Gross, H.
Math. Annalen 150, 285—292 (1963)

On a Special Group of Isometries of an Infinite Dimensional Vectorspace

Professor Dr. B. L. van der Waerden dedicated to his 60th birthday

By

Herbert Gross in Bozeman (Mont.)

In the following note a subgroup of the group of all isometries of an infinite dimensional, non-singular isotropic k-vectorspace E (the characteristic of the field k is assumed to be different from 2) is investigated, viz. the group $\mathfrak{Z}_E$ of all isometries whose restrictions to non-singular subspaces of E of finite codimensions are the identities on those subspaces. The methods are those applied by Eichler[1]) in the case of finite dimension of E, i.e. subgroups of the group $\mathfrak{Z}_E$ generated by special isometries of rather interesting nature are introduced. It seems that by this approach information about the structure of the full group of isometries of E can be obtained without the use of "analytical" methods. For these investigations and a necessary generalization of the concept of dual modules we have to refer to a later paper.

1. Definitions and general remarks

By E we denote an infinite dimensional vectorspace over the field k. The characteristic of k is assumed to be different from 2. Let (x, y) denote the value of a symmetric, non degenerate bi-linear form B from $E \times E$ into k: (i) $(x, y) = (y, x)$, $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$; $\alpha, \beta \in k, x, y, z \in E$; (ii) if $(x, y) = 0$ for all $x \in E$ then $y = 0$. Because of condition (ii) the metric space E is called non-singular. E may contain isotropic vectors, i.e. vectors $x \neq 0$ of norm $(x, x) = 0$ but no vectors $x \neq 0$ orthogonal to the whole space E.

By an automorphism Φ of E we mean a mapping Φ of E onto itself with $\Phi(x + y) = \Phi x + \Phi y$, $\Phi \lambda x = \lambda \Phi x$, $(\Phi x, \Phi y) = (x, y)$; $\lambda \in k$, $x, y \in E$. The group of all automorphisms of E shall be denoted by $\mathfrak{O}_E$ or simply $\mathfrak{O}$ if there is no risk of confusion. For any non-isotropic vector $a \in E$ we denote by Ω_a the reflection about the hyperplane orthogonal to a: $\Omega_a x = x - \frac{2(x,a)}{(a,a)} a$. Obviously $\Omega_a \in \mathfrak{O}_E$.

Often the following lemma will be used in proofs:

Lemma 1. *Every non-isotropic vector $a \in E$ can be completed to a rectangle with sides a, b and non isotropic diagonals $a + b$, $a - b$. Such a rectangle exists in every prescribed 3-dimensional, non-singular subspace of E containing the vector a.*

[1]) M. Eichler, Quadratische Formen und orthogonale Gruppen, Berlin 1952 (Chapter 1).

Proof: If $\overline{E}$ is a finite dimensional, non-singular subspace of E then E is the direct sum of $\overline{E}$ and its orthogonal complement $\overline{E}^{\perp} : E = \overline{E} \oplus \overline{E}^{\perp}$, and the orthogonal complement $\overline{E}^{\perp}$ of $\overline{E}$ is non-singular too (since $\overline{E}$ is of finite dimension Schmidt's orthogonalization process can be applied). Because of the relation $(x, y) = \frac{1}{4}\{(x + y, x + y) - (x - y, x - y)\}$ and since in a non-singular vectorspace by definition not all values (v, w) are zero, there is always a non-isotropic vector in a non-singular vectorspace (this reasoning immediately yields the theorem that every non-singular subspace of finite dimension posesses a basis of non-isotropic, mutually orthogonal vectors). If therefore $\{a\}$ denotes the 1-dimensional, non-singular subspace spanned by the non-isotropic vector a, we have $E = \{a\} \oplus \{a\}^{\perp}$ and $\{a\}^{\perp}$ is non-singular and contains a non-isotropic vector $u : \{a\}^{\perp} = \{u\} \oplus \{u\}^{\perp}$. Let v denote a non-isotropic vector of the non-singular space $\{u\}^{\perp}$. We then have the three vectors a, u, v all of them non-isotropic and mutually orthogonal. Not all of the three vectors $u, v, u + v$ have therefore the same length, for suppose that $(u, u) = (v, v)$, then $(u + v, u + v) = (u, u) + (v, v) = 2(u, u) \neq 0$ since u is non-isotropic and char $k \neq 2$, thus $(u + v, u + v) \neq (u, u)$. Since $u, v, u + v$ are not all of the same length but all orthogonal to a, not all vectors $a + u$, $a + v$, $a + (u + v)$ are of the same length; especially there is a non-isotropic vector among them. The difference between that vector and the vector a (which is either u or v or $u + v$) can be taken as b.

If E is isotropic, i.e. contains an isotropic vector, E can be written as a direct sum $E = E_0 \oplus E_2$, $E_2 \perp E_0$ where E_2 has a basis e_1, e_2 with $(e_1, e_1) = (e_2, e_2) = 0$, $(e_1, e_2) = 1$. Proof: Since E is non-singular there is a vector b to a given isotropic vector a with $(a, b) \neq 0$ and one can solve the equation $(b - \lambda a, b - \lambda a) = 0$. Put $e_1 = a$, $e_2 = \frac{b - \lambda a}{(a, b)}$.

A corollary of this remark is that an isotropic space E has a basis consisting of isotropic vectors only. For, let x be an arbitrary vector in this case. The equation $(x - \lambda e_1 - \mu e_2, x - \lambda e_1 - \mu e_2) = 0$ has always at least one solution, which means that x is the sum of three isotropic vectors:

$$x = (x - \lambda e_1 - \mu e_2) + \lambda e_1 + \mu e_2 .$$

Definition: Let $\Phi \in O_E$. Φ is called an almost identical automorphism if Φ induces an orthogonal decomposition of $E : E = E_{\Phi}^{\perp} \oplus E_{\Phi}$ where E_{Φ} is a non-singular, finite dimensional subspace of E and the restriction of Φ onto $E_{\Phi}^{\perp}$ is the identical automorphism of $E_{\Phi}^{\perp}$.

We want to prove that the product $\Phi \circ \Psi$ of two almost identical automorphisms Φ and Ψ is almost identical again. This is mainly a consequence of the following

Lemma 2: *The space $\overline{E} = \{E_{\Lambda}, E_{\Gamma}\}$ generated by finite dimensional, non-singular subspaces E_{Λ} and E_{Γ} of E is always contained in a finite dimensional subspace of E which is non-singular.*

Proof: All vectors a contained in $\overline{E}$ and perpendicular to the whole space $\overline{E}$ form a subspace of $\overline{E}$, the "radical" of $\overline{E}$. Since $\overline{E}$ is of finite dimension, its radical which we denote by $R_{\overline{E}}$ has a finite basis, say $e_1, \ldots, e_n$, $n = \dim R_{\overline{E}}$. Since the whole space E is non-singular, there exists a vector $a_1 \in E$ such that $(e_1, a_1) \neq 0$. Let E_1 denote the finite dimensional subspace of E generated by $\overline{E}$ and a_1. Clearly $\overline{E} \subset E_1$ and $R_{E_1} \subset R_{\overline{E}}$, and $e_1 \notin R_{E_1}$ since $(e_1, a_1) \neq 0$ whence $R_{E_1} \neq R_{\overline{E}}$. Since the dimension of $R_{\overline{E}}$ is finite, $R_{E_1} \subset R_{\overline{E}}$ and $R_{E_1} \neq R_{\overline{E}}$ entail $\dim R_{E_1} < \dim R_{\overline{E}}$. Thus by repeating the adjunction of conveniently chosen vectors $a_i \in E$ to the space $\overline{E}$ one ends up after at most n ($n = \dim R_{\overline{E}}$) steps with a non-singular vectorspace of finite dimension containing $\overline{E}$.

Let now Φ, Ψ be two almost identical automorphisms, $E = E_{\Phi}^{\perp} \oplus E_{\Phi}$ and $E = E_{\Psi}^{\perp} \oplus E_{\Psi}$ the corresponding decompositions. The space $\overline{E}$ generated by E_{Φ} and E_{Ψ} is contained in a finite dimensional, non-singular subspace $\overline{\overline{E}}$ of E and we have the orthogonal decomposition $E = \overline{\overline{E}}^{\perp} \oplus \overline{\overline{E}}$. Since $\overline{E} \subset \overline{\overline{E}}$ we have $\overline{\overline{E}}^{\perp} \subset \overline{E}^{\perp} = \{E_{\Phi} \cup E_{\Psi}\}^{\perp} \subset E_{\Phi}^{\perp} \cap E_{\Psi}^{\perp}$, which shows that the restriction of the automorphism $\Phi \circ \Psi$ to the space $\overline{\overline{E}}^{\perp}$ is the identical automorphism of $\overline{\overline{E}}^{\perp}$.

Further, if $X \in \mathfrak{O}_E$ the decomposition $E = (XE_{\Phi})^{\perp} \oplus (XE_{\Phi})$ shows that the automorphism $X \circ \Phi X^{-1}$ is almost identical too for every $X \in \mathfrak{O}_E$. Hence

Theorem 1: *The almost identical automorphisms of the space E form an invariant subgroup of $\mathfrak{O}_E$.*

This group will be denoted by $\mathfrak{Z}_E$.

Let $E = E_{\Phi} \oplus E_{\Phi}^{\perp}$ be any admissible decomposition for $\Phi \in \mathfrak{Z}_E$ and let $\Phi^{\perp}$ denote the restriction of Φ to $E_{\Phi}^{\perp}$ and $\{e_1, \ldots, e_n\}$ a basis of the subspace $E_{\Phi}^{\perp}$. To $\Phi^{\perp}$ then corresponds a matrix $(\alpha_k^i): \Phi e_k = \Sigma\, \alpha_k^i e_i\ \alpha_k^i \in k$, whose determinant does not depend on the basis chosen nor on the particular decomposition. For, suppose that we have the two decomposition for Φ: $E = E_{\Phi} \oplus E_{\Phi}^{\perp}$ and $E = \overline{E}_{\Phi} \oplus \overline{E}_{\Phi}^{\perp}$. A third decomposition is $E = \{E_{\Phi}, \overline{E}_{\Phi}\} \oplus (E_{\Phi}^{\perp} \cap \overline{E}_{\Phi}^{\perp})$. A basis of $E_{\Phi}^{\perp} \cap \overline{E}_{\Phi}^{\perp}$ can be completed to a basis of $E_{\Phi}^{\perp}$ and $\overline{E}_{\Phi}^{\perp}$ which shows the independence of the determinant of the decomposition chosen. Since $(\Phi e_i, \Phi e_k) = (e_i, e_k)$ we have $\det(\alpha_k^i) = \pm 1$. If $\det(\alpha_k^i) = 1$ we call Φ a proper automorphism of E. The proper automorphisms of E obviously form an invariant subgroup of $\mathfrak{Z}_E$ of index 2. This group will be denoted by $\mathfrak{Z}_E^+$.

2. The irreducibility of $\mathfrak{Z}_E$

Theorem 2: *The group $\mathfrak{Z}_E$ operates irreducibly on E, i.e. the vectors Φa span the whole space E for any fixed vector $a \in E$, $a \neq 0$, if Φ runs through $\mathfrak{Z}_E$.*

Proof: i) Suppose $(a, a) = 0$. By Witt's Theorem[2]) the space I generated by the vectors Φa, $\Phi \in \mathfrak{Z}_E$ contains all isotropic vectors and therefore also a basis of E. ii) Suppose $(a, a) \neq 0$. Let b be any non-isotropic vector of E orthogonal to a with non-isotropic sum $a + b$. Since $\Omega_{a+b} a = a - \frac{2(a+b, a)}{(a+b, a+b)} \times$ $\times (a + b)$, the subspace I contains both a and b. Suppose now that there exist a vector $c \in E$, $c \notin I$. We may also suppose that c is orthogonal to a and b.

[2]) Let F be a finite dimensional, non-singular vectorspace, F_1 and F_2 two subspaces of F of the same dimension. A necessary and sufficient condition that there exists an orthogonal transformation of F_1 onto F_2 is that the restrictions of the metric form on F to F_1 and F_2 are equivalent. See e.g. J. Dieudonné, Sur les groupes classiques. Paris 1958 (18).

If c is non-isotropic there is a non-isotropic vector e in the two-dimensional subspace spanned by a and b with non-isotropic sum $e + c$ (Lemma 1) and $\Omega_{e+c} a = a - \frac{2(e, a)}{(e + c, e + c)} (e + c)$, whence $c \in I$, which is a contradiction. If c is isotropic then $\Omega_{a+c} a = a - \frac{2(a + c, a)}{(a + c, a + c)} (a + c) = - a - 2c$ gives rise to the same contradiction.

Corollary: *The group $\mathfrak{J}_E^+$ operates irreducibly on E.*

Proof: We show that there is a $\Omega \in \mathfrak{J}_E$ with $\Omega \notin \mathfrak{J}_E^+$ such that $\Omega a = a$. Let $\bar{E}$ be any non-singular finite dimensional subspace of E containing a and let c be a non-isotropic vector orthogonal to $\bar{E}$. We have $\Omega_c a = a$. If therefore a vector $b \in I$ is an image $b = \Phi a$ with $\Phi \notin \mathfrak{J}_E^+$ we have $b = (\Phi \circ \Omega_c) a$ and $\Phi \circ \Omega_c \in \mathfrak{J}_E^+$.

3. Eichler's automorphisms of an isotropic vectorspace

Let E be isotropic. We then have $E = E_0 \oplus E_2$ where the basis e_1, e_2 of E_2 has the properties $(e_1, e_1) = (e_2, e_2) = 0$, $(e_1, e_2) = 1$. To every isotropic vector $a \in E_0$ we form the two automorphisms

$$\Lambda_a^{(1)} = \Omega_{e_1 - e_2 - \frac{a}{2}} \circ \Omega_{e_1 + e_2 + \frac{a}{2}} \circ \Omega_{e_1 + e_2} \circ \Omega_{e_1 - e_2},$$

$$\Lambda_a^{(2)} = \Omega_{e_1 - e_2 + \frac{a}{2}} \circ \Omega_{e_1 + e_2 + \frac{a}{2}} \circ \Omega_{e_1 + e_2} \circ \Omega_{e_1 - e_2}.$$

If $a \in E_0$ is non-isotropic, we define

$$\Lambda_a^{(1)} = \Omega_{a - \frac{(a,a)}{2} e_1} \circ \Omega_a, \quad \Lambda_a^{(2)} = \Omega_{a - \frac{(a,a)}{2} e_2} \circ \Omega_a .$$

The image of a vector $x \in E$ under $\Lambda_a^{(i)}$ $(i = 1, 2)$ is in both cases given by

$$\Lambda_a^{(i)} x = x - (x, a)\, e_i + (x, e_i)\, a - \frac{1}{2} (x, e_i)\, (a, a)\, e_i \tag{1}$$

and for $a, b \in E_0$ the equation $\Lambda_a^{(i)} \circ \Lambda_b^{(i)} = \Lambda_{a+b}^{(i)}$ holds. The automorphisms $\Lambda_a^{(1)}$, $a \in E_0$ and the automorphisms $\Lambda_a^{(2)}$, $a \in E_0$ form therefore two abelian groups $\mathfrak{L}_1, \mathfrak{L}_2$ isomorphic to the additive group E_0. Obviously $\mathfrak{L}_i \subset \mathfrak{J}_E^+$ $(i = 1, 2)$. We further consider the automorphisms

$$P_\varkappa = \Omega_{e_1 + \varkappa e_2} \circ \Omega_{e_1 + e_2} = \Omega_{e_1 - \varkappa e_2} \circ \Omega_{e_1 - e_2} \quad \text{for every } 0 \neq \varkappa \in k . \tag{2}$$

We have $P_\varkappa e_1 = \varkappa^{-1} e_1$, $P_\varkappa e_2 = \varkappa e_2$, $P_\varkappa a = a$, $a \in E_0$ and $P_\varkappa \circ P_\lambda = P_{\varkappa\lambda}$ $(\varkappa, \lambda \in k)$. These automorphisms form a group $\mathfrak{R}$ isomorphic to the multiplicative group of the field k. Finally the automorphism $\Psi = \Omega_{e_1 - e_2}$ interchanges e_1 and e_2 and we have $\Psi \in \mathfrak{J}_E$.

The following relations follow immediately from the above definitions by using trivial facts such as $\Omega_a^{-1} \circ \Omega_b \circ \Omega_a = \Omega_{\Omega_a b}$, $\Omega_a = \Omega_{\lambda a}$ which rule the computations with reflections.

$$\begin{gathered} \Psi^{-1} \circ \Lambda_a^{(1)} \circ \Psi = \Lambda_a^{(2)}, \ \Psi^{-1} \circ P_\varkappa \circ \Psi = P_{\varkappa^{-1}}, \ P_\varkappa \circ \Lambda_a^{(1)} \circ P_\varkappa^{-1} = \Lambda_{\varkappa a}^{(1)}, \\ P_\varkappa^{-1} \Lambda_a^{(2)} P_\varkappa = \Lambda_{\varkappa a}^{(2)}, \ \Lambda_a^{(2)} \circ \Psi = \Lambda^{(1)}_{-\frac{2a}{(a,a)}} \circ \Lambda_{-a}^{(2)} \circ \Omega_a \circ P_{-\frac{(a,a)}{2}} \end{gathered} \tag{3}$$

(for non-isotropic a.)

We next want to show that the subgroup $\mathfrak{L}$ of $\mathfrak{J}_E^+$ generated by the groups $\mathfrak{L}_1$ and $\mathfrak{L}_2$ is the commutator subgroup of $\mathfrak{J}_E^+$. In paragraph 6 we will discuss the structure of the factor group $\mathfrak{J}_E^+/\mathfrak{L}$.

4. The connection between $\mathfrak{J}_E$ and $\mathfrak{J}_{E_0}$

We show that by multiplication of an arbitrary automorphism $\Phi \in \mathfrak{J}_E$ ($E = E_0 \oplus E_2$) by elements of the groups $\mathfrak{L}$ and $\mathfrak{R}$ an automorphism can be obtained whose restriction to E_2 is the identity of E_2.

Let i be the number 1 or 2. A vector x which is not orthogonal to e_i is mapped into E_2 by $\Lambda^{(i)}_{-\frac{x}{(x,e_i)}}$ as can be seen from (1). Suppose x to be an isotropic vector of E with $(x, e_i) \neq 0$ and $\Lambda_a^{(i)} x = \alpha_1 e_1 + \alpha_2 e_2$. Since $(\Lambda_a^{(i)} x, \Lambda_a^{(i)} x) = (x, x) = 2\alpha_1\alpha_2 = 0$ and $0 \neq (x, e_i) = (\Lambda_a^{(i)} x, \Lambda_a^{(i)} e_i) = (\Lambda_a^{(i)} x, e_i) = \alpha_j$ we have $\alpha_i = 0$ $(j \neq i)$: An isotropic vector x not orthogonal to e_i is mapped onto a multiple of e_j $(j \neq i)$ by some $\Lambda_a^{(i)}$. If therefore $\Phi \in \mathfrak{J}_E$ is an automorphism with $(e_1, \Phi e_2) \neq 0$ there is a $\Lambda_a^{(1)}$ such that $\Lambda_a^{(1)} \circ \Phi e_2 = \lambda e_2$. Since also $(e_2, \Lambda_a^{(1)} \circ \Phi e_1) = \frac{1}{\lambda} (\Lambda_a^{(1)} \circ \Phi e_2, \Lambda_a^{(1)} \circ \Phi e_1) = \frac{1}{\lambda} (e_2, e_1) = \frac{1}{\lambda} \neq 0$, there is a $\Lambda_b^{(2)}$ such that $\Lambda_b^{(2)}(\Lambda_a^1 \circ \Phi e_1) = \mu e_1$. Therefore, since $\Lambda_a^{(i)} e_i = e_i$ we have $(\Lambda_b^{(2)} \circ \Lambda_a^{(1)} \circ \Phi)\, e_1 = \mu e_1$, $(\Lambda_b^{(2)} \circ \Lambda_a^{(1)} \circ \Phi)\, e_2 = \lambda e_2$, whence $\lambda\mu = 1$. The restriction of the automorphism $P_\mu \circ \Lambda_b^{(2)} \circ \Lambda_a^{(1)} \circ \Phi$ to the subspace E_2 therefore is the identity on E_2. If $\Phi \in \mathfrak{J}_E$ is an automorphism with $(e_1, \Phi e_2) = 0$ there is a vector $c \in E_0$ with $(e_1, \Lambda_{-c}^{(2)} \circ \Phi \circ \Lambda_c^{(2)} e_2) \neq 0$. For, suppose that for every $c \in E_0$ $(e_1, \Lambda_{-c}^{(2)} \circ \Phi \circ \Lambda_c^{(2)} e_2) = (\Lambda_c^{(2)} e_1, \Phi e_2) = \left(c + e_1 - \frac{1}{2}(c, c)\, e_2, \Phi e_2\right) = 0$, then we conclude that $\Phi e_2 \in E_2$ since the vectors $\Lambda_c^{(2)} e_1, c \in E_0$ span the whole space E_0. But if $\Phi e_2 = \alpha e_1 + \beta e_2$ we have for every non-isotropic $c \in E_0$ $\left(c + e_1 - \frac{1}{2}(c, c)\, e_2, \Phi e_2\right) \neq \left(-c + e_1 - \frac{1}{2}(-c, -c)\, e_2, \Phi e_2\right)$ since char $k \neq 2$, which is a contradiction. If we denote by $\mathfrak{J}_{E_0}$ the subgroup of $\mathfrak{J}_E$ containing all automorphisms of $E = E_0 \oplus E_2$ whose restrictions to E_2 are the identity on E_2, we have the result:

Theorem 3: *To every automorphism $\Phi \in \mathfrak{J}_E$, there exist automorphisms $\Lambda_a^{(1)}, \Lambda_b^{(2)}, \Lambda_c^{(2)} \in \mathfrak{L}$ $P_\varkappa \in \mathfrak{R}$, $\Phi_0 \in \mathfrak{J}_{E_0}$ such that*[3])

$$\Phi = \Lambda_c^{(2)} \circ \Lambda_a^{(1)} \circ \Lambda_b^{(2)} \circ \Phi_0 \circ P_\varkappa \circ \Lambda_{-c}^{(2)}. \tag{4}$$

5. The commutator subgroup of $\mathfrak{J}_E$ ($\mathfrak{J}_E^+$)

The reflections Ω_a generate the group $\mathfrak{J}_E$ and Ω_a^2 is the identity, from which one concludes simply by group theoretical arguments that the commutator subgroup of $\mathfrak{J}_E$ is generated by the special squares $(\Omega_a \circ \Omega_b)^2 = \Omega_a \circ \Omega_b \circ \Omega_a^{-1} \circ \Omega_b^{-1}$ and that it contains all squares Φ^2, $\Phi \in \mathfrak{J}_E$. The commutator subgroup is therefore equally generated by all squares Φ^2, $\Phi \in \mathfrak{J}_E$ since a

[3]) Or $\Lambda_u^{(2)}, \Lambda_v^{(1)}, \Lambda_w^1, P_\sigma$; $\Sigma_0 \in \mathfrak{J}_{E_0}$ such that $\Phi = \Lambda_w^{(1)} \circ \Lambda_u^{(2)} \circ \Lambda_v^{(1)} \circ \Sigma_0 \circ P_\sigma \circ \Lambda_{-w}^{(1)}$.

commutator always is a product of squares. The commutator subgroup of $\mathfrak{Z}_E^+$ is also the commutator subgroup of $\mathfrak{Z}_E$[4]).

Since $\Lambda_a^{(i)} = \Lambda^{(i)}_{\frac{a}{2}+\frac{a}{2}} = (\Lambda^{(i)}_{\frac{a}{2}})^2$ the group $\mathfrak{L} = \{\mathfrak{L}_1, \mathfrak{L}_2\}$ is contained in the commutator subgroup of $\mathfrak{Z}_E^+$. To show that it is equal to that group reduces to showing that every square Φ^2, $\Phi \in \mathfrak{Z}_E^+$ is contained in $\mathfrak{L}$. To this end we apply (3) to the identity $\Lambda_a^{(2)} \circ \Lambda_b^{(1)} = \Lambda_a^{(2)} \circ \Psi^{-1} \circ \Lambda_b^{(2)} \circ \Psi$ with non-isotropic $a \in E_0$, $b \in E_0$ and obtain $\Lambda_a^{(2)} \circ \Lambda_b^{(1)} = (\Lambda^{(1)}_{-\frac{2}{(a,a)}} \circ \Lambda^{(2)}_{-a} \circ \Omega_a \circ P_{-\frac{(a,a)}{2}}) \circ \Lambda_b^{(2)} \circ \Psi$ $= \Lambda^{(1)}_{\frac{-2}{(a,a)}} \circ \Lambda_d^{(2)} \circ \Psi \circ \Omega_a \circ P_{\frac{-2}{(a,a)}}$ for $d = (1 - (a, b))\, a + \frac{1}{2}(a, a)\, b \in E_0$.
If d is non-isotropic we substitute again for $\Lambda_d^{(2)} \circ \Psi$ in the last equation and obtain

(5) $\Lambda_a^{(2)} \circ \Lambda_b^{(1)} = \Lambda_e^{(1)} \circ \Lambda_d^{(2)} \circ \Omega_d \circ \Omega_a \circ P_{1-(a,b)+\frac{1}{4}(a,a)(b,b)}$

or

(6) $\Omega_d \circ \Omega_a \circ P_\varkappa = \Lambda^{(2)}_{-d} \circ \Lambda^{(1)}_{-e} \circ \Lambda_a^{(2)} \circ \Lambda_b^{(1)}$, $\varkappa = 1 - (a,b) + \frac{1}{4}(a,a)(b,b) = \frac{(d,d)}{(a,a)}$

for some vector e depending on a and b. Since we can solve with respect to b and e for prescribed non-isotropic d and a in E_0, all automorphisms $\Omega_d \circ \Omega_a \circ P_\varkappa$ are contained in $\mathfrak{L}$:

$$\Omega_d \circ \Omega_a \circ P_\varkappa \in \mathfrak{L}, \quad \varkappa = \frac{(d,d)}{(a,a)}\,. \tag{7}$$

From (7) we conclude as a partial result that $\mathfrak{L}$ contains the commutator subgroup of the group of automorphisms $\Phi \in \mathfrak{Z}_E$ which leave the two subspaces E_0, E_2 invariant: $(\Omega_d \circ \Omega_a \circ P_\varkappa)^2 = (\Omega_d \circ \Omega_a)^2 \circ P_{\varkappa^2} \in \mathfrak{L}$ for arbitrary non-isotropic $a, d \in E_0$. We also have $\Omega_{\mu e_1 + \lambda e_2} = \Psi \circ P_{\frac{\mu}{\lambda}}$ for non-isotropic $\mu e_1 + \lambda e_2$ i.e. $\lambda\mu \neq 0$). Every product $\Omega_a \circ \Omega_b$ for non-isotropic vectors $a, b \in E_2$ therefore is of the form $\Omega_a \circ \Omega_b = P_\nu$, hence $(\Omega_a \circ \Omega_b)^2 = P_{\nu^2}$, $a, b \in E_2$. But the subgroup $\mathfrak{R}_2$ of $\mathfrak{R}$ generated by all $P_{\varkappa^2}$, $\varkappa \in k$ is contained in $\mathfrak{L}$ as can be seen from (5) in the special case $a = \lambda b$;

$$P_{\nu^2} \in \mathfrak{L},\ \nu \in k\,. \tag{8}$$

This proves our remark.

Let now Φ be an arbitrary element of $\mathfrak{Z}_E^+$. According to (4) we have $\Phi = \Lambda_c^{(2)} \circ \Lambda_a^{(1)} \circ \Lambda_b^{(2)} \circ \Phi_0 \circ P_\varkappa \circ \Lambda^{(2)}_{-c}$ for certain vectors $a, b, c \in E_0$ and $\varkappa \in k$. Since $\Lambda_a^{(i)}$, $P_\varkappa \in \mathfrak{Z}_E^+$ the automorphism Φ_0 is a product of an even number of reflections, $\Phi_0 = \Omega_{a_1} \circ \cdots \circ \Omega_{a_{2m}}$. According to (7) the automorphisms $\Sigma_i = \Omega_{a_{2i-1}} \circ \Omega_{a_{2i}} \circ P_{\nu_i}$ with $\nu_i = \frac{(a_{2i-1}, a_{2i-1})}{(a_{2i}, a_{2i})}$ $(i = 1, \ldots, m)$ are elements of $\mathfrak{L}$. We write $\Phi = \Lambda_c^{(2)} \circ \Lambda_a^{(1)} \circ \Lambda^{(2)}_{b-c} \circ (\Lambda_c^{(2)} \circ \Sigma_1 \circ \Lambda^{(2)}_{-c}) \circ \cdots \circ (\Lambda_c^2 \circ \Sigma_m \circ \Lambda^2_{-c}) \circ P_\lambda$ with $\lambda = \frac{\varkappa}{\nu_1 \cdots \nu_m}$ and have thus the

Theorem 4: *An element Φ of the group $\mathfrak{Z}_E^+$ can always be written in the form*

$$\Phi = \Phi_* \circ P_\lambda,\ \Phi_* \in \mathfrak{L}\,. \tag{9}$$

[4]) See e.g. J. DIEUDONNÉ, loc. cit. (23).

By Theorem 4 and (8) we see that for every $\Phi \in \mathfrak{J}_E^+$ the square Φ^2 is an element of $\mathfrak{L}$ ($\Phi^2 = \Phi_* \circ P_\lambda \circ \Phi_* \circ P_\lambda = \Phi_* \circ (P_\lambda \circ \Phi_* \circ P_\lambda^{-1}) \circ P_{\lambda^2}$), whence

Theorem 5: *The commutator subgroup of $\mathfrak{J}_E^+$ ($\mathfrak{J}_E$) is generated by the automorphisms $\Lambda_a^{(1)}$, $\Lambda_a^{(2)}$, a varying over E_0.*

Dieudonné proved that for an isotropic non-singular k-vectorspace F of finite dimension n the commutator subgroup $\mathfrak{K}_n$ of the group $\mathfrak{O}_F$ has no invariant proper subgroup besides $\mathfrak{K}_n \cap \mathfrak{Z}_n$ where the center $\mathfrak{Z}_n$ of $\mathfrak{O}_F$ consists of the identity 1 and the reflection -1, except in the following three cases: a) $n = 2$, b) $n = 3$ and k is the prime field of three elements, c) $n = 4$ and $E = E_2 \oplus E_2$ (E_2 as defined on page 2).

Since the subgroup of $\mathfrak{J}_E^+$ of all automorphisms of E whose restrictions to the orthogonal complement E'' of a finite dimensional non-singular subspace $E' \subset E$ are the identity on E'', is isomorphic to $\mathfrak{O}_{E'}$, the above mentioned theorem entails in an obvious manner the

Theorem 6: *The commutator subgroup of $\mathfrak{J}_E^+$ ($\mathfrak{J}_E$) is simple.*

Note that the center of $\mathfrak{J}_E$ consists only of the identical automorphism of E. We thus have found all invariant subgroups of the group $\mathfrak{J}_E$.

6. The factor group $\mathfrak{J}_E^+/\mathfrak{L}$

In the field k we define an equivalence relation by declaring $\varkappa \sim \nu$, $\varkappa, \nu \in k$ if and only if $\varkappa\nu^{-1}$ is a square in k. By g_k we denote the multiplicative group whose elements are the equivalence classes. The element λ in (9) cannot be uniquely determined by Φ since $P_{\varkappa^2} \in \mathfrak{L}$. But we shall prove that the equivalence class of λ in (9) is uniquely determined by Φ.

In order to prove this assertion we make use of the representation of $\mathfrak{J}_E^+$ in the Clifford algebras $C(B)$ and $C_+(B)$ associated with our metric form B. The Clifford algebra $C(B)$ is the factor algebra of the tensor algebra $T = \sum_0^\infty \otimes^n E$ modulo the two-sided ideal I generated by the elements of the form $x \otimes x - \frac{1}{2} B(x, x)\, 1$ (1 stands for the neutral element of k). The image of $x \otimes y$ under the canonical mapping will be denoted by $x \circ y$. Elements $\varkappa \in k$ and vectors $x \in E$ will be identified with their images under the canonical mapping into $C(B)$. For the values of our metric form we now write more precisely $B(x, y)$ instead of (x, y). We have $x \circ y + y \circ x = B(x, y)\, 1$. If T_+ is the algebra $\sum_0^\infty \otimes^{2n} E \subset T$ we denote by $C_+(B)$ the factor algebra $T_+/T_+ \cap I$.

Let $E = E_\Phi \oplus E_\Phi^\perp$ be an admissible decomposition of E for the automorphism $\Phi \in \mathfrak{J}_E^+$, $\{e_1, \ldots, e_n\}$ an orthogonal basis of $E_\Phi^\perp$ and $\{e_\tau\}_{\tau \in J}$, where J is ordered, a basis of E_Φ. Let further B_1, B_2 denote the restrictions of B to the subspaces E_Φ, $E_\Phi^\perp$ respectively. The elements $e_K = e_{\tau_1} \circ \cdots \circ e_{\tau_m}$, $\tau_1 < \tau_2 \cdots < \tau_m$, corresponding to the finite subsets K of J form a basis of $C(B_1)$. Let i_1, i_2 denote the canonical mappings of $C(B_1)$, $C(B_2)$ into $C(B)$. The bilinear mapping $(a, b) \to (i_1 a) \circ (i_2 b)$ from $C(B_1) \times C(B_2)$ into $C(B)$ induces a linear bijection p from $C(B_1) \otimes C(B_2)$ onto $C(B)$. If we denote the product of two elements $x, y \in C(B_1) \otimes C(B_2)$ induced by the bijection p by $x \circ y$ too, we have $(x_1 \otimes x_2) \circ (y_1 \otimes y_2) = \varepsilon(x_1 \circ y_1) \otimes (x_2 \circ y_2)$ where $\varepsilon = 1$ except when both x_2 and y_1 are odd, i.e. images of elements of $T_- = \sum_0^\infty \otimes^{2n+1} E$, in which case $\varepsilon = -1$.

Every $\Phi \in \mathfrak{Z}_E^+$ induces an automorphism φ of $C(B)$ and $C_+(B)$ by the definition $\varphi(x_1 \circ \cdots \circ x_l) = (\Phi^{-1} x_1) \circ \cdots \circ (\Phi^{-1} x_l)$. If a is a non-isotropic vector, a has an inverse in $C(B): a^{-1} = \frac{2}{B(a, a)} a$ and one finds $a^{-1} \circ x \circ a = \frac{2}{B(a,a)} a \circ x \circ a = \frac{2}{B(a,a)} (-x \circ a \circ a + x \circ a \circ a + a \circ x \circ a) = \frac{2}{B(a,a)} \times \times \left(B(a, x)\, a - x \frac{B(a,a)}{2}\right) = -\left(x - \frac{2\, B(a,x)}{B(a,a)}\, a\right) = -\,\Omega_a x$ whence: To every automorphism $\Phi \in \mathfrak{Z}_E^+$ there exists an invertible element $t_\Phi \in C^+(B)$ with $p^{-1}(t_\Phi) \in C(B_1)$ such that $\varphi(x_1 \circ \cdots \circ x_l) = (\Phi^{-1} x_l) \circ \cdots \circ (\Phi^{-1} x_l) = t_\Phi^{-1} \circ \circ (x_1 \circ \cdots \circ x_l) \circ t_\Phi$. Let t_Φ and t'_Φ be two elements with that property. $t'_\Phi \circ t_\Phi^{-1}$ then commutes with every vector $a \in C(B)$, it therefore commutes with every element of $C(B)$ and $C_+(B)$. $p^{-1}(t'_\Phi \circ t_\Phi^{-1})$ therefore is contained in the centers of $C(B_2)$ and $C_+(B_2)$ whose intersection is k for even and odd dimensions of $E_\Phi^\perp$.

If α is the main antiautomorphism of $C_+(B_2)$[5] we assign to every $\Phi \in \mathfrak{Z}_E^+$ its spinor norm $L(\Phi) = \alpha(p(t_\Phi)) \circ p(t_\Phi)$. Since t_Φ is uniquely determined by Φ up to a factor out of k, the spinor norm is a mapping from $\mathfrak{Z}_E^+$ into g_k. We have $L(\Phi \circ \Psi) = L(\Phi)\, L(\Psi)$; $\Phi, \Psi \in \mathfrak{Z}_E^+$.

From the definitions of $\Lambda_a^{(i)}$ and $P_\varkappa$ as products of reflections (see p. 288) we immediately read off: $L(\Lambda_a^{(i)}) = 1$ for every $a \in E_0$ and $L(P_\varkappa) = \varkappa$. All elements of the commutator subgroup of $\mathfrak{Z}_E^+$ have the spinor norm 1, whence by (9) we obtain the

Theorem 7: *The factor group $\mathfrak{Z}_E^+/\mathfrak{L}$ is isomorphic to the group g_k.*

(Received December 12, 1962)

[5]) See e.g. C. C. CHEVALLEY, The Algebraic Theory of Spinors, New York 1955 (38).