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Lower Bounds for the Condition Number of Vandermonde Matrices*

Walter Gautschi[1,**] and Gabriele Inglese[2]

[1] Department of Computer Sciences, Purdue University, West Lafayette, IN 47907, USA
[2] CNR-Istituto Analisi Globale e Applicazioni, Via S. Marta 13/A, I-50139 Florence, Italy

Summary. We derive lower bounds for the ∞-condition number of the $n \times n$-Vandermonde matrix $V_n(x)$ in the cases where the node vector $x^T = [x_1, x_2, \ldots, x_n]$ has positive elements or real elements located symmetrically with respect to the origin. The bounds obtained grow exponentially in n, with $O(2^n)$ and $O(2^{n/2})$, respectively. We also compute the optimal spectral condition numbers of $V_n(x)$ for the two node configurations (including the optimal nodes) and compare them with the bounds obtained.

Subject Classifications: AMS (MOS): 15A12, 49D15, 65F35; CR: G1.3, G1.6.

1.1. Introduction

The condition of Vandermonde matrices

$$V_n(x) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{bmatrix}, \qquad x^T = [x_1, x_2, \ldots, x_n], \qquad n > 1, \tag{1.1}$$

where the nodes x_ν are real or complex numbers, and the related question of estimating the norm of $[V_n(x)]^{-1}$ have been studied in [2–5]. In [4] we considered the problem of minimizing the condition number

$$\kappa_{n,p}(x) = \operatorname{cond}_p V_n(x) = \| V_n(x) \|_p \, \| V_n^{-1}(x) \|_p, \tag{1.2}$$

where $p = \infty$, over all positive node vectors $x \in \mathbb{R}_+^n$, or all real symmetric node vectors $x \in \mathbb{R}^n$, $x_\nu + x_{n+1-\nu} = 0$ $(\nu = 1, 2, \ldots, n)$. We managed to obtain certain necessary conditions for optimality, computed optimal node configurations for

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$n=2$ and $n=3$ in the case of positive nodes, and for $2 \leqq n \leqq 6$ in the case of symmetric nodes, but did not address the question of how fast

$$\kappa_{n,p} = \inf_x \kappa_{n,p}(x) \tag{1.3}$$

grows with n (when $p=\infty$). While the exact growth rate is still unknown, we now derive lower bounds for $\kappa_{n,\infty}$ which show that the growth of $\kappa_{n,\infty}$ is exponential, namely at least $O(2^n)$ and $O(2^{n/2})$ in the two respective cases. We also compute $\kappa_{n,2}$ for $2 \leqq n \leqq 10$ in the former, and for $2 \leqq n \leqq 16$ in the latter case, and depict the optimal nodes graphically.

We first recall from [4] some key formulas that will be needed. In the case of nonnegative nodes

$$x_1 > x_2 > \ldots > x_n \geqq 0, \tag{1.4}$$

we have

$$\kappa_{n,\infty}(x) = \max\{n, g_n(x)\} \cdot \max_{1 \leqq \nu \leqq n} g_{n,\nu}(x), \tag{1.5}$$

where

$$g_n(x) = \sum_{\mu=1}^{n} x_\mu^{n-1}, \tag{1.6}$$

$$g_{n,\nu}(x) = \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{n} \frac{1+x_\mu}{|x_\nu - x_\mu|}, \qquad \nu = 1, 2, \ldots, n. \tag{1.7}$$

For real symmetric nodes

$$\begin{aligned} &x_\nu + x_{n+1-\nu} = 0, \qquad \nu = 1, 2, \ldots, n, \\ &x_1 > x_2 > \ldots > x_{[n/2]} > 0 \end{aligned} \tag{1.8}$$

(note that $x_{(n+1)/2} = 0$ if n is odd), we have

$$\kappa_{n,\infty}(x) = \max\left\{\frac{n}{2}, f_n(x)\right\} \cdot \max_{1 \leqq \nu \leqq [(n+1)/2]} f_{n,\nu}(x), \tag{1.9}$$

where

$$f_n(x) = \sum_{\mu=1}^{[n/2]} x_\mu^{n-1}, \tag{1.10}$$

$$f_{n,\nu}(x) = \left(1 + \frac{1}{x_\nu}\right) \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{n/2} \frac{1+x_\mu^2}{|x_\nu^2 - x_\mu^2|}, \qquad \nu = 1, 2, \ldots, n/2 \quad (n \text{ even}), \tag{1.11}$$

$$\left.\begin{aligned} &f_{n,\nu}(x) = \frac{1+x_\nu}{x_\nu^2} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{(n-1)/2} \frac{1+x_\mu^2}{|x_\nu^2 - x_\mu^2|}, \qquad \nu = 1, 2, \ldots, (n-1)/2, \\ &f_{n,(n+1)/2}(x) = 2 \prod_{\mu=1}^{(n-1)/2} \left(1 + \frac{1}{x_\mu^2}\right), \end{aligned}\right\} \quad (n \text{ odd}). \tag{1.12}$$

Empty products in (1.11), (1.12), when $n=2$ or $n=3$, are understood to have the value 1.

2. Positive Nodes

Although the following Theorem 2.1 will subsequently be sharpened, we state and prove it here because of its simplicity and elementary proof.

Theorem 2.1. *Let $\kappa_{n,\infty}$ be the infimum in* (1.3) (*for* $p=\infty$) *taken over all nonnegative nodes* (1.4). *Then, for* $n \geqq 2$,

$$\kappa_{n,\infty} > 2^{n-1}. \tag{2.1}$$

Proof. The optimal point is known to be finite (cf. the remarks preceding Theorem 3.1 of [4]). Letting

$$E_C = \{x \in \mathbb{R}^n : C = x_1 > x_2 > \dots > x_n \geqq 0\},$$

it suffices therefore to show that

$$\kappa_{n,\infty}(x) > 2^{n-1}, \quad \text{all } x \in E_C, \text{ all } C > 0. \tag{2.2}$$

At the heart of the proof is the elementary observation that

$$\inf_{0 \leqq v < u \leqq C} \frac{1+u}{u-v} = 1 + \frac{1}{C}, \tag{2.3}$$

where the infimum is attained for $u=C$, $v=0$.

Assume first $C>1$. Since, by (1.6), $g_n(x) \geqq C^{n-1}$ for $x \in E_C$, we have from (1.5), (1.7)

$$\begin{aligned}\kappa_{n,\infty}(x) &\geqq C^{n-1} g_{n,n}(x) = C^{n-1} \prod_{\mu=1}^{n-1} \frac{1+x_\mu}{x_\mu - x_n} \\ &\geqq C^{n-1} \inf_{E_C} \prod_{\mu=1}^{n-1} \frac{1+x_\mu}{x_\mu - x_n} \geqq C^{n-1} \prod_{\mu=1}^{n-1} \inf_{0 \leqq v < u \leqq C} \frac{1+u}{u-v} \\ &= C^{n-1}\left(1+\frac{1}{C}\right)^{n-1} = (1+C)^{n-1} > 2^{n-1},\end{aligned}$$

where (2.3) has been used to evaluate the last infimum. Similarly, if $C \leqq 1$,

$$\kappa_{n,\infty}(x) \geqq n \cdot \prod_{\mu=1}^{n-1} \frac{1+x_\mu}{x_\mu - x_n} \geqq n\left(1+\frac{1}{C}\right)^{n-1} \geqq 2 \cdot 2^{n-1} > 2^{n-1}$$

if $n \geqq 2$. □

We now improve upon Theorem 2.1 by establishing the following

Theorem 2.2. *Let $\kappa_{n,\infty}$ be as in Theorem* 2.1, *Then, for* $n \geqq 2$,

$$\kappa_{n,\infty} \geqq (n-1)\left\{1+\left(1-\frac{1}{n}\right)^{-1/(n-1)}\right\}^{n-1}. \tag{2.4}$$

In particular,

$$\kappa_{n,\infty} > (n-1)\cdot 2^{n-1}, \qquad n \geqq 2. \tag{2.5}$$

Proof. By Theorems 5.2 and 5.3 of [4], if $x=a$ is a minimum point of $\kappa_{n,\infty}(x)$, then

$$a_n = 0, \qquad g_n(a) = \sum_{\mu=1}^{n-1} a_\mu^{n-1} = n, \tag{2.6}$$

and, by (1.5),

$$\kappa_{n,\infty}(x) \geqq \kappa_{n,\infty}(a) = n \cdot \max_{1 \leqq v \leqq n} g_{n,v}(a).$$

In particular, therefore,

$$\kappa_{n,\infty}(x) \geqq n \cdot g_{n,n}(a) = n \prod_{\mu=1}^{n-1} \frac{1+a_\mu}{a_\mu}. \tag{2.7}$$

To get a lower bound, we minimize the product in (2.7) subject to the constraint in (2.6) (thereby changing the meaning of the variables a_μ). Using Lagrange multipliers, we obtain the necessary conditions

$$-\frac{1}{a_v^2} \prod_{\substack{\mu=1 \\ \mu \neq v}}^{n-1} \frac{1+a_\mu}{a_\mu} + \lambda(n-1)\, a_v^{n-2} = 0, \qquad v = 1, 2, \ldots, n-1,$$

or, equivalently,

$$\prod_{\mu=1}^{n-1} \frac{1+a_\mu}{a_\mu} = \lambda(n-1)\, a_v^{n-1}(1+a_v), \qquad v = 1, 2, \ldots, n-1.$$

This implies $a_1 = a_2 = \ldots = a_{n-1} = \alpha$, hence, by (2.6),

$$(n-1)\,\alpha^{n-1} = n, \qquad \alpha = \left(1 - \frac{1}{n}\right)^{-1/(n-1)}.$$

Substituting in (2.7) gives

$$\kappa_{n,\infty}(x) \geqq n \left(\frac{1+\alpha}{\alpha}\right)^{n-1},$$

which is (2.4). The corollary (2.5) is an immediate consequence of (2.4). □

Expanding the lower bound in (2.4) in powers of n^{-1}, we can also write

$$\kappa_{n,\infty} \geqq n \cdot 2^{n-1}\left(1 - \tfrac{1}{2}n^{-1} - \tfrac{1}{8}n^{-2} + \tfrac{1}{16}n^{-3} + \tfrac{19}{128}n^{-4} + \ldots\right), \qquad n \geqq 2. \tag{2.4'}$$

The five terms shown provide an accuracy of about 2 correct significant decimal digits when $n=2$, and 7 correct digits when $n=16$.

3. Symmetric Nodes

Since the infimum of $\kappa_{n,\infty}(x)$ over all $x\in\mathbb{R}^n$ is attained at a symmetric node configuration, if it is unique (see [4, Thm. 3.1]), the study of symmetric nodes is particularly appropriate. We have, in this case, results analogous to those in Theorems 2.1 and 2.2. Since the proofs are similar, we try to be brief.

Theorem 3.1. *Let $\kappa_{n,\infty}$ be the infimum in (1.3) (for $p=\infty$) taken over all nodes satisfying (1.8). Then, for $n\geqq 2$,*

$$\kappa_{n,\infty}\geqq 2^{n/2}. \tag{3.1}$$

If $n>2$, then (3.1) holds with strict inequality.

Proof. We now let

$$E_C=\{x\in\mathbb{R}^n\colon x_1=C>x_2>\ldots>x_{[n/2]}>0,\ x_\nu+x_{n+1-\nu}=0 \text{ all } \nu\}$$

and consider first the case n even. If $C>1$, then by (1.9)–(1.11),

$$\kappa_{n,\infty}(x)\geqq C^{n-1}\max_{1\leqq\nu\leqq n/2} f_{n,\nu}(x)\geqq C^{n-1}f_{n,n/2}(x)$$
$$=C^{n-1}\left(1+\frac{1}{x_{n/2}}\right)\prod_{\mu=1}^{(n/2)-1}\frac{1+x_\mu^2}{x_\mu^2-x_{n/2}^2},$$

and thus, by (2.3),

$$\kappa_{n,\infty}(x)\geqq C^{n-1}\left(1+\frac{1}{C}\right)\left(1+\frac{1}{C^2}\right)^{(n/2)-1}=(1+C)(1+C^2)^{(n/2)-1}>2^{n/2}.$$

Likewise, for $C\leqq 1$, if $n\geqq 4$,

$$\kappa_{n,\infty}(x)\geqq\frac{n}{2}f_{n,n/2}(x)\geqq 2\cdot 2\left(1+\frac{1}{C^2}\right)^{(n/2)-1}\geqq 2\cdot 2^{n/2}>2^{n/2}.$$

For $n=2$ one has $\kappa_{2,\infty}(x)\geqq 2$ (see [4, Eq. (4.1)]).

Consider now $n(\geqq 3)$ odd. Then, for $C>1$, by (1.9), (1.10), and (1.12),

$$\kappa_{n,\infty}(x)\geqq C^{n-1}\max_{1\leqq\nu\leqq(n+1)/2} f_{n,\nu}(x)\geqq C^{n-1}f_{n,(n+1)/2}(x)$$
$$=2C^{n-1}\prod_{\mu=1}^{(n-1)/2}\left(1+\frac{1}{x_\mu^2}\right)\geqq 2C^{n-1}\left(1+\frac{1}{C^2}\right)^{(n-1)/2}$$
$$=2(1+C^2)^{(n-1)/2}>2^{n/2},$$

and, for $C\leqq 1$,

$$\kappa_{n,\infty}(x)\geqq\frac{n}{2}f_{n,(n+1)/2}(x)=n\prod_{\mu=1}^{(n-1)/2}\left(1+\frac{1}{x_\mu^2}\right)\geqq n\cdot 2^{(n-1)/2}>2^{n/2}.\quad\square$$

Theorem 3.2. *Let $\kappa_{n,\infty}$ be as in Theorem* 3.1. *Then, for $n \geqq 4$,*

$$\kappa_{n,\infty} > \begin{cases} (n-2)\left\{1+\left(1-\frac{2}{n}\right)^{-2/(n-1)}\right\}^{(n-2)/2}, & n \text{ even}, \\ (n-3)\left\{1+\left(1-\frac{3}{n}\right)^{-2/(n-1)}\right\}^{(n-3)/2}, & n \text{ odd}. \end{cases} \tag{3.2}$$

In particular,

$$\kappa_{n,\infty} > \begin{cases} (n-2)\cdot 2^{(n-2)/2}, & n \text{ even}, \\ (n-3)\cdot 2^{(n-3)/2}, & n \text{ odd}. \end{cases} \tag{3.3}$$

Proof. By [4, Thm. 3.3], if $x=a$ is a minimum point, then

$$\kappa_{n,\infty}(x) \geqq \frac{n}{2} \max_{1 \leqq \nu \leqq [(n+1)/2]} f_{n,\nu}(a), \tag{3.4}$$

where

$$\sum_{\mu=1}^{[n/2]} a_\mu^{n-1} = \frac{n}{2} \tag{3.5}$$

and

$$a_1 > a_2 > \ldots > a_{[n/2]} > 0.$$

We assume first $n(\geqq 4)$ even. Then, by (3.4),

$$\kappa_{n,\infty}(x) \geqq \frac{n}{2} f_{n,n/2}(a) = \frac{n}{2}\left(1+\frac{1}{a_{n/2}}\right) \prod_{\mu=1}^{(n/2)-1} \frac{1+a_\mu^2}{a_\mu^2 - a_{n/2}^2} > \frac{n}{2}\cdot 2 \cdot \prod_{\mu=1}^{(n/2)-1} \frac{1+a_\mu^2}{a_\mu^2}. \tag{3.6}$$

We have used here $a_{n/2} \leqq 1$, which must certainly hold if (3.5) is to be true. We now minimize the last product in (3.6), subject to

$$\sum_{\mu=1}^{(n/2)-1} a_\mu^{n-1} = \frac{n}{2} - a_{n/2}^{n-1}. \tag{3.7}$$

(We may assume here that $a_{n/2} > 0$ is fixed.) Using Lagrange multipliers, we get

$$-\frac{2}{a_\nu^3} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{(n/2)-1} \frac{1+a_\mu^2}{a_\mu^2} + \lambda(n-1)\, a_\nu^{n-2} = 0, \qquad \nu = 1, 2, \ldots, \frac{n}{2}-1,$$

or, equivalently,

$$\prod_{\mu=1}^{(n/2)-1} \frac{1+a_\mu^2}{a_\mu^2} = \frac{1}{2}\lambda(n-1)\, a_\nu^{n-1}(1+a_\nu^2), \qquad \nu = 1, 2, \ldots, \frac{n}{2}-1,$$

which implies $a_1 = a_2 = \ldots = a_{(n/2)-1} = \alpha$. By (3.7),

$$\left(\frac{n}{2}-1\right)\alpha^{n-1} = \frac{n}{2} - a_{n/2}^{n-1} < \frac{n}{2},$$

hence

$$\alpha<\left(1-\frac{2}{n}\right)^{-1/(n-1)}.$$

Therefore, by (3.6),

$$\kappa_{n,\infty}>n\cdot\left(\frac{1+\alpha^2}{\alpha^2}\right)^{(n-2)/2}>n\cdot\frac{\left\{1+\left(1-\frac{2}{n}\right)^{-2/(n-1)}\right\}^{(n-2)/2}}{\left(\frac{n}{n-2}\right)^{(n-2)/(n-1)}},$$

which, by increasing the denominator to $n/(n-2)$, yields the first inequality in (3.2).

Assuming now $n(\geqq 5)$ odd, we have

$$\begin{aligned}\kappa_{n,\infty}(x)&\geqq\frac{n}{2}\cdot f_{n,(n-1)/2}(a)=\frac{n}{2}\frac{1+a_{(n-1)/2}}{a_{(n-1)/2}^2}\prod_{\mu=1}^{(n-1)/2-1}\frac{1+a_\mu^2}{a_\mu^2-a_{(n-1)/2}^2}\\&>\frac{n}{2}\cdot 2\cdot\prod_{\mu=1}^{(n-1)/2-1}\frac{1+a_\mu^2}{a_\mu^2}.\end{aligned}\tag{3.8}$$

We are led to the same problem as before, namely to minimize the last product in (3.8) subject to

$$\sum_{\mu=1}^{(n-1)2-1}a_\mu^{n-1}=\frac{n}{2}-a_{(n-1)/2}^{n-1}.$$

We find $a_1=a_2=\ldots=a_{(n-3)/2}=\alpha$, with

$$\alpha<\left(1-\frac{3}{n}\right)^{-1/(n-1)},$$

hence, by (3.8),

$$\begin{aligned}\kappa_{n,\infty}(x)&>n\cdot\left(\frac{1+\alpha^2}{\alpha^2}\right)^{(n-3)/2}>n\cdot\frac{\left\{1+\left(1-\frac{3}{n}\right)^{-2/(n-1)}\right\}^{(n-3)/2}}{\left(\frac{n}{n-3}\right)^{(n-3)/(n-1)}}\\&>(n-3)\left\{1+\left(1-\frac{3}{n}\right)^{-2/(n-1)}\right\}^{(n-3)/2}.\quad\square\end{aligned}$$

For $n=2$ and $n=3$, we have trivially $\kappa_{2,\infty}=2$, $\kappa_{3,\infty}=5$ ([4, Eqs. (4.1), (4.2)]). We can write (3.2), in expanded form, as

$$\kappa_{n,\infty}>\begin{cases}n\cdot 2^{(n-2)/2}\left(1-n^{-1}-\frac{3}{2}n^{-2}-\frac{1}{2}n^{-3}-\frac{7}{24}n^{-4}+\ldots\right), & n\ (\text{even})\geqq 4,\\ n\cdot 2^{(n-3)/2}\left(1-\frac{3}{2}n^{-1}-\frac{33}{8}n^{-2}-\frac{39}{16}n^{-3}+\frac{75}{128}n^{-4}+\ldots\right), & n\ (\text{odd})\geqq 5.\end{cases}\tag{3.2'}$$

The accuracy provided by the five terms shown is about 3 correct significant decimal digits, when $n=4$, and increases to 6 correct digits for $n=15$.

4. Numerical Results

In order to assess the quality of the bounds obtained, it would be desirable to compute the optimum condition number $\kappa_{n,\infty}$ numerically. This would require the solution of nonlinearly constrained optimization problems [4, Eqs. (3.13), (5.10)] or nonlinear programming problems [4, Eqs. (3.15), (5.12)]. Since, at this time, there seems to be no easy access to reliable software in this area, we decided to minimize the spectral condition number,

$$\kappa_{n,2}(x)=\operatorname{cond}_2 V_n(x)=\frac{\sigma_1(V_n(x))}{\sigma_n(V_n(x))}, \tag{4.1}$$

where $\sigma_\nu=\sigma_\nu(V_n)$, $\sigma_1>\sigma_2>\ldots>\sigma_n$, are the singular values of V_n. This requires only unconstrained optimization and singular value decomposition, for which there exists standard software. Having found $\kappa_{n,2}=\inf\kappa_{n,2}(x)$, we can use the inequality

$$\kappa_{n,\infty}>\frac{1}{n}\kappa_{n,2} \tag{4.2}$$

to get a lower bound for $\kappa_{n,\infty}$. Indeed, if $\kappa_{n,\infty}=\operatorname{cond}_\infty V_n(a)$ and $\kappa_{n,2}=\operatorname{cond}_2 V_n(b)$, then, since $\|A\|_2\leqq\sqrt{n}\,\|A\|_\infty$ (see, e.g., [6, Eq. (2.2–14)]), we get $\kappa_{n,2}=\operatorname{cond}_2 V_n(b)<\operatorname{cond}_2 V_n(a)\leqq n\operatorname{cond}_\infty V_n(a)=n\kappa_{n,\infty}$. In the case of nonnegative nodes, we make the usual substitution

$$x_\nu=a_\nu+(b_\nu-a_\nu)\sin^2 t_\nu, \qquad \nu=1,2,\ldots,n,$$

where a_ν, b_ν are lower and upper bounds for x_ν, in order to reduce the problem to an unconstrained problem in the variables t_ν. In our case, $a_\nu=0$, and we took for b_ν variously $b_\nu=2$, 2.5, and 3. Using the IMSL routine ZXMIN (cf. [7, pp. ZXMIN 1–4]) for minimization (with initial approximations $t_\nu=\nu\pi/(2n+2)$, $\nu=1, 2, \ldots, n$), and the EISPACK routine SVD (cf. [1, p. 265]) for singular value decomposition, to compute $\kappa_{n,2}$, we obtained for the lower bound in (4.2) the results in the second column of Table 1. (The computation was done in single precision on the CDC 6500. Integers in parentheses denote decimal exponents.) The routine ZXMIN, for n beyond 10, was unable to produce reliable answers. In the third column of Table 1 we show the lower bounds

Table 1. Lower bounds for $\kappa_{n,\infty}$ in (4.2) and Theorem 2.2 for $n=2(1)10$

n	(4.2)	Theorem 2.2	n	(4.2)	Theorem 2.2
			6	3.715 (2)	1.754 (2)
2	1.207	3.000	7	1.812 (3)	4.150 (2)
3	4.250	9.899	8	9.062 (3)	9.582 (2)
4	1.764 (1)	2.781 (1)	9	4.621 (4)	2.173 (3)
5	7.892 (1)	7.167 (1)	10	2.393 (5)	4.858 (3)

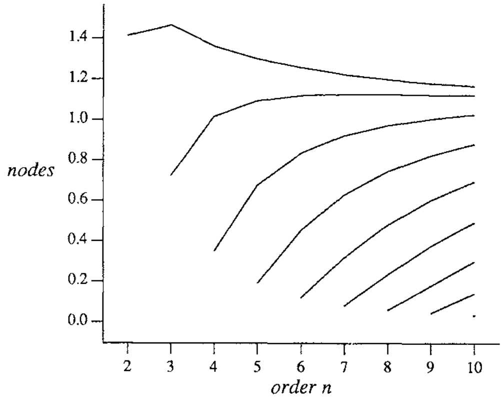

Fig. 1. Optimal positive nodes for $n=2(1)10$

computed from Theorem 2.2, Eq. (2.4). It can be seen that the bound from Theorem 2.2 is competitive with the one from (4.2) for about $n \leqq 5$, but then gradually weakens. Both bounds should be compared for $n=2$ and $n=3$ with the values $\kappa_{2,\infty}=3$, $\kappa_{3,\infty}=12.708$ computed in [4, Sect. 5].

The optimal nodes, as computed for the spectral norm, were found to have $x_n=0$. The positive nodes are depicted in Fig. 1, where the largest, second-largest, etc. are connected by straight lines for visual effect.

In the case of symmetric nodes, we used the same routines as above, with the Chebyshev nodes on $[-1, 1]$ as initial approximations. The results are shown in the second column of Table 2. We compare them in the third column with the lower bounds computed from Theorem 3.2, Eq. (3.2). In this case it was possible to go as far as $n=16$. Again, the bound in (3.2) is competitive with the one from (4.2) for about $n \leqq 10$, but then slowly deteriorates. Note also from [4, Sect. 4] that $\kappa_{2,\infty}=2$, $\kappa_{3,\infty}=5$, $\kappa_{4,\infty}=11.776$, $\kappa_{5,\infty}=21.456$, and $\kappa_{6,\infty}=51.330$.

The nonnegative optimal nodes in the symmetric case are shown graphically in Fig. 2.

Table 2. Lower bounds for $\kappa_{n,\infty}$ in (4.2) and Theorem 3.2 for $n=2(1)16$

n	(4.2)	Theorem 3.2	n	(4.2)	Theorem 3.2
			9	4.644 (1)	5.610 (1)
2	0.500		10	9.607 (1)	1.415 (2)
3	1.049		11	2.119 (2)	1.457 (2)
4	1.465	5.175	12	4.522 (2)	3.479 (2)
5	2.904	5.162	13	1.012 (3)	3.574 (2)
6	5.216	1.894 (1)	14	2.204 (3)	8.250 (2)
7	1.092 (1)	1.945 (1)	15	4.986 (3)	8.457 (2)
8	2.149 (1)	5.444 (1)	16	1.102 (4)	1.908 (3)

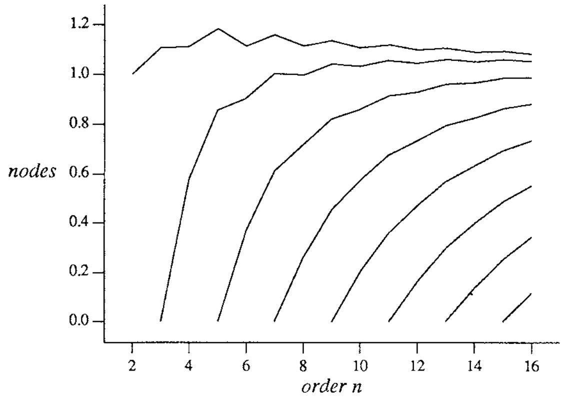

Fig. 2. Optimal symmetric nodes for $n = 2(1)16$

Interestingly, the same results were obtained if the initial approximations were chosen to be nonsymmetric, for example the Chebyshev points on [0, 1]. (Since the routine takes considerably longer to converge in this case, we verified this only for $2 \leqq n \leqq 10$.) This seems to indicate that the optimally conditioned Vandermonde matrix (in the spectral norm) indeed has symmetric nodes.

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