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Extended Iterative Methods for the Solution of Operator Equations

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Summary. Given an iterative method M_0, characterized by $x^{(k+1)} = G_0(x^{(k)})\,(k \geqq 0)$ ($x^{(0)}$ prescribed), for the solution of the operator equation $F(x)=0$, where $F: X \to X$ is a given operator and X is a Banach space, it is shown how to obtain a family of methods M_p characterized by $x^{(k+1)} = G_p(x^{(k)})\,(k \geqq 0)$ ($x^{(0)}$ prescribed), with order of convergence higher than that of M_0. The infinite dimensional multipoint methods of Bosarge and Falb [2] are a special case, in which M_0 is Newton's method.

Analogues of Theorems 2.3 and 2.36 of [2] are proved for the methods M_p, which are referred to as extensions of M_0. A number of methods with order of convergence greater than two are discussed and existence-convergence theorems for some of them are proved.

Finally some computational results are presented which illustrate the behaviour of the methods and their extensions when used to solve systems of nonlinear algebraic equations, and some applications currently being investigated are mentioned.

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1. Introduction

Let X be a Banach space, and let $F: D_F \subset X \to X$ be given. Suppose that $P: D_F \subset X \to X$ is defined by

$$F(x) = x - P(x) \qquad (x \in D_F). \tag{1.1}$$

In this paper we consider the equivalent problems of finding a zero x^* of F or a fixed point x^* of P.

Suppose that $G_0: X \to X$ and $\omega: X \to X$ are given operators. We consider an iterative procedure M_p defined by

$$x^{(k+1)} = G_p(x^{(k)}) \qquad (k \geqq 0) \tag{1.2}$$

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with $x^{(0)}$ prescribed, where $G_p: X \to X$ is defined recursively by

$$G_i(x) = G_0(x) - \sum_{j=1}^{i} F'(\omega(x))^{-1} F(G_{j-1}(x)) \quad (1 \leqq i \leqq p) \tag{1.3}$$

in which $F': X \to L(X)$ is the Fréchet derivative of F. Bosarge and Falb [2] have studied methods of the form M_p for the special case in which

$$G_0(x) = x - F'(x)^{-1} F(x) \tag{1.4}$$

and

$$\omega(x) = x. \tag{1.5}$$

The method M_1 defined by (1.2), (1.3) with $p = 1$ and G_0 and ω defined by (1.4), (1.5) respectively, has been discussed by Bosarge and Falb in [1].

Brent [3] has studied methods of the form M_p for the special case in which $X = \mathbb{R}^n$, $\omega(x) = x$, and G_0 corresponds to a class of secant methods, or to a class of methods related to Brown's methods for solving systems of nonlinear equations.

The method M_p defined by (1.2), (1.3) will be referred to as an extension of method M_0. Thus, for example, the infinite dimensional multipoint method M_p of Bosarge and Falb, with G_0 and ω defined by (1.4), (1.5) respectively, is an extension of Newton's method. In this paper we consider some methods having a higher order of convergence than that of Newton's method, and we consider also their extensions.

2. Notation

In this section we introduce some notation which will be used subsequently. The symbol X will always denote an arbitrary Banach space. Let $I: X \to X$ be the identity operator in X. Then we write (1.1) as

$$F(x) = (I - P)(x). \tag{2.1}$$

If $P: X \to X$ is Fréchet differentiable then we say that P is F-differentiable. If P is F-differentiable at $x \in D_F$ then by (2.1),

$$F'(x) = I - P'(x). \tag{2.2}$$

It is easily shown by induction that if $\{I - P'(\omega(x))\}^{-1}$ exists, then

$$G_i(x) = [\{I - P'(\omega(x))\}^{-1} \{P - P'(\omega(x))\}]^i (G_0(x)) \quad (1 \leqq i \leqq p) \tag{2.3}$$

where G_i is defined by (1.3). Suppose that $\{I - P'(\omega(x))\}^{-1}$ exists $(\forall x \in D)$, and that $P(y)$ is defined $(\forall y \in E)$. Define $Q: D \times E \to X$ by

$$Q(x, y) = [\{I - P'(\omega(x))\}^{-1} \{P - P'(\omega(x))\}](y). \tag{2.4}$$

Then by (2.3),

$$G_{i+1}(x)=Q(x, G_i(x)) \qquad (\forall x \in D) \tag{2.5}$$

and method M_p corresponding to (1.2), (1.3) is equivalent to

$$x^{(k+1)}=Q(x^{(k)}, G_{p-1}(x^{(k)})). \tag{2.6}$$

If $\Omega \subset X$ is a given set then by $F \in C^k(\Omega)$ we mean that F is k times continuously F-differentiable in Ω. We denote the open and closed balls with centre x and radius r by $B(x,r)$ and $B[x,r]$ respectively. The Banach space of all bounded linear operators from the Banach space X to the Banach space Y is denoted by $L(X, Y)$, and by $L(X)$ if $X=Y$.

3. Order of Convergence

In this section, a theorem about the order of convergence of M_p will be proved. The following definition is required.

Definition 3.1. Let $G: X \to X$ be given and suppose that the sequence $\{x^{(k)}\}$ generated from $x^{(k+1)}=G(x^{(k)})\,(k \geqq 0)$, ($x^{(0)}$ prescribed) converges to $x^* \in X$ such that $F(x^*)=0$, where $F: X \to X$ is given. Then the algorithm which the sequence $\{x^{(k)}\}$ is generated has order of convergence $q \geqq 1$ if

$$\lim_{k \to \infty} \frac{\|G(x^{(k)})-x^*\|}{\|G(x^{(k-1)})-x^*\|^q}=c,$$

where $c>0$ is a constant. □

The following theorem corresponds to Theorem 2.3 of [2] and to Theorem 1 and Theorem 2 of [3].

Theorem 3.2. *Suppose that* (i) $P: X \to X$ *is given;* (ii) $x^*=P(x^*)$; (iii) $P \in C^2(S)$ *where* $S=B[x^*,r]$; (iv) $(I-P'(x))^{-1}$ *exists* $(\forall x \in S)$, *with*

$$\sup_{x \in S} \|(I-P'(x))^{-1}\| \leqq B; \tag{3.1}$$

(v) $P'': X \to L(X, L(X))$ *is such that*

$$\sup_{x \in S} \|P''(x)\| \leqq K; \tag{3.2}$$

(vi) $x^{(0)} \in S$; (vii) $\omega: X \to X$ *is such that*

$$\|\omega(x)-x^*\| \leqq a\|x-x^*\|^{\mu} \qquad (\forall x \in S) \tag{3.3}$$

for some constant $a>0$ *and some* $\mu \geqq 1$; (viii) $G_0: X \to X$ *is such that*

$$\|G_0(x)-x^*\| \leqq b\|x-x^*\|^{\nu} \qquad (\forall x \in S) \tag{3.4}$$

for some constant $b>0$ and some $\nu \geqq \mu$; (ix) $BKr<\frac{2}{5}$; (x) $ar^{\mu-1}<1$; (xi) $br^{\nu-1}<1$. *Then the sequence $\{x^{(k)}\}$ generated from* (1.2) *with G_p defined by* (1.3) *lies in S and converges to x^* with order of convergence at least $\nu+p\mu$. Moreover, the rate of convergence is given by*

$$\|x^{(k+1)}-x^*\| \leqq c_p \|x^{(k)}-x^*\|^{\nu+p\mu} \qquad (\forall k \geqq 0) \tag{3.5}$$

where c_p is defined recursively by

$$\begin{aligned} &c_0=b, \\ &c_i=BKc_{i-1}(a+\tfrac{3}{2}c_{i-1}r^{\nu+(i-2)\mu}) \qquad (i \geqq 1). \end{aligned} \tag{3.6}$$

Proof. The theorem is proved by induction on p. By hypotheses (vi), (vii), (viii), (x), (xi), $x^{(0)} \in S$, $\omega(x^{(0)}) \in S$, and $G_0(x^{(0)}) \in S$. Therefore by hypotheses (iv), (ii), (viii), (iii), (v),

$$\|x^*-G_1(x^{(0)})\| \leqq \tfrac{1}{2}BKb^2\|x^{(0)}-x^*\|^{2\nu}+BKb\|x^{(0)}-x^*\|^{\nu}\|G_0(x^{(0)})-\omega(x^{(0)})\|. \tag{3.7}$$

Now by hypotheses (vii), (viii),

$$\|G_0(x^{(0)})-\omega(x^{(0)})\| \leqq (a+br^{\nu-\mu})\|x^{(0)}-x^*\|^{\mu}. \tag{3.8}$$

By (3.6)–(3.8) therefore,

$$\|x^*-G_1(x^{(0)})\| \leqq c_1\|x^{(0)}-x^*\|^{\mu+\nu}. \tag{3.9}$$

Furthermore, by hypotheses (ix)–(xi), $c_1 r^{\mu+\nu-1}<1$, whence by (3.9), $G_1(x^{(0)}) \in S$.

Suppose that $x^{(k)}=G_1(x^{(k-1)}) \in S$ for some $k \geqq 1$. By a similar argument to that used in going from $k=0$ to $k=1$, we deduce that $x^{(k+1)} \in S$, and that

$$\|x^{(k+1)}-x^*\| \leqq c_1\|x^{(k)}-x^*\|^{\mu+\nu}. \tag{3.10}$$

Therefore by induction on k, $x^{(k)} \in S$ $(\forall k \geqq 0)$ and (3.10) holds $(\forall k \geqq 0)$. Therefore the theorem holds for $p=1$.

Suppose that for some $p \geqq 1$, the theorem holds for each $m \leqq p$. Since $x^{(0)}$, $G_p(x^{(0)})$, and $\omega(x^{(0)})$ are in S, then by (1.3),

$$\begin{aligned} \|x^*-G_{p+1}(x^{(0)})\| \leqq{} & \tfrac{1}{2}BKc_p^2\|x^{(0)}-x^*\|^{2(\nu+p\mu)} \\ & +BKc_p\|x^{(0)}-x^*\|^{\nu+p\mu}\|G_p(x^{(0)})-\omega(x^{(0)})\|. \end{aligned} \tag{3.11}$$

But by (3.3) and (3.5),

$$\|G_p(x^{(0)})-\omega(x^{(0)})\| \leqq (a+c_p r^{\nu+(p-1)\mu})\|x^{(0)}-x^*\|^{\mu}. \tag{3.12}$$

Therefore by (3.11), (3.12), and (3.6),

$$\|x^*-G_{p+1}(x^{(0)})\| \leqq c_{p+1}\|x^{(0)}-x^*\|^{\nu+(p+1)\mu}. \tag{3.13}$$

Suppose that

$$c_m r^{\nu+m\mu-1}<1 \qquad (m=1,\ldots,p). \tag{3.14}$$

This is certainly true for $p=1$. Then by (3.6) and hypotheses (ix)–(xi), (3.14) is true for $p+1$, whence by (3.13), $G_{p+1}(x^{(0)})\in S$. Now suppose that for some $k\geq 0$, $x^{(i)}\in S$ $(i=0, \ldots, k)$. Then by a similar argument to that which was used in going from $k=0$ to $k=1$ with $p=1$, $G_{p+1}(x^{(k)})\in S$ and

$$\|x^* - G_{p+1}(x^{(k)})\| \leqq c_{p+1} \|x^{(k)} - x^*\|^{v+(p+1)\mu}.$$

Therefore $x^{(k+1)} = G_{p+1}(x^{(k)})\in S$. Therefore by induction on k, $G_{p+1}(x^{(k)})\in S$ $(\forall k\geqq 0)$. Clearly by (3.14), $x^{(k)}\to x^*$ $(k\to\infty)$. Therefore the theorem holds for $p+1$ if it holds for p. Therefore by induction on p, the theorem holds $(\forall p\geqq 1)$. □

4. A Convergence Theorem

In this section a convergence theorem for the extended methods defined by (1.2), (1.3) is given. In order to prove this theorem, three lemmas similar to Lemmas 2.16, 2.19, and 2.32 of [2] are required.

Lemma 4.1. *Suppose that* (i) $P\colon X\to X$, $\omega\colon X\to X$, *and* $G_0\colon X\to X$ *are given operators;* (ii) $x^* = P(x^*)$*;* (iii) $x^* = G_0(x^*)$*;* (iv) $\{I - P'(\omega(x^*))\}^{-1}$ *exists;* (v) $G_p\colon X\to X$ $(p\geqq 1)$ *is defined by* (1.3). *Then* $x^* = G_p(x^*)$ $(\forall p\geqq 1)$.

Proof. The proof is similar to that of Lemma 2.16 in [2] and is therefore omitted. □

Lemma 4.2. *Suppose that* (i) $P\colon X\to X$ *is such that* $P\in C^2(S_0)$, *where* $S_0 = B[x^{(0)}, r]$*;* (ii) $\omega\colon X\to X$ *is such that* $\omega\in C^1(S_0)$*;* (iii) $\{I-P'(\omega(x))\}^{-1}$ *exists* $(\forall x\in S_0)$ *and*

$$\sup_{x\in S_0} \|\{I-P'(\omega(x))\}^{-1}\| \leqq D; \tag{4.1}$$

(iv) $P''\colon X\to L(X, L(X))$ *is such that*

$$\sup_{x\in S_0} \|P''(x)\| \leqq M; \tag{4.2}$$

(v) $\omega'\colon X\to L(X)$ *is such that*

$$\sup_{x\in S_0} \|\omega'(x)\| \leqq \gamma. \tag{4.3}$$

Then the mapping $Q\colon X\times X\to X$ *defined by* (2.4) *has partial F-derivatives* $Q_1(x, y)(.)$, $Q_2(x, y)(.)$ *with respect to* x *and* y *respectively, and these are given,* $(\forall x, y\in S_0)$, *by*

$$Q_1(x, y)(.) = \{I-P'(\omega(x))\}^{-1} P''(\omega(x))\,\omega'(x)(.)\,\{I-P'(\omega(x))\}^{-1}(P-I)(y), \tag{4.4}$$

and

$$Q_2(x, y)(.) = \{I-P'(\omega(x))\}^{-1}\{P'(y)-P'(\omega(x))\}(.). \tag{4.5}$$

Proof. The proof is similar to that of Lemma 2.19 in [2] and is therefore omitted. □

Corollary 4.3. *Suppose that* (i) *hypotheses* (i)–(v) *of Lemma* 4.2 *are valid;* (ii) $x \in S_0$ *is such that* $G_{p-1}(.)$ *is F-differentiable at* x *and* $G_{p-1}(x) \in S_0$. *Then* $G_p(.)$ *is F-differentiable at* x *and*

$$G'_p(x)(.) = Q_1(x, G_{p-1}(x))(.) + Q_2(x, G_{p-1}(x))\, G'_{p-1}(x)(.). \tag{4.6}$$

Proof. The proof follows immediately from (2.4) and Lemma 4.2. □

Lemma 4.4. *Suppose that* (i) $V: X \to X$ *is such that* $V \in C^1(S_0)$ *where* $S_0 = B[x^{(0)}, r]$*;* (ii) $V': X \to L(X)$ *is such that*

$$\sup_{x \in S_0} \|V'(x)\| \leqq \delta < 1; \tag{4.7}$$

(iii) $\exists \eta > 0$,

$$\|V(x^{(0)}) - x^{(0)}\| \leqq \eta; \tag{4.8}$$

(iv) η, δ, *and* r *are such that*

$$\eta/(1-\delta) \leqq r. \tag{4.9}$$

Then the sequence $\{x^{(k)}\}$ *generated from* $x^{(k+1)} = V(x^{(k)})$ $(k \geqq 0)$ *converges to the unique fixed point* x^* *of* V *in* S_0 *and*

$$\|x^{(k)} - x^*\| \leqq \frac{\delta^k}{(1-\delta)} \|x^{(1)} - x^{(0)}\| \qquad (\forall k \geqq 1). \tag{4.10}$$

Proof. This lemma is just Lemma 2.32 of [2]. For a proof see [7] or [10]. □

The following theorem holds for algorithms defined by (1.2), (1.3), and contains Theorem 2.3 of [2] as a special case.

Theorem 4.5. *Suppose that* (i) $P: X \to X$ *and* $\omega: X \to X$ *are such that* $P \in C^2(S_0)$, $\omega \in C^1(S_0)$ *where* $S_0 = B[x^{(0)}, r]$*;* (ii) $\{I - P'(\omega(x))\}^{-1}$ *exists* $(\forall x \in S_0)$, *and*

$$\sup_{x \in S_0} \|\{I - P'(\omega(x))\}^{-1}\| \leqq B; \tag{4.11}$$

(iii) $P'': X \to L(X, L(X))$ *is such that*

$$\sup_{x \in S_0} \|P''(\omega(x))\| \leqq K; \tag{4.12}$$

(iv) $\omega': X \to L(X)$ *is such that*

$$\sup_{x \in S_0} \|\omega'(x)\| \leqq \gamma; \tag{4.13}$$

(v) $G_0: X \to X$ *is such that* $G_0 \in C^1(S_0)$ *and*

$$\sup_{x \in S_0} \|G'_0(x)\| \leqq h_0; \tag{4.14}$$

(vi) $G_0(x) \in S_0$ *whenever* $x \in S_0$; (vii) $(P-I)$ *is uniformly bounded on* S_0 *with*

$$\sup_{x \in S_0} \|(P-I)(x)\| \leqq M; \tag{4.15}$$

(viii) $G_0 - \omega$ *is uniformly bounded on* S_0 *with*

$$\sup_{x \in S_0} \|G_0(x) - \omega(x)\| \leqq E; \tag{4.16}$$

(ix) $\exists\, \eta_p > 0$ $(p = 0, \ldots, v)$ *such that*

$$\|G_p(x^{(0)}) - x^{(0)}\| \leqq \eta_p; \tag{4.17}$$

(x) h_p $(p = 1, \ldots, v)$ *is defined recursively by*

$$h_p = BK[B\gamma M + \{(p-1)BM + E\}\, h_{p-1}] \qquad (1 \leqq p \leqq v), \tag{4.18}$$

and

$$\eta_p \leqq (1 - h_p)\, r \qquad (p = 0, \ldots, v); \tag{4.19}$$

(xi) $h_p < 1$ $(p = 0, \ldots, v)$. *Then for* $p = 0, \ldots, v$, *the sequence* $\{G_p(x^{(k)})\}$ *converges to the unique fixed point* x^* *of* P *in* S_0, *and*

$$\|G_p(x^{(k)}) - x^*\| \leqq \frac{h_p^{k+1}}{(1 - h_p)} \|x^{(1)} - x^{(0)}\| \qquad (\forall\, k \geqq 0).$$

Proof. By (2.5),

$$G_1(x) = Q(x, G_0(x)). \tag{4.20}$$

By Corollary 4.3, G_1 is F-differentiable on S_0 and

$$\sup_{x \in S_0} \|G_1'(x)\| \leqq \sup_{x \in S_0} \|Q_1(x, G_0(x))\| + \sup_{x \in S_0} \|Q_2(x, G_0(x))\| \sup_{x \in S_0} \|G_0'(x)\|,$$

so by (4.4), (4.5), (4.11)–(4.16), and (x),

$$\begin{aligned} \sup_{x \in S_0} \|G_1'(x)\| &\leqq BK(B\gamma M + E h_0) \\ &= h_1. \end{aligned} \tag{4.21}$$

Then $(\forall\, x \in S_0)$, by (4.21), (4.17), (4.19),

$$\begin{aligned} \|G_1(x) - x^{(0)}\| &\leqq \|G_1(x) - G_1(x^{(0)})\| + \|G_1(x^{(0)}) - x^{(0)}\| \\ &\leqq h_1 r + (1 - h_1)\, r \\ &= r. \end{aligned}$$

Therefore $G_1(x) \in S_0$ $(\forall\, x \in S_0)$. Suppose that for some $p \geqq 1$, G_i is F-differentiable on S_0, that $G_i(x) \in S_0$ $(\forall\, x \in S_0)$, and that

$$\sup_{x \in S_0} \|G_i'(x)\| \leqq h_i \qquad (i = 1, \ldots, p < v). \tag{4.22}$$

Then by Corollary 4.3, G_{p+1} is F-differentiable on S_0 and, as for (4.21),

$$\sup_{x\in S_0} \|G'_{p+1}(x)\| \leqq B^2 K\gamma M + BK \sup_{x\in S_0} \|G_p(x)-\omega(x)\|\, h_p. \tag{4.23}$$

Now

$$\|G_p(x)-\omega(x)\| \leqq \|G_p(x)-G_{p-1}(x)\| + \cdots + \|G_0(x)-\omega(x)\|, \tag{4.24}$$

and for $i=1, \ldots, p$, by (vi), (4.15), and (4.11),

$$\begin{aligned}\sup_{x\in S_0} \|G_i(x)-G_{i-1}(x)\| &= \sup_{x\in S_0} \|\{I-P'(\omega(x))\}^{-1}(P-I)(G_{i-1}(x))\| \\ &\leqq BM. \end{aligned}\tag{4.25}$$

Therefore by (4.23)–(4.25),

$$\begin{aligned}\sup_{x\in S_0} \|G'_{p+1}(x)\| &\leqq B^2 K\gamma M + BK(pBM+E)\, h_p \\ &= h_{p+1}. \end{aligned}\tag{4.26}$$

Therefore by (4.26), (4.17), and (4.19), $(\forall x\in S_0)$,

$$\begin{aligned}\|G_{p+1}(x)-x^{(0)}\| &\leqq \|G_{p+1}(x)-G_{p+1}(x^{(0)})\| + \|G_{p+1}(x^{(0)})-x^{(0)}\| \\ &\leqq r h_{p+1} + (1-h_{p+1})\, r \\ &= r. \end{aligned}$$

Therefore $G_{p+1}(x)\in S_0$ $(\forall x\in S_0)$. Therefore by induction on p, (4.22) holds for $p=0, \ldots, v$ and $G_p(x)\in S_0$ $(\forall x\in S_0)$ for $p=0, \ldots, v$. Therefore by Lemma 4.4, $\{G_p(x^{(k)})\}$ converges to the unique fixed point x_p^* of G_p in S_0, for $p\in 0, \ldots, v$.

Now by Lemma 4.1, the unique fixed point x_0^* of G_0 in S_0 is also a fixed point of G_p in S_0 $(\forall p\geqq 0)$. Therefore since for $p=0, \ldots, v$ x_p^* is the unique fixed point of G_p in S_0, then $x_p^*=x_0^*$ $(p=1, \ldots, v)$. Therefore with $x^*=x_0^*$, the theorem is proved. □

5. Some Third Order Methods

In this section a number of iterative methods for the solution of $F(x)=0$ or equivalently, of $x=P(x)$, where $F=I-P$, are discussed. We consider firstly some third order methods. The method corresponding to

$$y^{(k)} = x^{(k)} - F'(x^{(k)})^{-1} F(x^{(k)}), \tag{5.1}$$

$$x^{(k+1)} = y^{(k)} - F'(x^{(k)})^{-1} F(y^{(k)}) \tag{5.2}$$

has been discussed by several authors, and in particular by Bosarge and Falb [1], who gave a convergence theorem for it which shows essentially that if $x^{(0)}$ is chosen suitably, then the sequence $\{x^{(k)}\}$ generated from (5.1), (5.2) converges to a solution x^* of $F(x)=0$ with order at least three. The procedure corresponding to (5.1), (5.2) requires one evaluation and one inversion of F' and two evaluations of F per iteration. This makes the procedure very attractive for solving, in particular, systems of nonlinear algebraic equations, nonlinear integral equations, and nonlinear two-point boundary value problems, as discussed by Bosarge and Falb in [2]. Henceforth, the method corresponding to (5.1), (5.2) will be referred to as the Bosarge-Falb method (BF), for convenience. The extension of the method BF will be denoted by EBF. The following lemmas will be required.

Lemma 5.1. *Suppose that* (i) $F\colon D_F \subset X \to X$ *is given and* $\exists x^* \in D_F$, $F(x^*)=0$; (ii) $F \in C^2(S)$ *where* $S = B[x^*, r] \subset D_F$; (iii) $F'(x)^{-1}$ *exists* ($\forall x \in S$), *with*

$$\sup_{x\in S} \|F'(x)^{-1}\| \leqq B; \tag{5.3}$$

(iv) $F''\colon X \to L(X, L(X))$ *is such that*

$$\sup_{x\in S} \|F''(x)\| \leqq K; \tag{5.4}$$

(v) B, K, *and* r *satisfy*

$$BKr < 1; \tag{5.5}$$

(vi) $x^{(0)} \in S$. *Then the sequences* $\{y^{(k)}\}$, $\{x^{(k)}\}$ *generated from* (5.1), (5.2) *respectively remain in* S *and converge to* x^*. *Furthermore,*

$$\|y^{(k)} - x^*\| \leqq \tfrac{1}{2} BK \|x^{(k)} - x^*\|^2 \qquad (\forall k \geqq 0), \tag{5.6}$$

and

$$\|x^{(k+1)} - x^*\| \leqq \tfrac{5}{8} (BK)^2 \|x^{(k)} - x^*\|^3 \qquad (\forall k \geqq 0). \tag{5.7}$$

Proof. By (5.1), (i), (ii), (5.3), (5.4), and (vi),

$$\|y^{(0)} - x^*\| \leqq \tfrac{1}{2} BK \|x^{(0)} - x^*\|^2. \tag{5.8}$$

Therefore by (5.5), $y^{(0)} \in S$. Therefore by (5.2), (5.8),

$$\begin{aligned} \|x^{(1)} - x^*\| &\leqq \tfrac{1}{2} BK \|y^{(0)} - x^*\|^2 + K \|x^{(0)} - x^*\| \, \|y^{(0)} - x^*\| \\ &\leqq \tfrac{5}{8} (BK)^2 \|x^{(0)} - x^*\|^3. \end{aligned} \tag{5.9}$$

Therefore by (5.5), $x^{(1)} \in S$. Suppose that for some $k \geqq 1$, $x^{(k)} \in S$. Then by an argument similar to that used in going from $k=0$ to $k=1$, it follows that $y^{(k)} \in S$, $x^{(k+1)} \in S$,

$$\|y^{(k)} - x^*\| \leqq \tfrac{1}{2} BK \|x^{(k)} - x^*\|^2,$$

and

$$\|x^{(k+1)} - x^*\| \leqq \tfrac{5}{8} (BK)^2 \|x^{(k)} - x^*\|^3.$$

Therefore by induction on k, the lemma is proved. □

Lemma 5.2. *Suppose that* (i) *the hypotheses of Lemma* 5.1 *are valid;* (ii) ω: $X \to X$ *is defined by*

$$\omega(x) = x \qquad (x \in S); \tag{5.10}$$

(iii) G_0: $X \to X$ *is defined by*

$$G_0(x) = \omega(x) - F'(x)^{-1}\,\omega(x) \qquad (x \in S). \tag{5.11}$$

Then

$$\|\omega(x) - x^*\| \leqq a\|x - x^*\|^{\mu} \qquad (\forall\, x \in S), \tag{5.12}$$

and

$$\|G_0(x) - x^*\| \leqq b\|x - x^*\|^{\nu} \qquad (\forall\, x \in S), \tag{5.13}$$

where $a = 1$, $\mu = 1$, $b = \frac{5}{8}(BK)^2$, *and* $\nu = 3$.

Proof. The proof is immediate from the proof of Lemma 5.1. □

The following theorem holds.

Theorem 5.3. *Suppose that* (i) F: $X \to X$ *is given and* $\exists\, x^* \in X$, $F(x^*) = 0$; (ii) $F \in C^2(S)$ *where* $S = B[x^*, r]$; (iii) $F'(x)^{-1}$ *exists* $(\forall\, x \in S)$ *and* (5.3) *holds;* (iv) (5.4) *holds;* (v) $x^{(0)} \in S$; (vi) ω: $X \to X$ *is defined by* (5.10) *and* (5.12) *holds;* (vii) G_0: $X \to X$ *is defined by* (5.11) *and* (5.13) *holds;* (viii) $BKr < \frac{2}{5}$. *Then the sequence* $\{x^{(k)}\}$ *generated from* (1.2), (1.3) *lies in* S *and converges to* x^*, *with order of convergence at least* $3 + p$. *Moreover,*

$$\|x^{(k+1)} - x^*\| \leqq c_p \|x^{(k)} - x^*\|^{3+p} \qquad (\forall\, k \geqq 0)$$

where

$$c_0 = \tfrac{5}{8}(BK)^2,$$

and

$$c_i = \tfrac{1}{2} BK c_{i-1}(2 + 3c_{i-1} r^{i+1}) \qquad (1 \leqq i \leqq p).$$

Proof. The theorem is an immediate consequence of Theorem 3.2 with $P = I - F$. □

The preceding theorem is equivalent to Theorem 2.3 of [2], and shows that the method EBF has order of convergence at least $3 + p$, $(p \geqq 1)$.

The method corresponding to

$$y^{(k)} = x^{(k)} - \tfrac{1}{2} F'(x^{(k)})^{-1} F(x^{(k)}), \tag{5.14}$$

$$x^{(k+1)} = x^{(k)} - F'(y^{(k)})^{-1} F(x^{(k)}) \tag{5.15}$$

attributed by Collatz [4] to Pasquali and henceforth referred to as method P, also has order of convergence at least three as is shown by the following lemma.

Lemma 5.4. *Suppose that* (i) F: $D_F \subset X \to X$ *is given and* $\exists x^* \in D_F$, $F(x^*)=0$; (ii) $F \in C^3(S)$ *where* $S = B[x^*, r] \subset D_F$; (iii) $F'(x)^{-1}$ *exists* ($\forall x \in S$) *and* (5.3) *holds;* (iv) (5.4) *holds;* (v) F': $X \to L(X)$ *is such that*

$$\sup_{x \in S} \|F'(x)\| \leqq D; \tag{5.16}$$

(vi) F''': $X \to L(X, L(X, L(X)))$ *is such that*

$$\sup_{x \in S} \|F'''(x)\| \leqq L; \tag{5.17}$$

(vii) B, K, r, D, *and* L *satisfy*

$$BKr < 1, \tag{5.18}$$

$$\tfrac{1}{2} B^2 K r D < 1, \tag{5.19}$$

and

$$b r^2 < 1, \tag{5.20}$$

where

$$b = \frac{B}{(8 - 4KB^2 Dr)} \left[\tfrac{4}{3} L + 2BK^2 + B^3 D^3 L\right]; \tag{5.21}$$

(viii) $x^{(0)} \in S$. *Then the sequence* $\{x^{(k)}\}$ *generated from* (5.14), (5.15) *remains in* S *and converges to* x^*. *Furthermore,*

$$\|y^{(k)} - x^*\| \leqq \tfrac{3}{4} \|x^{(k)} - x^*\| \qquad (\forall k \geqq 0), \tag{5.22}$$

and

$$\|x^{(k+1)} - x^*\| \leqq b \|x^{(k)} - x^*\|^3 \qquad (\forall k \geqq 0). \tag{5.23}$$

Proof. The first Newton iterate $x_N^{(1)}$ is defined by $x_N^{(1)} = x^{(0)} - F'(x^{(0)})^{-1} F(x)^{(0)}$. Therefore by (5.8),

$$\begin{aligned} \|y^{(0)} - x^*\| &\leqq \tfrac{1}{2}(\|x^{(0)} - x^*\| + \|x_N^{(1)} - x^*\|) \\ &\leqq \tfrac{3}{4} \|x^{(0)} - x^*\|. \end{aligned}$$

Therefore $y^{(0)} \in S$, and $F'(y^{(0)})^{-1}$ exists. Let $h^{(0)} = x_N^{(1)} - x^{(0)}$. Then

$$\|\tfrac{1}{2} F''(x^{(0)}) h^{(0)}\| \, \|F'(x^{(0)})^{-1}\| \leqq \tfrac{1}{2} KB^2 D \|x^{(0)} - x^*\|. \tag{5.24}$$

Let $H^{(0)} = F'(x^{(0)}) + \tfrac{1}{2} F''(x^{(0)}) h^{(0)}$. Then by (5.24), (5.19), and the Banach perturbation lemma, $H^{(0)-1}$ exists and

$$\|H^{(0)-1}\| \leqq \frac{2B}{(2 - KB^2 Dr)}. \tag{5.25}$$

Therefore

$$\|x^{(1)}-x^*\| \leqq \|H^{(0)-1}\| \, \|H^{(0)}(x^{(0)}-x^*)-F(x^{(0)})+F(x^*)\| \\ + \|H^{(0)-1}-F'(y^{(0)})^{-1}\| \, \|F(x^{(0)})-F(x^*)\|. \tag{5.26}$$

Now by definition of $H^{(0)}$,

$$\|H^{(0)}(x^{(0)}-x^*)-F(x^{(0)})+F(x^*)\| \leqq (\tfrac{1}{6}L+\tfrac{1}{4}BK^2)\, \|x^{(0)}-x^*\|^3. \tag{5.27}$$

Also by (5.25), the definition of $H^{(0)}$, and (5.17),

$$\|H^{(0)-1}-F'(y^{(0)})^{-1}\| \leqq \frac{B^4D^2L}{4(2-KB^2Dr)} \|x^{(0)}-x^*\|^2. \tag{5.28}$$

Therefore by (5.26)–(5.28), we have $\|x^{(1)}-x^*\| \leqq b\|x^{(0)}-x^*\|^3$, and so by (5.20), $x^{(1)}\in S$.

Suppose that for some $k\geqq 1$, $x^{(k)}\in S$. Then by a similar argument to that which was used in going from $k=0$ to $k=1$, it follows that $y^{(k)}\in S$, $x^{(k+1)}\in S$, and (5.22), (5.23) hold for k. Therefore by induction on k, the lemma is proved. □

Lemma 5.5. *Suppose that* (i) *the hypotheses of Lemma* 5.4 *are valid;* (ii) ω: $X\to X$ *is defined by*

$$\omega(x)=x-\tfrac{1}{2}F'(x)^{-1}F(x) \qquad (x\in S); \tag{5.29}$$

(iii) G_0: $X\to X$ *is defined by*

$$G_0(x)=x-F'(\omega(x))^{-1}F(x) \qquad (x\in S). \tag{5.30}$$

Then (5.12) *and* (5.13) *hold, with* $a=\frac{3}{4}$, $\mu=1$, b *defined by* (5.21), *and* $v=3$.

Proof. The proof is immediate from Lemma 5.4. □

Theorem 5.6. *Suppose that* (i) *the hypotheses of Lemma* 5.4 *are valid;* (ii) $BKr<\frac{2}{5}$. *Then the sequence* $\{x^{(k)}\}$ *generated from* (1.2), (1.3) *lies in S and converges to* x^* *with order of convergence at least* $3+p$. *Moreover*

$$\|x^{(k+1)}-x^*\| \leqq c_p\|x^{(k)}-x^*\|^{3+p} \qquad (\forall k\geqq 0)$$

where

$$c_0=b$$

and

$$c_i=\tfrac{3}{4}BKc_{i-1}(1+2c_{i-1}r^{i+1}) \qquad (1\leqq i\leqq p).$$

Proof. The theorem is an immediate consequence of Theorem 3.2. □

The preceding theorem shows that the extended Pasquali method (EP) has, like EBF, an order of convergence at least $3+p$ $(p\geqq 1)$.

Bosarge and Falb [1] have proved a theorem giving sufficient conditions for the existence of a solution x^* of $F(x)=0$ to which the sequence generated from

BF will converge, and in [2] they prove a similar theorem for EBF. It is difficult to prove an existence-convergence theorem for the general class of methods defined by (1.2), (1.3) because the special nature of G_0 must be exploited, as was done by Bosarge and Falb for EBF. It is however, easy to prove a theorem for the method P by using the following result which is due to Rheinboldt [11].

Theorem 5.7. *Suppose that* (i) $F\colon D_F \subset X \to X$ *is F-differentiable on an open convex set* $D_0 \subset D_F$, *and*

$$\|F'(x)-F'(y)\| \leqq \gamma \|x-y\| \qquad (\forall x, y \in D_0); \tag{5.31}$$

(ii) $A\colon D_0 \subset X \to L(X)$ *has a bounded inverse* $A(x)^{-1}$ *and*

$$\sup_{x \in D_0} \|A(x)^{-1}\| \leqq \beta; \tag{5.32}$$

(iii) A *is such that*

$$\sup_{x \in D_0} \|F'(x)-A(x)\| \leqq \delta; \tag{5.33}$$

(iv) $x^{(0)} \in D_0$ *is such that*

$$\|A(x^{(0)})^{-1} F(x^{(0)})\| \leqq \alpha; \tag{5.34}$$

(v) α, β, γ, δ *are such that*

$$h = \tfrac{1}{2}\beta\gamma\alpha + \beta\delta < 1; \tag{5.35}$$

(vi) $B[x^{(0)}, r] \subset D_0$, *where*

$$r = \alpha/(1-h). \tag{5.36}$$

Then the sequence $\{x^{(k)}\}$ *defined by*

$$x^{(k+1)} = x^{(k)} - A(x^{(k)})^{-1} F(x^{(k)}) \qquad (k \geqq 0)$$

lies in $B[x^{(0)}, r]$ *and converges to a solution* x^* *of* $F(x)=0$. □

The following theorem gives sufficient conditions for the existence of a solution of $F(x)=0$ to which a sequence of iterates generated from method P converges.

Theorem 5.8. *Suppose that* (i) $F\colon D_F \subset X \to X$ *is a given mapping and* $\exists x^{(0)} \in D_F$ *and* $R>0$ *such that* $D = B(x^{(0)}, R) \subset D_F$; (ii) F *is F-differentiable in* D *and*

$$\|F'(x)-F'(y)\| \leqq \gamma \|x-y\| \qquad (\forall x, y \in D); \tag{5.37}$$

(iii) $F'(x)^{-1}$ *exists* $(\forall x \in D)$ *and*

$$\sup_{x \in D} \|F'(x)^{-1}\| \leqq \beta; \tag{5.38}$$

(iv) F *is uniformly bounded in* D *with*

$$\sup_{x \in D} \|F(x)\| \leqq \nu; \tag{5.39}$$

(v) $\beta^2\gamma\nu<1$; (vi) $0<r<R-\frac{1}{2}\beta\nu$, *where*

$$r=\beta\nu/(1-\beta^2\gamma\nu). \tag{5.40}$$

Then the sequence $\{x^{(k)}\}$ *defined by* (5.14), (5.15) *lies in* $B[x^{(0)}, r]$, *and converges to a solution* x^* *of* $F(x)=0$.

Proof. Let $\rho=R-\frac{1}{2}\beta\nu$ and let $D_0=B(x^{(0)},\rho)$. Also let $y=x-\frac{1}{2}F'(x)^{-1}F(x)$ $(x\in D_0)$. Then by (5.38), (5.39) and (vii), $\|y-x^{(0)}\|\leqq R$ $(\forall x\in D_0)$. Therefore $y\in D$ whenever $x\in D_0$ and so by (iii), $F'(y)^{-1}$ exists and $\|F'(y)^{-1}\|\leqq\beta$, whenever $x\in D_0$. Also, by (5.37), $\|F'(y)-F'(x)\|\leqq\gamma\|y-x\|\leqq\frac{1}{2}\gamma\beta\nu$ $(\forall x\in D_0)$, and $\|F'(y^{(0)})^{-1}F(x^{(0)})\|\leqq\beta\nu$. The theorem now follows from Theorem 5.7. □

The method corresponding to

$$y^{(k)}=x^{(k)}-F'(x^{(k)})^{-1}F(x^{(k)}), \tag{5.41}$$

$$x^{(k+1)}=x^{(k)}-2(F'(x^{(k)})+F'(y^{(k)}))^{-1}F(x^{(k)}) \tag{5.42}$$

discussed by Traub [12] and henceforth referred to as method T, has, like method P, an order of convergence at least three, as shown by the following lemma.

Lemma 5.9. *Suppose that* (i) F: $D_F\subset X\to X$ *is given and* $\exists x^*\in D_F$, $F(x^*)=0$; (ii) $F\in C^3(S)$ *where* $S=B[x^*, r]\subset D_F$; (iii) $F'(x)^{-1}$ *exists* $(\forall x\in S)$ *and* (5.3) *holds;* (iv) (5.4) *holds;* (v) (5.16) *holds;* (vi) (5.17) *holds;* (vii) (5.18) *and* (5.19) *hold;* (viii) $br^2<1$ *where*

$$b=\frac{B}{(2-B^2KrD)}\left[\frac{1}{3}L+\frac{1}{2}BK^2+\frac{B^3LD^3}{(2-B^2KrD)}\right]; \tag{5.43}$$

(ix) $x^{(0)}\in S$. *Then the sequence* $\{x^{(k)}\}$ *defined by* (5.41), (5.42) *lies in* S *and converges to* x^*. *Moreover,*

$$\|y^{(k)}-x^*\|\leqq\tfrac{1}{2}BK\|x^{(k)}-x^*\|^2 \qquad (\forall k\geqq 0), \tag{5.44}$$

and

$$\|x^{(k+1)}-x^*\|\leqq b\|x^{(k)}-x^*\|^3 \qquad (\forall k\geqq 0). \tag{5.45}$$

Proof. The proof is similar to that of Lemma 5.4 and is therefore omitted. □

Lemma 5.10. *Suppose that* (i) *the hypotheses of Lemma* 5.9 *are valid;* (ii) ω: $X\to X$ *is defined by*

$$\omega(x)=x-F'(x)^{-1}F(x) \qquad (x\in S); \tag{5.46}$$

(iii) G_0: $X\to X$ *is defined by*

$$G_0(x)=x-2(F'(x)+F'(\omega(x)))^{-1}F(x) \qquad (x\in S). \tag{5.47}$$

Then (5.12) *and* (5.13) *hold, with* $a=\frac{1}{2}BK$, $\mu=2$, b *defined by* (5.43), *and* $\nu=3$.

Proof. The proof is immediate from Lemma 5.9. □

Theorem 5.11. *Suppose that* (i) *the hypotheses of Lemma* 5.9 *are valid;* (ii) $BKr<\frac{2}{5}$. *Then the sequence* $\{x^{(k)}\}$ *generated from* (1.2), (1.3) *with* ω *and* G_0 *defined by* (5.46) *and* (5.47) *respectively, lies in* S *and converges to* x^* *with order of convergence at least* $3+2p$. *Moreover.*

$$\|x^{(k+1)}-x^*\| \leqq c_p \|x^{(k)}-x^*\|^{3+2p} \qquad (\forall k \geqq 0),$$

where

$$c_0=b,$$

and

$$c_i=\tfrac{1}{2}BKc_{i-1}(BK+3c_{i-1}r^{2i-1}) \qquad (1\leqq i\leqq p).$$

Proof. The theorem is an immediate consequence of Theorem 3.2. □

The following theorem gives sufficient conditions for the existence of a solution of $F(x)=0$ to which a sequence of iterates generated from method T converges.

Theorem 5.12. *Suppose that* (i) *hypotheses* (i), (ii), (iv) *of Theorem* 5.8 *are valid; (ii)* $F'(x)^{-1}$ *exists* $(\forall x \in D)$ *and*

$$\sup_{x\in D}\|F'(x)^{-1}\| \leqq B; \tag{5.48}$$

(iii) $B^2\gamma v<\frac{1}{2}$; (iv) $0<r<R-Bv$, *where* $r=\alpha/(1-h)$, *in which* $\alpha=Bv/(1-\frac{1}{2}B^2\gamma v)$, *and*

$$h=\frac{B^2\gamma v}{(2-B^2\gamma v)}\left[1+\frac{1}{(2-B^2\gamma v)}\right].$$

Then the sequence $\{x^{(k)}\}$ *generated from* (5.41), (5.42) *lies in* $B[x^{(0)},r]$ *and converges to a solution* x^* *of* $F(x)=0$.

Proof. In Theorem 5.7 set $D_0=B(x^{(0)},\rho)$ where $\rho=R-Bv$, and define A: $D_0\subset X\to X$ by $A(x)=\frac{1}{2}(F'(x)+F'(y))$, where $y=x-F'(x)^{-1}F(x)$ $(x\in D_0)$. □

By Hypothesis (iii) of Theorem 5.12, $h<\frac{7}{9}$, so $r<6Bv$. Therefore since $\rho=R-Bv$, the condition $R>7Bv$ ensures that $r<\rho$ whence $B[x^{(0)},r]\subset D_0$. Therefore if v is sufficiently small, then Hypothesis (iv) is automatically valid.

6. Some Methods of Higher Order

Iterative methods of order higher than three for the solution of $F(x)=0$ can be constructed in a number of well known ways, many of which are described by Ortega and Rheinboldt [9]. Many high order methods are computationally expensive, requiring several evaluations of F or F', or even F'' per iteration. The methods BF and EBF described in Section 5 are of interest because only one evaluation and inversion of F' per iteration is required for either method. The methods P and T require two evaluations and inversions of F' per iteration and have the same order of convergence as BF, and should therefore be more computationally expensive than BF.

Suppose that a method could be found which has order of convergence greater than three, which requires not more than two evaluations and inversions of F' and not more than two evaluations of F, and which is such that for each iteration k, one of the points at which F is evaluated and inverted is $\omega(x^{(k)})$. Then Theorem 3.2 indicates that if $\mu>1$, then the corresponding extended method should be efficient. Computational experience suggests that the following method is attractive. Generate $\{x^{(k)}\}$ from

$$y^{(k)}=x^{(k)}-F'(x^{(k)})^{-1}F(x^{(k)}), \tag{6.1}$$

$$z^{(k)}=y^{(k)}-\tfrac{1}{2}F'(x^{(k)})^{-1}F(y^{(k)}), \tag{6.2}$$

$$x^{(k+1)}=y^{(k)}-F'(z^{(k)})^{-1}F(y^{(k)}), \tag{6.3}$$

with $x^{(0)}$ prescribed. This method requires two evaluations and inversions of F', and two evaluations of F and so requires the same computational labour per iteration as two Newton iterations. Whereas, however, the method obtained by combining two Newton iterations has order of convergence four, the method corresponding to (6.1)–(6.3) has order of convergence at least five, as is shown by the following lemma.

Lemma 6.1. *Suppose that* (i) $F\colon D_F\subset X\to X$ *is given and* $\exists x^*\in D_F$, $F(x^*)=0$; (ii) $F\in C^3(S)$ *where* $S=B[x^*,r]\subset D_F$; (iii) $F'(x)^{-1}$ *exists* ($\forall x\in S$), *with*

$$\sup_{x\in S}\|F'(x)^{-1}\|\leqq B; \tag{6.4}$$

(iv) $F'\colon X\to L(X)$ *is such that*

$$\sup_{x\in S}\|F'(x)\|\leqq D; \tag{6.5}$$

(v) $F''\colon X\to L(X,L(X))$ *is such that*

$$\sup_{x\in S}\|F''(x)\|\leqq K; \tag{6.6}$$

(vi) $F'''\colon X\to L(X,L(X,L(X)))$ *is such that*

$$\sup_{x\in S}\|F'''(x)\|\leqq L; \tag{6.7}$$

(vii) $BKr<1$; (viii) $\frac{2}{9}(\frac{1}{3}BL+\frac{289}{64}B^2LD)r^2<1$; (ix) $x^{(0)}\in S$. *Then the sequences* $\{x^{(k)}\}$, $\{y^{(k)}\}$, *and* $\{z^{(k)}\}$ *generated from* (6.1)–(6.3) *respectively lie in S and converge to* x^*. *Furthermore,* ($\forall k\geqq 0$),

$$\|y^{(k)}-x^*\|\leqq\tfrac{1}{2}BK\|x^{(k)}-x^*\|^2, \tag{6.8}$$

$$\|z^{(k)}-x^*\|\leqq\tfrac{9}{16}BK\|x^{(k)}-x^*\|^2, \tag{6.9}$$

and

$$\|x^{(k+1)}-x^*\| \leqq b\|x^{(k)}-x^*\|^5, \tag{6.10}$$

where

$$b=\tfrac{1}{7}B^3K^2[\tfrac{1}{3}L+\tfrac{5}{2}BK^2+\tfrac{289}{64}BLD]. \tag{6.11}$$

Proof. The proof is similar to that of Lemma 5.9 although rather mote tedious, and is therefore omitted. □

The coefficients in hypotheses (vii) and (viii) are not optimal but are chosen for convenience.

Lemma 6.2. *Suppose that* (i) *the hypotheses of Lemma* 6.1 *are valid;* (ii) ω: $X \to X$ *is defined by*

$$\omega(x)=y-\tfrac{1}{2}F'(x)^{-1}F(y), \tag{6.12}$$

where

$$y=x-F'(x)^{-1}F(x); \tag{6.13}$$

(iii) G_0: $X \to X$ *is defined by*

$$G_0(x)=y-F'(\omega(x))^{-1}F(y). \tag{6.14}$$

Then (5.12) *and* (5.13) *hold, with* $a=\frac{9}{16}BK$, $\mu=2$, b *defined by* (6.11), *and* $v=5$. *Furthermore,* $ar^{\mu-1}<1$ *and* $br^{v-1}<1$.

Proof. The proof is immediate from Lemma 6.1. □

Theorem 6.3. *Suppose that* (i) F: $X \to X$ *is given and* $\exists x^* \in X$, $F(x^*)=0$; (ii) $F \in C^3(S)$ *where* $S=B[x^*, r]$; (iii) $F'(x)^{-1}$ *exists* ($\forall x \in S$) *and* (6.4) *holds;* (iv) (6.5) *holds;* (v) (6.6) *holds;* (vi) (6.7) *holds;* (vii) $BKr<\frac{2}{5}$; (viii) $\frac{2}{9}(\frac{1}{3}BL+\frac{289}{64}B^2LD)r^2<1$; (ix) $x^{(0)} \in S$; (x) ω: $X \to X$ *is defined by* (6.12) *and* (5.12) *holds;* (xi) G_0: $X \to X$ *is defined by* (6.14) *and* (5.13) *holds. Then the sequence* $\{x^{(k)}\}$ *generated from* (1.2), (1.3) *lies in* S *and converges to* x^* *with order of convergence at least* $5+2p$. *Moreover,*

$$\|x^{(k+1)}-x^*\| \leqq c_p\|x^{(k)}-x^*\|^{5+2p} \qquad (\forall k \geqq 0),$$

where $c_0=b$ *is defined by* (6.11) *and*

$$c_i=BKc_{i-1}(\tfrac{9}{16}BK+\tfrac{3}{2}c_{i-1}r^{2i+1}) \qquad (1 \leqq i \leqq p).$$

Proof. The proof is immediate from Theorem 3.2 and Lemmas 6.1 and 6.2. □

The method corresponding to (6.1)–(6.3) will be denoted by W1, and its extension by EW1. It is possible to construct several other methods of high order. For example, the method corresponding to

$$y^{(k)}=x^{(k)}-F'(x^{(k)})^{-1}F(x^{(k)}), \tag{6.15}$$

$$z^{(k)} = x^{(k)} - 2(F'(x^{(k)}) + F'(y^{(k)}))^{-1} F(x^{(k)}), \tag{6.16}$$

$$x^{(k+1)} = z^{(k)} - F'(y^{(k)})^{-1} F(z^{(k)}) \tag{6.17}$$

with $x^{(0)}$ prescribed, may be shown to have order of convergence at least five under appropriate conditions. Indeed, it is equivalent to ET with $p=1$. Furthermore, if ω: $X \to X$ is defined by $\omega(x) = x - F'(x)^{-1} F(x)$ and G_0: $X \to X$ is defined by $G_0(x) = y - F'(\omega(x))^{-1} F(y)$ where $y = x - 2(F'(x) + F'(\omega(x)))^{-1} F(x)$, then the extension M_p of the method M_0, defined by (1.2), (1.3) can be shown to have order of convergence at least $5+2p$. The method corresponding to (6.15)–(6.17) will be denoted by W2 and its extension by EW2.

7. Computational Results

In this section some computational results for the methods N (Newton's method), BF, P, T, W1, W2, EBF, EP, ET, EW1, and EW2 are given. The relative computational efficiencies of these methods clearly depend, to some extent, on the kind of equation which is being solved. If the evaluation and inversion of F' is computationally very expensive compared with the evaluation of F, then it may be better to use a method such as EBF which requires one evaluation and inversion of F' per iteration, than it would be to use a method such as ET or EW1 which require two evaluations and inversions of F' per iteration.[1] This is especially true when the initial iterate is so close to the required solution that the convergence criterion is likely to be satisfied after only one iteration of EBF. If, however, the computational labour of evaluating and inverting F' is not much greater than that of evaluating F, then ET, EW1, and EW2 may be preferable to EBF.

If $F(x)=0$ is a system of nonlinear algebraic equations for which Jacobian is sparse, then F' may not be much more computationally expensive to evaluate than is F. Furthermore, tri-diagonal or five-diagonal matrices may, in general, be inverted with less computational labour than may full matrices. Systems of nonlinear algebraic equations with tri-diagonal or five-diagonal Jacobians are of particular importance in connection with the numerical solution of two-point boundary value problems.

The problem of finding a local minimizer of a given nonlinear function f: $\mathbb{R}^n \to \mathbb{R}^1$ with or without constraints involves the solution of a system of nonlinear algebraic equations. Work currently in progress on this problem indicates that extended iterative methods may be used with advantage in a manner similar to that in which Gill and Murray [5] have used Newton's method.

If f: $\mathbb{R}^n \to \mathbb{R}^1$ is a sum of squares of nonlinear functions then several special methods exist for finding local minimizers of f; these methods are usually modifications of Newton's method. An example is the modified damped least squares (MDLS) method of Meyer and Roth [8]. Wolfe [13] has shown how BF may be used to improve the efficiency of MDLS. Gill and Murray [6] have

[1] Method ET actually requires three inversions of F'

Table 1

i	p						
	0	1	2	3	4	5	6
1	W2	EBF EW2	EBF	EBF	EBF	EBF	EBF
2	BF W2	EW1	EW1 EBF	EW1	EW1	EW1	EW1
3	BF	EW1 EW2	EBF	EBF	EBF	EBF	EBF
4	W2	ET EW1	EW1 W2	EW1	EW1	EW1	EW1
5	BF	EW1 EW2	EW1 EW2 ET	EW1 EW2 ET, EBF	EW1 EW2 ET, EBF	EW1 EW2 ET, EBF	EW1 EW2 ET, EBF
6	N T	EW2	ET	ET	/	/	/
7	W1	W1 EBF	W1 EBF	EW1 EW2	EW1 EW2 ET	EW1 EW2 ET	EW1 EW2 ET
8	W1	EBF	EW1 EW2	EW1 EW2 ET	EW1 EW2 ET	EW1 EW2 ET	EBF
9	W1	EBF	EW1 EW2	EW1 EW2 ET	EW1 EW2 ET	EW1 EW2 ET	EW1 EW2 ET
10	BF	BF	EBF	BF	BF EBF	BF	BF

described a modification of the Gauss-Newton method for minimizing a sum of squares of nonlinear functions. Work currently in progress indicates that certain extended iterative methods used in conjunction with the algorithm of Gill and Murray give promising results.

In order to illustrate the relative computational efficiencies of various iterative methods and their extensions, the ten systems of nonlinear algebraic equations given in the appendix to this paper were solved by using methods N, BF, P, T, W1, W2, EBF, EP, ET, EW1, and EW2. For each method, convergence was considered to have been attained when $\|F(x)^{(k)})\| \leqq 10^{-5}$ and $\|u-v\| \leqq 10^{-5}\|v\|$ where u, $v \in \mathbb{R}^n$ are two consecutive estimates of a solution x^*. For method N, $u=x^{(k)}$, and $v=x^{(k+1)}$, while for method W1, possible values of u and v are $u=x^{(k)}$, $v=y^{(k)}$, or $u=y^{(k)}$, $v=z^{(k)}$, or $u=z^{(k)}$, $v=x^{(k+1)}$. Similar possibilities exist for the other methods. All calculations were performed in double precision FORTRAN on the IBM 360/44 computer at St. Andrews. Two indices of computational labour were used, namely n_1 and n_2, where $n_1=n_F+nn_D$ in which n_F is the number of evaluations of F and n_D is the number of evaluations of F' required for convergence, and where $n_2=nn_F+mn_D$ in which m is the

Table 2

i	p						
	0	1	2	3	4	5	6
1	W2	W2 EW2 ET	W2 EW2 ET	W2 EW2 ET	W2 EW2 ET	W2 EW2 ET	W2 EW2 ET
2	W2 BF	EW1	EW1 EBF	EW1	EW1	EW1	EW1
3	BF	EW1 EW2	EBF	EBF	EBF	EBF	EBF
4	W2	EW1 W2 ET	EW1 W2	EW1	EW1	EW1	EW1
5	BF	EW1 EW2	EW1 EW2 ET	EW1 EW2 ET, EBF	EW1 EW2 ET, EBF	EW1 EW2 ET, EBF	EW1 EW2 ET, EBF
6	N T	EW2	ET	ET	/	/	/
7	W1	W1 EBF	W1 EBF	EW1 EW2	EW1 EW2 ET	EW1 EW2 ET	EW1 EW2 ET
8	W1	EBF	EW1 EW2	EW1 EW2 ET	EW1 EW2 ET	EW1 EW2 ET	EW1 EW2 ET
9	W1	EBF	EW1 EW2	EW1 EW2	EW1 EW2 ET	EW1 EW2 ET	EW1 EW2 ET
10	BF	BF	BF	BF	BF	BF	BF

number of elements of the Jacobian F' which require at least one arithmetical operation for their evaluation. If the $n \times n$ matrix F' is tri-diagonal, then $m = 3n - 2$. The index n_1 is adequate when F' is a full matrix, but n_2 is a more informative index when F' is sparse.

Table 1 shows which methods are the most efficient for $p = 0, \ldots, 6$ when n_1 is used as an index of computational labour. Table 2 gives the same information when n_2 is used.

In both Table 1 and Table 2, the index i denotes the relevant system of equations in the order given in the appendix. When, in Tables 1 and 2, several methods are listed under one value of i and one value of p, this means that they require the same amount of computational labour.

Let

$$N_j = \sum_{\substack{i=1 \\ i \neq 6}}^{10} n_{ji} \qquad (j = 1, 2),$$

where n_{ji} is the value of n_j corresponding to system i. Then N_1 and N_2 are indices of total computational labour for the nine systems $i = 1, \ldots, 10$ $(i \neq 6)$.

Table 3

p	0	3
N_1	179	122
N_2	475	358

Table 4

p	0	3
N_1	176	111
N_2	447	302

The results for System 6 are omitted because neither EBF nor EW1 converge for this system when $p \geqq 3$. Tables 3 and 4 show the results for BF and W1 respectively.

For Method N, $N_1 = 166$ and $N_2 = 494$. Therefore the percentage saving in computational labour for EBF relative to N is 16% and that for EW1 is 29%, when $p = 3$. If only systems with a tri-diagonal Jacobian are considered, then EW1 is even more favourable relative to EBF in many cases. For systems 8 and 9, for example, which have tri-diagonal Jacobians, the savings are 15% for EBF and 35% for EW1, relative to N, when $p = 3$. The preceding figures are based on N_2.

For systems of nonlinear equations in up to ten variables, a value of $p = 3$ appears to be satisfactory for all extended methods which have been described. For systems of nonlinear algebraic equations in a larger number of variables, with band-type Jacobians, such as would be encountered when solving nonlinear two point boundary value problems, optimum efficiency may be obtained with larger values of p. It would probably then be desirable to devise a technique for estimating optimal values of p similar to that suggested in [3]. It is not, however, the intention in this section to describe efficient algorithms for solving specific types of problems, but to illustrate the saving in computational labour which can be obtained, for at least one kind of operator equation, when certain extended methods replace Newton's method or the Bosarge-Falb methods.

Appendix

1. $F_1(x) = x_1^2 - x_2$
   $F_2(x) = 1 - x_1$
   $x^{(0)} = (-1.2, 1.0)^T$.

2. $F_1(x) = x_1 - 0.7 \sin x_1 - 0.2 \cos x_2$
   $F_2(x) = x_2 - 0.7 \cos x_1 + 0.2 \sin x_2$
   $x^{(0)} = (0, 0)^T$.

3. $F_1(x) = x_1 - x_1^2 - x_2^2$
   $F_2(x) = x_2 - x_1^2 + x_2^2$
   $x^{(0)} = (0.8, 0.4)^T$.

4. $F_1(x) = x_1 + 13 \log x_1 - x_2^2$
   $F_2(x) = 2 x_1^2 - x_1 x_2 - 5 x_1 + 1$
   $x^{(0)} = (4.0, 4.0)^T$.

5. $F_1(x) = 4 x_1^3 - 27 x_1 x_2^2 + 25$
   $F_2(x) = 4 x_1^2 - 3 x_2^3 - 1$
   $x^{(0)} = (1.0, 1.0)^T$.

6. $F_1(x)=3x_1+x_2+2x_3^2-3$
$F_2(x)=-3x_1+5x_2^2+2x_1x_3-1$
$F_3(x)=25x_1x_2+20x_3+12$
$x^{(0)}=(0,0,0)^T$.

7. $F_1(x)=\frac{1}{2}\sin(x_1x_2)-x_2/4\pi-x_1/2$
$F_2(x)=(1-1/4\pi)\{\exp(2x_1)-e\}+(e/\pi)x_2-2ex_1$
$x^{(0)}=(0.6, 3.0)^T$.

8. $F_1(x)=-x_1(3-x_1/10)+2x_2-1$
$F_i(x)=x_{i-1}-x_i(3-x_i/10)+2x_{i+1}-1 \quad (i=2,\dots,4)$
$F_5(x)=x_4-x_5(3-x_5/10)-1$
$x^{(0)}=(-1,\dots,-1)^T$.

9. $F_1(x)=-x_1(3-x_1/2)+2x_2-1$
$F_i(x)=x_{i-1}-x_i(3-x_i/2)+2x_{i+1}-1 \quad (i=2,\dots,9)$
$F_{10}(x)=x_9-x_{10}(3-x_{10}/2)-1$
$x^{(0)}=(-1,\dots,-1)^T$.

10. $F_1(x)=16x_1^4+16x_2^4+x_3^4-16$
$F_2(x)=x_1^2+x_2^2+x_3^2-3$
$F_3(x)=x_1^3-x_2$
$x^{(0)}=(1,1,1)^T$.

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