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A Class of Simple Exponential B-Splines and their Application to Numerical Solution to Singular Perturbation Problems

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Summary. We shall consider an application of simple exponential splines to the numerical solution of singular perturbation problem. The computational effort involved in our collocation method is less than that required for the other methods of exponential type.

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1. Introduction and Description of Method

In the present paper we shall consider an application of simple exponential splines to the numerical solution of the singular perturbation problem:

$$\begin{aligned} &\varepsilon y'' + b(x)\, y' - d(x)\, y = f(x) \qquad (0 \leqq x \leqq 1) \\ &y(0) = \alpha, \qquad y(1) = \beta \end{aligned} \tag{1.1}$$

for small $\varepsilon > 0$ and for smooth data functions b, d, and f subject to the conditions $d(x) \geqq 0$ and $b(x) \geqq B > 0$ on $[0, 1]$ for a fixed positive constant B. It is well known that the usual centered $O(h^2)$ difference scheme may not give an accurate approximate solution for a small value of ε.

Now, by making use of an indicator χ of $(0, 1]$ and an exponential function ϕ_p with nonzero real p:

$$\phi_p(x) = \begin{cases} p \exp(px)/\{\exp(p) - 1\} & (0 < x \leqq 1) \\ 0 & (\text{otherwise}), \end{cases} \tag{1.2}$$

we may define the exponential B-spline $U_{m+1,p}$:

$$U_{m+1,p}(x) = (\underbrace{\chi * \chi * \ldots * \chi * \phi_p}_{m})(x) \tag{1.3}$$

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where $*$ means the convolution of two functions. This analogous to the convolution formulation of the polynomial B-spline Q_{m+1}:

$$Q_{m+1}(x)=(\underbrace{\chi*\chi*\ldots*\chi}_{m+1})(x). \tag{1.4}$$

Since $U_{m+1,p}(x)=(Q_m*\phi_p)(x)$, from $Q_m(x)=Q_m(m-x)$ and $\phi_p(x)=\phi_{-p}(1-x)$ follows the symmetry

$$U_{m+1,p}(m+1-x)=U_{m+1,-p}(x). \tag{1.5}$$

This symmetry with respect to the transition $p\to -p$ can be used to advantage for an evaluation of $U_{m+1,p}$. By means of the convolution formulation of $U_{m+1,p}$, we may easily check the following properties similar to those of the polynomial B-spline Q_{m+1} [5, 6]:

(i) $U_{m+1,p}(x)\in C^{m-1}(-\infty,\infty)$

(ii) the support of $U_{m+1,p}=[0,m+1]$ and $U_{m+1,p}(x)>0$ on $(0,m+1)$

(iii) $\lim_{p\to 0} U_{m+1,p}(x)=Q_{m+1}(x)$, $\lim_{p\to -\infty} U_{m+1,p}(x)=Q_m(x)$

(iv) $U'_{m+1,p}(x)=U_{m,p}(x)-U_{m,p}(x-1) \quad (m\geqq 2)$

(v) $\sum_{j=-\infty}^{\infty} U_{m+1,p}(x-j)=1$ (a partition of unity)

(vi) $1, x, \ldots, x^{m-1}$, and $\exp(px)\in\mathrm{Span}\{U_{m+1,p}(x-j)\}_{j=-\infty}^{\infty}$

(vii) for $s\in\mathrm{Span}\{U_{m+1,p}(x-j)\}_{j=-\infty}^{\infty}$,

$$\sum_{j=-\infty}^{\infty} a_{m+1-j}\, s^{(k)}(j)=\sum_{j=-\infty}^{\infty} a^{(k)}_{m+1-j}\, s(j) \qquad (1\leqq k\leqq m-1)$$

where $a_j^{(k)}=U^{(k)}_{m+1,p}(j)$.

For later use, we give the explicit forms of the consistency relations (vii) $(m=3)$. First, by (1.3) we have for $0\leqq x\leqq 1$:

$$U_{4,p}(x)=\frac{1}{\exp(p)-1}\left\{\frac{\exp(px)-1}{p^2}-\frac{x}{p}-\frac{x^2}{2}\right\}. \tag{1.6}$$

Next, by (1.5)–(1.6) and (v), from (vii) we have

$$\begin{aligned} &A(p)\,s''_{j+1}+\{1-A(p)-A(-p)\}\,s''_j+A(-p)\,s''_{j-1}\\ &\quad=s_{j+1}+2s_j+s_{j-1} \end{aligned} \tag{1.7}$$

$$\begin{aligned} &A(p)\,s'_{j+1}+\{1-A(p)-A(-p)\}\,s'_j+A(-p)\,s'_{j-1}\\ &\quad=B(p)\,(s_{j+1}-s_j)+B(-p)\,(s_j-s_{j-1}) \end{aligned} \tag{1.8}$$

where $s_j^{(k)}=s^{(k)}(j)$,

$$A(p)\,(=U_{4,p}(1))=1/p^2-(\tfrac{1}{2}+1/p)\,\{\exp(p)-1\}$$

$$B(p)\,(=U'_{4,p}(1))=1/p-1/\{\exp(p)-1\}.$$

Now, by making use of the above stated B-spline $U_{4,p}$, we consider a spline function s of the form

$$s(x)=\sum_{j=-3}^{n-1} \alpha_j U_{4,p}(x/h-j) \qquad (nh=1) \tag{1.9}$$

with undetermined coefficients $(\alpha_{-3}, \alpha_{-2}, \ldots, \alpha_{n-1})$. For an appropriate value of p, the above s will be an approximate solution to the problem (1.1)–(1.2) if it satisfies

$$\varepsilon s_j'' + b_j s_j' - d_j s_j = f_j \qquad (0 \leqq j \leqq n) \tag{1.10}$$

$$s_0 = \alpha, \qquad s_n = \beta \tag{1.11}$$

where $s_j^{(k)} = s^{(k)}(jh)$.

By a simple calculation, Eq. (1.10)–(1.11) are equivalent to the following system of $n+3$ equations:

$$A(p)\alpha_{-1} + \{1 - A(p) - A(-p)\}\alpha_{-2} + A(-p)\alpha_{-3} = \alpha$$

$$\frac{\varepsilon}{h^2}(\alpha_{j-1} - 2\alpha_{j-2} + \alpha_{j-3}) + \frac{b_j}{h}\{B(p)(\alpha_{j-1} - \alpha_{j-2})$$

$$+ B(-p)(\alpha_{j-2} - \alpha_{j-3})\} - d_j[A(p)\alpha_{j-1} + \{1 - A(p) \tag{1.12}$$

$$- A(-p)\}\alpha_{j-2} + A(-p)\alpha_{j-3}] = f_j \qquad (0 \leqq j \leqq n) \tag{1.13}$$

$$A(p)\alpha_{n-1} + \{1 - A(p) - A(-p)\}\alpha_{n-2} + A(-p)\alpha_{n-3} = \beta. \tag{1.14}$$

Since $U_{4,p}$ converges to Q_4 and Q_3 as p goes to 0 and $-\infty$, respectively, our collocation method is situated between the collocation ones with the usual cubic and quadratic splines. In Sect. 2, we shall show that the difference scheme derived from our method ($p \to -\infty$) is almost the same one presented and analyzed in [1]. In Sect. 3, we shall determine an appropriate value of p and analyze our method (1.10)–(1.11).

2. Limiting case ($p \to -\infty$) of our Method

Let us take a spline s of the form:

$$r(x)=\sum_{j=-2}^{n-1} \alpha_j Q_3(x/h-j). \tag{2.1}$$

Then, by (iii) letting $p \to -\infty$ we have

$$s^{(k)}(x) \to r^{(k)}(x) \qquad (k=0, 1) \tag{2.2}$$

$$s_j'' \to r_{j-}'' \qquad (1 \leqq j \leqq n). \tag{2.3}$$

Hence, we have

Theorem 1. *The limiting case* $(p\to-\infty)$ *of our collocation method* (1.10)–(1.11) *reduces to the collocation one with the usual quadratic spline, i.e.,*

$$\varepsilon r''_{j-}+b_j r'_j-d_j r_j=f_j \quad (1\leqq j\leqq n). \tag{2.4}$$

$$r_0=\alpha, \qquad r_n=\beta \tag{2.5}$$

with $r(x)=\sum_{j=-2}^{n-1}\alpha_j Q_3(x/h-j)$.

By means of the consistency relations for polynomial quadratic spline r:

$$\tfrac{1}{2}(r'_j+r'_{j-1})=(r_j-r_{j-1})/h$$
$$(r'_j-r'_{j-1})/h=r''_{j-},$$

we see that Eq. (2.4) are equivalent to

$$\begin{aligned}L_h r_j=&\frac{\varepsilon}{h^2}(r_{j+1}-2r_j+r_{j-1})+\frac{(b_{j+1}+b_j)}{2h}\kappa(r_{j+1}-r_j)\\&-\frac{1}{2}\left(\frac{D_j}{D_{j+1}}d_{j+1}r_{j+1}+d_jr_j\right)-\frac{1}{2}\left(\frac{D_j}{D_{j+1}}f_{j+1}+f_j\right)=0\\&(1\leqq j\leqq n-1)\end{aligned} \tag{2.6}$$

where $D_j=b_j+2\varepsilon/h$

$$\kappa=1+\frac{b_{j+1}(b_j-b_{j+1})}{(b_{j+1}+b_j)(b_{j+1}+2\varepsilon/h)}.$$

Here, the above difference scheme is exactly the same upwind one for b constant on [0, 1] presented and analyzed in [1]. In addition, by property (vi) the truncation error of our scheme (2.6) is zero for $r=1, x, x^2$, while the one of the difference scheme in [1] is zero for only $r=1, x$.

3. Analysis of our Method

We assume that $b(x)$ is constant on [0, 1] and $h\gg\varepsilon$.

By (1.7)–(1.8) and (1.10)–(1.11), we have a system of equations in s_j $(0\leqq j\leqq n)$:

$$\begin{aligned}L_h s_j=&\frac{\varepsilon}{h^2}(s_{j+1}-2s_j+{}_{j-1})+\frac{b}{h}\{B(p)(s_{j+1}-s_j)\\&+B(-p)(s_j-s_{j-1})\}-[A(p)d_{j+1}s_{j+1}+\{1-A(p)\\&-A(-p)\}d_js_j+A(-p)d_{j-1}s_{j-1}]-[A(p)f_{j+1}+\{1\\&-A(p)-A(-p)\}f_j+A(-p)f_{j-1}]=0 \quad (1\leqq j\leqq n-1)\end{aligned} \tag{3.1}$$

$$s_0-\alpha=0, \qquad s_n-\beta=0 \tag{3.2}$$

where $b(x)=b$.

First we notice that the difference operator L_h satisfies the discrete maximum principle [3. p. 700] if

$$\frac{\varepsilon}{h^2}+\frac{b}{h}B(p)-A(p)\,d_{j+1}\geqq 0 \tag{3.3}$$

$$\frac{\varepsilon}{h^2}-\frac{b}{h}B(-p)-A(-p)\,d_{j-1}\geqq 0 \tag{3.4}$$

or if,

$$\frac{\varepsilon}{h^2}+\frac{b}{h}\left(\frac{1}{p}+1\right)>\left(\frac{1}{p^2}+\frac{1}{p}+\frac{1}{2}\right)\max_{0\leqq x\leqq 1} d(x) \tag{3.5}$$

$$\frac{\varepsilon}{h^2}+\frac{b}{ph}>\frac{1}{p^2}\max_{0\leqq x\leqq 1} d(x) \tag{3.6}$$

for $-p$ sufficiently large. By a simple calculation, we have

Lemma 1. *For $p=-\mu\kappa(\kappa=bh/\varepsilon)$, the discrete maximum principle is valid if*

$$\mu\geqq 1+(\varepsilon/b^2)\max_{0\leqq x\leqq 1} d(x) \tag{3.7}$$

$$0<h\leqq 2b/\max_{0\leqq x\leqq 1} d(x) \tag{3.8}$$

provided that κ is sufficiently large, i.e., $h\gg\varepsilon$.

Before we estimate the truncation error, we shall determine an appropriate value of p under (3.7) which would approximately make it be minimum. Be means of Lemma 2.4 in [4], the solution y of (1.1)–(1.2) can be written in the form:

$$y(x)=v(x)+w(x) \tag{3.9}$$

where $v(x)=C_1\exp(-bx/\varepsilon)\quad (b=b(0))$

$$|w^{(k)}(x)|\leqq C_2[1+\varepsilon^{-k+1}\exp(-\delta x/\varepsilon)]$$

(C_1, C_2 and δ are positive constants independent of ε). Then, since $L_h y_j=0$ for $y=1, x, x^2$ by (vi), there exists a "Peano kernel" k_j such that

$$L_h y_j=\int_{x_{j-1}}^{x_{j+1}} k_j(x)\,y^{(3)}(x)\,dx. \tag{3.10}$$

By (3.9), $L_h y_j$ can be approximated by $L_h v_j$ or

$$\int_{x_{j-1}}^{x_j} k_j(x)\,v^{(3)}(x)\,dx\sim\frac{C_1\varepsilon}{h^2}\Phi(\mu)\exp\left\{-\frac{bh}{\varepsilon}(j-1)\right\} \tag{3.11}$$

where $\Phi(\mu)=1-1/\mu$. Since $\Phi(\mu)$ is monotone increasing on $[1, \infty)$, by (3.7) we should take $\mu=1+(\varepsilon/b^2)\max_{0\leqq x\leqq 1} d(x)$, i.e.,

$$p=-\frac{bh}{\varepsilon}\left\{1+\frac{\varepsilon}{b^2}\max_{0\leqq x\leqq 1} d(x)\right\}. \tag{3.12}$$

Then, by an elementary and a little long calculation we have

Lemma 2. *For p defined by* (3.12),

$$L_h y_j=O\left(\frac{\varepsilon^2}{h^2}\right)\exp\{-(\sigma h/\varepsilon)(j-1)\}+O(h^2)$$

where σ is a positive constant independent of h and ε.

Before we complete the error estimate, we shall prove the following two lemmas that are required for a comparison method [3, p. 717].

Lemma 3. $L_h(x_j-2)_j\geqq b\ \ (x_j=jh)$.

Lemma 4. *For p by* (3.12),

$$L_h(-\theta^{-j})_j\geqq\theta^{-j}/O(h)\,(\theta=1+O(h/\varepsilon^2)).$$

Since the proof of Lemma 3 is easy, here we only prove Lemma 4. By a simple calculation, we have

$$L_h(-\theta^{-j})_j/\theta^{-j}\geqq\frac{\varepsilon(\theta-1)}{\theta h^2}\left\{1-\frac{1}{\mu}+\frac{\kappa}{\exp(\mu\kappa)-1}\right\}\cdot\{\Psi(\mu)-\theta\}$$

where

$$\Psi(\mu)=\left\{1-\frac{1}{\mu}+\frac{\exp(\mu\kappa)}{\exp(\mu\kappa)-1}\right\}\Big/\left\{1-\frac{1}{\mu}+\frac{\kappa}{\exp(\mu\kappa)-1}\right\}.$$

Since $\Psi(1+(\varepsilon/b^2)\max_{0\leqq x\leqq 1} d(x))=O(h/\varepsilon^2)$, we have the desired result.

Combining Lemmas 2–4, we have

Theorem 2. *Assume* (3.8) *and* (3.12). *Then, the solutions* $s_j (0\leqq j\leqq n)$ *of* (3.1)–(3.2) *satisfy*

$$|s_j-y_j|\leqq O(h^2)+O\left(\frac{\varepsilon^2}{h}\right)\left\{1+O\left(\frac{h}{\varepsilon^2}\right)\right\}^{-(j-1)}.$$

For b not always constant on [0, 1], the solution of the problem (1.1)–(1.2) would be largely determined by $C_1\exp(-b(0)\,x/\varepsilon)$, and so it would be sufficient to decrease parameter p, starting at $-b(0)\,h/\varepsilon$, until the approximate solution is satisfactory.

Table 1 (Example 1)

Method	$\varepsilon = h^{1.5}$		$\varepsilon = h^2$	
	Ours	OCI	Ours	OCI
$n = 32$	7.4−5	6.3−3	7.0−6	3.0−5
64	6.1−6	2.8−3	1.9−6	3.7−6
128	1.4−7	9.6−4	4.8−7	6.2−7
256	6.0−8	2.8−4	1.2−7	1.3−7
512	2.3−8	7.6−5	3.1−8	3.2−8
1024	6.7−9	2.0−5	7.8−9	7.9−9
2048	1.8−9	5.3−6	1.9−9	2.0−9

Table 2 (Example 2)

Method	$\varepsilon = h^{1.5}$		$\varepsilon = h^2$	
	Ours	EW scheme	Ours	EW scheme
$n = 32$	4.9−4	2.6−4	2.4−4	2.5−4
64	1.3−4	6.3−5	6.2−5	6.6−5
128	3.6−5	1.6−5	1.6−5	1.7−5
256	9.4−6	4.1−6	3.9−6	4.3−6
512	2.5−6	1.1−6	1.0−6	1.1−6

4. Numerical Illustration

In this section, we consider an application of our collocation method to the numerical solution of the following two examples.

Example 1 [3].

$$\varepsilon y'' + (x+1)^3 y' = f \qquad (0 \leqq x \leqq 1)$$
$$y(0) = 2, \qquad y(1) = (1/8)\exp\{-15/(4\varepsilon)\} + \exp(-\tfrac{1}{2})$$

where the solution is given by

$$y(x) = \frac{1}{(x+1)^3}\exp\left[-\frac{1}{4\varepsilon}\{(x+1)^4 - 1\}\right] + \exp(-\tfrac{1}{2}x).$$

Example 2 [2].

$$\begin{aligned} &\varepsilon y'' + (x+1)\, y' - 0.31(x+1)^5 y \\ &\quad = -0.23x^2 - 0.29x - 0.43 \qquad (0 \leqq x \leqq 1) \\ &y(0) = 2.7, \qquad y(1) = 0.53. \end{aligned}$$

The notation $a - b = a \times 10^{-b}$ is used throughout. The columns in the following tables give the maximum absolute errors at all the mesh points. For comparison, we also give some numerical results by using particular schemes, i.e., the generalized OCI scheme and the El-Mistikawy and Werle scheme [2, 3]. The results show that the numerical rate of convergence would be $O(h^2)$ even for the case when b is not always constant on [0, 1]. It should be noted that computational effort involved in our scheme (1.10)–(1.11) (i.e., (1.12)–(1.14)) is significantly less than that required for the generalized OCI scheme and the El-Mistikawy and Werle one. In addition, our method gives twice continuously differentiable approximate solutions, while El-Mistikawy and Werle method gives a continuously differentiable one.

References

1. Abrahamsson, L.R., Keller, H.B., Kreiss, H.-O.: Difference approximations for singular perturbations of systems of ordinary differential equations. Numer. Math. **22**, 367–391 (1974)
2. Berger, A., Solomon, J., Ciment, M.: An analysis of a uniformly accurate difference method for a singular perturbation problem. Math. Comput. **37**, 79–94 (1981)
3. Berger, A., Solomon, J., Ciment, M., Leventhal, S., Weinberg, B.: Generalized OCI schemes for boundary layer problems. Math. Comput. **35**, 695–731 (1980)
4. Kellogg, R., Tsan, A.: Analysis of some difference approximations for a singular perturbation problem without turning points. Math. Comput. **32**, 1025–1039 (1978)
5. Sakai, M., López de Silanes, M.C.: A simple rational spline and its application to monotonic interpolation to monotonic data. Numer. Math. **50**, 171–182 (1986)
6. Schmumaker, L.: Spline functions: Basic theory. New York: Wiley 1981

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