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Mixed Finite Elements for Second Order Elliptic Problems in Three Variables

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Summary. Two families of mixed finite elements, one based on simplices and the other on cubes, are introduced as alternatives to the usual Raviart-Thomas-Nedelec spaces. These spaces are analogues of those introduced by Brezzi, Douglas, and Marini in two space variables. Error estimates in L^2 and H^{-s} are derived.

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1. Introduction

We introduce two families of spaces of mixed finite elements to approximate the solutions of second order elliptic equations in three space variables. These families are the reasonable analogues of the spaces recently described by Brezzi et al. [3, 4] for two-dimensional problems. For the simplicial elements our space of index j lies between the spaces of index $j-1$ and j of Nedelec [15] and provides approximation of the vector variable of the same order of accuracy as does Nedelec's space of index j. Our cubic elements of index j are based on polynomials of total degree j for the vector variable and total degree $j-1$ for the scalar variable; hence, the local dimension of this space is about half that of the Raviart-Thomas [18] space over cubes (i.e., rectangular parallelepipeds) of equivalent accuracy for the vector variable. Nedelec [16] has recently considered the same tetrahedral spaces.

In Sect. 2 we describe the simplicial elements and locally defined projections that enable us to use the theory of Douglas and Roberts [10] to obtain error estimates for the Dirichlet problem in L^2 and H^{-s}, which will be derived in Sect. 4. In Sect. 3 we discuss the cubic elements and the corresponding projections. In Sect. 5 a hybridization of our mixed method is introduced, and its properties with respect to linear algebra and seperconvergence for the scalar variable are studied. In Sect. 6 an Arrow-Hurwitz-type alternating-direction iterative technique is described briefly.

2. Simplicial Elements

Denote by $P_j(K)$ the set of restrictions of polynomials of total degree not greater than j to the set K; $\mathbf{P}_j(K)$ is the vector analogue of $P_j(K)$ consisting of three copies of $P_j(K)$. Vector functions will be indicated by bold face.

Let j be a positive integer and let K be a simplex. Set

(2.1 a) $$\mathbf{V}(j, K)=\mathbf{P}_j(K),$$

(2.1 b) $$W(j, K)=P_{j-1}(K),$$

(2.1 c) $$\mathbf{M}(j, K)=\mathbf{V}(j, K)\times W(j, K);$$

$\mathbf{M}(j, K)$ is the simplicial element of index j. Note that the local degree of the scalar space $W(j, K)$ is one less than that of the vector space $\mathbf{V}(j, K)$. If $\mathbf{N}(j, K)$ denotes the simplicial element of Nedelec [15] of index j, then

$$\mathbf{N}(j-1, K)\subset\mathbf{M}(j, K)\subset\mathbf{N}(j, K), \quad j>0.$$

In fact, it is easy to see that

(2.2) $$\dim\{\mathbf{M}(j, K)\}=[3(j+3)(j+2)(j+1)+(j+2)(j+1)j]/6 =(4j+9)(j+2)(j+1)/6,$$

while

(2.3) $$\dim\{\mathbf{N}(j, K)\}=[3(j+4)(j+2)(j+1)+(j+3)(j+2)(j+1)]/6 =(4j+15)(j+2)(j+1)/6,$$

so that the dimension of $\mathbf{N}(j, K)$ exceeds that of $\mathbf{M}(j, K)$ by $(j+2)(j+1)$.

Denote by $(\cdot,\cdot)_K$ the inner product in $L^2(K)$ or in $\mathbf{L}^2(K)$ and by $\langle\cdot,\cdot\rangle_e$ that in $L^2(e)$, where e is a face of K. We wish to define a projection from $\mathbf{H}^1(K)$ to $\mathbf{V}(j, K)$ that can be extended to a projection from $\mathbf{H}^1(G)$ to the subspace of $H(\mathrm{div}, G)$ of vector functions having restrictions to each element of a simplicial decomposition $\{K\}$ of G. A vector function that is a piecewise-polynomial function over $\{K\}$ lies in $H(\mathrm{div}, G)$ if and only if its component normal to an interior face is continuous across each such face; consequently, the degrees of freedom for the projection on K must determine these normal components. The remaining degrees of freedom will be chosen to ensure a convenient commuting diagram property.

Let K have flat faces and let $\Pi^j=\Pi(j, K)$: $\mathbf{H}^1(K)\to\mathbf{V}(j, K)$ be defined by the following relations:

(2.4 a) $$\langle(\mathbf{q}-\Pi^j\mathbf{q})\cdot\mathbf{n}_e, p\rangle_e=0, \quad p\in P_j(e), \text{ for each face } e \text{ of } K,$$

(2.4 b) $$(\mathbf{q}-\Pi^j\mathbf{q}, \mathbf{grad}\, w)_K=0, \quad w\in P_{j-1}(K),$$

(2.4 c) $$(\mathbf{q}-\Pi^j\mathbf{q}, \mathbf{v})_K=0, \quad \mathbf{v}\in\{\mathbf{u}\in\mathbf{P}_j(K)\colon \mathbf{u}\cdot\mathbf{n}=0 \text{ on } \partial K \text{ and } (\mathbf{u}, \mathbf{grad}\, w)=0,\ w\in P_{j-1}(K)\}.$$

Assume for the moment that the conditions (2.4a) and (2.4b) are independent. Then, to show existence of Π^j it is sufficient to prove that a vector in

$\mathbf{V}(j, K)$ having vanishing degrees of freedom must itself vanish, since the number of degrees of freedom equals the dimension of $\mathbf{V}(j, K)$. So, let $\mathbf{v} \in \mathbf{V}(j, K)$ have vanishing degrees of freedom. Since $\mathbf{v} \cdot \mathbf{n}_e \in P_j(e)$, (2.4a) implies that $\mathbf{v} \cdot \mathbf{n}_e = 0$. Then, (2.4b) and (2.4c) imply that $\mathbf{v} = \mathbf{0}$.

To prove independence of (2.4a) and (2.4b), it suffices to treat the case of the reference simplex $\mathbf{S}$ having vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Let $e_i = \mathbf{S} \cap \{x_i = 0\}$, $i = 1, 2, 3$, and $e_4 = \mathbf{S} \cap \{x + y + z = 1\}$. The independence is a consequence of the next two lemmas.

Lemma 2.1. *Let $p_i \in P_j(e_i)$, $i = 1, 2, 3, 4$. Then, there exists $\mathbf{q} \in \mathbf{P}_j(\mathbf{S})$ such that $\mathbf{q} \cdot \mathbf{n}_i = p_i$ on e_i, where $\mathbf{n}_i$ is the normal to e_i.*

Proof. Fix a face, say $z = 0$, and take a polynomial $p(x, y) \in P_j$. It is enough to prove that there exists at least one $\mathbf{q} \in \mathbf{P}_j$ such that $\mathbf{q} \cdot \mathbf{n} = p$ on $z = 0$ and $\mathbf{q} \cdot \mathbf{n} = 0$ elsewhere on $\partial \mathbf{S}$. To do this, first find a constant c and two polynomials $p_1(x, y)$ and $p_2(x, y)$ in P_{j-1} such that

$$-p(x, y) = c(1 - x - y) + x p_1(x, y) + y p_2(x, y).$$

Then the vector polynomial $\mathbf{q} = (x p_1(x, y), y p_2(x, y), p(x, y) + c z)$ satisfies the prescribed requirements.

Lemma 2.2. *Let $\{w_\alpha\colon \alpha = 1, \ldots, N\} = Basis\{w \in P_{j-1}(\mathbf{S})\colon \int w\, dx\, dy\, dz = 0\}$, and let $a_\alpha \in \mathbf{R}^N$. Then, there exists $\mathbf{q} \in \mathbf{P}_j(\mathbf{S})$ such that $\mathbf{q} \cdot \mathbf{n} = 0$ on $\partial \mathbf{S}$ and*

$$(\mathbf{q}, \mathbf{grad}\, w_\alpha) = a_\alpha, \qquad \alpha = 1, \ldots, N.$$

Proof. Let $p \in P_{j-1}(\mathbf{S})$ be such that $(p, 1)_{\mathbf{S}} = 0$ and $(p, w_\alpha) = -a_\alpha$. It suffices to show that there exists $\mathbf{q} \in \mathbf{P}_j(\mathbf{S})$ with $\mathbf{q} \cdot \mathbf{n}_i = 0$ and $\operatorname{div} \mathbf{q} = p$. To do this, first take [14] $\mathbf{u} \in \mathbf{H}_0^1(\mathbf{S})$ such that $\operatorname{div} \mathbf{u} = p$. Then let $\mathbf{q}$ be the unique polynomial in $\mathbf{P}_{j-1}(\mathbf{S}) + \mathbf{x} P_{j-1}(\mathbf{S})$ [15] for which

$$\langle (\mathbf{q} - \mathbf{u}) \cdot \mathbf{n}_i, t \rangle_e = 0, \qquad t \in P_{j-1}(e),\ e \in \{e_1, e_2, e_3, e_4\},$$
$$(\mathbf{q} - \mathbf{u}, \mathbf{v})_{\mathbf{S}} = 0, \qquad \mathbf{v} \in \mathbf{P}_{j-2}(\mathbf{S}).$$

The first set of these relations implies that $\mathbf{q} \cdot \mathbf{n}$ vanishes on $\partial \mathbf{S}$, and the second that

$$(\operatorname{div}(\mathbf{q} - \mathbf{u}), w)_{\mathbf{S}} = -(\mathbf{q} - \mathbf{u}, \mathbf{grad}\, w)_{\mathbf{S}} = 0, \qquad w \in P_{j-1}(\mathbf{S});$$

thus, $\operatorname{div} \mathbf{q} = p$, as was to be shown.

Since Π^j reproduces $\mathbf{P}_j(K)$, it follows from the Bramble-Hilbert lemma [11] that

$$\|\mathbf{q} - \Pi^j \mathbf{q}\|_{0,K} < C \|\mathbf{q}\|_{r,K} (\operatorname{diam}(K))^r, \qquad 1 \leq r \leq j+1, \tag{2.5}$$

with the constant C depending only on the minimum vertex angle of K; in the above, $\| \ \|_{r,K}$ indicates the standard norm in $\mathbf{H}^r(K)$. Note also that, for $w \in W(j, K) = P_{j-1}(K)$,

$$(\operatorname{div}(\mathbf{q} - \Pi^j \mathbf{q}), w)_K = -(\mathbf{q} - \Pi^j \mathbf{q}, \mathbf{grad}\, w)_K + \langle (\mathbf{q} - \Pi^j \mathbf{q}) \cdot \mathbf{n}, w \rangle_{\partial K} = 0. \tag{2.6}$$

Boundary simplices are allowed to have one curved face, that face lying in the boundary of G. Consequently, it is necessary to modify the definition of Π^j on such simplices. Let K have flat faces e_1, e_2, and e_3, with e_4 possibly being curved. Then, let Π^j: $\mathbf{H}^1(K)\to\mathbf{V}(j,K)$ be determined by the requirements that

(2.7a) $$\langle(\mathbf{q}-\Pi^j\mathbf{q})\cdot\mathbf{n}_e, p\rangle_e=0, \quad p\in P_j(e),\ e\in\{e_1,e_2,e_3\},$$

(2.7b) $$(\operatorname{div}(\mathbf{q}-\Pi^j\mathbf{q}), w)_K=0, \quad w\in P_{j-1}(K),$$

(2.7c) $$(\mathbf{q}-\Pi^j\mathbf{q},\mathbf{v})_K=0, \quad \mathbf{v}\in\{\mathbf{u}\in\mathbf{P}_j(K)\colon\ \mathbf{u}\cdot\mathbf{n}_e=0 \text{ on } e_1\cup e_2\cup e_3 \text{ and } \operatorname{div}\mathbf{u}=0\}.$$

Again, it is easy to see that Π^j is uniquely determined by (2.7) and that (2.5) and (2.6) hold for boundary simplices. The constant C of (2.5) now depends solely on the ratio of the diameter of K and that of the inscribed ball, a condition that generalizes the vertex angle constraint.

Let $R^{j-1}=R(j-1,K)$: $L^2(K)\to W(j,K)$ be L^2-projection:

(2.8) $$(w-R^{j-1}w,p)_K=0, \quad p\in P_{j-1}(K).$$

Since $\operatorname{div}\mathbf{V}(j,K)=W(j,K)$,

(2.9) $$(\operatorname{div}\mathbf{q}, w-R^{j-1}w)_K=0, \quad \mathbf{q}\in\mathbf{V}(j,K).$$

We have now the local properties of our space of index j. In order to construct the space globally, let $\mathcal{T}_h=\{K\}$ be a decomposition of the domain G into nonoverlapping simplices such that

(2.10a) the intersection of two distinct K's in $\mathcal{T}_h$ is either a face, an edge, a vertex, or void;

(2.10b) if $K\subset G$, K has flat faces;

(2.10c if K is a boundary simplex, the boundary face can be curved;

(2.10d) if $\operatorname{diam}(K)=h_K$, $h_K\leq h$;

(2.10e) if r_K is the radius of the ball inscribed in K, $h_K/r_K<$constant.

Set

(2.11a) $$\mathbf{V}_h=\mathbf{V}_h^j=\mathbf{V}(j,\mathcal{T}_h)=\{\mathbf{v}\in H(\operatorname{div},G)\colon\ \mathbf{v}|_K\in\mathbf{V}(j,K),\ K\in\mathcal{T}_h\},$$

(2.11b) $$W_h=W_h^j=W(j,\mathcal{T}_h)=\{w\in L^2(G)\colon\ w|_K\in W(j,K),\ K\in\mathcal{T}_h\},$$

(2.11c) $$\mathbf{M}_h=\mathbf{M}_h^j=\mathbf{V}_h^j\times W_h^j.$$

Extend the projections $\Pi(j,K)$ and $R(j-1,K)$ to $\mathbf{H}^1(G)$ and $L^2(G)$ respectively, as follows:

(2.12a) $$\Pi_h=\Pi_h^j\colon\ \mathbf{H}^1(G)\to\mathbf{V}_h^j \quad \text{satisfies } \Pi_h|_{\mathbf{H}^1(K)}=\Pi(j,K);$$

(2.12b) $$R_h=R_h^j\colon\ L^2(G)\to W_h^j \quad \text{satisfies } R_h=R(j-1,K) \text{ on } L^2(K).$$

The following properties of Π_h and R_h result immediately from their local properties:

$$(2.13\text{a}) \qquad (\operatorname{div}(\mathbf{q}-\Pi_h\mathbf{q}), w)=0, \quad w\in W_h,\ \mathbf{q}\in H^1(G),$$

$$(2.13\text{b}) \qquad (\operatorname{div}\mathbf{q}, w-R_h w)=0, \quad w\in L^2(G),\ \mathbf{q}\in \mathbf{V}_h;$$

i.e.,

$$(2.13\text{c}) \qquad \operatorname{div}\Pi_h=R_h\operatorname{div}:\ \mathbf{H}^1(G)\to W_h.$$

The relations (2.13) can be expressed succinctly by stating that the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{H}^1(G) & \xrightarrow{\operatorname{div}} & L^2(G) & & \\ \downarrow \Pi_h^j & & \downarrow R_h^j & & \\ \mathbf{V}_h & \xrightarrow{\operatorname{div}} & W_h & \longrightarrow & 0. \end{array}$$

The approximation properties of Π_h and R_h also can be read out of their local properties:

$$(2.14\text{a}) \qquad \|\mathbf{q}-\Pi_h^j\mathbf{q}\|_0\leqq C\|\mathbf{q}\|_r h^r, \qquad \mathbf{q}\in\mathbf{H}^r(G),\ 1\leqq r\leqq j+1,$$

$$(2.14\text{b}) \qquad \|w-R_h^j w\|_{-s}\leqq C\|w\|_r h^{r+s}, \qquad w\in H^r(G),\ 0\leqq r,s\leqq j.$$

3. Cubic Elements

As in the case of the two-dimensional elements of Brezzi et al. [3], our elements over rectangular parallelepipeds differ significantly from those of Raviart and Thomas [18], in that our elements are based on augmenting the space of vector polynomials of *total* degree j by $3j+3$ additional vectors in place of augmenting the space of vector *tensor products* of polynomials of degree j by $3j+3$ polynomials of higher degree. We also use a space of total degree $j-1$ for the scalar variable instead of a tensor product of polynomials of degree j. We shall again obtain approximation of vector functions of the same order of accuracy as do Raviart and Thomas.

Let K be a rectangular parallelepiped and j a positive integer. Let

$$(3.1\text{a}) \qquad \begin{aligned} \mathbf{V}(j,K)=\mathbf{P}_j(K)+\operatorname{Span}[&\mathbf{curl}(0,0,x^{j+1}y), \mathbf{curl}(0,xz^{j+1},0),\\ &\mathbf{curl}(y^{j+1}z,0,0);\ \mathbf{curl}(0,0,xy^{i+1}z^{j-i}),\\ &\mathbf{curl}(0,x^{i+1}y^{j-i}z,0), \mathbf{curl}(x^{j-i}yz^{i+1},0,0),\ i=1,\dots,j], \end{aligned}$$

$$(3.1\text{b}) \qquad W(j,K)=P_{j-1}(K).$$

Lemma 3.1. *The dimension of* $\mathbf{V}(j,K)$ *is that of* $\mathbf{P}_j(K)$ *plus* $3j+3$; i.e.,

$$(3.2) \qquad \begin{aligned} \dim\mathbf{V}(j,K)&=(j+3)(j+2)(j+1)/2+3(j+1)\\ &=(j^3+6j^2+17j+12)/2. \end{aligned}$$

Proof. It suffices to prove that the $3(j+1)$ polynomial vectors of degree $j+1$ added to $\mathbf{P}_j(K)$ are independent. Suppose that

$$\sum_{i=1}^{j}[a_i((i+1)x y^i z^{j-i}, -y^{i+1}z^{j-i}, 0)+b_i(-x^{i+1}y^{j-i}, 0, (i+1)x^i y^{j-i} z)$$
$$+c_i(0, (i+1)x^{j-i} y z^i, -x^{j-i}z^{i+1})]$$
$$+d(x^{j+1}, -(j+1)x^j y, 0)+e(-(j+1)x z^j, 0, z^{j+1})$$
$$+f(0, y^{j+1}, -(j+1)y^j z)=0.$$

Summing on the first component shows that $e=0$, $a_1=\ldots=a_j=0$, $b_1=\ldots=b_{j-1}=0$, and $d=b_j$. Then summing on the third component implies that $d=f=0$ and $c_1=\ldots=c_j=0$, so that all coefficients above vanish. Thus, the lemma has been proved.

The space $\mathbf{V}(1, K)$ has dimension 18 and consists of $\mathbf{P}_1(K)$ plus the span of the six vectors

$$(x^2, -2xy, 0), (2xy, -y^2, 0), (-2xz, 0, z^2),$$
$$(-x^2, 0, 2xz), (0, y^2, -2yz), (0, 2yz, -z^2), \tag{3.3}$$

while $\mathbf{V}(2, K)$ has dimension 39 and consists of $\mathbf{P}_2(K)$ and the span of the nine vectors

$$(x^3, -3x^2 y, 0), (2xyz, -y^2 z, 0), (3xy^2, -y^3, 0),$$
$$(x^3, 0, -3x^2 z), (-yx^2, 0, 2xyz), (3xz^2, 0, -z^3), \tag{3.4}$$
$$(0, y^3, -3y^2 z), (0, 3yz^2, -z^3), (0, 2xyz, -xz^2).$$

The projection Π^j: $\mathbf{H}^1(K)\to\mathbf{V}(j, K)$ can be defined in the following way:

$$\langle(\mathbf{q}-\Pi^j\mathbf{q})\cdot\mathbf{n}_e, p\rangle_e=0, \quad p\in P_j(e), \text{ for each face of } K, \tag{3.5a}$$

$$(\mathbf{q}-\Pi^j\mathbf{q}, \mathbf{v})_K=0, \quad \mathbf{v}\in\mathbf{P}_{j-2}(K). \tag{3.5b}$$

The number of degrees of freedom is easily seen to be equal to the dimension of $\mathbf{V}(j, K)$; thus, it suffices to establish unisolvence to show existence of Π^j. Moreover, it is sufficient to treat the unit cube. The first component of $\mathbf{q}$ has the form

$$q_1=\sum_{i=1}^{j}(a_i(i+1)x y^i z^{j-i}-b_i x^{i+1}y^{j-i})+d x^{j+1}-e(j+1)x z^j+r_1,$$

$$q_2=\sum_{i=1}^{j}(-a_i y^{i+1}z^{j-i}+c_i(i+1)x^{j-i} y z^i)-d(j+1)x^j y+f y^{j+1}+r_2,$$

$$q_3=\sum_{i=1}^{j}(b_i(i+1)x^i y^{j-i} z-c_i x^{j-i}z^{i+1})+e z^{j+1}-f(j+1)y^j z+r_3,$$

where $r_i\in P_j(K)$. Assume that the degrees of freedom of $\mathbf{q}$ vanish. Then, (3.5a) applied to the face $x=0$ implies that $r_1=x s_1$. Next, (3.5a) applied on $x=1$ gives $a_i=e=0$. Similarly, $b_i=c_i=d=f=0$. Thus, $\mathbf{q}\in\mathbf{P}_j(K)$. Since then $q_1=0$ on $x=0$ or 1, $q_1=x(1-x)t_1$, $t_1\in P_{j-2}(K)$. Hence, (3.5b) implies that q_1 vanishes, and the existence of Π^j has been determined for cubic elements with flat faces.

Boundary elements are allowed to have at most one curved face; this introduces a geometric constraint on a cubic decomposition of G. The projection Π^j can be defined in a manner analogous to (2.7):

(3.6a) $\langle (\mathbf{q} - \Pi^j \mathbf{q}) \cdot \mathbf{n}_e, p \rangle_e = 0, \quad p \in P_j(e)$, for each flat face e of K,

(3.6b) $(\operatorname{div}(\mathbf{q} - \Pi^j \mathbf{q}), w)_K = 0, \quad w \in P_{j-1}(K)$,

(3.6c) $(\mathbf{q} - \Pi^j \mathbf{q}, \mathbf{v}) = 0, \quad \mathbf{v} \in \{\mathbf{u} \in \mathbf{P}_j(K)\colon \operatorname{div} \mathbf{u} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on all flat faces of } K\}$.

That Π^j is well defined on boundary elements is easy to see.

Let $\{K\}$ be a decomposition $\mathcal{T}_h$ of G into simplices and cubes satisfying (2.10). Construct a global projection Π_h: $\mathbf{H}^1(G) \to \mathbf{V}_h^j$, where the restriction of an element $\mathbf{v}$ of $\mathbf{V}_h^j$ to $K \in \mathcal{T}_h$ is a simplicial or cubic element as appropriate, by piecing together the appropriate $\Pi^j = \Pi(j, K)$'s. The commuting diagram properties (2.13) and the approximation properties (2.14) follow clearly from the corresponding local ones.

4. The Dirichlet Problem

Consider the Dirichlet problem

(4.1 a) $$Lu = -\operatorname{div}(a(x) \,\mathbf{grad}\, u) = f, \qquad x \in G,$$

(4.1 b) $$u = -g, \qquad x \in \partial G,$$

where G is a domain in $\mathbf{R}^3$ having a smooth boundary ∂G and $a(x)$ is a smooth, positive function on the closure of G. Let

(4.2) $$\mathbf{q} = -a \,\mathbf{grad}\, u, \qquad c(x) = a(x)^{-1},$$

and factor (4.1 a) into the first order system

(4.3 a) $$c\,\mathbf{q} + \mathbf{grad}\, u = 0,$$

(4.3 b) $$\operatorname{div} \mathbf{q} = f,$$

for $x \in G$. The weak form of (4.3)-(4.1 b) appropriate for mixed finite element methods is given by seeking $\{\mathbf{q}, u\} \in H(\operatorname{div}, G) \times L^2(G)$ such that

(4.4 a) $$(c\,\mathbf{q}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, u) = \langle g, \mathbf{v} \cdot \mathbf{n} \rangle, \qquad \mathbf{v} \in H(\operatorname{div}, G),$$

(4.4 b) $$(\operatorname{div} \mathbf{q}, w) = (f, w), \qquad w \in L^2(G).$$

Let $\mathcal{T}_h$ be a decomposition of G into simplices and rectangular parallelepipeds, and assume that $\mathcal{T}_h$ satisfies (2.10). The mixed finite element approximation $\{\mathbf{q}_h, u_h\} \in \mathbf{V}_h \times W_h = \mathbf{V}(j, \mathcal{T}_h) \times W(j, \mathcal{T}_h)$ is defined as the solution of the equations

(4.5 a) $$(c\,\mathbf{q}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, u_h) = \langle g, \mathbf{v} \cdot \mathbf{n} \rangle, \qquad \mathbf{v} \in \mathbf{V}_h,$$

(4.5 b) $$(\operatorname{div} \mathbf{q}_h, w) = (f, w), \qquad w \in W_h.$$

The existence and uniqueness of $\{\mathbf{q}_h, u_h\}$ follow immediately from the general argument of Douglas and Roberts [10] in exactly the same manner as in the paper of Brezzi et al. [3]. Moreover, the error analysis of [3] applies without modification to the spaces $\mathbf{M}_h^j$ of (2.11), whether formed from simplicial or cubic elements or a combination of the two kinds, since the derivation of the estimates of [3] depended solely on the properties of the projections Π_h^j and R_h^j.

Let $\| \ \|_{-s}$ denote the norm in the space $H^s(G)'$ or $[H^s(G)^3]'$. Then, the errors in the scalar approximation u_h, the vector approximation $\mathbf{q}_h$, and its divergence satisfy the inequalities

$$\|u-u_h\|_{-s} \leqq K(\|f\|_{r-2}+|g|_{r-1/2})\,h^{r+s}, \quad 2\leqq r\leqq j,\ 0\leqq s\leqq j, \tag{4.6a}$$

$$\|\mathbf{q}-\mathbf{q}_h\|_{-s} \leqq K(\|f\|_{r-1}+|g|_{r+1/2})\,h^{r+s}, \quad 1\leqq r\leqq j+1,\ 0\leqq s\leqq j-1, \tag{4.6b}$$

$$\|\operatorname{div}(\mathbf{q}-\mathbf{q}_h)\|_{-s} \leqq K\,\|\operatorname{div}\mathbf{q}\|_r\,h^{r+s} = K\,\|f\|_r\,h^{r+s}, \quad 0\leqq r\leqq j,\ 0\leqq s\leqq j, \tag{4.6c}$$

where $| \ |_t$ is the norm in $H^t(\partial G)$. In particular,

$$\|u-u_h\|_{-j}+\|\mathbf{q}-\mathbf{q}_h\|_{-j+1}+\|\operatorname{div}(\mathbf{q}-\mathbf{q}_h)\|_{-j} \leqq K(\|f\|_j+|g|_{j+3/2})\,h^{2j}. \tag{4.7}$$

It also is a consequence of the argument of [3] that

$$\|u_h-R_h u\|_0 \leqq K(\|f\|_j+|g|_{j+3/2})\,h^{\min(j+2,\,2j)}. \tag{4.8}$$

5. Hybridization of the Mixed Method

The solution of the algebraic equations generated by (4.5) can possibly be simplified by the introduction of a Lagrange multiplier to enforce the continuity of the normal component of $\mathbf{q}_h$ across interelement boundaries [1-3, 12, 13]. We shall see also that this multiplier can be used in a postprocessing procedure to give a higher-order correct approximation of the scalar variable u.

Let $\{e\}$ denote the set of all faces of the elements of $\mathscr{T}_h$, and set

$$\mathscr{L}_h=\mathscr{L}_h^j=\{m\colon m|_e\in P_j(e) \text{ if } e\subset G \text{ and } m|_e=0 \text{ if } e\subset\partial G\}. \tag{5.1}$$

Let

$$\mathscr{V}_h=\mathscr{V}_h^j=\{\mathbf{v}\colon \mathbf{v}|_K\in\mathbf{V}(j,K),\ K\in\mathscr{T}_h\}, \tag{5.2}$$

where each K can be either a simplex or a cube. Note that $\mathbf{v}\in\mathscr{V}_h$ belongs to $\mathbf{V}_h$ if and only if

$$\sum_K \langle \mathbf{v}\cdot\mathbf{n}_K, m\rangle_{\partial K}=0, \qquad m\in\mathscr{L}_h. \tag{5.3}$$

The Fraeijs de Veubeke [12, 13] hybridization of (4.5) consists of finding the triple $\{\mathbf{q}_h, u_h, m_h\}\in\mathscr{V}_h\times W_h\times\mathscr{L}_h$ satisfying

$$(c\,\mathbf{q}_h, v)-\sum_K(\operatorname{div}\mathbf{v}, u_h)_K+\sum_K\langle\mathbf{v}\cdot\mathbf{n}_K, m_h\rangle_{\partial K}=\langle\mathbf{v}\cdot\mathbf{n}, g\rangle, \qquad \mathbf{v}\in\mathscr{V}_h, \tag{5.4a}$$

$$\sum_K(\operatorname{div}\mathbf{q}_h, w)_K=(f, w), \qquad w\in W_h, \tag{5.4b}$$

(5.4c) $$\sum_K \langle \mathbf{q}_h \cdot \mathbf{n}_K, p \rangle_{\partial K} = 0, \qquad p \in \mathscr{L}_h.$$

The *function* $\mathbf{q}_h$ given by (5.4) coincides with that of (4.5) as a result of (5.3), although the *parameters* defining it are different, since the dimension of $\mathscr{V}_h$ is greater than that of $\mathbf{V}_h$; u_h is also the same as before.

The original object of the hybridization was to produce a system of equations in which it is easy to eliminate the parameters defining first $\mathbf{q}_h$ and then u_h. The matrix $\mathscr{D}$ for the remaining equations for m_h is positive-definite and has a sparsity structure essentially equivalent to that of the matrix of the nonconforming finite element method using piecewise polynomials of degree $j+1$ for the scalar function u. As is the case with any three-dimensional problem, the equations associated with $\mathscr{D}$ are not solved inexpensively by Gaussian elimination techniques; however, it is much easier to find effective preconditioners for a conjugate gradient iteration for these equations than it is for those coming from (4.5).

Arnold and Brezzi [1] found in studying the hybridization of the Raviart-Thomas mixed method in two space variables that u_h and m_h can be postprocessed element-by-element to produce a new approximation u_h^* to u that is more rapidly convergent than u_h; their ideas were extended and completed in Brezzi et al. [3], both for the Raviart-Thomas spaces and for the spaces introduced there in two-space. The development of [3] can be carried over to the mixed elements of this paper with little difficulty and most of the details of the proofs can be omitted.

Let $Q_h = Q_h^j$ be the $L^2(e)$-projection into $P_j(e)$ for a face $e \subset G$. If the proof of Lemma 4.1 of [3] is modified by using the degrees of freedom given by (2.4), (2.7), (3.5), or (3.6), as appropriate for K, to define the vector $\mathbf{v}$ arising in that proof, then it follows that

(5.5) $$\|m_h - Q_h u\|_{0,e} \leqq C\{h_K^{1/2} \|\mathbf{q} - \mathbf{q}_h\|_{0,K} + h_K^{-1/2} \|u_h - R_h u\|_{0,K}\}.$$

We would like to show that (5.5) enables us to construct a new approximation u_h^* of u that converges to u with a better order as, for instance, in [1] or [3]. We present here a general strategy for constructing u_h^*. In each particular case one can then find different convenient choices. Without going into the details, let us, for each element K, denote by $\mathscr{M}(\partial K)$ the set of functions in $L^2(\partial K)$ that are polynomials of degree $\leqq j$ on each face e. We *assume* now that we are given, for each element K, a space of polynomials $\mathscr{P}(K)$ such that: 1) $P_{j+1}(K) \subseteq \mathscr{P}(K)$ and that: 2) for every $\zeta \neq 0$ in $\mathscr{M}(\partial K)$ there exists a $v \neq 0$ in $\mathscr{P}(K)$ such that

(5.6) $$\langle \zeta, v \rangle_{\partial K} \geqq c h_K^{\frac{1}{2}} |\zeta|_{0,\partial K} \{h_K^{-1} |v|_{0,K} + |v|_{1,K}\}.$$

Note that (5.6) is a kind of local inf-sup condition related to the mesh dependent norms $h_K^{\frac{1}{2}} |\zeta|_{0,\partial K}$ and $\{h_K^{-1} |v|_{0,K} + |v|_{1,K}\}$. Therefore (5.6) will be easily satisfied if we take $\mathscr{P}(K)$ rich enough (how rich will depend on the particular case). We consider now the following auxiliary problem:

(5.7) $$\begin{cases} \text{find } u_h^* \in \mathscr{P}(K) \text{ and } \zeta_h^* \in \mathscr{M}(K) \text{ such that} \\ (a\, \mathbf{grad}\, u_h^*, \mathbf{grad}\, v)_K - \langle \zeta_h^*, v \rangle_{\partial K} = (f, v)_K, \quad v \in \mathscr{P}(K) \\ \langle \zeta, u_h^* \rangle_{\partial K} = \langle \zeta, m_h \rangle_{\partial K}, \quad \zeta \in \mathscr{M}(\partial K). \end{cases}$$

It is clear that u_h^* will be an approximation of u and that ζ_h^* will be an approximation of $a\partial_n u$, the co-normal derivative of u with respect to ∂K. Standard arguments in the approximation of saddle points now yield

$$(5.8)\qquad |||u-u_h^*|||+|||a\partial_n u-\zeta_h^*||| \leqq c\{\inf_{v\in\mathscr{P}(K)}|||u-v|||+\inf_{\zeta\in\mathscr{M}(\partial K)}|||a\partial_n u-\zeta|||+\sup_{\zeta\in\mathscr{M}(\partial K)}(\langle\zeta, m_h-u\rangle_{\partial K})/|||\zeta|||\}$$

where the triple bars indicate the mesh dependent norms (in K and on ∂K, respectively) appearing in (5.6). Using now (5.8), (5.5) and standard approximation results we obtain:

$$(5.9)\qquad \|u-u_h^*\|_0\leqq c\{\|f\|_j+|g|_{j+3/2}\}\,h^{j+2}$$

for $j>1$ and

$$(5.10)\qquad \|u-u_h^*\|_0\leqq c\{\|f\|_1+|g|_{5/2}\}\,h^2$$

for $j=1$. It is worth noting that in particular cases one can often find a set of degrees of freedom in $\mathscr{P}(K)$ which includes explicitly the moments $\langle\zeta, v\rangle_{\partial K}$, $\zeta\in\mathscr{M}(\partial K)$. In this case one can set, for every $\chi\in L^2(\partial K)$,

$$(5.11)\qquad \mathscr{P}_\chi(K)=\{v\in\mathscr{P}(K),\ \langle\zeta, v-\chi\rangle_{\partial K}=0\ \ \forall\,\zeta\in\mathscr{M}(\partial K)\}$$

and solve (5.7) in the more convenient form

$$(5.12)\qquad u_h^*\in\mathscr{P}_{m_h}(K)\quad\text{and}\quad(a\,\mathbf{grad}\,u_h^*, \mathbf{grad}\,v)_K=(f, v)_K,\qquad v\in\mathscr{P}_0(K).$$

We have seen that, though nominally the mixed method using the elements of this paper produces approximation of u only to order $O(h^j)$ the computed solution $\{\mathbf{q}_h, u_h, m_h\}$ contains information sufficient to allow determining u with an accuracy $O(h^{j+2})$ when $j>1$. Since the algebraic problem associated with the matrix $\mathscr{D}$ is of the same complexity as that of the nonconforming Galerkin method that would produce the same order of accuracy, this is not altogether surprising.

6. Alternating-Direction Iteration

Recently there have been several alternating-direction iterative methods introduced to treat the solution of the algebraic equations arising from mixed methods [5, 7–9]. Brown [5] discussed a method for the Raviart-Thomas space [18] over rectangular elements in the plane; his method is an implicit, locally one-dimensional version of the Uzawa iterative technique for saddle-point problems. He extended essentially all of the known theoretical results for alternating-direction methods for finite difference or Galerking methods to the Raviart-Thomas mixed method in two variables; however, he did not cover the three-dimensional case. He also carried a reasonably extensive set of computational experiments that confirmed the observations that have been made over the past thirty years that the direct use of alternating-direction iteration on problems with variable coefficients on general domains leads to efficient convergence.

Douglas and Pietra [9] introduced a different alternating-direction method based on an implicit, locally one-dimensional Arrow-Hurwitz procedure; there is no finite difference or Galerkin analogue of this technique. There were two motivations for this procedure, the first being that it allows the taking advantage of initial information about the vector variable and the second being that the Uzawa iteration is not obviously applicable to the planar mixed elements of Brezzi et al. [3, 4] in an effective manner, while the Arrow-Hurwitz one is. When a better initial guess was known for the vector variable (as is quite often the case when the mixed method is applied to an elliptic equation arising in a transient problem described by a system of equations, usually nonlinear, containing one or more equations that would be elliptic if arising separately; most petroleum reservoir simulation models are of this character) than for the scalar variable, the Arrow-Hurwitz procedure significantly outperformed the Uzawa one when Raviart-Thomas elements were used. For the same set of test problems, the Arrow-Hurwitz scheme required almost exactly the same number of iterations to achieve a prescribed error reduction for the index one space of Brezzi, Douglas, and Marini as it did for the zero index Raviart-Thomas space, so that a second order correct approximation of the vector variable could be obtained, in place of a first order one, in less than twice as many arithmetic operations for the same grid.

Douglas et al. [7] formulated both Uzawa and Arrow-Hurwitz alternating-direction iterative methods for the three space variable Raviart-Thomas spaces. These procedures relate to an alternating-direction method of Douglas [6] for three space variable finite difference schemes. Again it was found that the Arrow-Hurwitz version outperformed the Uzawa one on problems for which good initial guesses were available for the vector variable. The same authors [8] have also considered an Arrow-Hurwitz iterative method for the spaces of this paper. We shall limit ourselves here to outlining a special case of this iterative technique applied to the space $\mathbf{M}_h^1$.

Let

$$-\Delta u = f, \quad x \in G = [0,1]^3, \tag{6.1 a}$$

$$u = 0, \quad x \in \partial G, \tag{6.1 b}$$

and take $\mathcal{T}_h$ to be the collection of cubes

$$K_{ijk} = X_i \times Y_j \times Z_k = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}], \tag{6.2}$$

where $x_i = ih$ and $h = N^{-1}$. We work with the space $\mathbf{V}_h$, not the hybrid space $\mathscr{V}_h$. A basis can be constructed as follows. Let $\mathbf{K} = [-1,1] \times [0,1] \times [0,1]$, and let

$$\mathbf{X}^1 = \begin{bmatrix} 1-|x| \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{X}^2 = \begin{bmatrix} 2(1-|x|)(y-0.5) \\ y(y-1)\operatorname{sgn} x \\ 0 \end{bmatrix}, \tag{6.3}$$

$$\mathbf{X}^3 = \begin{bmatrix} 2(1-|x|)(z-0.5) \\ 0 \\ z(z-1)\operatorname{sgn} x \end{bmatrix}.$$

The basis function $\mathbf{X}^1$ represents the value of $q_1(0, 0.5, 0.5)$, while $\mathbf{X}^2$ represents $\partial q_1/\partial y(0, 0.5, 0.5)$ and $\mathbf{X}^3$ represents $\partial q_1/\partial z(0, 0.5, 0.5)$. Similarly, let $\mathbf{Y}^1$, $\mathbf{Y}^2$, and $\mathbf{Y}^3$ represent $q_2(0.5, 0, 0.5)$, $\partial q_2/\partial x(0.5, 0, 0.5)$, and $\partial q_2/\partial z(0.5, 0, 0.5)$ on $[0,1]\times[-1,1]\times[0,1]$ and $\mathbf{Z}^1$, $\mathbf{Z}^2$, and $\mathbf{Z}^3$ represent $q_3(0.5, 0.5, 0)$, $\partial q_3/\partial x(0.5, 0.5, 0)$, and $\partial q_3/\partial y(0.5, 0.5, 0)$ on $[0,1]\times[0,1]\times[-1,1]$.

The vector $\mathbf{q}_h$ can be written in the form

(6.4)
$$\begin{aligned}\mathbf{q}_h = \sum_{ijk} \{&\alpha_{ijk}\mathbf{X}^1_{ijk} + \beta_{ijk}\mathbf{X}^2_{ijk} + \gamma_{ijk}\mathbf{X}^3_{ijk} \\ &+ \mu_{ijk}\mathbf{Y}^1_{ijk} + \nu_{ijk}\mathbf{Y}^2_{ijk} + \eta_{ijk}\mathbf{Y}^3_{ijk} \\ &+ \sigma_{ijk}\mathbf{Z}^1_{ijk} + \omega_{ijk}\mathbf{Z}^2_{ijk} + \theta_{ijk}\mathbf{Z}^3_{ijk}\},\end{aligned}$$

with $\mathbf{X}^m_{ijk}$ obtained from $\mathbf{X}^m$ by the usual affine transformations. Note that the indices $\{i, j, k\}$ can be considered to range over all of the integers by considering a periodic extension of the problem (6.1). The scalar u_h is constant on K_{ijk} and takes the value u_{ijk} there; for convenience, let $v_{ijk} = 6h^{-1}u_{ijk}$. Now set

(6.5a) $$\delta_1 v_{ijk} = v_{ijk} - v_{i-1,j,k}, \qquad d_1 v_{ijk} = v_{i+1,j,k} - v_{ijk},$$

(6.5b) $$m_1 v_{ijk} = 0.5(v_{ijk} + v_{i-1,j,k}), \qquad m_1^* v_{ijk} = 0.5(v_{ijk} + v_{i+1,j,k}),$$

(6.5c) $$S_1 \alpha_{ijk} = \alpha_{i-1,j,k} + 4\alpha_{ijk} + \alpha_{i+1,j,k},$$

(6.5d) $$T_1 \beta_{ijk} = 4\beta_{i-1,j,k} + 52\beta_{ijk} + 4\beta_{i+1,j,k},$$

with $\delta_2, \ldots, T_3$ defined analogously. Then, the equations for the mixed method for $\mathbf{M}_h^1$ take the form

(6.6a) $$S_1\alpha + m_1 d_2 \nu + m_1 d_3 \omega + \delta_1 v = 0,$$

(6.6b) $$S_2\mu + m_2 d_1 \beta + m_2 d_3 \theta + \delta_2 v = 0,$$

(6.6c) $$S_3\sigma + m_3 d_1 \gamma + m_3 d_2 \eta + \delta_3 v = 0,$$

(6.6d) $$T_2 \nu - 30\delta_2 m_1^* \alpha - 6\delta_2 d_3 \omega = 0,$$

(6.6e) $$T_3 \omega - 30\delta_3 m_1^* \alpha - 6\delta_3 d_2 \nu = 0,$$

(6.6f) $$T_1 \beta - 30\delta_1 m_2^* \mu - 6\delta_1 d_3 \theta = 0,$$

(6.6g) $$T_3 \theta - 30\delta_3 m_2^* \mu - 6\delta_3 d_1 \beta = 0,$$

(6.6h) $$T_1 \gamma - 30\delta_1 m_3^* \sigma - 6\delta_1 d_2 \eta = 0,$$

(6.6i) $$T_2 \eta - 30\delta_2 m_3^* \sigma - 6\delta_2 d_1 \gamma = 0,$$

(6.6j) $$d_1\alpha + d_2\mu + d_3\sigma = \varphi, \qquad \varphi_{ijk} = h^{-1}\int_Q f\,d\mathbf{x}, \qquad Q = K_{ijk}.$$

Equations (6.6a), (6.6f), and (6.6h) have leading parts $S_1\alpha$, $T_1\beta$, and $T_1\gamma$, respectively, which are locally one-dimensionally oriented in the x-direction. Similarly, (6.6b), (6.6d), and (6.6i) have leading parts locally one-dimensionally oriented in the y-direction and (6.6c), (6.6e), and (6.6g) in the z-direction. Each triple can be aligned with (6.6j) to lead to the following alternating-direction process:

x-sweep

(6.7a) $S_1[(\alpha^{n+1}-\alpha^n)/\xi+\alpha^{n+1}]+m_1 d_2 v^n+m_1 d_3 \omega^n+\delta_1 v^*=0,$

(6.7b) $T_1[(\beta^{n+1}-\beta^n)/\zeta+\beta^{n+1}]-30\delta_1 m_2^* \mu^n-6\delta_1 d_3 \theta^n=0,$

(6.7c) $T_1[(\gamma^{n+1}-\gamma^n)/\zeta+\gamma^{n+1}]-30\delta_1 m_3^* \sigma^n-6\delta_1 d_2 \eta^n=0,$

(6.7d) $(v^*-v^n)/\tau^n+d_1(\alpha^{n+1}+\alpha^n)/2+d_2\mu^n+d_3\sigma^n=\varphi;$

y-sweep

(6.8a) $S_2[(\mu^{n+1}-\mu^n)/\xi+\mu^{n+1}]+m_2 d_1 \beta^{n+1}+m_2 d_3 \theta^n+\delta_2 v^{**}=0,$

(6.8b) $T_2[(v^{n+1}-v^n)/\zeta+v^{n+1}]-30\delta_2 m_1^* \alpha^{n+1}-6\delta_2 d_3 \omega^n=0,$

(6.8c) $T_2[(\eta^{n+1}-\eta^n)/\zeta+\eta^{n+1}]-30\delta_2 m_3^* \sigma^n-6\delta_2 d_1 \gamma^{n+1}=0,$

(6.8d) $(v^{**}-v^n)/\tau^n+[d_1(\alpha^{n+1}+\alpha^n)+d_2(\mu^{n+1}+\mu^n)]/2+d_3\sigma^n=\varphi;$

z-sweep

(6.9a) $S_3[(\sigma^{n+1}-\sigma^n)/\xi+\sigma^{n+1}]+m_3 d_1 \gamma^{n+1}+m_3 d_2 \eta^{n+1}+\delta_3 v^{n+1}=0,$

(6.9b) $T_3[(\omega^{n+1}-\omega^n)/\zeta+\omega^{n+1}]-30\delta_3 m_1^* \alpha^{n+1}-6\delta_3 d_2 v^{n+1}=0,$

(6.9c) $T_3[(\theta^{n+1}-\theta^n)/\zeta+\theta^{n+1}]-30\delta_3 m_2^* \mu^{n+1}-6\delta_3 d_1 \beta^{n+1}=0,$

(6.9d) $(v^{n+1}-v^n)/\tau^n+[d_1(\alpha^{n+1}+\alpha^n)+d_2(\mu^{n+1}+\mu^n)+d_3(\sigma^{n+1}+\sigma^n)]/2=\varphi.$

No complete spectral analysis has been made for this iteration; however on the basis of experience with the two-dimensional analogue [9] of these elements, we conjecture that use of the same parameters for $\{\tau^n\}$ as for the Uzawa alternating-direction method for the three-dimensional Raviart-Thomas elements will lead to rapid convergence when combined with constant choices for ξ and ζ, which should be chosen such that $\xi>(\sin 0.5\pi h)^{-1}$ and $\zeta\approx 10\xi$. The cycle for $\{\tau^n\}$ is given [6, 7] by

(6.10a) $\rho_m=\rho_1\approx\pi^2h^2/6, \quad \rho_M=\rho_N\approx 2, \quad N=1/h;$

(6.10b) $\tau^1=2\beta/\rho_m; \quad \tau^n=\beta\gamma^{-1}\tau^{n-1}, \quad n=2,\dots,NC;$

(6.10c) $NC=[\log(\rho_M/\rho_m)\div\log(\beta\gamma^{-1})]+1;$

here, $\beta<1<\gamma$ and $\Upsilon(\beta,0,0)=\Upsilon(\gamma,\gamma,\gamma)$ and

(6.11) $\Upsilon(a,b,c)=1-2(a+b+c)(1+a)^{-1}(1+b)^{-1}(1+c)^{-1}.$

In the Uzawa version in three space it was found [7] that the optimal choice for γ was about 1.71 for $h=0.1$, and the corresponding cycle length was four. For a more extensive treatment of the iterative procedure given above, see [8].

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