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Geometric Quantization and Multiplicities of Group Representations

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§ 1. Introduction

The Heisenberg uncertainty principle says that it is impossible to determine simultaneously the position and momentum of a quantum-mechanical particle. This can be rephrased as follows: the smallest subsets of classical phase space in which the presence of a quantum-mechanical particle can be detected are its Lagrangian submanifolds. For this reason it makes sense to regard the Lagrangian submanifolds of phase space as being its true "points"; see Weinstein [17].

Now let G be a compact Lie group and $G \times X \to X$ a Hamiltonian action of G on X (see § 2 for definitions). It is well-known that the fixed points of this action form a symplectic submanifold of X. (See for instance Guillemin and Sternberg [5].) However, what can one say about the fixed "points" of G? We will show that they are also the "points" of a symplectic manifold, X_G. This manifold is the Marsden-Weinstein reduction of X with respect to the zero orbit in $\mathfrak{g}^*$, and will be described in Sect. 2. (It was introduced in a completely different context from ours by Marsden and Weinstein [12].)

Problems in classical mechanics can often be reduced to the study of Hamiltonian systems on symplectic manifolds and problems in quantum mechanics to the study of linear operators on Hilbert space. This fact has inspired a number of efforts to "quantize" symplectic geometry by devising schemes for associating Hilbert space to symplectic manifolds. The "no-go" theorems of Groenwald and Van Hove impose some embarrassing limitations on all such schemes; however, it seems to be a useful idea heuristically to think of every symplectic manifold, $X_{\text{classical}}$, as being symbiotically associated with a Hilbert space, X_{quantum}, in such a way that the classical observables on the first space correspond to quantum observables on the second space. The heuristics further suggests that if G is a group of symmetries of $X_{\text{classical}}$, it should also be a group of symmetries of X_{quantum}. In this heuristic spirit, we will state the main conjecture of this paper:

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"Theorem" Fixed "Point". *Let $X_{\text{classical}}$ be a symplectic manifold possessing a compact Lie group of symmetries, G. Let $(X_{\text{quantum}})_G$ be the set of fixed points of G in X_{quantum}. Then*

$$(X_{\text{quantum}})_G = (X_G)_{\text{quantum}}, \tag{1.1}$$

X_G being the fixed "point" set of G in $X_{\text{classical}}$.

By pursuing the heuristics of (1.1) we have been led to a number of rather interesting results. In this article we will describe one such result in detail. We will assume that $X = X_{\text{classical}}$ is a compact symplectic manifold and G a compact Lie group acting in a Hamiltonian fashion on X. To quantize X we will use the machinery of geometric quantization developed by Kostant and Souriau. Namely we will assume

A) X is pre-quantizable,

and

B) X possesses a positive-definite complex G-invariant polarization.

(Apropos of B, it is usually the case that if compact symplectic manifolds are polarizable at all it is by means of complex polarizations.) Let X_{quantum} be the Hilbert space obtained by setting up the machinery of geometric quantization on X and turning the crank: in other words, X_{quantum} = sections of the pre-quantum line bundle which are covariant constant along leaves of the polarization. We will prove the following under some genericity assumptions to be stated precisely later:

Theorem 1. a) *X_G inherits from X pre-quantum data and a positive-definite complex polariztion.*

b) *The identity* (1.1) *holds providing we take for $(X_G)_{\text{quantum}}$ the Hilbert space obtained by applying the machinery of geometric quantization to X_G.*

Remarks. 1. In the course of proving this theorem we have discovered a remarkable connection between the Marsden-Weinstein construction and Mumford's construction of a moduli space for the "stable" orbits of an algebraic group acting on a projective variety. This connection was also observed in a somewhat different setting by Kempf and Ness in [8].

2. Theorem 1 gives a formula for the multiplicity with which the zero representation of G occurs in X_{quantum}. Also, appropriately adapted, it gives a formula for the multiplicities of other irreducible representations of G as well. See § 6.

3. In the course of proving Theorem 1 we prove an old conjecture of Kirillov: By the Borel-Weil theorem, there is a one-one correspondence between integral co-adjoint orbits of G in $\mathfrak{g}^*$ and irreducible representations of G. Given an irreducible representation, ρ, of G let O_ρ be the corresponding-co-adjoint orbit.

4. We will prove Theorem 1 under hypotheses that guarantee that X_G is a manifold. For many interestingly these hypotheses are too strong and X_G is only a V manifold in the sense of Satake [20], see also Weinstein [21] and

Kawasaki [19]. The proof of Theorem 1 carries over to this more general case, but, for the sake of simplicity we will not present this more general version here.

Theorem 2. *The representation* ρ *occurs in* X_{quantum} *only if* O_ρ *occurs in the image of the moment mapping* Φ: $X_{\text{classical}} \to \mathfrak{g}^*$.

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§ 2. The Marsden-Weinstein Construction

Let X be a symplectic manifold with symplectic form, Ω. The space of smooth functions on X is a Lie algebra under the Poisson bracket operation. Moreover, there is a morphism of Lie algebras

$$C^\infty(X) \to \text{Symplectic vector fields} \tag{2.1}$$

which to functions associates their Hamiltonian vector fields. Let G be a connected Lie group and $G \times X \to X$ an action of G on X which preserves Ω. Let $\mathfrak{g}$ be the Lie algebra of G. To each element, ξ, of $\mathfrak{g}$ corresponds a symplectic vector field, $\xi^\#$, on X. Moreover, the mapping

$$\mathfrak{g} \to \text{Symplectic vector fields} \tag{2.2}$$

sending ξ to $\xi^\#$ is a Lie algebra morphism. The action of G on X is said to be Hamiltonian if (2.2) factors through (2.1); i.e. if there is given a Lie algebra morphism

$$\mathfrak{g} \to C^\infty(X), \qquad \xi \to \phi^\xi, \tag{2.3}$$

such that (2.2) is the composition of (2.1) and (2.3). The existence of (2.3) is equivalent to the pair of conditions

$$\xi^\# \lrcorner\, \Omega = d\phi^\xi, \qquad \{\phi^\xi, \phi^\eta\} = \phi^{[\xi,\eta]}. \tag{2.4}$$

(The first of these conditions determines ϕ^ξ up to an additive constant.) To each point, x, in X we can associate an element, $\Phi(x)$, of $\mathfrak{g}^*$ by the formula

$$\langle \Phi(x), \xi \rangle = \phi^\xi(x). \tag{2.5}$$

As we vary x, this gives us a smooth mapping:

$$\Phi: X \to \mathfrak{g}^* \tag{2.6}$$

This mapping is by definition the *moment mapping* associated with the action of G on X. From the second of the two Eqs. (2.4) it is easy to see that it is equivariant, i.e. intertwines the action of G on X and the co-adjoint action of

G on $\mathfrak{g}^*$. Before describing some of its other properties it will be useful to compute its derivative at points, $x \in X$. By evaluating $\xi^\#$ at x we get a linear mapping

$$\mathfrak{g} \to T_x. \tag{2.7}$$

The symplectic form, Ω_x, gives us an identification

$$T_x \cong T_x^*, \tag{2.8}$$

and composing (2.7) and (2.8) we get a linear mapping

$$\mathfrak{g} \to T_x^*. \tag{2.9}$$

On the other hand, the derivative of Φ at x is a mapping $d\Phi_x \colon T_x \to \mathfrak{g}^*$.

Lemma 2.1. *The derivative of Φ at x is the dual of* (2.9).

Proof. This is just a restatement of the first of the identities (2.4).

This lemma has a number of interesting corollaries which we leave as trivial exercises.

Corollary 2.2. *Let $\mathfrak{g}_x$ be the Lie algebra of the stabilizer group of x. Then the image of $d\Phi_x$ is the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$.*

Corollary 2.3. *The derivative of Φ is surjective at x if and only if the stabilizer group of x is discrete.*

Corollary 2.4. *The kernel of $d\Phi_x$ is the set of all $v \in T_x$ such that $\Omega(v, \xi_x^\#) = 0$ for all $\xi \in \mathfrak{g}$.*

Now let $X_0 = \{x \in X, \Phi(x) = 0\}$. Because of the equivariance of Φ, X_0 is a G-invariant subset of X. If the origin in $\mathfrak{g}^*$ is a regular value of Φ, X_0 is a submanifold of X. Moreover, by Corollary 2.4, the tangent space of X_0 at $x \in X_0$ is

$$\{v \in T_x, \Omega(v, \xi_x^\#) = 0\}. \tag{2.10}$$

Because of the G-invariance the vectors, $\xi_x^\#$, are tangent to X_0 at x; therefore, by (2.10) the tangent space to X_0 at x is co-isotropic and its null-space is $\{\xi_x^\#, \xi \in \mathfrak{g}\}$. Finally notice that since the origin in $\mathfrak{g}^*$ is a regular value of Φ the derivative of Φ is surjective at x; so by Corollary 2.3 the stabilizer group of x is a discrete subgroup of G. Summarizing we have proved the following result of Marsden-Weinstein [12]:

Theorem 2.5. *If the origin in $\mathfrak{g}^*$ is a regular value of Φ, then X_0 is a G-invariant co-isotropic submanifold of X. Moreover, the action of G on X_0 is locally free and the orbits of G are the leaves of the null-foliation.*

Suppose now that G is compact. Then the stabilizer group of $x \in X_0$ is a finite subgroup of G. We will henceforth assume that *for all $x \in X_0$ the stabi-*

lizer group of x is trivial.[1] This assumption implies that G acts freely on X_0; so the orbit space

$$X_G = X_0/G$$

is a C^∞ Hausdorff manifold and the projection mapping

$$\pi: X_0 \to X_G \tag{2.11}$$

is a principal G-fibration. Since the fibers are the leaves of the null-foliation, there exists a unique symplectic form, Ω_G, on X_G such that

$$\pi^* \Omega_G = \iota^* \Omega, \tag{2.12}$$

ι being the inclusion mapping of X_0 into X. X_G is called the *Marsden-Weinstein reduction of X with respect to the zero orbit in* $\mathfrak{g}^*$. (The Marsden-Weinstein reduction of X with respect to an arbitrary orbit in $\mathfrak{g}^*$ will be defined in §6.) We will now prove that the fixed "points" of G in X are identical with the "points" of X_G. To get the cleanest statement possible of this result we will assume that the Lie algebra of G has the property,

$$\mathfrak{g} = \text{its own commutator} = [\mathfrak{g}, \mathfrak{g}]. \tag{2.13}$$

Theorem 2.6. *Let Λ be a Lagrangian submanifold of X_G. Then the pre-image of Λ in X_0 is a G-invariant Lagrangian submanifold of X. Moreover, every G-invariant Lagrangian submanifold of X is of this form, i.e. there is a one-one correspondence between Lagrangian submanifolds of X_G and G-invariant Lagrangian submanifolds of X.*

Proof. The first statement is a simple consequence of (2.12). To prove the second statement, let Λ be a G-invariant Lagrangian submanifold of X. Then for all $\xi \in \mathfrak{g}$ and all $x \in \Lambda$, $\xi^\#$ is tangent to Λ at x; so for all $v \in T_x\Lambda$, $\Omega(\xi_x^\#, v) = 0$. By (2.4) this implies that $d\phi_x^\xi(v) = 0$; so ϕ^ξ is constant on connected components of Λ. Since $\xi^\#$ is tangent to Λ, $\xi^\# \phi^\eta = 0$ for all ξ, η in $\mathfrak{g}$. By the second half of (2.4) $\phi^{[\xi,\eta]} = 0$ on Λ; so by (2.13) $\phi^\xi = 0$ on Λ. This shows that Λ is contained in X_0. Since it is G-invariant it is the pre-image of a subset, Λ_1, of X_G. It is easy to see that Λ_1 is a submanifold and is itself Lagrangian. Q.E.D.

Remark. The condition (2.13) is a necessary and sufficient condition that the moment map be uniquely defined. If (2.13) fails to hold, then if $\Phi: X \to \mathfrak{g}^*$ satisfies (2.4), so does $\Phi + c$, where c is in the annihilator of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}^*$. To get all the G-invariant Lagrangian submanifolds of X, one has to perform the construction above for *all* $\Phi + c$'s.

§ 3. Geometric Quantization

The first few paragraphs of this section are a brief review of the material in the last two sections of Kostant [10]. Let X be a symplectic manifold with symplectic form, Ω. Let

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[1] Most of the results described in the next few paragraphs are true without this hypothesis. However X_G (defined below) is then not a manifold but only a V-manifold

$$[\Omega] \in H^2_{\text{DeRham}}(X)$$

be its cohomology class. One says that Ω is *integral* if $[\Omega]$ is in the image of the map

$$H_{\text{Cech}}(X, Z) \xrightarrow{\text{Weil}} H_{\text{DeRham}}(X).$$

If Ω is integral there exists a line bundle L on X whose Chern class is $[\Omega]$, a connection ∇ on L whose curvature form is Ω, and a Hermitian inner product $\langle\ ,\ \rangle$ on L which is invariant under parallel transport. The data L, ∇ and $\langle\ ,\ \rangle$ are called *pre-quantum data* on X.

Now let G be a connected Lie group and $G \times X \to X$ a Hamiltonian action of G on X. Let Φ: $X \to \mathfrak{g}^*$ be the associated moment mapping. There is a canonical representation of the Lie algebra, $\mathfrak{g}$, on smooth sections of L given by the operators

$$\nabla_{\xi^\#} + 2\pi i \phi^\xi, \qquad \xi \in \mathfrak{g}. \tag{3.1}$$

The pre-quantum data are said to be *G-invariant* if there exists a global action of G on L such that the induced action of $\mathfrak{g}$ is given by (3.1). The obstruction to extending (3.1) from $\mathfrak{g}$ to G is topological in nature. For instance it is always possible to do this if G is simply-connected.

Example. (See pages 176–207 of [11].) Let f be an element of $\mathfrak{g}^*$ and $X = O_f$ = the co-adjoint orbit through f. Let G_f be the stabilizer group of f and $\mathfrak{g}_f$ its Lie algebra. Consider the linear functional

$$\rho_f\colon \xi \in \mathfrak{g}_f \to 2\pi i \langle f, \xi \rangle \in \sqrt{-1}\mathbf{R}. \tag{3.2}$$

It is easy to see that ρ_f is an "infinitesimal character" of G_f, that is, vanishes on the commutator, $[\mathfrak{g}_f, \mathfrak{g}_f]$. One says that f is *integral* if there exists a global character χ_f: $G_f \to S^1$ such that $d\chi_f = \rho_f$.

Proposition 3.1. *O_f possesses a G-invariant pre-quantization if and only if f is integral.* (This is Theorem 5.7.1 of [11].)

Now let G be a compact Lie group, X a Hamiltonian G-space and Φ: $X \to \mathfrak{g}^*$ the moment mapping. Let $X_0 = \{x \in X \mid \Phi(x) = 0\}$. If G acts freely on X_0 we can, as in Sect. 2, form the reduced space

$$X_G = X_0/G.$$

We will show that if L, ∇ and $\langle\ ,\ \rangle$ are G-invariant pre-quantum data on X, then there are associated pre-quantum data on X_G. Let

$$\pi\colon X_0 \to X_G$$

be the projection map and

$$\iota\colon X_0 \to X$$

the inclusion map.

Theorem 3.2. *There is a unique line bundle with connection, (L_G, ∇_G) on X_G such that*

$$\pi^* L_G = \iota^* L \quad \text{and} \quad \pi^* \nabla_G = \iota^* \nabla. \tag{3.3}$$

Proof. To define L_G it is enough to define the sheaf of sections of L_G. We will take this to be the sheaf of G-invariant sections of $\iota^* L$. Let us now show how to define a connection on L_G. Let U be an open subset of X_G and s_U a non-vanishing G-invariant section of $\iota^* L$ on $\pi^{-1}(U)$. The covariant derivative of s_U is the tensor product of s_U and a one-form, α_U on X. We will first show:

Lemma 3.3. *There is a unique one-form β_U on $U \subset X_G$ such that $\pi^* \beta_U = \alpha_U$.*

Proof. It is obvious that α_U is G-invariant since s_U is G-invariant. Since $\phi^\xi = 0$ on X_0, s_U is covariant constant along the fibers, $\pi^{-1}(m)$, $m \in U$, by (3.1); so, for each $m \in U$, the restriction of α_U to the fiber, $\pi^{-1}(m)$, is zero. These two facts together imply that α_U pushes down to a well-defined one-form, β_U, on X_G. Q.E.D.

Let s_U and s_V be non-vanishing G-invariant sections on $\pi^{-1}(U)$ and $\pi^{-1}(V)$. Then there is a non-vanishing G-invariant function, f_{UV}, on $\pi^{-1}(U \cap V)$ such that

$$s_U = f_{UV} s_V.$$

A simple computation shows that α_U and α_V satisfy the standard "gauge" conditions

$$\alpha_U = \alpha_V + d \log f_{UV}.$$

Since f_{UV} is G-invariant, it is the pull-back of a function, g_{UV}, on $U \cap V$ and hence,

$$\beta_U = \beta_V + d \log g_{UV};$$

i.e. the β's also satisfy the standard "gauge" conditions, and so define a global connection, ∇_G, on L_G. Q.E.D.

Comparing (3.3) with (2.12) we obtain

Corollary 3.4. *The curvature of the connection, ∇_G, is the symplectic form Ω_G.*

Since the Hermitian inner product, $\langle\ ,\ \rangle$, is G-invariant, there is a unique Hermitian inner product on L_G such that $\pi^* \langle\ ,\ \rangle_G = \iota^* \langle\ ,\ \rangle$. By Corollary 3.4, L_G, ∇_G and $\langle\ ,\ \rangle_G$ are pre-quantum data on X_G, (so we have accomplished what we set out to prove.)

Next, we will review a few facts about polarizations. For a more detailed account of the material below, see, for instance, [15]. Let V be a $2n$-dimensional real vector space and Ω a symplectic form on V. Let $\Omega_{\mathbf{C}}$ be the $\mathbf{C}$-linear extension of Ω to $V \otimes \mathbf{C}$. An n-dimensional complex subspace, F, of $V \otimes \mathbf{C}$ is *Lagrangian* if it satisfies $\Omega_{\mathbf{C}}(v, w) = 0$ for all $v, w \in F$. It is *positive-definite* if, in addition, the Hermitian form

$$\sqrt{-1}\,\Omega(v, \bar{w})$$

is positive-definite on F. Now let X be a $2n$-dimensional symplectic manifold and T its tangent bundle. A *polarization* of X is an integrable Lagrangian sub-bundle, F, of $T\otimes\mathbf{C}$. It is *positive-definite*, if, for all $x\in X$, F_x is a positive-definite Lagrangian sub-space of $T_x\otimes\mathbf{C}$.

Example. Let G be a compact Lie group. Let B be a positive-definite Ad_G-invariant bilinear form on $\mathfrak{g}$. By means of B we get a G-equivariant identification of $\mathfrak{g}$ with $\mathfrak{g}^*$; so we can identify co-adjoint orbits in $\mathfrak{g}^*$ with adjoint orbits in $\mathfrak{g}$. Let O be an adjoint orbit, let ξ be a point of O and let T_ξ be the tangent space to O at ξ. The map, $\mathrm{ad}\,\xi\colon \mathfrak{g}\to\mathfrak{g}$ maps T_ξ onto itself and is skew-adjoint with respect to B; so its eigenvalues are pure imaginary and half of them lie on the positive imaginary axis. Let $F_\xi\subset T_\xi\otimes\mathbf{C}$ be the space spanned by the eigenvectors corresponding to these positive eigenvalues. F_ξ varies smoothly as one varies ξ, and so defines a vector subbundle, F, of the complex tangent bundle of O. One can show that F is a G-invariant positive-definite polarization.

Now let G be a connected, compact Lie group, X a Hamiltonian G-space and F a G-invariant, positive-definite polarization of X. We will prove:

Theorem 3.5. *There is canonically associated with F a positive-definite polarization, F_G, of the reduced space, X_G.*

Proof. For each point, $x\in X_0$, let $F'_x=(T_xX_0)\otimes\mathbf{C}\cap F_x$. Let W_x be the tangent space to the G-orbit through x. We will show below that

$$F_x\cap(W_x\otimes\mathbf{C})=0. \tag{3.4}$$

Assuming this for the moment we show that

$$\dim F'_x=(\dim X_G)/2. \tag{3.5}$$

Since $(F'_x)^\perp=F_x+W_x\otimes\mathbf{C}$, by (2.10), and the sum is direct, by (3.4), $\dim(F'_x)^\perp=(\dim X)/2+\dim G$, from which one easily deduces (3.5). It follows from (3.5) that F'_x varies smoothly as x varies on X_0 and so defines a vector subbundle, F', of the complex tangent bundle of X_0. Now let m be a point of X_G and x a point on the fiber above m. The derivative of π, $d\pi_x\colon T_xX_0\to T_m$ maps F'_x onto a subspace of $T_m\otimes\mathbf{C}$. By (3.4) this map is a bijection; so the image of F'_x is of dimension equal to $(\dim X_G)/2$ and is consequently Lagrangian by (2.12). Since F is G-invariant this image is the same for all x in the fiber above m. Let us denote it by $(F_G)_m$. It is clear that $(F_G)_m$ varies smoothly as we vary m; so it defines a Lagrangian subbundle, F_G, of the complexified tangent bundle of X_G. To show that it is integrable, let Ξ_1 and Ξ_2 be sections of F_G and let Ξ'_1 and Ξ'_2 be the unique G-invariant sections of F' sitting above them in X_0. Because F is integrable, $[\Xi'_1,\Xi'_2]$ is also a G-invariant section of F', and its projection down in X_G is $[\Xi_1,\Xi_2]$. Thus $[\Xi_1,\Xi_2]$ is also a section of F_G. Q.E.D.

We must still prove (3.4). This is a consequence of the following elementary fact.

Lemma 3.6. *Let V be a $2n$-dimensional real vector space and Ω a symplectic form on V. Let W be an isotropic subspace of V and F a positive-definite Lagrangian subspace of $V\otimes\mathbf{C}$. Then $(W\otimes\mathbf{C})\cap F=0$.*

Proof. Suppose w_1 and w_2 are in W and $v = w_1 + \sqrt{-1}\, w_2$ is in F. Then

$$\Omega(v, \bar{v}) = 2\sqrt{-1}\,\Omega(w_1, w_2) = 0$$

since W is isotropic. But if F is positive-definite this implies $v = 0$. Q.E.D.

Let $X = X_{\text{classical}}$ be a symplectic manifold. Let L, ∇, and $\langle\ ,\ \rangle$ be pre-quantum data on X and F a positive-definite polarization. A section $s: X \to L$ is said to be *polarized* if $\nabla_{\Xi} s = 0$ for all sections, Ξ, of $\bar{F}$. If X is compact, the set of polarized sections forms a finite dimensional vector space. Using the Hermitian inner product, $\langle,\rangle$, on L and the Liouville measure on X, this vector space becomes a finite dimensional Hilbert space which we denote by X_{quantum}. If X is a Hamiltonian G-space, and the pre-quantum data and the polarization are G-invariant, there is a natural unitary representation of G on X_{quantum}.

Example. Let G be a compact, connected Lie group and f an integral element of $\mathfrak{g}^*$. Let O be the co-adjoint orbit through f. We saw above how to polarize and pre-quantize O in a G-invariant fashion. Let ρ_0 be the representation of G which we have just described.

Theorem 3.7. *ρ_0 is irreducible. Moreover, the correspondence, $O \to \rho_0$, is a bijective correspondence between integral orbits in $\mathfrak{g}^*$ and irreducible unitary representations of G.*

This is the Borel-Weil theorem in a guise due to Kostant. See [10]. We will come back to it in §6.

Now let X be a compact Hamiltonian G-space. We will assume that X can be pre-quantized and admits a positive-definite polarization. Let X_{quantum} be the Hilbert space described above and $(X_{\text{quantum}})_G$ the set of G-fixed vectors in it. We have proved that the reduced space, X_G, is pre-quantizable and admits a positive definite polarization, so it also possesses its quantum counterpart, $(X_G)_{\text{quantum}}$. We will conclude this section by showing

Theorem 3.8. *There is a canonical map*

$$(X_{\text{quantum}})_G \to (X_G)_{\text{quantum}}. \tag{3.6}$$

Proof. By restricting a G-invariant section of L to X_0 we get a section of L_G by definition. It is clear from Theorem 3.5 that polarizes sections go into polarized sections.

§4. The Group $G^{\mathbb{C}}$

Let G be a compact connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{g}^{\mathbb{C}}$ be the complexified Lie algebra, $\mathfrak{g} \oplus \sqrt{-1}\,\mathfrak{g}$. Our first result has to do with the existence of a "complex form" of G.

Proposition 4.1. *There exists a unique connected complex Lie group, $G^{\mathbb{C}}$, with the following two properties:*

i) *Its Lie algebra is* $\mathfrak{g}^{\mathbf{C}}$.

ii) G *is a maximal compact subgroup of* $G^{\mathbf{C}}$.

Proof. By structure theory for compact Lie groups, G is the product of a compact semi-simple group and a finite number of copies of S^1. If $G=S^1$, $C^{\mathbf{C}}=\mathbf{C}^*$. If G is semi-simple, its fundamental group is finite; so if G_1 is the universal covering group of G, there exists a finite central subgroup, K, of G_1 such that

$$G=G_1/K.$$

Let $G_1^{\mathbf{C}}$ be the unique simply-connected complex Lie group with $\mathfrak{g}^{\mathbf{C}}$ as its Lie algebra. It is clear that G_1 is a maximal compact subgroup of $G_1^{\mathbf{C}}$ and that the center of $G_1^{\mathbf{C}}$ is identical with the center of G_1. Let $G^{\mathbf{C}}=G_1^{\mathbf{C}}/K$.

For the general case let $G^{\mathbf{C}}$ be the product of the $G^{\mathbf{C}}$'s described above. Q.E.D.

We will now discuss some properties of G-actions on Kaehler manifolds.

Definition 4.2. A symplectic manifold is a (positive) Kaehler manifold if it possesses a positive-definite polarization.

The next well-known lemma will be used to reconcile this definition with the standard one:

Lemma 4.3. *Let* V *be a (real) symplectic vector space with symplectic form,* Ω. *Let* F *be a positive-definite Lagrangian subspace of* $V\otimes\mathbf{C}$. *Then there exists a unique linear mapping* $J\colon V\to V$ *such that*

i) $J^2=-I$.

ii) $F=\{v+\sqrt{-1}\,Jv,\ v\in V\}$.

iii) $\Omega(Jv,Jw)=\Omega(v,w)$.

iv) *The quadratic form* $B(v,w)=\Omega(v,Jw)$ *is symmetric and positive-definite.*

Proof. F is positive-definite if and only if the quadratic form $\sqrt{-1}\,\Omega(v,\bar{w})$ is positive-definite on F; so $F\cap\bar{F}=\{0\}$. From this fact it is easy to see that there exists a mapping, J, with properties i) and ii). Since F is Lagrangian

$$\Omega(v+\sqrt{-1}\,Jv,\ w+\sqrt{-1}\,Jw)=0 \tag{4.1}$$

for all $v,w\in V$. By evaluating the real and imaginary parts of (4.1) one obtains iii) and the fact that, in iv), B is symmetric. Finally B is positive-definite since

$$\sqrt{-1}\,\Omega(u,\bar{u})=2B(v,v)$$

for $u=v+\sqrt{-1}\,Jv$. Q.E.D.

Let X be a symplectic manifold and F a positive definite polarization. By the lemma we get for each $x\in X$ a mapping

$$J_x\colon T_x\to T_x$$

with the properties i), ii), and iii) and a positive-definite quadratic form, B_x, on T_x. J and B vary smoothly with x; so J defines an almost-complex structure on X and B a Riemannian structure. The integrability of F implies that the almost-complex structure is complex. Therefore, the quadruple (X, J, B, Ω) is Kaehler manifold in the usual sense.

Let (X, F) be a compact Kaehler manifold and G a compact connected Lie group which acts on X, preserving F. We will prove.

Theorem 4.4. *The action of G can be canonically extended to an action of $G^{\mathbf{C}}$, preserving F.*

Proof. Let ξ_1 be a vector field on X. We will say that ξ_1 preserves F if, for every section, Ξ, of F, $[\xi_1, \Xi]$ is also a section of F. It is clear that ξ_1 preserves F if and only if

$$[\xi_1, J\xi_2] = J[\xi_1, \xi_2] \tag{4.2}$$

for all vector fields, ξ_2. Suppose now that ξ_1 preserves F. Then $J\xi_1$ preserves F. Indeed, for all vector fields, ξ_1 and ξ_2,

$$J([\xi_1, \xi_2] - [J\xi_1, J\xi_2]) = [J\xi_1, \xi_2] + [\xi_1, J\xi_2]$$

by the integrability of F. If ξ_1 preserves F, this becomes

$$[J\xi_1, J\xi_2] = J[J\xi_1, \xi_2]$$

for all vector fields, ξ_2; so $J\xi_1$ preserves F as claimed. In particular if both ξ_1 and ξ_2 preserve F,

$$[J\xi_1, J\xi_2] = -[\xi_1, \xi_2]. \tag{4.3}$$

Now for every $\xi \in \mathfrak{g}$ let $\xi^\#$ be the corresponding vector field on X. Let

$$\tau\colon \mathfrak{g}^{\mathbf{C}} \to \text{(real) vector fields on } X$$

be the mapping, $\xi_1 + \sqrt{-1}\,\xi_2 \to \xi_1^\# + J\xi_2^\#$. By (4.2) and (4.3), τ is a morphism of Lie algebras. Moreover, by (4.2), if $\eta \in \mathfrak{g}^{\mathbf{C}}$, $\tau(\eta)$ is a vector field preserving F. Let $\mathrm{Diff}(X)_F$ be the group of analytic diffeomorphisms of X which preserve F. By [9], $\mathrm{Diff}(X)_F$ is a (finite dimensional) Lie group; therefore, if $G^{\mathbf{C}}$ is simply-connected, τ can be extended uniquely to a morphism of Lie groups:

$$G^{\mathbf{C}} \to \mathrm{Diff}(X)_F. \tag{4.4}$$

If $G^{\mathbf{C}}$ is not simply-connected, let G_1 and $G_1^{\mathbf{C}}$ be the universal covering groups of G_1 and $G_1^{\mathbf{C}}$ respectively. Then there exists a discrete subgroup, K, of G_1, contained in the center of $G_1^{\mathbf{C}}$, such that

$$G = G_1/K \quad \text{and} \quad G^{\mathbf{C}} = G_1^{\mathbf{C}}/K.$$

By the same reasoning as before, τ can be extended uniquely to a morphism of Lie groups

$$G_1^{\mathbf{C}} \to \mathrm{Diff}(X)_F.$$

Moreover, restricted to G_1, this map factors through K since, by assumption, there is an action of G on X extending τ. Therefore K is in the kernel of this mapping. Q.E.D.

Suppose now that the action of G on X is Hamiltonian. Let Φ: $X \to \mathfrak{g}^*$ be the momentum mapping. Let $X_0 = \{x \in X, \Phi(x) = 0\}$. We will assume as in Sect. 2 that G acts freely on X_0; so that we can form the reduced space

$$X_G = X_0/G.$$

Let X_s be the saturation of X_0 with respect to $G^{\mathbf{C}}$; i.e.

$$X_s = \{gx;\ x \in X_0,\ g \in G^{\mathbf{C}}\}. \tag{4.5}$$

We will call the points of X_s *stable points* for the action of $G^{\mathbf{C}}$ on X.[2]

Theorem 4.5. *X_s is an open subset of X and $G^{\mathbf{C}}$ acts freely on it.*

Proof. Let V be a real symplectic vector space and F a positive-definite Lagrangian subspace of $V \otimes \mathbf{C}$. Let J and B be as in Lemma 4.3.

Lemma 4.6. *Let W be a subspace of V, and let $W^{\perp} = \{v \in V,\ \Omega(v, w) = 0$ for all $w \in W\}$. Then JW is the orthogonal complement of $W^{\perp}$ with respect to B.*

Proof. For all $v, w \in V$

$$B(Jv, w) = \Omega(Jv, Jw) = \Omega(v, w).$$

If $v \in W$ the last term is zero for all $w \in W^{\perp}$; so Jv is in the ortho-complement of $W^{\perp}$. Conversely if Jv is in the ortho-complement of $W^{\perp}$, $\Omega(v, w) = 0$ for all $w \in W^{\perp}$; so $w \in W$. Q.E.D.

We will use this lemma to prove that X_s is open in X. Let x be a point of X_0 and let W be the tangent space to the orbit of G through x. By Theorem 2.5

$$W^{\perp} = T_x X_0.$$

Therefore, by the lemma

$$\{\eta_x^{\#}, \eta \in \sqrt{-1}\,\mathfrak{g}\}$$

is a complementary space to $T_x X_0$ in $T_x X$. This shows that X_s contains an open neighborhood, U, of X_0. Since $X_s = \bigcup gU$, $g \in G^{\mathbf{C}}$, X_s is itself open. This argument also shows that the stabilizer algebra of x in $\mathfrak{g}^{\mathbf{C}}$ is zero; so the action of $G^{\mathbf{C}}$ on X_s is locally free. To show that $G^{\mathbf{C}}$ acts freely on X_s we need a refinement of this argument: If $\xi \in \mathfrak{g}$ then

[2] If X is a projective variety and $G^{\mathbf{C}}$ an algebraic group acting algebraically on X then, by a recent result of Kempf and Ness, X_s is the set of *stable points* of X in the sense of Mumford, [14]. Consequently X_G is the moduli space constructed by Mumford in §5.2 of [14]. We are indebted to Mumford for having spotted this fact. Several of the results described in the next two sections are either analogues or symplectic reformulations of results in [14]

$$\xi^{\#} \lrcorner \Omega = d\phi^{\xi}$$

by (2.4); i.e. $\xi^{\#}$ is the *Hamiltonian* vector field associated with the function, ϕ^{ξ}. Let $\eta = \sqrt{-1}\,\xi \in \sqrt{-1}\,\mathfrak{g}$. By definition $\eta^{\#} = J\xi^{\#}$.

Lemma 4.7. *$\eta^{\#}$ is the gradient vector field associated with the function, ϕ^{ξ}.*

Proof. We have to show that for all $x \in X$ and all $v \in T_x$,

$$B(\eta_x^{\#}, v) = \langle d\phi_x^{\xi}, v \rangle.$$

However, $B(\eta_x^{\#}, v) = \Omega(J\xi_x^{\#}, Jv) = \Omega(\xi_x^{\#}, v) = (\xi^{\#} \lrcorner \Omega)(v)$; so the assertion is clear. Q.E.D.

By Proposition 4.1, G is a maximal compact subgroup of $G^{\mathbf{C}}$. Let

$$G^{\mathbf{C}} = PG$$

be the Cartan decomposition of $G^{\mathbf{C}}$. It is clear that

$$\mathfrak{g}^{\mathbf{C}} = \sqrt{-1}\,\mathfrak{g} \oplus \mathfrak{g}$$

is the corresponding Cartan decomposition of $\mathfrak{g}^{\mathbf{C}}$, i.e. $\mathfrak{p} = \sqrt{-1}\,\mathfrak{g}$. The exponential map,

$$\exp\colon g^{\mathbf{C}} \to G^{\mathbf{C}},$$

therefore maps $\sqrt{-1}\,\mathfrak{g}$ bijectively onto P. Let x be a point of X_0 and g an element of the stabilizer group of x in $G^{\mathbf{C}}$. Then $g = (\exp \eta)k$ for some $\eta = \sqrt{-1}\,\xi \in \sqrt{-1}\,\mathfrak{g}$ and $k \in G$. Let $y = kx$. Since X_0 is G-invariant, $y \in X_0$; so $\phi^{\xi}(y) = 0$. The curve, $(\exp t\eta)\,y$, $-\infty < t < \infty$, is the integral curve through y of the gradient vector field of ϕ^{ξ}; so if $\xi \neq 0$, ϕ^{ξ} is strictly increasing along this curve, and in particular $\phi^{\xi} > 0$ at the point $(\exp \eta)\,y$. But $(\exp \eta)\,y = x \in X_0$, so we get a contradiction. Thus $\xi = 0$ and $g = k$. But since G acts freely on X_0, k has to be the identity element. Q.E.D.

By (4.5) X_G can be represented as the quotient space

$$X_G = X_s / G^{\mathbf{C}}. \tag{4.6}$$

We know from Theorem 3.5 that X_G is a Kaehler manifold. By Theorem 4.5, X_s is an open complex submanifold of X on which the complex group, $G^{\mathbf{C}}$ acts freely and holomorphically; so (4.6) provides another description of the complex structure on X_G.

§ 5. The Bijectivity of (3.6)

Now let L, V and $\langle\ ,\ \rangle$ be G-invariant pre-quantum data on X. We will first prove an analogue of Theorem 4.4.

Theorem 5.1. *The action of G on the line bundle, L, can be canonically extended to an action of $G^{\mathbf{C}}$ on L.*

Proof. We will describe how the Lie algebra, $\mathfrak{g}^{\mathbf{C}}$, acts on sections of L. If $\xi \in \mathfrak{g}$ then by (3.1)

$$\xi s = \nabla_{\xi^{\#}} s + 2\pi\sqrt{-1}\,\phi^{\xi} s \tag{5.1}$$

for all sections, s, of L. If s is holomorphic (or polarized) we will define

$$\eta s = \sqrt{-1}\,\xi s \tag{5.2}$$

for $\eta = \sqrt{-1}\,\xi \in \sqrt{-1}\,\mathfrak{g}$. Since s is holomorphic and $\eta^{\#} = J\xi^{\#}$

$$\nabla_{\xi^{\#} - \sqrt{-1}\eta^{\#}} s = 0;$$

so $\nabla_{\xi^{\#}} s = \sqrt{-1}\,\nabla_{\eta^{\#}} s$. Therefore, by (5.1)

$$\eta s = -(\nabla_{\eta^{\#}} s + 2\pi\phi^{\xi} s). \tag{5.3}$$

If f is a smooth function and $s' = fs$, one expects to have

$$\eta s' = (\eta^{\#} f)\, s + f\eta s,$$

which will be the case if we take (5.3) to be our definition of ηs for *all* sections, s, of L. We let the reader check that (5.1) and (5.3) define a representation of $\mathfrak{g}^{\mathbf{C}}$ on sections of L. The proof that this representation corresponds to a global action of $G^{\mathbf{C}}$ on L is identical with the proof of the analogous result in Section 4, and we will omit it. Q.E.D.

Let s be a section of L, and let $\langle s, s\rangle(x)$ be the norm of s with respect to the Hermitian inner product, $\langle\ ,\ \rangle_x$ on L_x. By definition $\langle s, s\rangle$ is a non-negative real-valued function. By assumption, $\langle\ ,\ \rangle$ is invariant with respect to parallel transport; so for all $\eta = \sqrt{-1}\,\xi \in \sqrt{-1}\,\mathfrak{g}$,

$$\eta^{\#}\langle s, s\rangle = \langle \nabla_{\eta^{\#}} s, s\rangle + \langle s, \nabla_{\eta^{\#}} s\rangle. \tag{5.4}$$

Suppose now that s is $G^{\mathbf{C}}$-invariant. Then by (5.3)

$$\nabla_{\eta^{\#}} s = -2\pi\phi^{\xi} s;$$

so

$$\zeta^{\#}\langle s, s\rangle = -4\pi\phi^{\xi}\langle s, s\rangle. \tag{5.5}$$

This equation, as we will shortly see, plays a crucial role in the proof of the bijectivity of (3.6).

Now let X_{quantum} be the space of holomorphic sections of L over X and $(X_s)_{\text{quantum}}$ the space of holomorphic sections of L over X_s. Let

$$[X_{\text{quantum}}]_G \quad \text{and} \quad [(X_s)_{\text{quantum}}]_G$$

be the set of G-fixed vectors in these two spaces. Let $(X_G)_{\text{quantum}}$ be the space of holomorphic sections of L_G over X_G.

Theorem 5.2. *The canonical mapping*

$$[(X_s)_{\text{quantum}}]_G \to (X_G)_{\text{quantum}}$$

is bijective.

Proof. Let s: $X_s \to L$ be G-invariant and holomorphic. By (5.2) it is $G^{\mathbf{C}}$-invariant. Since X_s is the saturation of X_0 by $G^{\mathbf{C}}$, s is determined by its restriction, s', to X_0. But s' is G-invariant; so it is, by definition, a section of L_G. Since $G^{\mathbf{C}}$ acts freely on X_s it is clear that, given a G-invariant section, s: $X_0 \to L$, one can extent it uniquely to a $G^{\mathbf{C}}$-invariant section, s: $X_s \to L$. Finally if s' is polarized, so is s since $G^{\mathbf{C}}$ preserves the polarization. Q.E.D.

It is clear that the restriction mapping

$$[X_{\text{quantum}}]_G \to [(X_s)_{\text{quantum}}]_G, \tag{5.6}$$

is injective; so, by Theorem 5.2, to prove that (3.6) is bijective, it is enough to prove that (5.6) is surjective. We will do so below; however, first we will prove a special case of the Kirillov conjecture mentioned in the introduction.

Theorem 5.3. *If zero is not in the image of the moment mapping, there are no non-zero global G-invariant holomorphic sections of L.*

Proof. Let s be a global, holomorphic G-invariant section. Suppose $s(x) \neq 0$. Let Z be the closure of the orbit of $G^{\mathbf{C}}$ through x, and let z be a point on Z at which $\langle s, s\rangle$ takes on a maximum value. Clearly Z is $G^{\mathbf{C}}$ invariant; so for all $\eta = \sqrt{-1}\,\xi \in \sqrt{-1}\,\mathfrak{g}$, the vector field, $\eta^{\#}$, is tangent to Z at z. Hence, by (5.5)

$$\eta^{\#} \langle s, s\rangle = -4\pi\phi^{\xi}\langle s, s\rangle = 0$$

for all $\xi \in \mathfrak{g}$ at z. In particular, $\Phi(z) = 0$; so zero is in the image of the moment mapping. Q.E.D.

We can actually prove a somewhat stronger result.

Theorem 5.4. *Let s be a G-invariant holomorphic section of L and $x \in X$ a point where $s(x) \neq 0$. Then $x \in X_s$.*

Proof. We have just shown that the closure of $G^{\mathbf{C}}x$ intersects X_0 non-trivially. Therefore, since X_s is an open neighborhood of X_0, $G^{\mathbf{C}}x$ intersects X_s non-trivially. Since X_s is $G^{\mathbf{C}}$-invariant, $x \in X_s$. Q.E.D.

Let n be a positive integer. Applying Theorem 5.4 to the Kaehler manifold $\{X, n\Omega, F\}$, we get

Theorem 5.5. *Let x be a G-invariant holomorphic section of the line bundle, $\bigotimes^n L$. Then if $s(x) \neq 0$, $x \in X_s$. (See the remarks at the beginning of Sect.* 6.)

In the appendix we will prove the following existence theorem.

Theorem 5.6. *If the set $X_0 = \{x \in X, \Phi(x) = 0\}$ is non-empty and zero is a regular value of Φ, then for some n, there exists a global non-vanishing holomorphic G-invariant section of $\bigotimes^n L$.*

Combining this with Theorem 5.5 we obtain[3]

Theorem 5.7. *The set* $X_u = X - X_s$ *is contained in a complex subvariety of* X *of (complex) codimension* $\geqq 1$.

Finally we will prove

Theorem 5.8. *Let* $s: X_s \to L$ *be a holomorphic* G*-invariant section of* L. *Then* $\langle s, s\rangle$ *is bounded and takes its maximum value on* X_0.

Before we prove this we note that it implies the surjectivity of (5.6). Indeed if x is a point of X_u then we can find a neighborhood, U, of x in X and a non-vanishing holomorphic section, $s_0: U \to L$. Then $s = f s_0$ on $U \cap X_s$, f being a *bounded* holomorphic function. Since $X_u \cap U$ is of complex codimension $\geqq 1$ in U, the singularity of f at x is removable. Thus s extends to a holomorphic section of L over all of X.

We will now prove Theorem 5.8. Let x be a point of X_s. Then $y = g x_0$ with $x_0 \in X_0$ and $g \in G^{\mathbf{C}}$. As in Sect. 4, we will make use of the Cartan decomposition, $G^{\mathbf{C}} = PG$ of $G^{\mathbf{C}}$, and we will write g as $(\exp \eta) k$ with $\eta = \sqrt{-1}\,\xi \in \sqrt{-1}\,\mathfrak{g}$ and $k \in G$. Replacing x_0 by $k x_0$ we can assume that $x = (\exp \eta) x_0$. We will prove that

$$\langle s, s\rangle (x) \leqq \langle s, s\rangle (x_0). \tag{5.7}$$

To see (5.7) consider the behavior of $\langle s, s\rangle$ along the curve $\gamma(t) = (\exp t\eta) x_0$, $-\infty < t < \infty$. By (5.5)

$$(d/dt)\langle s, s\rangle = -4\pi \phi^{\xi} \langle s, s\rangle \tag{5.8}$$

along $\gamma(t)$. By Lemma 4.7, ϕ^{ξ} is strictly increasing along $\gamma(t)$; so it is positive for $t > 0$ and negative for $t < 0$. Therefore, by (5.8), $\langle s, s\rangle$ has a unique maximum at $t = 0$, and this establishes (5.7). Q.E.D.

§ 6. Multiplicities

Let (X, Ω) be a symplectic manifold and λ a non-zero real number. Then $\lambda\Omega$ is also a symplectic form on X. In other words one can view the pair $(X, \lambda\Omega)$ as a new symplectic manifold. (In particular one often denotes the manifold, $(X, -\Omega)$ by X^-.) If (X, Ω) is a Hamiltonian G-space and $\Phi: X \to \mathfrak{g}^*$ is its moment mapping, then $(X, \lambda\Omega)$ is also a Hamiltonian G-space and its moment mapping is $\lambda\Phi$. (In particular, X^- is a Hamiltonian G-space and its moment mapping is $-\Phi$.)

Let X_i, $i = 1, 2$, be a symplectic manifold with symplectic form, Ω_i. Let π_i: $X_1 \times X_2 \to X_i$ be the projection onto X_i. Then $\pi_1^* \Omega_1 + \pi_2^* \Omega_2$ is a symplectic form on $X_1 \times X_2$. If X_i, $i = 1, 2$, is a Hamiltonian G-space and Φ_i: $X_i \to \mathfrak{g}^*$ is its

[3] It would be nice to have a direct geometric proof of Theorem 5.7 which avoids the existence theorem 5.6. If G is a torus, we can prove Theorem 5.7 this way using the convexity ideas of [1, 5] and [6].

moment mapping then $X_1 \times X_2$ is a Hamiltonian G-space and its moment mapping is $\Phi_1 \circ \pi_1 + \Phi_2 \circ \pi_2$. In particular let X be a Hamiltonian G-space and Φ: $X \to \mathfrak{g}^*$ its moment mapping. Let 0 be a co-adjoint orbit in $\mathfrak{g}^*$. Then the product symplectic manifold, $X \times O^-$ is a Hamiltonian G-space and its moment mapping, Ψ: $X \times O^- \to \mathfrak{g}^*$, is the mapping:

$$\Psi(x,f) = \Phi(x) - f. \tag{6.1}$$

The set

$$(X \times O^-)_0 = \{(x,f), \Psi(x,f) = 0\} \tag{6.2}$$

is identical with the set

$$\{x \in X, \Phi(x) \in O\} \tag{6.3}$$

by (6.1). Moreover, G acts freely on (6.2) if and only if for some (and hence for all) $f \in 0$, the stabilizer group of f, G_f, acts freely on the set

$$X_f = \{x \in X, \ \Phi(x) = f\}. \tag{6.4}$$

When this happens one can, as in Sect. 2, form the reduced space

$$X_0 = (X \times O^-)_0 / G. \tag{6.5}$$

This space is called the *Marsden-Weinstein reduction of X with respect to O*. By (2.12) X_0 is a symplectic manifold. Note that if $f \in O$, then, set-theoretically, (6.5) is just the space

$$X_f / G_f. \tag{6.6}$$

(in fact (6.6) is the definition of X_0 given in [12].) We will prove analogues of the theorems of §3 for X_0. First, however, we will review some standard facts about line bundles and connections:

Let X be a manifold, L a line bundle on X and ∇ a connection on L. Let $\overset{n}{\otimes} L$ be the n-th tensor product of L. Then there is a unique connection, $\nabla^{(n)}$, on $\overset{n}{\otimes} L$ with the property

$$\nabla^{(n)}(\overset{n}{\otimes} s) = n(\overset{n-1}{\otimes} s) \otimes \nabla s$$

for all sections, s, of L. The curvature of this connection is $n(\operatorname{curv} \nabla)$. If $\langle \ , \ \rangle$ is a Hermitian inner product on L there is a unique Hermitian inner product $\langle \ , \ \rangle^{(n)}$ on $\overset{n}{\otimes} L$ such that if $s^n = \overset{n}{\otimes} s$ then

$$\langle s^n, s^n \rangle^{(n)} = (\langle s, s \rangle)^n.$$

In particular, if L, ∇ and $\langle \ , \ \rangle$ are pre-quantum data on the symplectic manifold (X, Ω) then $\overset{n}{\otimes} L$, $\nabla^{(n)}$ and $\langle \ , \ \rangle^{(n)}$ are pre-quantum data on $(X, n\Omega)$. We note also, in passing, that if F is a polarization of (X, Ω) it is also a polarization of $(X, n\Omega)$.

Next let L^* be the dual bundle of L. There is a unique connection, ∇^*, on L^* such that for all sections, s, of L and s' of L^*

$$(\nabla s, s') + (s, \nabla^* s') = 0.$$

The curvature of the connection ∇^* is $-\operatorname{curv} \nabla$. If $\langle\ ,\ \rangle$ is a Hermitian inner product on L, there is a Hermitian inner product, $\langle\ ,\ \rangle^*$, dual to $\langle\ ,\ \rangle$ on L^*. In particular, if L, ∇ and $\langle\ ,\ \rangle$ are pre-quantum data on the symplectic manifold, X, L^*, ∇^* and $\langle\ ,\ \rangle^*$ are pre-quantum data on the symplectic manifold, X^-. Note also that if F is a (positive-definite) polarization of X, $\bar{F}$ is a (positive-definite) polarization of X^-.

Let X_1 and X_2 be manifolds and L_i a line bundle on X_i. Let π_i be the projection of $X_1 \times X_2$ on X_i. One denotes by $L_1 \boxtimes L_2$ the line bundle, $\pi_1^* L_1 \otimes \pi_2^* L_2$, on $X_1 \times X_2$. If ∇_i is a connection on L_i there is a unique connection, ∇, on $L_1 \boxtimes L_2$ such that if s_i, $i=1,2$, is a section of L_i then

$$\nabla(\pi_1 s_1 \otimes \pi_2^* s_2) = \pi_1^*(\nabla_1 s_1) \otimes \pi_2^* s_2 + \pi_1^* s_1 \otimes \pi_2^* \nabla s_2.$$

The curvature of this connection is $\pi_1^* \operatorname{curv} \nabla_1 + \pi_2^* \operatorname{curv} \nabla_2$. If $\langle\ ,\ \rangle_i$ is a Hermitian form on L_i, there is a unique Hermitian form, $\langle\ ,\ \rangle$ on $L_1 \boxtimes L_2$ such that

$$\langle \pi_1^* s_1 \otimes \pi_2^* s_2, \pi_1^* s_1 \otimes \pi_2^* s_2 \rangle = \pi_1^* \langle s_1, s_1 \rangle \, \pi_2^* \langle s_2, s_2 \rangle.$$

In particular, if, for $i=1,2$, L_i, ∇_i and $\langle\ ,\ \rangle_i$ are pre-quantum data on X_i, then $L_1 \boxtimes L_2$, ∇ and $\langle\ ,\ \rangle$ are pre-quantum data on $X_1 \times X_2$. Finally note that if F_1 and F_2 are polarizations of X_1 and X_2 then $\pi_1^* F_1 + \pi_2^* F_2$ is a polarization of $X_1 \times X_2$.

Combining these remarks with the results of Sect. 3 we obtain:

Theorem 6.1. *Let X be a Hamiltonian G-space and O a co-adjoint orbit in $\mathfrak{g}^*$. Then to every G-invariant polarization, F, of X corresponds a polarization, F_0, of X_0. If O is integral, then to every G-invariant set of pre-quantum data, L, ∇ and $\langle\ ,\ \rangle$ on X corresponds a set of pre-quantum data, L_0, ∇_0 and $\langle\ ,\ \rangle_0$ on X_0.*

Now let L, ∇ and $\langle\ ,\ \rangle$ be G-invariant pre-quantum data on X and F a positive-definite G-invariant polarization. Let X_{quantum} be the space of polarized sections of L, and τ the unitary representation of G on this space. Let ρ be an irreducible representation of G. By the Borel-Weil theorem (see Theorem 3.7) there is a unique integral co-adjoint orbit, O, in $\mathfrak{g}^*$ such that ρ is the canonical representation of G on O_{quantum}. Let $V_1 = X_{\text{quantum}}$, $V_2 = O_{\text{quantum}}$ and

$$\operatorname{Hom}_G(V_2, V_1)$$

the set of linear mappings from V_2 to V_1 which intertwine the representations, ρ and τ. Let L_0, ∇_0, $\langle\ ,\ \rangle_0$ and F_0 be as in Theorem 6.1, and let $(X_0)_{\text{quantum}}$ be the space of polarized sections of L_0.

Theorem 6.2. *There is a canonical isomorphism of vector spaces*

$$(X_0)_{\text{quantum}} \cong \operatorname{Hom}_G(V_2, V_1).$$

Proof. Let $X_1 = X$ and $X_2 = 0$. Let L_i, V_i, $\langle\ ,\ \rangle_i$ and F_i, $i = 1, 2$, be the "quantum" data described above. Then $X_1 \times X_2^-$ is equipped with the quantum data

$$L_1 \boxtimes L_2^*, \quad V, \langle\ ,\ \rangle \text{ and } \pi_1^* F_1 + \pi_2^* \bar{F}_2.$$

A polarized section of $L_1 \boxtimes L_2^*$ is by definition a section of $L_1 \boxtimes L_2^*$ which is holomorphic with respect to X_1 and anti-holomorphic with respect to X_2. Because of the bijectivity of (3.6) we can identify $(X_0)_{\text{quantum}}$ with the space of G-invariant polarized sections of $L_1 \boxtimes L_2^*$. Let $s(x_1, x_2)$ be such a section, and let dx_2 be the Liouville measure on X_2. Then the operator

$$T_s \colon C^\infty(L_2) \to C^\infty(L_1)$$

defined by

$$(T_s f)(x_1) = \int s(x_1, x_2) f(x_2)\, dx_2$$

maps V_2 equivariantly onto V_1, and so defines an element of $\mathrm{Hom}_G(V_2, V_1)$. Conversely every element of $\mathrm{Hom}_G(V_2, V_1)$ can be uniquely expressed as an integral operator of this form. Q.E.D.

A direct corollary of this theorem is the Kirillov conjecture mentioned in the introduction:

Theorem 6.3. *Let O be an integral co-adjoint orbit in $\mathfrak{g}^*$. If O is not in the image of the moment mapping, then the irreducible representation of G corresponding to O does not occur in X_{quantum}.*

Another corollary is the following:

Theorem 6.4. *Let O be an integral co-adjoint orbit in $\mathfrak{g}^*$. Suppose G acts freely and transitively on the set $\{x \in X, \Phi(x) \in 0\}$. Then the irreducible representation of G corresponding to O occurs in X_{quantum} with multiplicity one.*

Proof. If the hypothesis is satisfied, X_0 consists of a single point. Q.E.D.

If the polarization, F_0, is "sufficiently" positive-definite, the dimension of $(X_0)_{\text{quantum}}$ can be computed by the Riemann-Roch formula, and we get the following expression for the multiplicity with which the irreducible representation of G corresponding to O occurs in X_{quantum}:

$$e^{\omega} \tau [X_0], \tag{6.7}$$

τ being the Todd class of X_0 and ω its symplectic form. The Todd class is a symplectic invariant of X; so (6.7) is a *symplectic* recipe for the multiplicity in question. (Compare with [3], § 15.)

Let n be a positive integer. Let X_n be the symplectic manifold $(X, n\Omega)$ and O_n the co-adjoint orbit $\{nf, f \in 0\}$. One can show that with X replaced by X_n and O replaced by O_n, the induced polarization on the reduced space is "sufficiently" positive-definite when n is sufficiently large; so from (6.7) we obtain

Theorem 6.5. *For n sufficiently large the multiplicity with which the irreducible representation of G corresponding to O_n occurs in $(X_n)_{\text{quantum}}$ is given by the characteristic number*

$$e^{n\omega}\tau[X_0]. \tag{6.8}$$

Remark. 1. For n large (6.8) is approximately equal to n^k volume X_0 where k is half the dimension of X_0 because. This estimate on the "asymptotic multiplicity of O" is closely related to some recent results of Gerrit Heckman. (See [6].)

2. As we have already mentioned the above results can be generalized to the case where X_G is a V-manifold. Then the Todd class must be replaced by an equivariant Todd class as defined by Atiyah and Singer [18]. Then (6.8) follows from Kawasaki's Riemann-Roch formula for V-manifolds of [19].

Appendix:

An Existence Theorem

Let W be an $(n+1)$-dimensional compact, complex domain with a smooth, strictly pseudoconvex boundary. Let r: $W \to R$ be a smooth function which is positive in the interior of W, zero on the boundary and has no critical points on the boundary. Let ι: $\partial W \to W$ be the inclusion map and let

$$\alpha = \sqrt{-1}\,\iota^* \bar{\partial} r. \tag{A.1}$$

Because of the pseudo-convexity, $\alpha \wedge (d\alpha)^n$ is non-vanishing; so α is a contact form on ∂W. It is not intrinsically defined, since (A.1) depends on the choice of r; however, the manifold

$$Y = \{(m, \lambda\alpha_m);\ m \in \partial W,\ \lambda \in \mathbf{R}^+\} \tag{A.2}$$

is an intrinsically defined submanifold of $T^*\partial W$. The following is elementary to verify:

Proposition A.1. *The condition that $\alpha \wedge (d\alpha)^n$ be non-vanishing is equivalent to the condition that Y be a symplectic submanifold of $T^*\partial W$.*

We will denote by Ω the restriction to Y of the standard symplectic form on T^*M. Note that in addition to being symplectic, Y is positively homogeneous. If $(m, \xi) \in T^*M$ and $\lambda \in \mathbf{R}^+$ then

$$(m, \xi) \in Y \Leftrightarrow (m, \lambda\xi) \in Y.$$

Now let G be a Lie group and $G \times W \to W$ a holomorphic action of G on W. Then G acts on ∂W and on Y. We will prove below

Proposition A.2. *The action of G on Y is Hamiltonian. Moreover the moment mapping, Ψ: $Y \to \mathfrak{g}^*$ is positively homogeneous: $\Psi(m, \lambda\xi) = \lambda\psi(m, \xi)$ for $(m, \xi) \in Y$ and $\lambda \in \mathbf{R}^+$.*

Let B^2 be the L^2 closure of the space of holomorphic functions on W. Let B^2_G be the space of G-fixed vectors in B^2. The main result of this section is the following:

Theorem A.3. a) *If zero is not in the image of the moment mapping, Ψ: $Y \to \mathfrak{g}^*$, then* $\dim B^2_G < \infty$.

b) *Let zero be in the image of the moment mapping, and in addition, be a regular value of the moment mapping. Then* $\dim B^2_G = \infty$.

Proof of Proposition A.2. Let M be a manifold and G a Lie group acting on M. Let $T^+M = T^*M$ $-$(zero section). We will show that the induced action of G on T^+M is Hamiltonian. To every element, ξ of $\mathfrak{g}$ corresponds a vector field, ξ^*, on M. Let Ψ^ξ: $T^+M \to \mathbf{R}$ be the function

$$\Psi^{\xi}(z)=(\xi^{\#}(m),\mu) \tag{A.3}$$

at $z=(m,\mu)\in T^+M$. It is easy to see that Ψ^{ξ} satisfies the analogue of (2.4); so the action of G on T^+M is Hamiltonian and its moment mapping, Ψ: $T^+M\to\mathfrak{g}^*$ has (A.3) as its ξ-th coefficient. It is clear from (A.3) that Ψ is homogeneous. If Y is a G-invariant symplectic submanifold of T^+M then by restricting (A.3) to Y we see that the action of G on Y is Hamiltonian as well. Q.E.D.

Before proving Theorem A.3 we will first prove a more primitive version of it. Let M be a compact manifold and G a compact Lie group acting on M. Let Ψ: $T^+M\to\mathfrak{g}^*$ be the moment mapping, (A.3). Let $(T^+M)_0=\{z\in T^+M, \Psi(z)=0\}$. Suppose zero is a regular value of Ψ. Then by Theorem 2.4, $(T^+M)_0$ is a co-isotropic submanifold of T^+M and the leaves of its null-foliation are the orbits of G. It follows that the orbit relation

$$\Gamma=\{(z_1,z_2);\ z_i\in(T^+M)_0,\ z_1=gz_2\ \text{ for } g\in G\} \tag{A.4}$$

is a canonical relation. (See for instance, [4] Proposition 2.2.) Now let μ be a positive smooth G-invariant measure on M and let $L^2(M)$ be the L^2 space of M with respect to μ. Let $L^2(M)_G$ be the space of G-fixed vectors in $L^2(M)$ and let P_G be orthogonal projection of $L^2(M)$ onto $L^2(M)_G$.

Theorem A.4. *P_G is a zeroth order elliptic Fourier integral operator associated with the canonical relation, Γ.*

Proof. Let ρ: $G\times M\to M$ be the mapping, $(g,m)\to gm$ and τ: $G\times M\to M$ the mapping, $(g,m)\to m$. Let

$$\rho^*: L^2(M)\to L^2(G\times M)$$

be the bounded linear operator, $f\to f\circ\rho$, and

$$\tau^*: L^2(M)\to L^2(G\times M)$$

the bounded linear operator, $f\to f\circ\tau$. Let $(\tau^*)^t$: $L^2(G\times M)\to L^2(M)$ be the transpose of τ^*. Then

$$P_G=(\tau^*)^t\rho^*. \tag{A.5}$$

Both τ^* and ρ^* are Fourier integral operators. To describe their underlying canonical relations, let us use the right action of G on itself to identify T^*G with $G\times\mathfrak{g}^*$. The underlying canonical relation of τ^* is then

$$((g,0),z,z),\qquad g\in G,\ z\in T^+M \tag{A.6}$$

in $T^*G\times T^*M\times T^*M$, and the underlying canonical relation of ρ^* is the "moment Lagrangian"

$$((g,\Psi(z)),z,gz),\qquad g\in G,\ z\in T^+M. \tag{A.7}$$

(See [16], p. 21. Here Ψ: $T^+M\to\mathfrak{g}^*$ is the moment mapping.)

The transpose of A.6 is composible with A.7 in the sense of Hörmander, [7], Sect. 4 if and only if zero is a regular value of Ψ; and if this is the case, the composite relation is (A.4). Since τ^* and ρ^* are elliptic (have non-vanishing symbols) the same is true of P_G. Q.E.D.

Let $M=\partial W$. If f is a holomorphic function on W its restriction to M is a C^∞ function satisfying the boundary Cauchy-Riemann equations. Let H^2 be the L^2 closure of the space of all such functions in $L^2(M)$. It is sufficient to prove Theorem A.3 with B^2 replaced by H^2. (See [2].) Let P_S be the orthogonal projection of $L^2(M)$ onto H^2 (the "Szegö projector".) For the following, see [2].

Theorem A.5. *P_S is an elliptic Fourier integral operator (with complex phase). Its associated canonical relation is the diagonal in $Y\times Y$.*

Since G leaves H^2 fixed, $P_S P_G = P_G P_S$ is orthogonal projection onto H^2_G.

Combining the previous two theorems with known facts about the compositions of Fourier integral operators with complex phase (see [13]) we obtain:

Theorem A.6. *Let $Y_0 = \{y \in Y, \Psi(y) = 0\}$. Suppose zero is a regular value of Ψ. Then $P_G P_S$ is an elliptic Fourier integral operator with complex phase. Its underlying canonical relation is the orbit relation*

$$\{(y, gy), \ y \in Y_0, \ g \in G\} \tag{A.8}$$

in $Y \times Y$.

Corollary 1. *If zero is not in the image of Ψ, $P_G P_S$ is a smoothing operator.*

This corollary proves the first part of Theorem A.3 because if $P_G P_S$ is smoothing, the space

$$\{f \in L^2(M), \ P_G P_S f = f\}$$

is finite dimensional by the Fredholm theorem.

Corollary 2. *If zero is in the image of Ψ, the range of $P_G P_S$ is infinite dimensional.*

Indeed if the range were finite dimensional, $P_G P_S$ would be smoothing. In particular its leading symbol would have to be zero. However $P_G P_S$ is elliptic. This proves the second part of Theorem A.3.

We will now prove Theorem 5.6. Let (X, F) be a compact symplectic manifold with a positive-definite polarization and let L, ∇ and $\langle \ , \ \rangle$ be pre-quantum data. Let L^*, ∇^* and $\langle \ , \ \rangle^*$ be the associated pre-quantum data for X^-. Let

$$W = \{(x, v); \ x \in X, \ v \in L_x^*, \ \langle v, v\rangle^* \leqq 1\}$$

and

$$M = \{(x, v); \ x \in X, \ v \in L_x^*, \ \langle v, v\rangle = 1\}.$$

W is a compact, complex domain with boundary M. Moreover, since F is positive definite, W is strictly pseudoconvex. (See, for instance, [3].) If X is a Hamiltonian G-space and the "quantum" data above are G-invariant, then G acts holomorphically on W. The circle group, S^1, also acts holomorphically on W by the action

$$(x, v) \in W \to (x, e^{i\theta} v) \in W.$$

By Proposition A.2 we get a Hamiltonian action of $G \times S^1$ on Y. The relationship between the action of G on X and the action of G on Y is easy to describe. The Hamiltonian action of S^1 on Y gives rise to a moment mapping

$$\Phi_1 \colon Y \to \mathbf{R}$$

which, by Proposition A.2, is positively homogeneous. Let $Y_1 = \{y \in Y, \Phi_1(y) = 1\}$ and let Y_{S^1} be the reduced space:

$$Y_{S^1} = U_1 / S^1.$$

Since the action of G on Y commutes with the action of S^1 on Y, G acts in a Hamiltonian fashion on Y_{S^1}. We will prove

Theorem A.7. *X and Y_{S^1} are isomorphic as Hamiltonian G-spaces.*

Proof. Let $r \colon W \to \mathbf{R}$ be the function, $(x, v) \to 1 - \|v\|$, and let α be the form (A.1) on $M = \partial W$. M is a principal S^1 bundle over X, and the connection, ∇, on L is associated with a "principal-bundle" connection on M. It is easy to see that α is the connection form for this connection; so, in particular, if Ω_X is the symplectic form on X and $\rho \colon M \to X$ the projection of M on X then

$$d\alpha = \rho^* \operatorname{curv}(\nabla) = \rho^* \Omega_X. \tag{A.9}$$

Let ξ_θ be the infinitesimal generator of S^1. Since α is a connection form, $(\xi_\theta^\#, \alpha) = 1$; so by (A.3) the mapping

$$\iota: M \to Y, \quad m \to (m, \alpha_m)$$

maps M diffeomorphically onto Y_1, intertwining the two actions of S^1. Thus

$$X \cong M/S^1 \cong Y_1/S^1 \cong Y_{S^1}. \tag{A.10}$$

Moreover, if Ω_Y is the symplectic form on Y then by (A.8), $\iota^*\Omega_Y = \rho^*\Omega_X$; so by (2.12), the symplectic forms on X and Y_{S^1} are the same. Finally it is clear that (A.9) intertwines the two G-actions. Q.E.D.

Let $\Psi: Y \to \mathfrak{g}^*$ be the moment mapping associated with the action of G on Y and $\Phi: X \to \mathfrak{g}^*$ the moment mapping associated with the action of G on X. With ι and ρ as above we get as a corollary of Theorem A.7 the identity

$$\Psi \circ \iota = \Phi \circ \rho. \tag{A.11}$$

Let π be the projection of Y onto X. Since Ψ is positively homogeneous, we conclude from (A.11):

Proposition A.8. *Let* $X_0 = \{x \in X, \Phi(x) = 0\}$ *and* $Y_0 = \{y \in Y, \Psi(y) = 0\}$, *then* $Y_0 = \pi^{-1}(X_0)$. *In particular if zero is a regular value of* Φ, *it is a regular value of* Ψ.

Hence by Theorem A.3, if X_0 is non-empty and zero is a regular value of Φ, $\dim B_G^2 = \infty$. Since the action of S^1 on W commutes with the action of G on W, S^1 acts as a one-parameter unitary group on B_G^2, and we can decompose B_G^2 into a Hilbert space direct sum of the subspaces

$$k = \{f \in B_G^2; f(e^{i\theta} w) = e^{ik\theta} f(w)\}. \tag{A.12}$$

Therefore $(A.12)_k$ is non-zero for some k. However, $(A.12)_k$ is just the space of G-invariant holomorphic sections of $\bigotimes^k L$.

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