. \tag{2.6}$$

We need to know in which of our weighted Besov spaces certain functions are. Let x_i be one of the vertices of Ω, $1\leqq i\leqq M$ and let $(r_i.\phi_i)$ be polar coordinates which are centered at x_i. Let $u(x)=r_i^\alpha\Phi(\phi_i)\eta(r_i)$, where $\alpha>0$ and Φ, η are smooth functions of their arguments be defined on Ω. Obviously $u(x)\in H^1(\Omega)$. We have, assuming that $u=0$ on Γ_D.

Lemma 2.1. *Assume $k-1<\alpha<m-1-\beta_i$ where k and m are non negative integers. Then $u\in X_D(k,m,\beta,\theta,\infty)$ with $\theta=(\alpha-k+1)/(m-k-\beta_i)$.*

Proof. Since $\alpha>k-1$, we easily see that $u\in H^k(\Omega)$. We estimate the function $K(u,t)$. If $t\geqq 1$ setting $v=u$, $w=0$ we have $K(u,t)\leqq\|u\|_k$. To estimate K for $t<1$, we use a smooth function $\xi(s)$, $0\leqq s<\infty$ such that

$$\xi(s)=1,\quad 0\leqq s\leqq 1/2,$$

$$\xi(s)=0,\quad 1\leqq s<\infty,$$

and

$$\xi_\delta(s)=\xi\left(\frac{s}{\delta}\right),\quad 0<\delta\leqq 1.$$

Then obviously

$$\left|\frac{d^j\xi_\delta(s)}{ds^j}\right|\leqq C\delta^{-j}$$

with C independent of δ. Also we use the obvious relation

$$|D^j u(x)|\leqq C r_i^{\alpha-j}$$

with C depending on α, j but independent of x.

We now set

$$v=\xi_\delta(r_i)u,\qquad w=(1-\xi_\delta(r_i))u,$$

where $0<\delta\leqq 1$ will be chosen shortly. From the Leibnitz formula we have

$$\|v\|_k^2 \leqq C \int_\Omega \left[\sum_{j=0}^k |D^j \xi_\delta|^2 |D^{k-j} u|^2 + \xi^2 u^2 \right] dx$$
$$\leqq C \int_0^\delta \left[\sum_{j=0}^k \delta^{-2j} r^{2(\alpha-k+j)} + r^{2\alpha} \right] r\, dr$$
$$\leqq C[\delta^{2(\alpha-k+1)} + \delta^{2(\alpha+1)}]$$
$$\leqq C \delta^{2(\alpha-k+1)}.$$

To estimate

$$\|w\|_{m,\beta}^2 = \|w\|_{m-1}^2 + \int_\Omega \phi_\beta^2 |D^m w|^2\, dx$$

we note that $|D^j \xi_\delta(r_i)|^2 = 0$ for $j>0$ and $r_i \notin (\delta/2, \delta)$. Using the Leibnitz formula

$$\int_\Omega \phi_\beta^2 |D^m w|^2\, dx \leqq C \sum_{j=0}^m \int_\Omega [r_i^{2\beta_i} |D^j(1-\xi_\delta)|^2 |D^{m-j} u|^2]\, dx$$
$$\leqq C \left[\int_0^\delta r^{2(\beta_i+2\alpha-2m)} r\, dr + \sum_{j=1}^m \int_{\delta/2}^\delta \delta^{-2j} r^{2(\beta_i+\alpha-m+j)} r\, dr \right]$$
$$\leqq C \delta^{2(\alpha-m+\beta_i+1)}.$$

A similar calculation gives

$$\|w\|_{m-1}^2 = \|w\|_0^2 + \int |D^{m-1} w|^2\, dx$$
$$\leqq C + C\delta^{2(\alpha-m+2)}.$$

Hence we get

$$\|w\|_{m,\beta} \leqq C \delta^{\alpha-m+\beta_i+1}.$$

Combining the inequalities for v and w

$$K(u,t) \leqq C[\delta^{\alpha-k+1} + t\delta^{\alpha-m+\beta_i+1}].$$

Let δ be defined by $t = \delta^{m-k-\beta_i}$. Then for $0<t<1$ we have $\delta<1$ and we obtain

$$K(u,t) \leqq C[\delta^{\alpha-k+1} + \delta^{m-k-\beta_i}] \leqq C\delta^{\alpha-k+1} = Ct^\theta,$$

where $\theta = (\alpha-k+1)/(m-k-\beta_i) < 1$. The lemma follows now from these inequalities for K and the basic definition of the interpolated norm.

The spaces $X_D(k, m, \beta, \theta; q)$ are increasing functions of $q \in [1, \infty]$. If $q<\infty$ one can only assert that $u \in X_D(k, m, \beta\theta - \varepsilon q)$, $\varepsilon > 0$. For this reason the spaces with $q = \infty$ are needed to obtain sharp estimates.

In the appendix we derive some results on interpolation that will be of use to us.

3. The Model Problem

We consider the model problem

$$-\Delta u + u = f \quad \text{in } \Omega, \tag{3.1a}$$

$$\left.\begin{aligned} u &= 0 \quad \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_N \end{aligned}\right. \tag{3.1b}$$

and we understand it in the usual weak sense. It is well known that this problem has a unique solution in $H_D^1(\Omega)$ for each $f \in H_D^{-1}(\Omega)$. Further statements about the regularity of u depend on the regularity of f and on the interior angles at the vertices of Ω. We shall require the following two results from Kondrat'ev [9]. See also [7].

Theorem 3.1. *There are functions $v_i \in H_D^1(\Omega)$, $i \in \mathcal{M}$ which are zero outside a neighborhood of x_i and which have the form*

$$v_i(x) = \rho_i(r_i)\, r_i^{\alpha_i}\, \Phi_i(\theta_i),$$

where (r_i, θ_i) are polar coordinates with origin at x_i, $\rho_i \Phi_i$ are smooth functions of their arguments, $\rho_i(r) = 1$, $0 \leqq r \leqq a_i$, $a_i > 0$ such that if u is a solution of (3.1) *with $f \in H^0(\Omega)$ then*

$$u = \sum_{i \in \mathcal{M}} C_i v_i + w$$

with $w \in H^2(\Omega) \cap H_D^1(\Omega)$ and

$$\sum_{i \in \mathcal{M}} |C_i| + \|w\|_2 \leqq C \|f\|_0.$$

To state the next theorem, we first note that, using Holder's inequality and the inclusion $H^1(\Omega) \subset L_p(\Omega)$, $1 \leqq p < \infty$, we obtain

$$\|u\|_{0,-\beta} = \{\int_\Omega \phi_\beta^{-2} u^2\, dx\}^{1/2} \leqq C \|u\|_1, \qquad u \in H^1(\Omega). \tag{3.2}$$

Hence, if $f \in H^{0,\beta}(\Omega)$, $\int_\Omega f\, u dx$ defines a linear functional on $H_D^1(\Omega)$. With this understanding we have [9, Theorem 1.1]

Theorem 3.2. *Suppose $0 \leqq \beta_i < 1$, $1 \leqq i \leqq M$, and suppose $1 - \alpha_i < \beta_i < 1$, $i \in \mathcal{M}$. Then if $f \in H^{0,\beta}(\Omega)$, the solution u of* (3.1) *belongs to $H^{2,\beta}(\Omega)$, and there is a $c > 0$ independent of f such that*

$$\|u\|_{2,\beta} \leqq c \|f\|_{0,\beta}.$$

We derive now a corollary of Theorem 3.1.

Corollary 3.1. *Let* $0 \leqq \beta_i < 1$, $i \in \mathcal{M}$, *and suppose* $\theta = \min\{\alpha_i/(1-\beta_i), i \in \mathcal{M}\} < 1$. *If* $f \in H^0(\Omega)$, *then* $u \in X_D(1, 2, \beta, \theta, \infty)$ *and*

$$\|u\|_{X_D(1,2,\beta,\theta,\infty)} \leqq C\|f\|_0.$$

Proof. From Lemma 2.1 we have $v_i \in X_D(1, 2, \beta, \theta, \infty)$. The result then follows from Theorem 3.1.

4. Mesh Refinement

We define a family of triangulations which are refined in a systematic way near the vertices of Ω, governed by the function $\phi_\gamma(x)$ defined by (2.1) with $\gamma = (\gamma_1 \dots \gamma_M)$ $0 \leqq \gamma_i < 1$. By a triangulation $\mathcal{T}$ of Ω we mean a finite collection of closed triangles T whose union is $\bar{\Omega}$ and such that if $S, T \in \mathcal{T}$ then $S \cap T$ is either empty, a common side or a common vertex of S and T. For $T \in \mathcal{T}$ we let $d_T = \sup\{|x-y|; x, y \in T\}$.

We say that $\mathcal{T}$ is a triangulation of type (h, γ, L) if:

(i) if $T \in \mathcal{T}$ and θ is an angle of $T, \theta \geqq L^{-1}$;

(ii) if $\phi_\gamma \neq 0$ on T then $L^{-1} h\phi_\gamma(x) \leqq d_T \leqq Lh\,\phi_\gamma(x)$, $x \in T$;

(iii) if ϕ vanishes at some point of T then $L^{-1} h \sup \phi_\gamma(x) \leqq d_T \leqq Lh \sup_{x \in T} \phi_\gamma(x)$.

If $\gamma \equiv (0, \dots, 0)$ there is no refinement and the triangulation is quasiuniform. If $\gamma \neq 0$ then the amount of mesh refinement near the vertex x_i is determined by γ_i. One of the goals of this paper is to illustrate how the theory of quasiuniform meshes may be generalized to refined meshes through the use of weighted Sobolev spaces.

We shall measure the error in our finite element approximation in terms of the parameter h. To justify this we require an estimate on the number of vertices in a refined triangulation. We have

Lemma 4.1. *There is a constant* $c > 0$ *independent of* h *such that if* $\mathcal{T}$ *is a triangulation of type* (h, γ, L) *and if* N *is the number of vertices in* $\mathcal{T}$ *then*

$$N \leqq Ch^{-2}.$$

(C depends in general on Ω, γ, L.)

Proof. We have $N \leqq 3 \sum_T 1$ where the sum is taken over $T \in \mathcal{T}$. If ϕ_γ does not vanish in T

$$1 \leqq C d_T^{-2} \int_T dx \leqq C h^{-2} \int_T \phi_\gamma^{-2}(x)\, dx.$$

Since the number of triangles in which ϕ_γ vanishes is finite we have

$$N \leqq Ch^{-2} \int_\Omega \phi_\gamma^{-2}(x)\, dx$$

and the lemma follows immediately.

Given a triangulation $\mathscr{T}$ of Ω we let $\mathscr{S}(\mathscr{T}) \subset H^1(\Omega)$ denote the corresponding collection of continuous piecewise linear functions on Ω, which are linear on every $T \in \mathscr{T}$ and we let $\mathscr{S}_D(\mathscr{T}) \subset H_D^1(\Omega)$ denote the corresponding collection of functions which vanish on Γ_D. We require an approximation property for functions in $\mathscr{S}_D(\mathscr{T})$. We start with some inequalities.

Lemma 4.2. *One has the inequality*

$$\int_0^1 t^{\alpha-2}[z(t)-a]^2\,dt \leqq C(\alpha)\int_0^1 t^\alpha \left(\frac{dz}{dt}\right)^2 dt, \qquad \alpha \neq 1, \tag{4.1}$$

where $a=z(0)$ if $\alpha<1$, $a=z(1)$ if $\alpha>1$.

Proof. For $\alpha<1$ we start with inequality [8, Theorem 254]

$$\int_0^1 s^{-2} w(s)^2\,ds \leqq C\int_0^1 w'(s)^2\,ds + C w(1)^2, \qquad w(0)=0.$$

Using the inequality $w(1)^2 \leqq \int_0^1 w'^2\,ds$, making the change of variables $s=t^{1-\alpha}$, and setting $z(t)=w(t)-w(0)$, we obtain (4.1). For $\alpha>1$ we use the inequality [8, Theorem 255]

$$\int_0^\infty s^{-2} w(s)^2\,ds \leqq 4\int_0^\infty w'(s)^2\,ds, \qquad w(0)=0.$$

Making the change of variable $t=s^{1-\alpha}$, and setting $v(t)=w(t)-w(1)$ for $t<1$, $v(t)=0$ for $t>1$, we obtain (4.1) in this case.

Let T be a triangle with one vertex at 0. Then we have

Lemma 4.3. *Let $\alpha \neq 0$, and let u be defined on T with weak first derivatives which satisfy $\int_T |x|^\alpha |D^1 u|^2\,dx<\infty$. Then there is a constant a, depending on u and a constant $C>0$, independent of u but depending on α and the minimal angle θ_T of T, such that*

$$\int_T |x|^{\alpha-2}|u-a|^2\,dx \leqq C\int_T |x|^\alpha |D^1 u|^2\,dx, \tag{4.2}$$

and

$$|a| \leqq C\int_T r^\alpha |D^1 u|\,dx.$$

In addition if $\alpha<0$ and u is continuous then $a=u(0)$.

Proof. Let S be the finite sector defined in terms of polar coordinates (r,θ) by $0 \leqq \theta \leqq \theta_0$, $0 \leqq r \leqq 1$, where θ_0 is the angle of T at the origin. Then T may be mapped into S by a smooth map and so it suffices to prove (4.2) with T replaced by S. Let

$$\bar{u}(r)=\theta_0^{-1}\int_0^{\theta_0} u(r,\theta)\,d\theta.$$

It is easily seen that

$$\int_0^1 r^{\alpha+1} \left|\frac{d\bar{u}}{dr}\right|^2 dx \leqq C \int_S r^\alpha |D^1 u|^2 dx < \infty. \tag{4.3}$$

Using (4.1) and (4.3) we obtain

$$\begin{aligned} \int_0^1 r^{\alpha-1}[\bar{u}(r)-a]^2 dr &\leqq C \int_S r^{\alpha+1} \left|\frac{d\bar{u}}{dr}\right|^2 dx \\ &\leqq C \int_S r^\alpha |D^1 u|^2 dx, \qquad \alpha \neq 0. \end{aligned} \tag{4.4}$$

We remark that for $\alpha<0$, $\bar{u}(r)$ is continuous on $[0,1]$ and $a=\bar{u}(0)$ in (4.4). In addition when $u(r,\theta)$ is a continuous function on T then in the case $\alpha<0$ we get $a=u(0)$.

Integrating (4.4) over θ gives

$$\int_S r^{\alpha-2} |\bar{u}-a|^2 dx \leqq C \int_S r^\alpha |D^1 u|^2 dx. \tag{4.5}$$

Further for almost all r, ϕ we get

$$u(r,\phi)-u(r,\psi) = \int_\psi^\phi u_\theta(u,\theta)\, d\theta,$$

and therefore

$$\begin{aligned} |u(r,\phi)-\bar{u}(r)| &= \left|\theta_0^{-1} \int_0^{\theta_0} d\psi \int_\psi^\phi u_\theta(r,\theta)\, d\theta\right| \\ &\leqq C \left[\int_0^{\theta_0} |u_\theta(r,\theta)|^2 d\theta\right]^{1/2}, \end{aligned}$$

and hence

$$\int_0^{\theta_0} [u(r,\phi)-\bar{u}(r)]^2 d\phi \leqq C \int_0^{\theta_0} |u_\theta(r,\theta)|^2 d\theta. \tag{4.6}$$

Multiplying (4.6) by $r^{\alpha-1}$ and integrating over r we get

$$\int_S r^{\alpha-2} |u-\bar{u}|^2 dx \leqq C \int_S r^\alpha |D^1 u|^2 dx. \tag{4.7}$$

Using (4.5), (4.7) and the triangle inequality we get (4.2).

Lemma 4.4. *Let $\varepsilon>0$ and let $0<s<1$. Then there is a positive constant C, (dependent on ε and s) such that if T is any triangle with vertices $z^1=0$, z^2, z^3 and with minimal interior angle $\geqq\varepsilon$, if u satisfies*

$$\int_T [|u|^2 + |D^1 u|^2 + |x|^{2s} |D^2 u|] \, dx < \infty, \tag{4.8}$$

and if p is the linear interpolating polynomial to u on T, then

$$\begin{aligned}&\int_T [|x|^{2s-4}|u-p|^2+|x|^{2s-2}|D^1(u-p)|^2]\,dx\\&\quad\leqq C\int_T |x|^{2s}|D^2 u|^2\,dx.\end{aligned} \tag{4.9}$$

Proof. From (4.8) and (2.2), u is continuous function on T as shown in Sect. 2, so the interpolating polynomial p is well defined. To prove (4.9) we suppose that T has vertices $z^1=(0,0)$, $z^2=(1,0)$, $z^3=(0,1)$. Let $v_i=\frac{\partial u}{\partial x_i}$ $i=1,2$. Since $0<s<1$ we use (4.2) to obtain constants a_i such that

$$\int_T |x|^{2s-2}\left|\frac{\partial u}{\partial x_i}-a_i\right|^2 dx\leqq C\int |x|^{2s}|D^2 u|^2\,dx,\qquad i=1,2. \tag{4.10}$$

Now we set $v=u-a_1x_1-a_2x_2$. Since $s<1$ and v is continuous we use (4.2) and get

$$\int_T |x|^{2s-4}|v-v(0)|^2\,dx\leqq C\int_T |x|^{2s-2}|D^1 v|^2\,dx. \tag{4.11}$$

Combining these inequalities, we obtain (4.9) where p is replaced by the polynomial $q(x)=u(0)+a_1x_1+a_2x_2$. Let $u_0=u-q=v-v(0)$, $p_0=p-q$. Then p_0 vanishes at $z^1=(0,0)$ and equals u_0 at z^2 and z^3. We have, by (2.2), $u_0=v-v(0)$, so from (4.10), (4.11),

$$\begin{aligned}&\int_T [|x|^{2s-4}|p_0|^2+|x|^{2s-2}|D^1 p_0|^2\,dx\\&\quad\leqq C[|u_0(x^2)|^2+|u_0(x^3)|^2]\\&\quad\leqq C\int_T [|u_0|^2+|D^1 u_0|^2+|x|^{2s}|D^2 u_0|^2]\,dx\\&\quad\leqq C\int_T |x|^{2s}|D^2 u|^2.\end{aligned} \tag{4.12}$$

Using (4.12) and triangle inequality we obtain (4.9) for our triangle. If T_1 is another triangle with the area 1/2 we map T_1 onto T with a linear transformation. Using this transformation we get (4.9) for T_1 with C depending on ε. Since (4.9) remains unchanged with a change of scale we get (4.9) in full generality.

Using this lemma we give the approximation property for functions in $\mathscr{S}_D(\mathscr{T})$.

Lemma 4.5. *Let $\mathscr{T}$ be a triangulation of type (h,γ,L). Then there is a constant C depending only on γ and L such that for $u\in H^{2,\gamma}(\Omega)\cap H^1_D(\Omega)$ there is a $v\in\mathscr{S}_D(\mathscr{T})$ such that*

$$\|u-v\|_1\leqq Ch\,\|u\|_{2,\gamma}.$$

Proof. Let $u\in H^{2,\gamma}(\Omega)\cap H^1_D(\Omega)$ and let $v\in\mathscr{S}(\mathscr{T})$ be the piecewise linear interpolation to u. Since u is continuous in $\bar{\Omega}$, v is well defined. Let $T\in\mathscr{T}$ If none of the vertices of T coincides with a vertex x_i of Ω with $\gamma_i>0$, then $u\in H^2(T)$. Applying standard results we have

$$\int_T [|u-v|^2+|D^1(u-v)|^2]\,dx \leqq C\,d_T^2 \int_T |D^2 u|^2\,dx$$

when C depends only on the minimal angle of T. Hence we have

$$\int_T [|u-v|^2+|D^1(u-v)|^2]\,dx \leqq C \int_T \phi_\gamma^2\,|D^2 u|^2\,dx. \tag{4.13}$$

Suppose that one of the vertices of T is a vertex x_i of Ω with $\gamma_i>0$. We use (4.9) with $s=\gamma_i$, and the inequality $r\leqq d_T$, to obtain

$$\begin{aligned}&\int_T [|u-v|^2+|D^1(u-v)|^2]\,dx\\ &\quad \leqq C(d_T)^{2(1-\gamma_i)} \int |x-x_i|^{2\gamma_i}\,|D^2 u|^2\,dx.\end{aligned} \tag{4.14}$$

Since $\mathscr{T}$ is of type (h,γ,L)

$$\begin{aligned}d_T &\leqq L\,h \sup\{\phi_\gamma(x), x\in T\}\\ &\leqq C h\{\sup|x-x_i|^{\gamma_i}, x\in T\} \leqq C h(d_T)^{\gamma_i}.\end{aligned}$$

Hence $(d_T)^{1-\gamma_i}\leqq Ch$. Also, for $x\in T\,|x-x_i|^{\beta_i}\leqq C\phi_\gamma(x)$. Using these inequalities together with (4.14) we see that (4.13) is valid for all triangles. Summing (4.13) over all triangles we get the result.

In Sect. 6 we shall require an 'inverse property' of the functions in $\mathscr{S}_D(\mathscr{T})$. We have

Lemma 4.6. *Let $\mathscr{T}$ be a triangulation of type (h,γ,L). Then there is a constant $C>0$ depending only on γ and L such that for $w\in\mathscr{S}_D(\mathscr{T})$ and $0<\theta\leqq 1/2$*

$$\|u\|_{X_D(1,2,\gamma,\theta,\infty)} \leqq C\,h^{-\theta}\,\|u\|_1.$$

Proof. Let $u\in\mathscr{S}_D(\mathscr{T})$ and let

$$K(u,t)=\inf\{\|v\|_1+t\,\|w\|_{2,\gamma};\ u=v+w,\ t>0\}.$$

To prove the result we will show that

$$K(u,t)\leqq C\,t^\theta\,h^{-\theta}\,\|u\|_1,\qquad 0<\theta\leqq 1/2. \tag{4.15}$$

Setting $v=u$, $w=0$ we have for $t\geqq 1/4\,h$

$$K(u,t)\leqq \|u\|_1 \leqq C\,t^\theta\,h^{-\theta}\,\|u\|_1.$$

We now estimate $K(u,t)$ for $t\leqq \frac{h}{4}$. For $T\in\mathscr{T}$ let

$$N(T)=U\{T'\in\mathscr{T}, T'\cap T\neq 0\}.$$

For $t \leqq h/4$ we will construct a twice continuously differentiable function $w \in H_D^1(\Omega)$ such that if $\delta_T = \frac{t\, d_T}{h}$, $T \in \mathscr{T}$, we have

$$\sup_{x \in T} [|w(x)| + |D^1 w(x)|] \leqq C \sup_{x \in N(T)} [|u(x)| + |D^1 u(x)|], \tag{4.16}$$

$$\sup_{x \in T} |D^2 w(x)| \leqq C \delta_T^{-1} \sup_{x \in N(T)} [|u(x)| + |D^1 u(x)|], \tag{4.17}$$

$$\text{meas}\{x \in T;\ u(x) \neq w(x)\} \leqq C\, d_T\, \delta_T. \tag{4.18}$$

Using these properties we get

$$\|u - w\|_{H^1(T)} \leqq C\, \delta_T^{1/2}\, d_T^{-1/2}\, \|u\|_{H^1(N(T))} \tag{4.19}$$

because it is easy to see that for $u \in \mathscr{S}_D(\mathscr{T})$,

$$\sup_{x \in N(T)} [|u(x)| + |D^1 u(x)|] \leqq d_T^{-1}\, \|u\|_{H^1(N(T))}. \tag{4.20}$$

Further we have

$$\|\phi_\gamma |D^2 w|\|_{H^0(T)} \leqq C\, h^{-1}\, \delta_T^{-1/2}\, d_T^{1/2}\, \|u\|_{H^1(N(T))} \tag{4.21}$$

because $|D^2 u| = 0$, $w \neq u$ only on set of measure $\leqq C\, d_T\, \delta_T$ and because of (4.17) (4.20). Squaring (4.19) and (4.21) and summing over all $T \in \mathscr{T}$ we obtain

$$\|u - w\|_1 \leqq C\, t^{1/2}\, h^{-1/2}\, \|u\|_1,$$
$$\|\phi_\gamma D^2 w\|_0 \leqq C\, h^{-1/2}\, t^{-1/2}\, \|u\|_1.$$

From these two estimates (4.15) follows readily. Hence it suffices to satisfy (4.16), (4.17), (4.18).

Let us now construct the function w. Let S and T be two triangles of $\mathscr{T}$ with common side (ST) and let $x_{S,T}$ denote the distance from a point x to the common side. Let ε_{ST} be the $2\,\delta_{ST}$-neighborhood of (ST). If z is a vertex of $T \in \mathscr{T}$, let B_z be the $2\,\rho_z$-neighborhood of z. We choose δ_{ST} and ρ_z so that

$$\delta_{ST} \leqq C_1 [\delta_S + \delta_T], \qquad C_1\, \rho_z \leqq \delta_T \leqq C_2\, \rho_z,$$

where $C_1 > 0$, $C_2 > 0$ depends only on L. We also assume that the situation is as in Fig. 1.

For $T \in \mathscr{T}$, u is a linear function in T. We let u_T denote the extension of this linear function to Ω. We let $e(s)$, $-\infty < s < \infty$ denote a smooth function such that $e(s) = 1$ for $s \geqq 1$, $e(s) = -1$ for $s \leqq -1$ and $e(s) = 0$ for $|s| \leqq 1/2$. Now we define a function w on every $T \in \mathscr{T}$ as follows. Let T be as shown in the figure, with vertices z_i, $i = 1, 2, 3$, and with adjacent triangles S_i, $i = 1, 2, 3$. For $x \in T$, $|x - z_i| \geqq \rho_{z_i}$ $i = 1, 2, 3$, set

$$w_1(x) = \begin{cases} \frac{1}{2}[u(x) + u_{S_i}(x)] + \frac{1}{2} e(x_{TS_i}/\delta_{TS_i})[u(x) - u_{S_i}(x)], & x \in \varepsilon_{TS_i}, \quad i = 1, 2, 3, \\ u(x), \quad x \notin \varepsilon_{TS_i}, \quad i = 1, 2, 3. \end{cases}$$

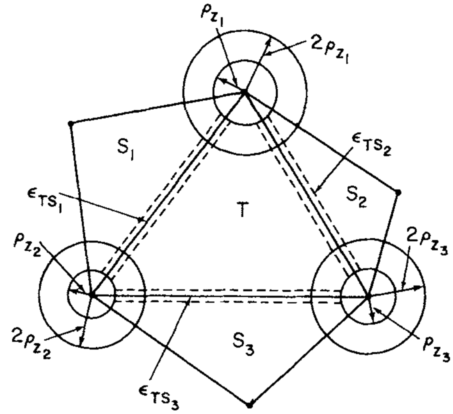

Fig. 1

For $x \in T$, define

$$w(x) = \begin{cases} w_1(x), & |x-z_i| \geqq 2\rho_{z_i}, \quad i=1,2,3 \\ u(x)+[w_1(x)-u(x)]\, e(|x-z_i|/2\rho_{z_i}), & \rho_{z_i} \leqq |x-z_i| \leqq 2\rho_{z_i}, \quad i=1,2,3, \\ u(x), & |x-z_1| \leqq \rho_{z_i}, \quad i=1,2,3. \end{cases}$$

Using the properties of the function e, we see that w is a smooth function and that (4.16), (4.17), (4.18) hold.

5. The Error Estimates

We now give error estimates for the standard finite element approximation to (3.1) using the subspace $\mathcal{S}_D(\mathcal{T})$. Let $\mathcal{T}$ be a triangulation of type (h, γ, L) and let u_h be the finite element approximation of u using $\mathcal{S}_D(\mathcal{T})$. Then

$$\|u-u_h\|_1 \leqq \|u-w\|_1, \qquad w \in \mathcal{S}_D(\mathcal{T}).$$

Our basic result is

Theorem 5.1. *If* $u \in H^{2,\gamma}(\Omega) \cap H_D^1(\Omega)$ *then*

$$\|u-u_h\|_1 \leqq C\, h\, \|u\|_{2,\gamma}.$$

If $u \in X_D(1,2,\gamma,\theta,\infty)$ *then*

$$\|u-u_h\|_1 \leqq C\, h^\theta\, \|u\|_{X_D(1,2,\gamma,\theta,\infty)}.$$

Proof. The first result follows immediately from Lemma 4.5. For the second result we define the operator E: $H_D^1(\Omega) \to H_D^1(\Omega)$ by $E u = u - u_h$, and we estimate the norm of E on the interpolation space $X_D(1,2,\gamma,\theta,\infty)$.

Now we state a corollary of this result in the case that $f \in L_2(\Omega)$.

Corollary 5.1. *Let $f \in L_2(\Omega)$. If $\gamma_i > 1-\alpha_i$, $i \in \mathcal{M}$, then*

$$\|u-u_h\|_1 \leqq C h \|f\|_0.$$

If $\gamma_i < 1-\alpha_i$ for some $i \in \mathcal{M}$ then setting

$$\mu = \min\{\alpha_i/1-\gamma_i; i \in \mathcal{M}\},$$
$$\|u-u_h\|_1 < C h^\mu \|f\|.$$

Proof. The first result follows from Theorem 3.2 and Theorem 5.1. The second result follows from Corollary 3.1.

We note that if there is no mesh refinement, i.e. if $\gamma \equiv 0$, then Corollary 5.1 gives

$$\|u-u_h\|_1 \leqq C h^\mu \|f\|_0$$

where $\mu = \min\{\alpha_i, i \in \mathcal{M}\}$.

We now consider an L_2 estimate for the error. Because of our mesh refinement near the vertices of Ω we obtain an error estimate in the (stronger) weighted L_2 norm, defined by the formula

$$\|v\|_{0,-\beta} = \left\{ \int_\Omega \phi_\beta^{-2} v^2 \, dx \right\}^{1/2}.$$

Theorem 5.2. *Let $1-\alpha_i < \gamma_i < 1$, $i \in \mathcal{M}$. Then*

$$\|u-u_h\|_{0,-\gamma} \leqq C h \|u-u_h\|_1.$$

Proof. From (3.2) we have

$$\|\phi_\gamma^{-2}(u-u_h)\|_{0,\gamma} = \|u-u_h\|_{0,-\gamma} \leqq \|u-u_h\|_1 < \infty. \tag{5.1}$$

Let z denote the solution of the problem (3.1) with right hand side

$$f = \phi_\gamma^{-2}(u-u_h) \in H^{0,\gamma}(\Omega).$$

Applying Theorem 3.2 we get $z \in H^{2,\gamma}(\Omega) \cap H_D^1(\Omega)$ and

$$\|z\|_{2,\gamma} \leqq C \|u-u_h\|_{0,-\gamma}. \tag{5.2}$$

Using this solution we have

$$\begin{aligned} \|u-u_h\|_{0,-\gamma}^2 &= (\phi_\gamma^{-2}(u-u_h), u-u_h)_0 \\ &= (z, u-u_h)_1 = (z-\xi, u-u_h)_1 \\ &\leqq \|z-\xi\|_1 \|u-u_h\|_1 \end{aligned} \tag{5.3}$$

where $\xi \in \mathcal{S}_D(\mathcal{T})$ is arbitrary. Choosing ξ as in Lemma 4.5 we obtain

$$\begin{aligned} \|u-u_h\|_{0,-\gamma}^2 &\leqq C h \|z\|_{2,\gamma} \|u-u_h\|_1 \\ &\leqq C h \|u-u_h\|_{0,-\gamma} \|u-u_h\|_1. \end{aligned}$$

The result follows immediately.

6. The Inverse Theorem

We shall establish two converse theorems to Theorem 5.1. For our first result, Theorem 6.1, we restrict ourselves to the case $0<\theta<1/2$ and we use the inverse property of $\mathscr{S}_D(\mathscr{T})$ stated in Lemma 4.6. In addition we require an assumption (H 1) on the existence of a family of triangulations of suitable type. Our second result, Theorem 6.2, allows $\theta\in(0,1)$ and does not use Lemma 4.6. On the other hand, we replace (H 1) by a considerably more complicated assumption (H 2).

For our first theorem we require the assumption

(H 1) there is a sequence $h_n\to 0$ and a family $\mathscr{T}_n$ of triangulations, such that

(i) $\mathscr{T}_n$ is a triangulation of type (h_n,γ,L),

(ii) $\mathscr{T}_{n+1}$ is a refinement of $\mathscr{T}_n$,

(iii) $C_1 2^{-n}\leqq h_n\leqq C_2 2^{-n}$ where $C_2>C_1>0$ are independent of n.

We let $\mathscr{S}^n=\mathscr{S}_D(\mathscr{T})$ and we shall let u_n denote the finite element solution of (3.1) using $\mathscr{S}^n$ as the set of trial and test functions. Because of (ii) we have $\mathscr{S}^{n+1}\supset\mathscr{S}^n$.

Theorem 6.1. *Suppose* (H 1) *holds. Let* $u\in H_D^1(\Omega)$. *Suppose there is a* $\theta\in(0,\frac{1}{2})$, *and a* $P>0$ *such that for each n, there is a* $u_n\in\mathscr{S}^n$ *such that*

$$\|u-u_n\|_1\leqq P\,h_n^\theta.$$

Then $u\in X_D(1,2,\gamma,\theta,\infty)$ *and for some* $C>0$ *independent of* u

$$\|u\|_{X_D(1,2,\gamma,\theta,\infty)}\leqq C[\|u\|_1+P].$$

Proof. Let $\theta_1\in(0,1/2)$, and let

$$K(u,t)=\inf\{\|v\|_1+t\,\|w\|_{X_D(1,2,\gamma,\theta_1,\infty)};\,u=v+w\},\qquad t>0.$$

We set

$$w=u_n=u_1+\sum_2^n(u_j-u_{j-1}),$$

$$v=u-w.$$

By our assumption we have

$$\|v\|_1\leqq P\,h_n^\theta\leqq CP\,2^{-\theta n}.$$

Using Lemma 4.6 we get

$$\|u_n\|_{X_D(1,2,\gamma,\theta_1,\infty)}\leqq\|u_1\|_{X_D(1,2,\gamma,\theta_1,\infty)}+\sum_2^n\|u_j-u_{j-1}\|_{X_D(1,2,\beta,\theta_1,\infty)}$$

$$\leqq C\{\|u_1\|_1+\sum_2^n\|u_j-u_{j-1}\|\,2^{\theta_1 j}\}.$$

Since

$$\|u_j - u_{j-1}\|_1 \leqq \|u_{j-1} - u\|_1 + \|u_j - u\|_1 \leqq C\,2^{-\theta j}$$

we obtain

$$\|u_h\|_{X_D(1,2,\gamma,\theta_1,\infty)} \leqq C\{\|u\|_1 + 2^{n(\theta_1-\theta)}\}.$$

Hence

$$K(u,t) \leqq C\{P\,2^{-\theta n} + t\,\|u\|_1 + t\,2^{n(\theta_1-\theta)}\} \leqq C[\|u\|_1 + P][t\,2^{n(\theta_1-\theta)} + 2^{-\theta n}].$$

For $t<1$ we pick n so that $2^n \leqq t^{-1/\theta_1} < 2^{n+1}$. Then $2^{\theta_1 n} \leqq t^{-1} < 2.2^{\theta_1 n}$, so setting $\theta_2 = \theta/\theta_1 < 1$,

$$2^{-n\theta} = 2^{-n\theta_1\theta_2} \leqq C\,t^{\theta_2},$$
$$2^{n(\theta_1-\theta)} = 2^{n\theta_1(1-\theta_2)} \leqq C\,t^{-(1-\theta_2)}.$$

Hence we have for $t<1$

$$K(u,t) \leqq C[\|u\|_1 + P]\,t^{\theta_2}.$$

For $t \geqq 1$ we set $u=v$ and obtain

$$K(u,t) \leqq C\,\|u\|_1 \leqq C\,\|u\|_1\,t^{\theta_2}.$$

Using (2.6) we get our result.

Now we will formulate our second assumption.

(H2) there is a sequence h_n, an integer N and a family $\mathcal{T}_n^\mu$, $0 \leqq \mu \leqq N$, $n=1,\dots$ of triangulations such that

(i) $\mathcal{T}_n^\mu$ is a triangulation of type (h_n, γ, L);

(ii) for each $\mu = 0,\dots,N$, $\mathcal{T}_{n+1}^\mu$ is a refinement of $\mathcal{T}_n^\mu$;

(iii) for each $n=1,2,\dots$, $\mathcal{T}_n^0$ is a refinement of $\mathcal{T}_n^\mu$, $\mu=1,\dots,n$;

(iv) for each $T \in \mathcal{T}_n^0$, there is a $\mu>0$ and a $\tilde{T} \in \mathcal{T}_n^\mu$ such that $T \subset \tilde{T}$ and if $T_1 \in \mathcal{T}_n^0$, $T_1 \cap T \neq \emptyset$ then $T_1 \subset \tilde{T}$;

(v) $C_1 2^{-n} \leqq h_n \leqq C_2 h^{-n}$ where $C_2 > C_1 > 0$ are independent of n;

(vi) there is a finite collection of convex polygons $\Omega_m \subset \Omega$, $1 \leqq m \leqq R$ such that, setting

$$\Omega_m^* = U\{T;\, T \subset \Omega_m,\, T \in \mathcal{T}_1^0\},$$

we have

a) $\bigcup_{m=1}^{R} \Omega_m^* = \Omega$;

b) for each $m=0,\dots,R-1$, $\bigcup_{i=1}^{m} \Omega_i^*$ is a polygon which, together with Ω_m^* satisfies assumption (H) of the Appendix.

We shall set $\mathcal{S}_D^{n,\mu}=\mathcal{S}_D(\mathcal{T}_n^\mu)$ and $\mathcal{S}^{n,\mu}=\mathcal{S}(\mathcal{T}_n^\mu)$. Because of (ii) and (iii) we have

$$\mathcal{S}^{n,\mu}\subset\mathcal{S}^{n+1,\mu},\qquad \mathcal{S}^{n,\mu}\subset\mathcal{S}^{n,0}.$$

Theorem 6.2. *Suppose* (H 2) *holds. Let* $u\in H_D^1(\Omega)$. *Suppose there is a* $\theta\in(0,1)$ *and a* $P>0$ *such that for each* n *and* μ *there is a* $u_{n,\mu}\in\mathcal{S}^{n,\mu}$ *such that*

$$\|u-u_{n,\mu}\|_1\leqq P\,h_n^\theta. \tag{6.1}$$

Then $u\in X_D(1,2,\gamma,\theta,\infty)$ *and for some* $c>0$ *independent of* u

$$\|u\|_{X_D(1,2,\gamma,\theta,\infty)}\leqq C[\|u\|_1+P].$$

Proof. Let $T\in\mathcal{T}_n^0$ and let $\kappa_T(x)$ be the piecewise quintic function on $\mathcal{T}_n^0$ which is of class $C^1(\bar\Omega)$, defined in terms of the Argyris triangle [5, p. 71] by specifying the following values: for x a vertex of T, $\kappa_T(x)=\dfrac{1}{n(x)}$ where $n(x)$ is the number of triangles of $\mathcal{T}_n^0$ which have x as a vertex; for x a vertex of $T'\in\mathcal{T}$, $x\notin T$ let $\kappa_T(x)=0$; all derivatives of κ_T that may be specified are chosen to be zèro. The family of functions $\kappa_T(x)$, $T\in\mathcal{T}_n^0$ have the following properties:

$$\sum_T\kappa_T(x)=1,\qquad x\in\bar\Omega,$$
$$\kappa_T(x)=0,\qquad \text{if } x\in T'\in\mathcal{T}_n^0\quad\text{and}\quad S\cap T=\emptyset,$$
$$|D^i\kappa_T(x)|\leqq C\,d_T^{-i},\qquad i=0,1,2.$$

Now to every $T\in\mathcal{T}_n^0$ we associate $\lambda(T)=\mu$ and $\tilde T(T)\in\mathcal{T}_n^{\lambda(T)}$ given by (H 2) part (iv). Let us introduce

$$w_n=\sum_{T\in\mathcal{T}_n^0}\kappa_T\,u_{n,\lambda(T)}.$$

Since, for fixed T, $\kappa_T\,u_{n,\lambda(T)}\in H_D^1(\Omega)$ is a piecewise 6th degree polynomial on $\mathcal{T}_n^0$ which is of class $C^1(\bar\Omega)$, so is w_n.

Let $\tilde{\mathcal{S}}_D^n(\Omega)\subset H_D^1(\Omega)$(resp. $\mathcal{S}^n(\Omega)\subset H^1(\Omega)$) denote the class of all piecewise sextic functions on $\mathcal{T}_n^0$ which are in $C^1(\bar\Omega)$. Further for any $\Omega'\subset\Omega$ we let $\tilde{\mathcal{S}}^n(\Omega')$ be the restriction of $\tilde{\mathcal{S}}^n(\Omega)$ to Ω'.

Also, let $\kappa_n^\mu=\sum\kappa_T$ where the sums are taken over all $T\in\mathcal{T}_n^0$, such that $\lambda(T)=\mu$. The functions κ_n^μ then satisfy

$$\sum_{\mu=1}^N\kappa_n^\mu(x)=1,\qquad x\in\bar\Omega, \tag{6.2}$$
$$|D^i\kappa_n^\mu(x)|\leqq C\,h_n^{-i}\,\phi_\gamma^{-i}(x)\qquad i=0,1,2. \tag{6.3}$$

It can be readily seen that

$$w_n=\sum_{\mu=1}^N\kappa_n^\mu\,u_{n,\mu}.$$

Let $u^m_{n,\mu}$ be the best approximation of u in $H^1(\Omega_m)$ by functions from $\mathcal{S}^{n,\mu}(\Omega_m)$. Because of (6.1) we have

$$\|u-u^m_{n,\mu}\|_{H^1(\Omega_m)} \leqq P\, h_n^\theta. \tag{6.4}$$

Because Ω_m is a convex polygon we get in the same way as in Theorem 5.2,

$$\|u-u^m_{n,\mu}\|_{H^{0,-\gamma}(\Omega_m)} \leqq \|z^m_{n,\mu}-\xi\|_{H^1(\Omega_m)}\, \|u^m-u_{n,\mu}\|_{H^1(\Omega_m)},$$

where $z^m_{n,\mu}$ is the solution of (3.1) on Ω_m with right hand side $f=(u-u^m_{n,\mu})\,\phi_\gamma^{-2}$ and $\Gamma_D=\emptyset$, and where $\xi\in\mathcal{S}^{n,\mu}(\Omega_m)$ is arbitrary. Also we have

$$\|z^m_{n,\mu}\|_{H^{2,\gamma}(\Omega_m)} \leqq C\,\|u-u^m_{n,\mu}\|_{H^1(\Omega_m)}.$$

Using Theorem A.1 there exists $\tilde{z}^m_{n,\mu}\in H^{2,\gamma}(\Omega)$ with

$$\|\tilde{z}^m_{n,\mu}\|_{H^{2,\gamma}(\Omega)} \leqq \|z^m_{n,\mu}\|_{H^{2,\gamma}(\Omega_m)}$$

and $z^m_{n,\mu}=\tilde{z}^m_{n,\mu}$ on Ω_m. Using ξ as in Lemma 4.5 we get

$$\|u-u^m_{n,\mu}\|_{H^{0,-\gamma}(\Omega_m)} \leqq C P\, h_n^{1+\theta} \tag{6.5}$$

with the constant C depending only on Ω_m.

Define

$$w^m_n=\sum_{\mu=1}^{N} \kappa^\mu_n\, u^m_{n,\mu}.$$

Using (6.3), (6.4), (6.5) we have

$$\begin{aligned}\|\kappa_n{}^\mu(u-u^m_{n,\mu})\|^2_{H^1(\Omega_m)} &\leqq C\int_{\Omega_m} |D^1(u-u^m_{n,\mu})|^2\,dx\\ &+C\,h_n^{-2}\int_{\Omega_m}\phi_\gamma^{-2}\,|u-u^m_{n,\mu}|\,dx \leqq CP\,h_n^{2\theta}.\end{aligned} \tag{6.6}$$

Combining (6.5), (6.6) and (6.2) we get

$$\|u-w^m_n\|_{H^1(\Omega_m)} \leqq C\,h_n^\theta. \tag{6.7}$$

Let us show now that for any $T\in\mathcal{T}_n{}^0$ and any polynomial z of degree 6 on T we get

$$\|z\|_{H^{2,\gamma}(T)} \leqq C\,h_n^{-1}\,\|z\|_{H^1(T)}. \tag{6.8}$$

In fact by a scaling argument we get

$$d_T^2\int_T |D^2 z|^2\,dx \leqq C(L)\int_T |D^1 z|^2\,dx. \tag{6.9}$$

If ϕ_β does not vanish on T we obtain

$$h_n^2\int \phi_\gamma^2\,|D^2 z|^2\,dx \leqq C(L)\int_T |D^1 z|^2\,dx. \tag{6.10}$$

If ϕ_γ vanishes at a vertex x_i of T we use instead of (6.9) the easily derived inequality

$$d_T^{2(1-\gamma_i)} \int |x-x^i|^{2\gamma_i} |D^2 z|^2 \, dx \leqq C \int_T |D^1 z|^2 \, dx$$

and we again obtain (6.8).

Now we estimate the quantity

$$K(u,t) = \inf\{\|v\|_{H^1(\Omega_m^*)} + t\,\|w\|_{H^{2,\gamma}(\Omega_m^*)};\ u = v + w\}. \tag{6.10}$$

We set $w = w_n^m$, $v = u - w_n^m$ and using (6.7) we obtain

$$K(u,t) \leqq CP\,h_n^\theta + t\,\|w_n^m\|_{H^{2,\gamma}(\Omega_m^*)}. \tag{6.11}$$

Write

$$w_n^m = w_1^m + \sum_{j=2}^n (w_j^m - w_{j-1}^m).$$

By using the inclusion $w_j^m - w_{j-1}^m \in \tilde{\mathscr{S}}^j(\Omega_m^*)$ we get by help of (6.8), namely squaring it and adding over all T belonging to $\mathscr{T}_j^0$,

$$\|w_j^m - w_{j-1}^m\|_{H^{2,\gamma}(\Omega_m^*)} \leqq C\,h_j^{-1}\,\|w_j^m - w_{j-1}^m\|_{H^1(\Omega_m^*)}. \tag{6.12}$$

Using (6.7) and (6.12) we get

$$\begin{aligned}
\|w_n^m\|_{H^{2,\gamma}(\Omega_m^*)} &\leqq \|w_1^m\|_{H^{0,\gamma}(\Omega_m^*)} + \sum_{j=2}^n \|w_j^m - w_{j-1}^m\|_{H^{0,\beta}(\Omega_m^*)} \\
&\leqq C\,\|w_1^m\|_{H^1(\Omega_m^*)} + C\sum_{j=2}^n h_j^{-1}\,\|w_j^m - w_{j-1}^m\|_{H^1(\Omega_m^*)} \\
&\leqq C[P + \|u\|_1] + C\sum_{j=2}^n P\,2^{j-\theta j} \\
&\leqq C[P + \|u\|_1 + P\,2^{(1-\theta)n}].
\end{aligned}$$

Using this in (6.11),

$$K(u,t) \leqq CP\,h_n^\theta + t\,C[\|u\|_1 + P\,h_n^{-(1-\theta)}], \qquad t > 0.$$

Suppose $t < 1/2$ and pick n so that $2^{-n-1} < t < 2^{-n}$. Then we obtain

$$\begin{aligned}
K(u,t) &\leqq C\,t^\theta + C[\|u\|_1 + P\,t^{(1-\theta)}]\,t \\
&\leqq C[P + \|u\|_1]\,t^\theta.
\end{aligned}$$

For $t > 1/2$ we set $v = u$, $w = 0$ and obtain

$$K(u,t) \leqq \|u\|_1 \leqq C\,t^\theta\,\|u\|_1.$$

Combining these inequalities we find that

$$u \in [H^{2,\beta}(\Omega_m^*), H^1(\Omega_m^*)]_{\theta,\infty}.$$

Using Lemma 2.2 we get the desired result. Using (H2), part (vi) and Theorem A.2 successively for $m=1,\dots,R$, we find that $u\in X(1,2,\beta,\theta,\infty)$. Since $u\in H_D^1(\Omega)$, we find from Theorem A.4 that $u\in X(1,2,\beta,\theta,\infty)$, and the proof is complete.

Appendix

We prove two theorems on the extension of functions lying in weighted Besov spaces. In the first theorem we show that the Stein extension [13] may be used for this purpose. In the second theorem we show that the extension operator has a certain property with respect to the intersection of two domains. We follow closely the notation and arguments of [13, Chap. 6, Sect. 3].

Let h: $R^1 \to R^1$ be a function which satisfies the Lipschitz condition $|h(s) - h(t)| < M|s-t|$, and let $D=\{x\in R^2;\ x_2 > h(x_1)\}$. We call D a *special Lipschitz domain.* We further assume that $h(0)=0$. For $x\notin \bar{D}$, we let $\delta(x)$ denote the distance from x to D, and we let $\Delta(x)$ denote the regularized distance from x to D, as constructed in [13]. In particular, $C_1\Delta(x)\leqq\delta(x)\leqq C_2\Delta(x)$, where the positive constants C_1 and C_2 depend only on M. We let S_K denote the sector $S_K=\{x\colon x_2\leqq 0,\ K|x_1|\leqq|x_2|\}$. Thus, $S_M\cap\bar{D}=\{0\}$. We have

$$\delta(x)\geqq\frac{|x_2|}{2\sqrt{1+M^2}},\qquad x\in S_{2M},$$

for $\delta(x)$ is $\geqq$ the distance from x to S_M, and the result then follows from a computation (Fig. 2).

It is shown in [13] that there is a constant $C_3>0$ depending only on M such that $C_3\Delta(x)\geqq h(x_1)-x_2$. We set $C_4=\max(C_3, 4C_2(1+M^2)^{1/2})$, and we set $\delta^*(x)$

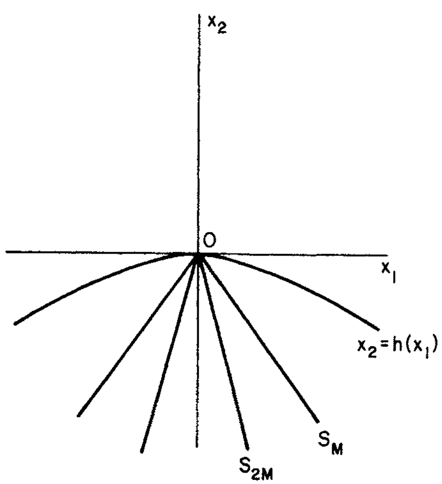

Fig. 2

$=C_1\,\Delta(x)$. Then δ^* is a regularized distance function that also satisfies

$$\begin{aligned}&\delta^*(x)\geqq h(x_1)-x_2,\\&\delta^*(x)\geqq 2|x_2|,\quad x\in S_{2M}.\end{aligned}\tag{A.1}$$

Let ψ be a smooth function defined on $[1,\infty)$ which satisfies

$$\int_1^\infty \psi(\lambda)\,d\lambda=1,\qquad \int_1^\infty \lambda^k\psi(\lambda)\,d\lambda=0,\quad k=1,2,\dots.\tag{A.2}$$

Let $f(x)$ be defined for $x\in\bar{D}$. We then define the extension Ef by $Ef(x)=f(x)$, $x\in\bar{D}$, and

$$Ef(x)=\int_1 f(x_1,x_2+\lambda\,\delta^*(x))\,\psi(\lambda)\,d\lambda,\qquad x\notin\bar{D}.\tag{A.3}$$

Let $\beta=(\beta_1)$, let $H^{2,\beta}(D)$ be the space of functions with

$$\|u\|_{H^{2,\beta}}=\{\int_D[|u|^2+|D^1u|^2+|x|^{2,\beta_1}|D^2u|^2]\,dx\}^{1/2}<\infty,$$

and let $H^{2,\beta}(R^2)$ have a similar definition. Then we have

Lemma A.1. E: $H^{2,\beta}(D)\to H^{2,\beta}(R^2)$ *is a bounded map.*

Proof. Suppose $f\in H^{2,\beta}(D)\cap C^2(D)$, and suppose that f and its first and second derivatives have continuous extensions to $\bar{D}$. Then as in [13], we conclude that $Ef\in H^1(R^2)$ and

$$\|Ef\|_{H^1(R^2)}\leqq C\,\|f\|_{H^1(D)}.$$

It remains to estimate the second derivatives of Ef. Let $x=(x_1,x_2)$, and $\tilde{x}=(x_1,x_2+\lambda\,\delta^*(x))$. We show that for $\lambda\geqq 1$, $x\notin\bar{D}$,

$$|x|\leqq C_3\,|\tilde{x}|,\tag{A.4}$$

where C_3 depends only on M. For if $x\in S_{2M}$, then using (A.1),

$$x_2+\lambda\,\delta^*(x)\geqq x_2+\delta^*(x)\geqq|x_2|,$$

so

$$\frac{|\tilde{x}|^2}{|x|^2}\geqq\frac{x_2^2}{x_1^2+x_2^2}\geqq\frac{4M^2}{1+4M^2},$$

and if $x\notin S_{2M}$ ·

$$\frac{|\tilde{x}|^2}{|x|^2}\geqq\frac{x_1^2}{x_1^2+x_2^2}\geqq\frac{1}{1+4M^2}.$$

Calculating the second derivatives of Ef and using (A.4) and the arguments of [13, p. 187] to estimate $\int |x|^{2,\beta} 1 \, |D^2 Ef|^2 \, dx$, we obtain

$$\int |x|^{2\beta_1} |D^2 Ef|^2 \, dx \leqq C \|f\|^2_{H^{2,\beta}(D)}.$$

The result follows by a limiting argument.

We now consider a polygon Ω with vertices x_i, $1 \leqq i \leqq M$, and weights $\beta_i \in [0, 1)$ as in Sect. 2. We have

Theorem A.1. *There is a bounded map* E: $H^1(\Omega) \to H^1(R^2)$, *such that* $Eu(x) = u(x)$, $x \in \Omega$. *The map* E *is also a bounded map from* $H^{2,\beta}(\Omega)$ *to* $H^{2,\beta}(R^2)$, *and from* $X(1, 2, \beta, \theta, p)$ *to* $[H^{2,\beta}(R^2), H^1(R^2)]_{\theta,p}$.

Proof. Following the argument of [13], we represent Ω as the union of a collection of special domains, and we use Lemma A.1 and a partition of unity to construct E. We find from this construction that E is a bounded map on $H^1(\Omega)$ and on $H^{2,\beta}(\Omega)$. We then use interpolation to complete the proof.

We now consider special properties of our extension operator when there are two domains present. Let h_i: $R^1 \to R^1$, $i = 1, 2$, be two Lipschitz continuous functions with Lipschitz constant M. Let $D_i = \{x \in R^2 : x_2 > h_i(x_1)\}$ be the corresponding special Lipschitz domains. Let E_1 be the extension operator corresponding to the domain D_1, constructed in Lemma A.1. Then we have

Lemma A.2. *If* $u \in H^1(D_1)$ *and* $u(x) = 0$, $x \in D_1 \cap D_2$, *then* $E_1(x) = 0$, $x \in D_2 \backslash D_1$.

Proof. We have as in (A.3),

$$E_1(x) = \int_1^\infty f(x_1, x_2 + \lambda \delta^*(x)) \, \psi(\lambda) \, d\lambda.$$

If $x \in D_2 \backslash D_1$, then $x_2 + \lambda \delta^*(x) > x_2 > h_2(x_1)$, so $\tilde{x} = (x_1, x_2 + \lambda \delta^*(x)) \in D_2$. By construction, $\tilde{x} \in D_1$, so $\tilde{x} \in D_1 \cap D_2$, and the integrand is always zero, so $E_1 f(x) = 0$, $x \in D_2 \backslash D_1$.

We now consider polygonal domains Ω_i, $i = 1$, 2. Let Ω_1 have vertices x_i, $1 \leqq i \leqq M$, and weights $\beta_i \in [0, 1)$. We suppose that Ω_1 and Ω_2 satisfy

(H) For each $x^* \in \partial\Omega_1 \cap \partial\Omega_2$ there is a neighborhood u of x^*, a linear transformation of the independent variables $x \to \tilde{x}$ of $U \to \tilde{U}$, and Lipshcitz continuous functions h_i: $R^1 \to R^1$, $i = 1$, 2, such that the image of $U \cap \Omega_i$ in $\tilde{U}$ is given by $\{\tilde{x} \in \tilde{U} : \tilde{x}_2 > h_i(\tilde{x}_1)\}$, $i = 1$, 2.

We then have

Theorem A.2. *Let the polygons* Ω_1, Ω_2 *satisfy* (*H*). *Then there is a bounded map* E: $H^1(\Omega_1) \to H^1(R^2)$, *such that* $Ef(x) = f(x)$, $x \in \Omega_1$, *such that* E *is also a bounded map from* $H^{2,\beta}(\Omega_1)$ *to* $H^{2,\beta}(R^2)$, *and from* $[H^{2,\beta}(\Omega_1), H^1(\Omega_1)]_{\theta,p}$ *to* $[H^{2,\beta}(R^2), H^1(R^2)]_{\theta,p}$ *and such that if* $f(x) = 0$ *for* $x \in \Omega_1 \cap \Omega_2$, *then* $Ef(x) = 0$ *for* $x \in \Omega_2 \backslash \Omega_1$.

Proof. We cover $\partial\Omega_1$ by a finite number of open neighborhoods U such that either $U \cap \partial\Omega_2 = \phi$, or that $\Omega_i \cap U, i = 1, 2$, can be represented as in (*H*). We use Lemma A.1, our linear change of variables, and a partition of unity, to construct E. Using the arguments of [13] and Lemma A.2, we obtain the result.

We now derive some consequences of these extension theorems.

Theorem A.3. *Let Ω_i, $i=1, 2$, be polygonal domains with non-empty intersection and which satisfy (H). Let $\Omega_3=\Omega_1\cup\Omega_2$. Let*

$$X_i=[H^{2,\beta}(\Omega_i), H^1(\Omega_i)]_{\theta,p}, \qquad i=1,2,3.$$

Then if $u\in X_i$, $i=1, 2$, we have $u\in X_3$, and $\|u\|_{X_3}\leqq C\{\|u\|_{X_1}+\|u\|_{X_2}\}$.

Proof. Using Theorems A.1, A.2, let E_i: $X_i\to X_3$, $i=1,2$, be a bounded map with the properties

(i) $(E_i u)(x)=u(x), \quad x\in\Omega_i, \ i=1,2,$
(ii) if $u\in X_i$ and $u=0$ on $\Omega_1\cap\Omega_2$, then

$$E_i u(x)=0, \ x\in\Omega_{3-i}, \qquad i=1,2.$$

Since $u\in X_1$, $E_1 u\in X_3$, so by restriction, $E_1 u\in X_2$. Hence $v=u-E_1 u\in X_2$, so $E_2 v\in X_3$. We claim that

$$u=E_1 u+E_2 v. \tag{A.5}$$

Since $v=0$ on $\Omega_1\cap\Omega_2$, $E_2 v=0$ on Ω_1, so (2.2) holds for $x\in\Omega_1$. If $x\in\Omega_2$, $E_2 v(x)=v(x)=u(x)-E_1 u(x)$, so (2.2) holds in Ω_2 also. Since the right side of (2.2) is in X_3, $u\in X_3$ and the proof is complete.

Let Ω be a polygonal domain. For our next result, we need the following lemma.

Lemma A.3. *There is a bounded map A: $H^1(\Omega)\to H_D^1(\Omega)$ such that $A=$the identity on $H_D^1(\Omega)$, and such that A: $H^{2,\beta}(\Omega)\to H^{2,\beta}(\Omega)$ is a bounded map.*

Proof. Let S be a disc with $S\supset\Omega$. Let x_i be a vertex of Ω with interior angle θ. We construct a linear transformation of R^2 to new variables $\tilde{x}$, and with a new vertex $\tilde{\theta}_i$ such that if $\kappa_i=1$,

$$\beta_i+\frac{\pi}{\tilde{\theta}_i}\neq 1,$$

and if $\kappa_i=1/2$, either $\tilde{\theta}_i<\pi/2$ or $2\pi-\tilde{\theta}_i<\pi/2$. We let $\tilde{S}_i$, $\tilde{\Omega}_i$ denote the image of S and Ω under the transformation. We now distinguish two cases. First, suppose that

$$\beta_i+\frac{\pi}{\tilde{\theta}_i}>1,$$

or, in the case of $\kappa_i=1/2$, $\tilde{\theta}_i<\pi/2$. Let $\tilde{A}_i$ denote the projection of $H^1(\tilde{\Omega})$ onto $H_D^1(\tilde{\Omega})$. Then we seen that if $\tilde{u}\in H^{2,\beta}(\tilde{\Omega})$, $\tilde{v}=\tilde{A}_i\tilde{u}$ satisfies the problem

$$-\Delta\tilde{v}+\tilde{v}=-\Delta\tilde{u}+\tilde{u} \quad \text{in } \tilde{\Omega}$$
$$v\in H_D^1(\tilde{\Omega}),$$

and from Theorem 3.1, we find that

$$\int_{\tilde{\Omega}_i}\tilde{r}^{2\beta_i}|\tilde{D}^2\tilde{v}|^2\,d\tilde{x}<\infty,$$

where $\tilde{u}_i$ is any neighborhood of $\tilde{x}_i$ which does not contain the other vertices of Ω. Transforming this operator back to the coordinate x, we obtain a bounded map

$$A_i\colon H^1(\Omega)\to H^1_D(\Omega)$$

such that $A_i u = u$, $u\in H^1_D(\Omega)$, and such that if U_i is any neighborhood of x_i which does not contain other vertices of Ω, then

$$\int_{U_i} r^{2\beta_i}|D^2 A_i u|^2\,dx<\infty.$$

Now suppose that

$$\beta_i+\frac{\pi}{\tilde{\theta}}<1,$$

or in the case of $\kappa_i=1/2$, $\tilde{\theta}_i>3\pi/2$. Let B_1 denote the extension operator, mapping $H^1(\Omega)\to H^1(S)$, let B_2 denote the projection of $H^1(S\setminus\Omega)\to H^1_D(S\setminus\Omega)$, and let B_3 denote the extension operator, mapping $H^1(S\setminus\Omega)\to H^1(\Omega)$. Set $\tilde{A}_i=B_3B_2B_1$. Then $\tilde{A}_3$ has the above properties, and we obtain a map

$$A_i\colon H^1(\Omega)\to H^1_D(\Omega)$$

having the above properties in this case also.

Let ζ_i be a smooth partition of unity such that $\zeta_i(x)\equiv 0$ near each x_k, $k\neq i$. Let $Au=\sum\zeta_i A_i u$. Then A is the desired operator.

To state our next theorem, we write

$$X=X(1,2,\beta,\theta,p),$$

$$X_D=X_D(1,2,\beta,\theta,p).$$

Then we have

Theorem A.4. $X_D=X\cap H^1_D(\Omega)$.

Proof. Suppose $u\in X_D$. Then $u\in X$, and since $u\in H^1_D(\Omega)$, $u\in X\cap H^1_D(\Omega)$. Now let $u\in X$. By interpolation, $A\colon X\to X_D$ is a bounded operator, so $Au\in X_D$. If also $u\in H^1_D(\Omega)$, then $Au=u\in X_D$, and the proof is complete.

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