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On the Asymptotic Stability of θ-Methods for Delay Differential Equations

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Summary. Stability regions of θ-methods for the linear delay differential test equations

$$\begin{aligned} y'(t) &= p\,y(t) + q\,y(t-\tau), \qquad && t>0,\\ y(t) &= \varphi(t), && t\in[-\tau, 0], \end{aligned}$$

where τ is a positive constant, are presented. In the case that p and q are real constant coefficients, necessary and sufficient conditions on the stepsize for the stability of a θ-method are obtained. Furthermore, when p and q are complex coefficients, sufficient conditions for the stability of the θ-methods are also given.

Subject Classifications: AMS(MOS): 65L20; CR: G1.7.

1. Introduction

In recent years, there has been a growing interest in numerical methods for the solution of initial value problems for delay differential equations (DDE). This is due to the fact that these equations arise in several fields of applied mathematics such as biomathematics, physics and control theory (see [3, 7–9, 11]).

Many numerical methods that were originally designed for solving initial value problems in ordinary differential equations have been adapted to solve DDEs of type

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)), \qquad && t>0,\\ y(t) &= \varphi(t), && t\in[-\tau, 0], \end{aligned} \tag{1.1}$$

where $\tau>0$ is a constant delay and φ is a given initial function ([15–18]). However, several difficulties occur in the application of usual Runge-Kutta or lineal multistep methods to the Eq. (1.1). First of all, as these methods require at each step one or more evaluations of the right hand side of the differential equation, approximations of the solution at several values of the retarded argument $t-\tau$ are needed. When $t-\tau$ does not coincide with a previous grid point, $y(t-\tau)$ is usually computed by an interpolation process that uses the values of the solution at neighbouring grid points. In this sense, it can be proved ([10, 19]) that if the original ODE method is consistent of order p, it is enough to take an interpolation process of order $p-1$ to have a method of order p. Further effects of the interpolation and jump discontinuities on the solution can be seen in the papers [1, 15], and [16].

Secondly, the stability behaviour of a numerical method can be quite different for an ordinary differential equation or a DDE equation. Therefore, a stability analysis of the numerical methods applied to solve DDEs is necessary. To study the asymptotic stability of a numerical method, it is usual to consider a family of differential equations (the test equations), comparing the behaviour of the analytical and numerical solutions of this family of equations. Thus, Cryer [6] studies the stability using the pure delay test equation

$$\begin{aligned} y'(t) &= \mu\, y(t-\tau), && t>0, \\ y(t) &= \varphi(t), && t\in[-\tau, 0], \end{aligned} \tag{1.2}$$

where μ is a real parameter and φ a given continuous function. Further stability results about multistep methods for the delay test Eq. (1.2), have been obtained by Van der Houwen and Sommeijer [20]. Other authors including Barwell [2], Bickart [4], Capdeville and Seguier [5], Jackiewicz [12], Wiederholt [22] and Watanabe and Roth [21] have studies the stability of some numerical methods with respect to the more general class of test equations

$$\begin{aligned} y'(t) &= p\, y(t) + q\, y(t-\tau), && t\geqq 0, \\ y(t) &= \varphi(t), && t\in[-\tau, 0], \end{aligned} \tag{1.3}$$

where p and q are real or complex constant coefficients.

Let us recall that the asymptotic behaviour of the exact solutions of (1.3) for real or complex p and q has been considered by several authors (see [3, 10, 18]). In the real case, the stability set S^*, i.e. the set of all values (p, q) such that, for any given continuous function $\varphi(t)$, the solution $y(t)$ of (1.3) tends to zero as t tends to infinity, is an open domain in the (p, q)-plane contained in the halfplane $p<1/\tau$ and determined by the straight line $q=-p$ and the curve $p=\varphi \cot(\varphi\tau)$, $q=-\varphi/\sin(\varphi\tau)$, $\varphi\in(0, \pi/\tau)$. However, it is difficult to analyze the stability of many numerical methods for all $(p, q)\in S^*$ and therefore it is usual to restrict this analysis to the convex cone $D^*=\{(p, q)\in R^2/|q|<-p\}$ which is a subset of S^*. In the complex case, also for the sake of simplicity, the stability studies are usually restricted to the complex set $D=\{(p, q)\in C^2/|q|<-\operatorname{Re} p\}$, which is contained in the complex stability region.

The aim of this paper is to analyze the asymptotic stability of θ-methods for the test Eq. (1.3), with the delay terms computed by linear interpolation. A brief outline of the rest of this paper is as follows: Sect. 2 is devoted to introduce the definitions and notations. In Sect. 3, some sufficient conditions for the stability of a θ-method applied to the complex test equation are given.

It must be remarked that Theorem 1 of this section, for the particular case of p and q real, modifies partially previous results on this subject (see Jackiewicz [12]). Finally, in Sect. 4 we prove that for the real test equation with $(p, q) \in S^*$, the conditions of Sect. 3 are not only necessary but also sufficient for the asymptotic stability of the θ-methods. For the complex test equation, a rigorous proof of the same fact has not been obtained. However, a large number of numerical experiments seem to confirm that the sufficient conditions will be also necessary in the complex case.

2. Definitions and Notations

Let $t_n = n\,h$, $n = 0, 1, \dots$ be a uniform grid with constant stepsize h and denote by $y_h(t_n)$ the numerical solution at the grid point t_n obtained when a θ-method is applied to the test Eq. (1.3) with stepsize h. The values $y_h(t_n)$ satisfy the difference equation

$$\begin{aligned} y_h(t_{n+1}) = y_h(t_n) + h[\theta(p\, y_h(t_{n+1}) + q\, y_h(t_{n+1} - \tau)) \\ + (1-\theta)(p\, y_h(t_n) + q\, y_h(t_n - \tau))]. \end{aligned} \tag{2.1}$$

If $t_n - \tau$ and $t_{n+1} - \tau$ are not grid points, the delay terms $y_h(t_n - \tau)$ and $y_h(t_{n+1} - \tau)$ will be evaluated by means of interpolation using an appropriate number of function values at grid points so that the order of the interpolation error does not modify the order of the discretization error of the θ-method. Since the θ-method has order two when $\theta = 1/2$ (trapezoidal rule) and order one for the other values of $\theta \in [0, 1]$, it is sufficient to compute the delay terms by linear interpolation between the two neighbouring points. Therefore, putting $\tau = (m-u)\,h$, with $u \in [0, 1)$ and m a positive integer we have

$$\begin{aligned} y_h(t_n - \tau) = (1-u)\, y_h(t_{n-m}) + u\, y_h(t_{n-m+1}), \\ y_h(t_{n+1} - \tau) = (1-u)\, y_h(t_{n-m+1}) + u\, y_h(t_{n-m+2}). \end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), we arrive to the difference equation

$$\begin{aligned} (1-\theta x)\, y_h(t_{n+1}) - (1 + (1-\theta)\, x)\, y_h(t_n) - \theta\, u\, y\, y_h(T_{n-m+2}) + \\ -(\theta + u - 2\theta u)\, y\, y_h(t_{n-m+1}) - (1-\theta)(1-u)\, y\, y_h(t_{n-m}) = 0, \end{aligned} \tag{2.3}$$

where $x = hp$ and $y = hq$.

From the stability theory of difference equations, it is known [14] that the solutions $y_h(t_n)$ of (2.3) tend to zero as t_n tends to ∞ for any given starting values if and only if the characteristic polynomial $W_m(z)$ of (2.3),

$$\begin{aligned} W_m(z) = (1-\theta x)\, z^{m+1} - (1 + (1-\theta)\, x)\, z^m - \theta u y\, z^2 + \\ -(\theta + u - 2\theta u)\, y\, z - (1-\theta)(1-u)\, y, \end{aligned} \tag{2.4}$$

is a Schur polynomial, i.e. if all its roots are inside the open unit disc. Consequently, given some values of p, q and h, the θ-method is asymptotically stable if and only if the corresponding polynomial $W_m(z)$ is a Schur polynomial.

Since the coefficients and degree of $W_m(z)$ are stepsize dependent, for each $h>0$ we have a stability set of a θ-method defined as the set of all (p, q)-values such that $W_m(z)$ is a Schur polynomial. However, as the coefficients of $W_m(z)$ depend on (p, q) in the form $(hp, hq)=(x, y)$, it is convenient to introduce the following:

Definition. For a fixed stepsize $h>0$ and $\theta\in[0, 1]$ the set

$$S_{\theta,h}=\{(x, y)\in C^2/W_m(z) \text{ is a Schur polynomial}\},$$

will be called the scaled complex stability set of the θ-method for the stepsize h. In addition, the set

$$S_\theta=\bigcap_{h>0} S_{\theta,h},$$

will be called the complex stability set of the θ-method. In the particular case that x and y are real variables, the corresponding sets will be denoted by $S^*_{\theta,h}$ and S^*_θ respectively.

3. Stability Regions for the Complex Test Equation

In this section we find some stability regions for the θ-methods when they are applied to the complex test Eq. (1.3) with $(p, q)\in D=\{(x, y)\in C^2/|y|<-\mathrm{Re}(x)\}$, this means that $x=hp$, $y=hq$ are complex variables and $(x, y)\in D$. To simplify the proof of the main result of this section, we give the following:

Lemma 1. *Let a, b, and c be real constants. Then the inequality*

$$a+b\,\omega>|c|(1-\omega^2)^{1/2}, \tag{3.1}$$

holds for all $\omega\in[-1, 1]$, *if and only if*

$$a>0 \quad and \quad a^2-b^2>c^2. \tag{3.2}$$

Proof. First we show that (3.1) implies (3.2). Taking $\omega=0$ in (3.1) we get $a>|c|$ and clearly $a>0$.

To show that $a^2-b^2>c^2$, let $r(\omega)=a+b\omega$ and $s(\omega)=|c|(1-\omega^2)^{1/2}$. The assumption (3.1) implies that the quadratic polynomial

$$P(\omega)=r^2(\omega)-s^2(\omega)=(b^2+c^2)\,w^2+2ab\omega+(a^2-c^2),$$

satisfies $P(\omega)>0$ for $\omega\in[-1, 1]$. It can be easily verified that $P(\omega)$ attains its minimum when $\omega=\omega^*=-ab/(b^2+c^2)$ and takes the value $P(\omega^*)=c^2(a^2-b^2-c^2)/(b^2+c^2)$. Hence, if $|\omega^*|\leq 1$ the assumption (3.1) implies that $P(\omega^*)>0$, and therefore $a^2-b^2>c^2$. For $|\omega^*|>1$, i.e. $|ab|>b^2+c^2$, putting in (3.1) $\omega=1$ and $\omega=-1$, it follows that $a>|b|$, and consequently

$$a^2>a|b|=|ab|>b^2+c^2.$$

Conversely, to prove that (3.1) follows from (3.2), note that $a^2-b^2>c^2$ implies $P(\omega)>0$ for all real ω and in particular,

$$r^2(\omega)>s^2(\omega) \quad \text{for } \omega\in[-1,1]. \tag{3.3}$$

Furthermore, from (3.2) we deduce $a>|c|$, i.e. $r(0)>s(0)$. This condition together with (3.3) and the continuity of $r(\omega)$ and $s(\omega)$ imply that $r(\omega)>s(\omega)$ for all $\omega\in[-1,1]$, and the proof is complete.

Theorem 1. *The asymptotic stability set* S_θ *of the* θ*-methods for the complex test Eq.* (1.3) *with linear interpolation satisfy*

$$S_\theta\cap D=D, \qquad \text{for } \theta\in[1/2,1],$$
$$S_\theta\cap D\supset D_\theta, \qquad \text{for } \theta\in[0,1/2),$$

where D_θ, $\theta\in[0,1/2)$, *is the set defined by*

$$D_\theta=\{(x,y)\in C^2/|x+1/\eta|+|y|<1/\eta\}, \qquad \eta=1-2\theta.$$

Proof. Consider the characteristic polynomial (2.4) of (2.3). It can be written in the form

$$W_m(z)=P_m(z)-Q(z),$$

with

$$P_m(z)=z^m(a_1 z+a_0), \qquad Q(z)=y(b_2 z^2+b_1 z+b_0),$$

and

$$\begin{aligned} &a_1=1-\theta x, && a_0=-(1+(1-\theta)x), \\ &b_2=\theta u, && b_1=\theta+u-2\theta u, \qquad b_0=(1-\theta)(1-u). \end{aligned} \tag{3.4}$$

Applying Rouche's Theorem [13], it follows that the conditions

$$|a_0| < |a_1| \tag{3.5}$$
$$|Q(\exp(i\varphi))|<|P_m(\exp(i\varphi))|, \qquad \varphi\in[0,2\pi),\ u\in[0,1), \tag{3.6}$$

imply that W_m is a Schur polynomial.

Since x is a complex variable, using (3.4) it is easy to show that (3.5) is equivalent to

$$\begin{aligned} &|x+1/\eta|<1/\eta, && \theta\in[0,1/2), \\ &\operatorname{Re} x<0, && \theta=1/2, \\ &|x+1/\eta|>-1/\eta, && \theta\in(1/2,1]. \end{aligned} \tag{3.7}$$

To study the inequality (3.6) we introduce the functions

$$\begin{aligned} f(\varphi)&=|P_m(\exp(i\varphi))|^2=|a_1\exp(i\varphi)+a_0|^2, \\ g(\varphi;u)&=|Q(\exp(i\varphi))|^2=|y|^2\, b_2\exp(iz\varphi)+b_1\exp(i\varphi)+b_0|^2, \end{aligned} \tag{3.8}$$

defined for $\varphi \in [0, 2\pi)$ and $u \in [0, 1)$, and with a_i, b_i given by (3.4). With this definition, (3.6) takes the form

$$f(\varphi) > g(\varphi; u), \qquad \varphi \in [0, 2\pi),\ u \in [0, 1). \tag{3.9}$$

Next, let us show that if (3.9) holds for $u=0$, then it holds for all $u \in [0, 1)$. Taking into account the values (3.4) of the coefficients b_j, the function $g(\varphi; u)$ can be written as

$$g(\varphi; u) = |y|^2 \{[\eta^2 (\cos\varphi - 1)^2 + (1 - \cos^2\varphi)](u^2 - u) + [(1-\eta^2)\cos\varphi + (1+\eta^2)]/2\}.$$

Since for all $\varphi \in (0, 2\pi)$, $g(\varphi; u)$ is a second degree polynomial in u with $g(\varphi; 0) = g(\varphi; 1)$ and its main coefficient is positive, we have

$$\sup\{g(\varphi; u),\ u \in [0, +1)\} = g(\varphi; 0),$$

for all $\varphi \in (0, 2\pi)$. Furthermore for $\varphi = 0$, $g(0; u) = |y|^2 = g(0; 0)$. Hence, (3.9) is equivalent to

$$f(\varphi) > g(\varphi; 0) = |y|^2 [(1-\eta^2)\cos\varphi + (1+\eta^2)]/2, \qquad \varphi \in [0, 2\pi). \tag{3.10}$$

On the other hand, putting $x = x_1 + i\, x_2$, from (3.8) and (3.4) we get

$$f(\varphi) = 2 + |x|^2 (1+\eta^2)/2 + 2\eta\, x_1 + (-2 + |x|^2 (1-\eta^2)/2 - 2\eta\, x_1) \cos\varphi - 2\, x_2 \operatorname{sen}\varphi.$$

Thus, (3.10) can be written as

$$a + b\cos\varphi > c \operatorname{sen}\varphi, \qquad \varphi \in [0, 2\pi), \tag{3.11}$$

where

$$\begin{aligned} &a = 2 + \delta(1+\eta^2)/2 + 2\eta\, x_1, \qquad && b = -2 + \delta(1-\eta^2)/2 - 2\eta\, x_1, \\ &c = 2x_2, && \delta = |x|^2 - |y|^2. \end{aligned} \tag{3.12}$$

In addition, putting $\omega = \cos\varphi$, it is clear that (3.11) is equivalent to

$$a + b\,\omega > |c|(1-\omega^2)^{1/2}, \qquad \omega \in [-1, 1]. \tag{3.13}$$

Now, we may apply Lemma 1 with a, b and c given by (3.12), and therefore (3.13) holds if and only if

$$\begin{aligned} &2 + \delta(1+\eta^2)/2 + 2\eta\, x_1 > 0, \\ &\delta(4 + 4\eta\, x_1 + \eta^2 \delta) > 4x_2^2, \end{aligned}$$

and these inequalities can be written in the form

$$|x + 2\eta/(1+\eta^2)|^2 > |y|^2 - 4/(1+\eta^2)^2, \tag{3.14}$$

$$|\eta\delta + 2x_1| > 2|y|. \tag{3.15}$$

To analyze these inequalities, we consider separately the following three cases:

i) $\eta=0\,(\theta=1/2)$.

Now (3.14) and (3.15) take the form $|x|^2>|y|^2$ and $|x_1|>|y|$, which are equivalent to $|x_1|>|y|$. On the other hand from (3.7), $x_1<0$; therefore we conclude that $|y|<-x_1$, and then the theorem holds for $\theta=1/2$.

ii) $\eta\in(0,1]$, $(\theta\in[0,1/2))$.

Since $1/\eta-|x+1/\eta|\leqq 1/\eta-\mathrm{Re}(x+1/\eta)=x_1$, it is clear that if $(x,y)\in C^2$ satisfy

$$|y|<1/\eta-|x+1/\eta|, \tag{3.16}$$

then $|y|<-x_1$ and therefore $D_\theta\subset D$. Consequently $D_\theta\subset S_\theta\cap D$ holds if and only if $D_\theta\subset S_\theta$. Next, to prove this inclusion, we show that (3.16) implies the inequalities (3.7) with $\theta\in[0,1/2)$, (3.14) and (3.15).

Firstly, it follows from (3.16) that $|x+1/\eta|<1/\eta$, which proves (3.7). Furthermore this inequality implies that $\eta\delta+2x_1<0$, and then (3.15) can be written in the form

$$|x+1/\eta|^2<(1/\eta-|y|)^2. \tag{3.17}$$

But it is clear that (3.16) implies (3.17), so that (3.15) is also a consequence of (3.16).

Finally, note that for all $\eta\in(0,1]$, we have $[(1-\eta^2)/(1+\eta^2)](x_1+1/\eta)\leqq |x+1/\eta|$. Since this inequality is equivalent to

$$(1/\eta-|x+1/\eta|)^2\leqq|x+2\eta/(1+\eta^2)|^2+4/(1+\eta^2)^2,$$

it follows that (3.16) implies (3.14).

iii) $\eta\in[-1,0)$, $\quad(\theta\in(1/2,1])$.

Note that $S_\theta\cap D=D$ is equivalent to $D\subset S_\theta$; therefore, it is enough to verify that

$$|y|<-x_1, \qquad x_1\leqq 0 \tag{3.18}$$

imply the inequalities (3.7) with $\theta\in(1/2,1]$, (3.14) and (3.15).

It follows from (3.18) that $\delta>0$; then we have $\eta\delta+2x_1<2x_1\leqq 0$, and the inequality (3.15) can be written as $|x+1/\eta|^2>(|y|-1/\eta)^2$. Using the fact that $|y|-1/\eta>0$, this inequality is equivalent to

$$|x+1/\eta|>|y|-1/\eta. \tag{3.19}$$

Now, assuming that (3.18) holds, we can write $|y|<-x_1\leqq 1/\eta+|x+1/\eta|$. Thus (3.18) implies (3.19) and also (3.15).

On the other hand, from (3.19) we have $|x+1/\eta|>-1/\eta$, so that (3.18) implies (3.7) for $\theta\in(1/2,1]$.

Finally, it only remains to show that (3.14) is a consequence of (3.18). Since $\eta<0$, it follows from (3.18) that $x_1\leqq -1/\eta$. Hence, it can be verified that

$$x_1^2-4/(1+\eta^2)^2\leqq(x_1+2\eta/(1+\eta^2))^2.$$

Consequently, we have $x_1^2-4/(1+\eta^2)^2\leqq|x+2\eta/(1+\eta^2)|^2$.

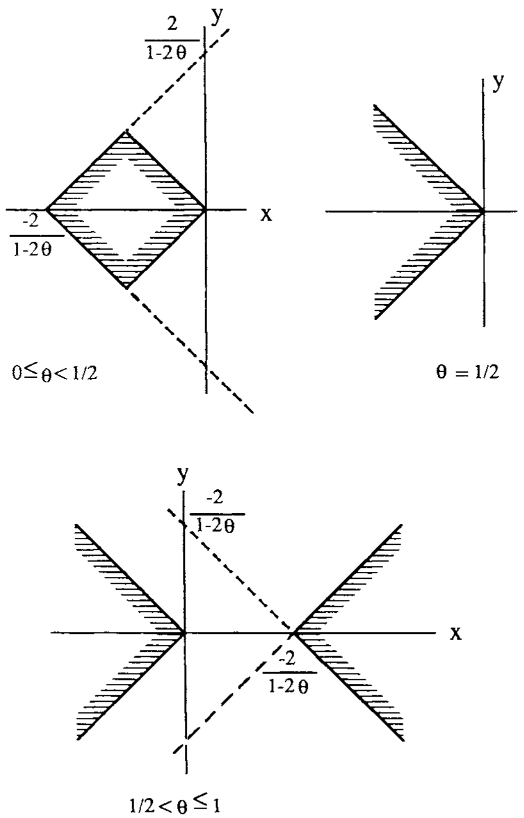

Fig. 1

But from (3.18), $|y|^2 < x_1^2$. Therefore (3.18) implies (3.14) and the proof is complete.

Theorem 1 is concerned with the stability sets of the θ-methods for the complex test Eq. (1.3) but it may be applied also to the real test equations. In fact, for the real case $x = hp, y = hq$ are real variables and $D = D^*$, $D_\theta = D_\theta^*$, where

$$D_\theta^* = \{(x, y) \in R^2 / |x + 1/\eta| + |y| < 1/\eta\}.$$

Therefore, we have

Corollary 1. *The stability sets S_θ^* of the θ-methods with linear interpolation for the real test Eq.* (1.3) *satisfy*

$$S_\theta^* \cap D^* = D^*, \quad \text{for } \theta \in [1/2, 1],$$
$$S_\theta^* \cap D^* \supset D_\theta^*, \quad \text{for } \theta \in [0, 1/2).$$

Figure 1 shows the sets defined by the sufficient stability conditions (3.7), (3.14) and (3.15) for several values of θ.

4. Necessary and Sufficient Conditions of the Stability Regions for the Complex Test Equation

As it has been proved in Theorem 1, for $\theta \in [1/2, 1]$ the complex stability sets satisfy $S_\theta \cap D = D$. Therefore this result provides us a necessary and sufficient condition for the stability of the θ-method for $\theta \in [1/2, 1]$ in the convex set D. However, for $\theta \in [0, 1/2)$, the theorem only proves that $S_\theta \cap D \supset D_\theta$ and it would be desirable to know whether or not the opposite inclusion is true. In this section, we demonstrate that in the real case, $S_\theta^* \cap D^* \subset D_\theta^*$. In the complex case we have the evidence that this assertion also holds true as it is infered from extensive numerical experiments, but unfortunately an analytical proof of this fact has not yet been obtained.

Theorem 2. *The stability sets S_θ^* of the θ-method, $\theta \in [0, 1/2)$, for the real test Eq.* (1.3) *with linear interpolation satisfy $S_\theta^* \cap D^* = D_\theta^*$.*

Proof. Let A be the set $A = \bigcap S_{\theta,h}^*$ with $h = \tau/(1-u)$, $u \in [0, 1)$ and let T be an appropriate set, defined below, satisfying $T \supset S_{\theta,h}^*$ for $h = \tau/2$. Then, it will be seen that $A \cap T \cap D^* = D_\theta^*$, and taking into account that $S_\theta^* \subset A \cap T$, it follows that $S_\theta^* \cap D^* \subset D_\theta^*$. Therefore, from Corallary 1, we have $S_\theta^* \cap D^* = D_\theta^*$.

The characteristic polynomial (2.4) for $m = 1$ takes the form

$$W_1(z) = c_2 z^2 + c_1 z + c_0,$$

where

$$c_2 = 1 - \theta x - \theta u y, \qquad c_1 = -[1 + (1-\theta) x + (\theta + u - 2\theta u) y],$$
$$c_0 = -(1-\theta)(1-u) y.$$

Since W_1 is a Schur polynomial if and only if $|c_0| < |c_2|$ and $|c_1| < |c_2 + c_0|$, taking into account the above values of c_0, c_1 and c_2, the conditions

$$(1-\theta)(1-u)|y| < |1 - \theta x - \theta u y|, \tag{4.1}$$
$$|1 + (1-\theta) x + (\theta + u - 2\theta u) y| < |1 - \theta x + (\theta + u - 2\theta u - 1) y| \tag{4.2}$$

are necessary and sufficient for W_1 to be a Schur polynomial.

The inequality (4.1) for $u = 0$ becomes $(1-\theta)|y| < |1 - \theta x|$ and this implies

$$(1-\theta)(1-u)|y| < (1-\theta)|y| - \theta u |y| < |1 - \theta x| - \theta u |y| < |1 - \theta x - \theta u y|,$$

for all $u \in [0, 1)$ and $\theta \in [0, 1/2)$, therefore (4.1) for all $u \in [0, 1)$ is equivalent to

$$(1-\theta)|y| < |1 - \theta x|. \tag{4.3}$$

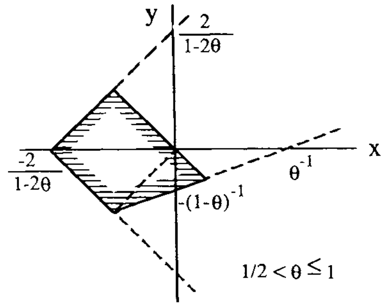

Fig. 2

On the other hand, (4.2) can be written in the form $|a+bu|<|a'+bu|$, where $a=1+(1-\theta)x+\theta y$, $a'=1-\theta x-(1-\theta)y$, $b=(1-2\theta)y$. Then, (4.2) will be satisfied for all $u\in[0,1)$ if and only if $|a|<|a'|$ and $|a+b|\leqq|a'+b|$. This means that

$$|1+(1-\theta)\,x+\theta\,y|<|1-\theta\,x-(1-\theta)\,y|, \tag{4.4}$$

$$|1+(1-\theta)\,x+(1-\theta)y|\leqq|1-\theta\,x-\theta\,y|. \tag{4.5}$$

Therefore, for all $\theta\in[0,1/2)$, the inequalities (4.3), (4.4) and (4.5) define a set A in the (x,y)-plane where W_1 is a Schur polynomial (Fig. 2).

Next, taking $m=2$ and $u=0$, the characteristic polynomial (2.4) becames

$$W_2(z)=(1-\theta x)\,z^3-(1+(1-\theta)\,x)\,z^2-y\theta z-y(1-\theta),$$

which has the root $z=-1$ for all real values of x and y on the straight line $L\colon 2+\eta x+\eta y=0$. Then, choosing as T the set $T=R^2-L$, it contains $S^*_{\theta,h}$ for $h=\tau/2$. Furthermore, it is clear from Fig. 2 that $A\cap T\cap D^*=D^*_\theta$ and this implies $S^*_\theta\cap D^*\subset D^*_\theta$, and the proof is complete.

As it was mentionated at the beginning of this section, an analytical proof of the fact $S_\theta\cap D\subset D_\theta$, $\theta\in[0,1/2)$ has not been obtained. However, after some numerical experiments we have the evidence that this result holds true. Next, let us briefly describe these numerical experiments. Putting $z=\exp(i\varphi)$; $\varphi\in[0,2\pi)$ in the characteristic polynomial (2.4) and solving for x, we have

$$x=[\exp(i\varphi(m+1))-\exp(i\varphi m)-\theta yu\exp(2i\varphi)-y(\theta+u-2\theta u)\exp(i\varphi)+ \\ -y(1-\theta)(1-u)]/[\theta\exp(i\varphi(m+1))+(1-\theta)\exp(i\varphi m)], \tag{4.6}$$

which defines in the x-complex plane, for given values of h and y, the boundary locus l_h of the characteristic polynomial (2.4), i.e. the set of all x-points such that (2.4) has at least one z-root with modulus one. As it is well known the boundary locus l_h determines in the (x_1,x_2)-plane a finite number of components such that all points of each component are either stable or unstable.

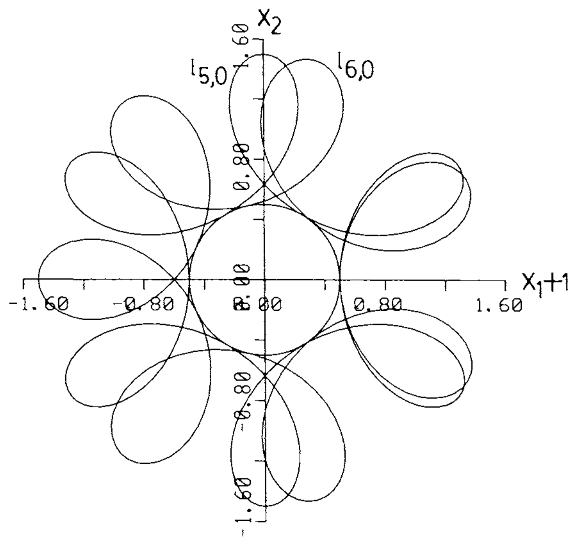

Fig. 3

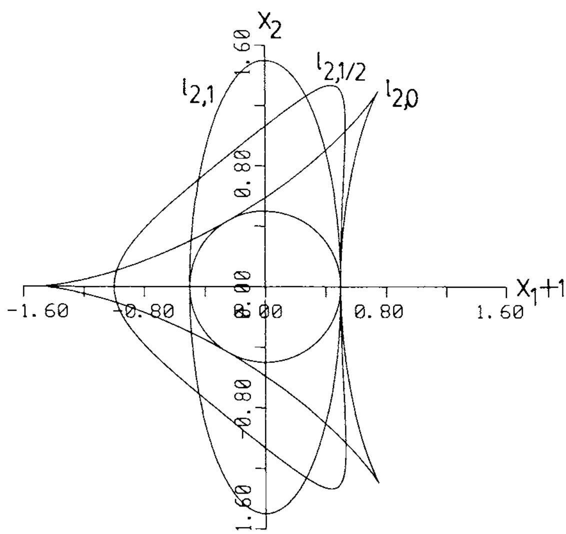

Fig. 4

Figures 3 and 4 show in the $(x_1+1/\eta,\ x_2)$-plane typical $l_h = l_{m,u}$-curves with $\theta=0\,(\eta=1)$, $y=1/2$ and several values of m and u where $h=\tau/(m-u)$. Also we show in these figures the two-dimensional set $D_\theta \cap \{|y|=1/2\}$ for $\theta=0$, which is given by $|x+1|<1/2$.

We have plotted the curves l_h for a large number of values of h and y. It is easy to verify that only the component R_h which contains the point

$(x_1, x_2) = (-1/\eta, 0)$ is a stability component. (In the above figures R_h is the component determined by $l_h = l_{m,u}$ that contains the origin.) For each $k \in [0, 1/\eta)$, denoting by Π_k the plane $|y| = k$ in the $(x_1, x_2, |y|)$-space, from our numerical experiments we have obtained that

$$\bigcap_{\substack{h>0\\|y|=k}} R_h \cap D = D_\theta \cap \Pi_k.$$

Taking into account the above assertions, it is clear that

$$\bigcap_{\substack{h>0\\|y|=k}} R_h = S_\theta \cap \Pi_k.$$

Therefore as k runs through the interval $[0, 1/\eta)$, we conclude that $S_\theta \cap D = D_\theta$, for all $\theta \in [0, 1/2)$ which is the desired result.

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