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The Method of Fractional Steps for Conservation Laws

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Summary. The stability, accuracy, and convergence of the basic fractional step algorithms are analyzed when these algorithms are used to compute discontinuous solutions of scalar conservation laws. In particular, it is proved that both first order splitting and Strang splitting algorithms always converge to the unique weak solution satisfying the entropy condition. Examples of discontinuous solutions are presented where both Strang-type splitting algorithms are only first order accurate but one of the standard first order algorithms is infinite order accurate. Various aspects of the accuracy, convergence, and correct entropy production are also studied when each split step is discretized via monotone schemes, Lax-Wendroff schemes, and the Glimm scheme.

Subject Classifications: AMS (MOS): 65 M10; 65 M05, 35L65.

Introduction

Most of the popular algorithms for the numerical computation of discontinuous solutions of the conservation laws,

$$\begin{aligned} u_t+(f_1(u))_x+(f_2(u))_y&=0 \\ u(x,y,0)&=u_0 \end{aligned} \tag{0.1}$$

involve the method of fractional steps either as introduced into gas dynamics by Godunov [10] or as modified by Strang [23]. These methods can be summarized as follows: Let $u(x,y,t)\equiv S(t)\,u_0$ denote the unique weak solution to (0.1) which satisfies the entropy conditions (see Sect. 2 for precise statements when u is a scalar function and [15] generally) and let $v(x,y,t)\equiv S^x(t)\,v_0$, $w(x,y,t)\equiv S^y(t)\,w_0$ denote the analogous weak solutions satisfying the one-dimensional conservation laws,

$$\begin{aligned} v_t+(f_1(v))_x&=0, & w_t+(f_2(w))_x&=0 \\ v(x,y,0)&=v_0(x,y) & w(x,y,0)&=w_0(x,y) \end{aligned} \tag{0.2}$$

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(where y is viewed as a parameter in $S^x(t)v_0$, etc.) Godunov's fractional step method is based on either of the approximations,

$$S(T)u_0 \cong (S^x(\Delta t)S^y(\Delta t))^n u_0, \quad n\Delta t = T$$

or (0.3)

$$S(T)u_0 \cong (S^y(\Delta t)S^x(\Delta t))^n u_0, \quad n\Delta t = T$$

while Strang's method instead uses the approximation

$$S(T)u_0 \cong \left(S^y\left(\frac{\Delta t}{2}\right)S^x\left(\frac{\Delta t}{2}\right)S^x\left(\frac{\Delta t}{2}\right)S^y\left(\frac{\Delta t}{2}\right)\right)^n u_0$$

$$\equiv \left(S^y\left(\frac{\Delta t}{2}\right)S^x(\Delta t)S^y\left(\frac{\Delta t}{2}\right)\right)^n u_0, \quad n\Delta t = T \tag{0.4}$$

or

$$S(T)u_0 \cong \left(S^x\left(\frac{\Delta t}{2}\right)S^y(\Delta t)S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0, \quad n\Delta t = T.$$

In practical computer calculations, next, one introduces one-dimensional nonlinear difference approximations,

$$G^x(\Delta t) \cong S^x(\Delta t)$$
$$G^y(\Delta t) \cong S^y(\Delta t) \tag{0.5}$$

into either of the splitting algorithms in (0.3), (0.4) to define a fully discrete splitting method. When Godunov introduced the algorithm in (0.3), he used the (monotone) Godunov scheme for G^x and G^y (see [22]). Recently, Chorin [3, 4] has implemented (0.3) in gas dynamics and combustion calculations by using the Glimm scheme for G^x and G^y; this algorithm has also been applied (see [6]) to the flow of immiscible fluids in porous media (the Buckley-Leavritt equation) with u in (0.1) a scalar function, $f_1(u) \equiv a f_2(u)$, and $f_2(u)$ a nonconvex flux function. The method of Strang defined in (0.4) has been implemented extensively in gas dynamics with Lax-Wendroff difference methods (see [17], for example) and other schemes.

Here we rigorously study the accuracy, stability, and convergence of the basic fractional step algorithms in (0.3) and (0.4) when *discontinuous* solutions are computed. We also study various aspects of the accuracy, stability, and entropy production of the fully discrete splitting schemes when G^x, G^y are given by monotone schemes, the Glimm scheme, and the Lax-Wendroff scheme. All of our rigorous results are necessarily restricted to the model problems where the function u in (0.1) is a *scalar* and $f_1(u), f_2(u)$ are smooth *scalar* functions. In a single space variable, the strictly convex or strictly concave conservation laws with $|f_1''(u)| \geqq \delta > 0$ have solutions with a discontinuous wave structure which yields a simplified model for the shock structure in one dimensional gas dynamics – $f_1(u) \equiv \frac{1}{2}u^2$ corresponds to the inviscid Burger's equation. On the other hand, in a single space variable, when $f_1(u)$ has inflection points, the wave structure of the solutions of (0.1) (with $f_2(u) \equiv 0$) is radically different from that of the convex case [17]. The equations of gas dynamics are rotationally invariant and the structure of shock solutions is essentially independent of direction. Thus, the multidimensional scalar equations which provide a simplified model for the shock structure occurring in polytropic gas dynamics are those so that given *any* unit direction $\vec{n}=(n_1, n_2)$, the function, $n_1 f_1(u)+n_2 f_2(u)$ is strictly convex, strictly concave, or constant. One verifies easily that this 'isotropic' condition is satisfied only when (with constants a, b, c and $\delta>0$)

$$|f_1''(u)|\geqq\delta, \quad f_2(u)\equiv a f_1(u)+bu+c. \tag{0.6}$$

The analogous models in two space variables of solutions of the one-dimensional inviscid Burger's equation are given by

$$\begin{gathered}\frac{\partial u}{\partial t}+\frac{\partial a\frac{1}{2}u^2}{\partial x}+\frac{\partial \frac{1}{2}u^2}{\partial y}=0, \quad a>0\\ u(x,y,0)=u_0(x,y).\end{gathered} \tag{0.7}$$

Obviously, there are two sources of error in the fully discrete fractional step algorithms - the intrinsic error involved in using (0.3) or (0.4) and the spatial discretization errors involved in the particular difference methods used in (0.5). In general these two sources of error interact in a complex fashion. Strang introduced the method in (0.4) because, even for systems, it is second-order accurate in time, i.e.,

when $S(t)u_0$ is a sufficiently smooth function

$$\left\|S(T)u_0-\left(S^x\left(\frac{\Delta t}{2}\right)S^y(\Delta t)S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0\right\|\leqq C(\Delta t)^2 \tag{0.8}$$

whereas the method in (0.3) is only first order accurate, when $S(t)u_0$ is smooth

$$\|S(T)u_0-(S^y(\Delta t)S^x(\Delta t))^n u_0\|\leqq C(\Delta t) \tag{0.9}$$

with similar estimates when the roles of x and y are reversed. In Sect. 1, in contrast to the results valid when $S(t)u_0$ is smooth, we construct a family of simple examples of *discontinuous* solutions of the model equations in (0.7) with $2>a\geqq 1$ so that both Strang-type splitting algorithms are at most first order accurate when computing these solutions but

$$(S^x(\Delta t)S^y(\Delta t))^n u_0\equiv S(T)u_0, \quad n\Delta t=T \tag{0.10}$$

i.e., one of the algorithms in (0.3) is infinite order accurate. In a single space dimension, the Glimm scheme keeps simple shock fronts perfectly sharp (see [2, 11]). A basic question is the following one. When the Glimm scheme is used in multi-dimensions in conjunction with splitting algorithms, are shock fronts smeared substantially? The example mentioned in (0.10) provides a good analytic test case and in Sect. 1 we explicitly compute the effect of the Glimm scheme on this solution.

In Sect. 2 we summarize some known properties of weak solutions of (0.1) satisfying the entropy conditions (see [13]) and list some simple estimates and notations useful in the subsequent sections. In Sect. 3, we prove the following result.

Theorem 1. *Assume* $u_0 \in L^1(R^2) \cap L^\infty(R^2)$ *and let* $S(t)u_0$ *denote the unique weak solution of* (0.1) *satisfying the entropy conditions, then both of the fractional step algorithms in* (0.3) *and* (0.4) *always converge to* $S(t)u_0$. *More precisely, if* $n\Delta t = T$, *then as* $n \to \infty$,

$$\max_{0 \leqq T \leqq T_0} \|S(T)u_0 - (S^y(\Delta t)S^x(\Delta t))^n u_0\|_{L^1(R^2)} \to 0$$

$$\max_{0 \leqq T \leqq T_0} \left\| S(T)u_0 - \left(S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t) S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0 \right\|_{L^1(R^2)} \to 0.$$

Similar results are valid with the roles of x and y reversed.

The results in Theorem 1 are also interesting from another point of view since they provide proofs in a singular case for the Trotter-Kato product formulas of functional analysis. The nonlinear semigroups studied here act on non-reflexive Banach spaces with a priori set-valued generators [7]; none of the abstract theorems in the current literature handle this generality (see [1, 5]).

Section 4 contains some brief remarks where we use the main result in [8] to prove that discrete splitting algorithms always converge to $S(t)u_0$ provided that G^x and G^y are monotone with conservation form. In particular, these results apply to the splitting algorithms using the Lax-Friedrichs, Godunov, and upwind schemes in any combination.

In Sect. 5 we study the discrete fractional step algorithms when G^x and G^y are given by the Lax-Wendroff scheme. As in [19], assuming this algorithm converges, we study when the limit function obeys the entropy condition. The same examples as constructed in a single space dimension (see [12, 20]) provide rigorous proof that the standard split Lax-Wendroff scheme can converge to weak solutions which violate the entropy condition even when $f_1 \equiv f_2$ and $|f_1''(u)| > 0$. Nevertheless, when the Lax-Wendroff scheme is modified with a systematic nonlinear artificial viscosity as used by Osher and Majda in [18, 19], we prove that this limit function satisfies a discrete entropy inequality provided that $|f_1''(u)|, |f_2''(u)| \geqq \delta > 0$. This entropy inequality is strong enough to prove that the limit solution is the unique physical solution provided that (0.6) is satisfied - in particular, this theorem applies to the models in (0.7). Perhaps surprisingly, we conjecture that this conclusion is false when (0.6) is violated even though $f_1(u), f_2(u)$ are both convex and give examples to support this reasoning. In the appendix, we give the proof of a convergence theorem for generadimensional splitting algorithms. Similar results hold 'mutatis mutandi' for general additive fractional step methods.

Section 1. The Accuracy of Splitting Algorithms when Discontinuous Solutions are Computed

A. Strang Splitting and First Order Splitting

We consider the solution, u, of the model equation

$$u_t + (a\tfrac{1}{2}u^2)_x + (\tfrac{1}{2}u^2)_y = 0,$$

with initial data

$$u(x,y,0)=u_0\equiv\begin{cases}1, & y>x\\ -1, & y\leqq x\end{cases}\tag{1.1}$$

for a fixed constant a, $1\leqq a\leqq 2$. One computes directly either by using the well-known entropy condition for the inviscid Burger's equation or by using the conditions in Sect. 2 that

$$u(x,y,t)\equiv S(t)\,u_0\equiv u_0.\tag{1.2}$$

For $1<a$, the exact solution is an oblique steady planar shock, while for $a\equiv 1$ the exact solution is an oblique steady contact discontinuity. We observe that $S^x(t)\,u_0\equiv u_0$ is a shock wave but $S^y(t)\,u_0$ is a rarefaction wave. Using these facts we calculate explicitly that for $t>a\Delta t$,

$$S^x(\Delta t)\,S^y(t)\,u_0\equiv\begin{cases}1, & x\leqq -t+a\Delta t+y\\ \dfrac{(x-y)}{a\Delta t-t}, & a\Delta t-t+y\leqq x\leqq t-a\Delta t+y\\ -1, & x\geqq t-a\Delta t+y\end{cases}\tag{1.3}$$

for $t\leqq a\Delta t$,

$$S^x(\Delta t)\,S^y(t)\,u_0\equiv u_0.$$

By applying the semigroup property and (1.3) we compute that

$$\begin{aligned}&\left(S^y\left(\frac{\Delta t}{2}\right)S^x(\Delta t)\,S^y\left(\frac{\Delta t}{2}\right)\right)^n u_0\\ &=S^y\left(\frac{\Delta t}{2}\right)(S^x(\Delta t)\,S^y(\Delta t))^{n-1}\left(S^x(\Delta t)\,S^y\left(\frac{\Delta t}{2}\right)\right)u_0\\ &\equiv S^y\left(\frac{\Delta t}{2}\right)u_0,\qquad \text{for any } 1\leqq a.\end{aligned}\tag{1.4}$$

In the same fashion, we consider the other Strang-type iterate and deduce that

$$\begin{aligned}&\left(S^x\left(\frac{\Delta t}{2}\right)S^y(\Delta t)\,S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0\\ &=S^x\left(\frac{\Delta t}{2}\right)S^y(\Delta t)(S^x(\Delta t)\,S^y(\Delta t))^{n-1}\,S^x\left(\frac{\Delta t}{2}\right)u_0\\ &\equiv S^x\left(\frac{\Delta t}{2}\right)S^y(\Delta t)\,u_0.\end{aligned}\tag{1.5}$$

If we restrict a to $1\leqq a<2$ and compute the L^1 error over the ball of radius one centered at the origin, it follows from (1.3), (1.4), and (1.5) that *both* of the *iterates* from *Strang-type* splitting have the explicit *first order errors,*

$$\left\|S(T)\,u_0-\left(S^y\left(\frac{\Delta t}{2}\right)S^x(\Delta t)\,S^y\left(\frac{\Delta t}{2}\right)\right)^n u_0\right\|_{L^1(B_1)}\geqq C\Delta t\tag{1.6}$$

for any $a \geqq 1$ and

$$\left\| S(T)u_0 - \left(S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t) S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0 \right\|_{L^1(B_1)} \geqq C\Delta t$$

for $1 \leqq a < 2$, with $C > 0$, $n\Delta t = T$.

On the other hand, we use (1.3) and compute that the splitting algorithm from (0.3) satisfies

$$(S^x(\Delta t) S^y(\Delta t))^n u_0 = u_0 \equiv S(T) u_0, \qquad n\Delta t = T \tag{1.7}$$

for $a \geqq 1$, thus this *algorithm* is *infinite order accurate* for the above discontinuous solution as asserted in the introduction! We remark that the other algorithm using (0.3) with the roles of x and y reversed has a first order error since by (1.3)

$$(S^y(\Delta t) S^x(\Delta t))^n u_0 = S^y(\Delta t)(S^x(\Delta t) S^y(\Delta t))^{n-1} u_0 \equiv S^y(\Delta t) u_0. \tag{1.8}$$

The examples in (1.6) and (1.7) indicate that under appropriate circumstances the second-order accurate Strang-type splittings can generate more dispersion near discontinuities than the first-order accurate algorithms from (0.3). How typical are the above examples in gas dynamics? The crucial facts needed to generate dispersion in splitting algorithms in the above examples were the following: (1) The exact solution to be computed contains an oblique sharp discontinuity (shock or contact); (2) $S^x(t)u_0$ contains sharp discontinuities; (3) $S^y(t)u_0$ contains only (dispersive) rarefaction fronts. One *computational situation in gas dynamics where conditions* (1)–(3) are satisfied *involves* the *numerical computation of* an *oblique* shear flow. Such shear flows can be produced when shock waves collide. To fix our ideas we consider a solution of the equations of isentropic gas dynamics

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) = \frac{-\partial}{\partial x}(p(\rho)),$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v^2) = -\frac{\partial}{\partial y}(p(\rho)),$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

with initial data,

$$\left.\begin{pmatrix} u \\ v \\ \rho \end{pmatrix}\right|_{t=0} = \begin{pmatrix} u_0 \\ v_0 \\ \rho_0 \end{pmatrix} = \begin{cases} \begin{pmatrix} a \\ a \\ \rho_0 \end{pmatrix}, & y \geqq x \\[2ex] \begin{pmatrix} b \\ b \\ \rho_0 \end{pmatrix}, & y < x, \ a > b. \end{cases} \tag{1.9}$$

We claim that the same conditions as (1)–(3) above are satisfied for the y-component, v, of velocity. The exact solution is the steady oblique contact discontinuity, $'(u_0, v_0, \rho_0)$; under the x-sweep in splitting, v_0 is transported as a sharp contact discontinuity; under the y sweep applied to the initial data, since $a>b$, $v(x, y, t)$ has two dispersive rarefaction fronts. Of course, this is only a plausibility argument since in this case exact closed-form solutions are essentially impossimble to compute for the splitting algorithms.

Next, we compute a family of discontinuous solutions where all the fractional step algorithms in (0.3) and (0.4) are infinite order accurate. These examples provide other good test problems to compute the amount of numerical dissipation and dispersion at discontinuities due to the choice of the difference operators G^x and G^y. We consider the equations in (0.7) with $a>0$ and initial data

$$u_0(x, y)=\tilde{u}(x+y) \quad \text{where} \quad \tilde{u}(q)=\begin{cases} u_L, & q\leqq 0 \\ u_R, & q>0 \\ u_L>u_R \end{cases} \tag{1.10}$$

The exact solution of (0.7) with $a>0$ is an oblique shock wave,

$$S(T)\,u_0\equiv\tilde{u}\left(x+y-(1+a)\left(\frac{u_L+u_R}{2}\right)T\right). \tag{1.10)(a}$$

Furthermore, because of the special choice of initial data, both the x and y sweeps are also shock waves,

$$\begin{aligned} S^x(\Delta t)\,u_0&\equiv\tilde{u}\left(x+y-a\left(\frac{u_L+u_R}{2}\right)\Delta t\right) \\ S^y(\Delta t)\,u_0&\equiv\tilde{u}\left(x+y-\left(\frac{u_L+u_R}{2}\right)\Delta t\right). \end{aligned} \tag{1.10)(b}$$

It follows easily from the above computations that

$$(S^y(\Delta t)\,S^x(\Delta t)^n\,u_0\equiv\left(S^x\left(\frac{\Delta t}{2}\right)S^y(\Delta t)\,S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0\equiv S(T)\,u_0 \tag{1.11}$$

with $n\Delta t=T$ and similarly with the roles of x and y reversed so that all splitting algorithms in (0.3) and (0.4) are infinite order accurate.

The above examples of solutions of (0.1) with planar discontinuities belie the complicated singular structure that can be present in solutions of (0.1). See [23] where examples with initial data given by three constant states forming a vertex are constructed.

B. Does Splitting With Glimm's Method Smear Discontinuities?

Here we present some closed form examples which (hopefully) give some insight into the above question. For simplicity in exposition, we only study initial data u_0 with $|u_0|\leqq 1$. First, we recall the definition of the deterministic Glimm scheme ([1]) in a single space variable when used to compute solutions of

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{a}{2}u^2\right)=0$$

$$u(x,0)=u_0, \qquad |u_0|\leqq 1.$$

Given a grid function, $u_j^n \cong u(j\Delta x, n\Delta t)$, piecewise constant for $(j-\frac{1}{2})\Delta x \leqq x<(j+\frac{1}{2})\Delta x$, the Glimm scheme, $G_{\alpha_n}^x$, is a staggered-grid method which computes the values of the grid function, $u_{j+\frac{1}{2}}^{n+1}$, piecewise constant on $(j\Delta x)\leqq x\leqq(j+1)\Delta x$, by finding the exact solution of the Riemann problem,

$$\frac{\partial u_j}{\partial t}+\frac{\partial}{\partial x}a(\tfrac{1}{2}u_j^2)=0$$

$$u_j(x,0)=\begin{cases}u_j^n, & x\leqq 0\\ u_{j+1}^n, & x>0\end{cases}$$

and defining $u_{j+\frac{1}{2}}^{n+1}\equiv(G_{\alpha_n}^x(u_j^n))_{j+\frac{1}{2}}$ via the recipe

$$u_{j+\frac{1}{2}}^{n+1}\equiv u_j(\alpha_n\Delta x, \Delta t)\equiv(G_{\alpha_n}^x(u_j^n))_{j+\frac{1}{2}} \tag{1.12}$$

where α_n is a number with $\alpha_n\in[-\frac{1}{2},\frac{1}{2}]$. Since $|u_0|\leqq 1$, this scheme is expected to be consistent provided that nearby Riemann problems do not interact during the time Δt, i.e. $a\frac{\Delta t}{\Delta x}\leqq\frac{1}{2}$. Liu has remarked that the above algorithm can be expected to converge only if the sequence, $\{\alpha_j\}$, is equidistributed in $[-\frac{1}{2},\frac{1}{2}]$, i.e., if I is any subinterval in $[-\frac{1}{2},\frac{1}{2}]$,

$$\lim_{N\to\infty}\frac{\#\{\alpha_j\in I, 1\leqq j\leqq N\}}{N}=m(I) \tag{1.13}$$

where $m(I)$ is the length of I.

We use the above preliminary information to define the discrete *Glimm splitting algorithm* for solutions of the two dimensional inviscid Burger's equation in (0.7) with $1\leqq a<2$ by

$$\begin{aligned} u_{j+\frac{1}{2},k+\frac{1}{2}}^n &= \prod_{r=1}^n (G_{\beta_r}^x G_{\alpha_r}^y)(u_{0,j,k}), \quad n \text{ odd}\\ u_{j,k}^n &= \prod_{r=1}^n (G_{\beta_r}^x G_{\alpha_r}^y)(u_{0,j,k}), \quad n \text{ even}\end{aligned} \tag{1.14}$$

where (for simplicity)

$$\Delta t\equiv\Delta x\gamma\equiv\Delta y\gamma \quad \text{with} \quad \gamma \text{ fixed}, \quad 0<a\gamma\leqq\tfrac{1}{2}. \tag{1.15}$$

Here $\{\alpha_j\}$, $\{\beta_j\}$ are two equidistributed sequences.

Next, we compute an iterate of the split Glimm scheme for the special initial data

$$u_{0,j,k}\equiv\begin{cases}1, & k>j\\ -1, & k\leqq j\end{cases} \tag{1.16}$$

which is a discretization of the initial data in (1.1). As we calculated in (1.7), the splitting algorithm in (0.3) is infinite order accurate on this data even though y sweeps are dispersive – thus, all smearing of shocks is caused by the method of discretizing $S^x(\Delta t)$ and $S^y(\Delta t)$. A tedious calculation yields that

$$(G^x_{\beta_1} G^y_{\alpha_1}(u_0))_{j+\frac{1}{2},k+\frac{1}{2}} = \begin{cases} 1, & k>j+1 \\ -1, & k=j+1,\ \alpha_1 \leqq -\gamma,\ \beta_1>0 \\ \dfrac{\alpha_1}{\gamma}, & k=j+1,\ -\gamma \leqq \alpha_1 \leqq \gamma,\ \beta_1 > a\dfrac{\gamma}{2}\left(1+\dfrac{\alpha_1}{\gamma}\right) \\ 1, & k=j+1, \quad \text{otherwise} \\ 1, & k=j,\ \gamma \leqq \alpha_1,\ \beta_1<0 \\ \dfrac{\alpha_1}{\gamma}, & k=j,\ -\gamma \leqq \alpha_1 \leqq \gamma,\ \beta_1 \leqq a\dfrac{\gamma}{2}\left(-1+\dfrac{\alpha_1}{\gamma}\right) \\ -1, & k=j, \quad \text{otherwise} \\ -1, & k<j. \end{cases} \tag{1.17}$$

From (1.17) it follows that for fixed γ the probability is $2\gamma(1-a\gamma)$ that the shock is smeared over one band around $k=j$ after a single iteration. This smearing is a new source of error due solely to the discretization via the split Glimm method with a structure different than that encountered in one-dimensional shock calculations with Glimm's method. For the remaining choices of α_1, β_1, as in a single space dimension, the shock front is kept perfectly sharp and only errors in location occur. One can observe that after N iterations there is a non-zero probability (quite small) that the shock is smeared over N bands around $K=j$ and a fairly large probability that it is smeared over at least one band. In fact, for simplicity set $a=1$ and let P_N denote the probability that the shock front is smeared over at least one band around $K=J$ after N iterations (ignoring errors due solely to location). It follows from an explicit argument (which we omit) and the fact, $\mathsf{P}_I = 2\gamma(1-\gamma)$, that

$$\mathsf{P}_{N+1} \geqq 2\gamma(1-\gamma)(1-\mathsf{P}_N) + 2\gamma\,\mathsf{P}_N + (1-2\gamma)(1-\gamma)\,\mathsf{P}_N$$

thus,

$$\mathsf{P}_N \geqq 2\gamma(1-\gamma)\frac{1-(2\gamma+(1-4\gamma)(1-\gamma))^N}{1-(2\gamma+(1-4\gamma)(1-\gamma))}, \qquad 0<\gamma\leqq\tfrac{1}{2}.$$

In particular, when $\gamma=\frac{1}{2}$, $\mathsf{P}_N = 1-(\frac{1}{2})^N$ so that for $\Delta t \to 0$, $N\Delta t = T_0$ fixed, numerical smearing over at least one band almost surely occurs. (Sharper quantitative estimates are interesting here but we have not attempted to derive any.)

In the above, general strategies with $\{\beta_j\}$, $\{\alpha_j\}$ chosen independently were analyzed and with probability always greater than $2\gamma(1-a\gamma)$, some numerical dispersion occurs. Next, we verify that for $a\equiv 1$ numerical *smearing* is completely *eliminated* by using the special strategy, $\alpha_j \equiv \beta_j$. From (1.17) we calculate in this case that

$$((G^x_{\beta_1} G^y_{\alpha_1})(u_0))_{j+\frac{1}{2},k+\frac{1}{2}} \equiv u_0((j+\tfrac{1}{2})\Delta x, (k+\tfrac{1}{2})\Delta y) \tag{1.18}$$

so that

$$\prod_{r=1}^{N} (G^x_{\beta_r} G^y_{\alpha_r})(u_0) \equiv u_0 = S(T)\, u_0 \tag{1.19}$$

and the split Glimm algorithm for this special strategy and initial data (with probability one) is infinite order accurate. Of course, there will be errors using this special strategy when the roles of x and y are reversed and the result in (1.19) is an accident.

The above examples in (1.17) and (1.19) indicate different behavior for the accuracy of the split Glimm scheme near discontinuities than that encountered for the Glimm scheme in a single space variable. When both the x and y sweeps are shock waves, the accuracy of the Glimm scheme near discontinuities is less sensitive to special strategies relating $\{\alpha_j\}$, $\{\beta_j\}$ and the structure of the error behaves as in a single space dimension [4, 15]. To illustrate this point, we compute the effect of the split Glimm scheme with the initial data in (1.10) and $0 < a \leqq 1$, $|\tilde{u}| \leqq 1$ so that the condition in (1.15) on the time steps, Δt, is satisfied – as computed in (1.11) all errors are due to discretization. For N even, it follows that

$$\left(\prod_{r=1}^{N} (G^x_{\beta_r} G^y_{\alpha_r})(u_0)\right)_{j,k} \equiv \tilde{u}(j\,\Delta x + k\,\Delta y - J_1^N\,\Delta y - J_2^N\,\Delta x) \tag{1.20}$$

where

$$\begin{aligned} J_1^N &= \#\left\{\alpha_r \,|\, 1 \leqq r \leqq N \,|\, \alpha_r < \gamma\, \frac{(u_L + u_R)}{2}\right\} \\ J_2^N &= \#\left\{\beta_r \,|\, 1 \leqq r \leqq N \,|\, \beta_r < a\gamma\, \left(\frac{u_L + u_R}{2}\right)\right\}. \end{aligned} \tag{1.21}$$

Thus, the front is kept perfectly sharp and all errors are due to location. As Lax [15] has observed in a single space dimension, the error in location can be made quite small by choosing special sequences $\{\alpha_j\}$, $\{\beta_j\}$ which are particularly well-distributed, i.e., such that

$$\begin{aligned} \frac{\#\{\alpha_r \in I,\ 1 \leqq r \leqq N\}}{N} &= m(I) + O\left(\frac{\log(N)}{N}\right) \\ \frac{\#\{\beta_r \in I,\ 1 \leqq r \leqq N\}}{N} &= m(I) + O\left(\frac{\log(N)}{N}\right). \end{aligned} \tag{1.22}$$

With sequences of this type, it follows from (1.20)–(1.22) that, with $N\,\Delta t = T$,

$$\left(\prod_{r=1}^{N} (G^x_{\beta_r} G^y_{\alpha_r})(u_0)\right)_{j,k} = \tilde{u}\left(j\,\Delta x + k\,\Delta y - T\left(1 + a\left(\frac{u_L + u_R}{2}\right)\right) + TO\left(\frac{\log N}{N}\right)\right) \tag{1.23}$$

Thus, not only is the front kept sharp but there is an L^1 error of only $O(\log((\Delta t)^{-1})\,\Delta t)$ when compared with the exact solution in (1.10)(a).

Section 2. Preliminaries and Background

We begin by summarizing some of the well-known properties of solutions of scalar conservation laws in several variables (see [13, 24, 8] for example). First, a bounded measurable function, u, is a *weak solution* of (0.1) if for all $\phi \in C_0^1(\overline{R^+} \times R^2)$,

$$\iiint (\phi_t u + \phi_x f_1(u) + \phi_y f_2(u))\, dx\, dy\, dt + \iint_{t=0} u_0 \phi\, dx\, dy = 0. \tag{2.1}$$

Weak solutions are not uniquely determined by their initial data and additional principles, entropy conditions, are needed to select the appropriate physical solution. Set $\operatorname{sgn}(s) = \frac{s}{|s|}$, $s \neq 0$ and $\operatorname{sgn}(0) = 0$. For the equation in (0.1), these entropy conditions take the following form: Choose k to be any constant and consider the 3-component vector,

$$(|u-k|,\ \operatorname{sgn}(u-k)(f_1(u)-f_1(k)),\ \operatorname{sgn}(u-k)(f_2(u)-f_2(k))).$$

A weak solution, u, of (0.1) is any *entropy* solution if for all $\phi \in C_2^1(R^+ \times R^2)$ with $\phi \geqq 0$ and any k,

$$\begin{aligned}\iiint_{R^+ \times R^2} & \phi_t |u-k| + \phi_x \operatorname{sgn}(u-k)(f_1(u)-f_1(k)) \\ & + \phi_y \operatorname{sgn}(u-k)(f_2(u)-f_2(k)) \geqq 0.\end{aligned} \tag{2.2}$$

Kruzkov has proved that the weak solution obeying the inequalities in (2.2) is the limit with vanishing viscosity of corresponding solutions of the viscous equation. For piecewise smooth solutions, the inequalities in (2.2) imply that Oleinik's Condition E is satisfied across discontinuities (see [14]).

Next, we list some function spaces, useful in our proof. The space $BV(R^2)$, denotes the locally integrable functions with distribution derivatives that are finite Borel measures under the semi-norm,

$$\begin{aligned}|u|_{BV(R^2)} = & \max_{0<|h|<\infty} \iint \frac{|u(x+h,y)-u(x,y)|}{|h|}\, dx\, dy \\ & + \max_{0<|h|<\infty} \iint \frac{|u(x,y+h)-u(x,y)|}{|h|}\, dy\, dx.\end{aligned} \tag{2.3}$$

Below, we use functions in $BV \cap L^\infty \cap L^1$ with the natural norm, $|u|_{BV \cap L^\infty} = |u|_{BV} + |u|_{L^1} + |u|_{L^\infty}$ and denote by $C([0,T], B)$, the space of continuous functions on $[0,T]$ with values in the Banach space, B. The following proposition summarizes several important properties of entropy solutions.

Proposition 2.1. *Suppose $u_0 \in L^1(R^2) \cap L^\infty(R^2)$, then there exists a unique weak solution, $u(x,y,t) \equiv S(t) u_0$, belonging to $C([0,T], L^1(R^2)) \cap L^\infty([0,T] \times R^2)$ and satisfying the entropy conditions in* (2.2). *The function $S(t) u_0$ has the following properties:*

(1) $\int S(t) u_0 = \int u_0$.

(2) $u_0 \leqq v_0$ a.e. $\Rightarrow S(t) u_0 \leqq S(t) v_0$ a.e.

(3) ess sup $S(t)u_0 \leqq$ ess sup u_0, ess inf $u_0 \leqq$ ess inf $S(t)u_0$

(4) $\|S(t)u_0 - S(t)v_0\|_{L^1} \leqq \|u_0 - v_0\|_{L^1}$.

(5) $|S(t)u_0|_{BV} \leqq |u_0|_{BV}$,

(6) *for* $u_0 \in BV \cap L^\infty$, $\|S(t_1)u_0 - S(t_2)u_0\|_{L^1} \leqq C|t_1 - t_2| \, |u_0|_{BV \cap L^\infty}$.

(7) *Consider* $c^2 = \max\limits_{|u| \leqq \|u_0\|_{L^\infty}} \left(\left(\frac{\partial f_1}{\partial u}\right)^2 + \left(\frac{\partial f_2}{\partial u}\right)^2\right)$, *the values of* $S(t)u_0$ *on* $|(x-x_0, y-y_0)| \leqq R$ *are uniquely determined by the values of* u_0 *on* $|(x-x_0, y-y_0)| \leqq R + ct$.

Of course, the same facts apply to $S^x(t)$ and $S^y(t)$. The important uniqueness theorem and many of the facts in the above proposition are due to Kruzkov [13]; the properties in (1)–(7) are all explicitly and constructively proved, for example – via finite difference techniques in [8] or by the viscosity method [13, 24].

We also define one-dimensional averaging operators which are useful for various discrete approximations.

Definition. Given $f \in L^1(R^2)$, the averages $I^{\Delta x} f$, $I^{\Delta y} f$ are well defined a.e. with respect to y and x respectively by

$$I^{\Delta x} f \equiv \sum_{j=-\infty}^{\infty} f_j(y)\chi_j(x), \qquad I^{\Delta y} f \equiv \sum_{k=-\infty}^{\infty} f_k(x)\chi_k(y)$$

where

$$f_j(y) = \frac{1}{\Delta x} \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} f(s, y)\, ds, \qquad f_k(x) = \frac{1}{\Delta y} \int_{(k-\frac{1}{2})\Delta y}^{(k+\frac{1}{2})\Delta y} f(x, s)\, ds$$

and χ_j, χ_k are the characteristic functions of $(j-\frac{1}{2})\Delta x \leqq x < (j+\frac{1}{2})\Delta x$, $(k-\frac{1}{2})\Delta y \leqq y \leqq (k+\frac{1}{2})\Delta y$, respectively.

In the following three sections, we study and use various discret-splitting algorithms when $S^x(\Delta t)$, $S^y(\Delta t)$ are approximated by conservation form difference schemes. Given a spatial lattice, $\{(j\Delta x, k\Delta y)\}$, the capital letter, U, will denote a function defined on this lattice with $U_{j,k}$, the values at respective lattice points (V, W denote corresponding one-dimensional functions defined on x and y lattices). Given a lattice function, U, we associate a function defined in all of R^2 via the piecewise constant interpolation formula,

$$\mathsf{J}(U) = \sum_{j,k=-\infty}^{\infty} U_{j,k}\chi_j(x)\chi_k(y)$$

where $\chi_j(x)$, $\chi_k(y)$ are the characteristic functions defined previously. Discrete analogues of the norms in (2.3) are defined by using $\|U\|_{L^1} = \|\mathsf{J}(u)\|_{L^1}$, $|U|_{BV} = |\mathsf{J}(U)|_{BV}$, etc. – these norms yield the same discrete function spaces used in [8]. We frequently abuse notation below by omitting J. Given grid-points $\left\{\left(j\Delta x, k\Delta y, \sum_{l=1}^{N} \Delta t^l\right)\right\}$, we use the standard notations, $U^n_{j,k} \cong u\left(j\Delta x, k\Delta y, \sum_{l=1}^{n-1} \Delta t^l\right)$, $\lambda^x_l = \frac{\Delta t^l}{\Delta x}$, $\lambda^y_l = \frac{\Delta t^l}{\Delta y}$, $\Delta^x_t U^n_{j,k} = U^n_{j+1,k} - U^n_{j,k}$, etc.

We will consider various *conservation-form*, one dimensional difference operators, $V^{n+1} \equiv \vec{G}^x(V^n)$, $W^{n+1} \equiv \vec{G}^y(W^n)$ defined at grid points by

$$\begin{aligned} V_j^{n+1} &= G^x(V_{j-p}^n, \ldots, V_{j+q+1}^n) = V_j^n - \lambda_n^x \Delta_+^x g_1(V_{j-p}^n, \ldots, V_{j+q}^n) \\ W_k^{n+1} &= G^y(W_{K-R}^n, \ldots, W_{K+S+1}^n) = W_k^n - \lambda_n^y \Delta_+^y g_2(W_{K-R}^n, \ldots, W_{K+S}^n) \end{aligned} \tag{2.4}$$

and consistent with the respective operators in (0.2), i.e.

$$g_1(v, \ldots, v) \equiv f_1(v), \qquad g_2(w, \ldots, w) \equiv f_2(w). \tag{2.5}$$

The functions g_1 and g_2 are called the numerical flux functions. The *discrete splitting* algorithm associated with the first approximation in (0.3) and the discretizations, $G^x \cong S^x(\Delta t)$, $G^y \cong S^y(\Delta t)$ is defined by

$$\begin{aligned} U_{j,k}^{n+\frac{1}{2}} &= U_{j,k}^n - \lambda_n^y \Delta_+^y g_2(U_{j,k-r}^n, \ldots, U_{j,k+s}^n) \\ U_{j,k}^{n+1} &= U_{j,k}^{n+\frac{1}{2}} - \lambda_n^x \Delta_+^x g_1(U_{j-p,k}^{n+\frac{1}{2}}, \ldots, U_{j+q,k}^{n+\frac{1}{2}}). \end{aligned} \tag{2.6}$$

It is an elementary general remark of Lax [2] that when the component schemes have the structure in (2.4), the scheme defined in (2.6) has conservation form and is consistent with (0.1). Similar formulae and remarks apply to the other discrete splitting algorithms defined when the schemes in (2.4) are used in conjunction with the approximations in (0.3) and (0.4).

Section 3. Proof of Theorem 1

Here we give a simple proof of Theorem 1. In the appendix, as a by-product of this method of proof, we obtain convergence results for general dimensional splitting algorithms.

We begin by recording two facts needed in the proof. First (see Proposition 2.2 of [8]), it is well-known that if $\{f_n\}$ is a sequence of functions in R^N with

$$\|f_n\|_{L^1 \cap L^\infty \cap BV} \leqq C, \quad f_n \text{ vanishing for } |x| > R_0 \tag{3.1}$$

then there is a function $f_0 \in L^1 \cap L^\infty \cap BV$ and a subsequence (still denoted by f_n) so that

$$f_n \text{ converges boundedly a.e. to } f_0 \text{ as } n \to \infty. \tag{3.2}$$

Next, assume $v_0 \in L^1 \cap L^\infty \cap BV$, $w_0 \in L^1 \cap L^\infty \cap BV$ and let $v(x, y, t) \equiv S^x(t)(v_0(\cdot, y))$, $w(x, y, t) \equiv S^y(t)(w_0(x, \cdot, t))$, denote the corresponding entropy solutions for the one-dimensional conservation laws in (0.2). Under the above assumption, we have the following lemma:

Lemma 3.1. *If* $\phi \in C_0^1(R^+ \times R^2)$, $\phi \geqq 0$ *and* $t \geqq s \geqq 0$, *then*

$$\begin{aligned} &\int_s^t \iint_{R^2} |v-k|\, \phi_t + \operatorname{sgn}(v-k)(f_1(v) - f_1(k))\, \phi_x \, dx\, dy\, dt \\ &\geqq \iint_{R^2} |v-k|\, \phi(x, y, t)\, dx\, dy - \iint_{R^2} |v-k|\, \phi(x, y, s)\, dx\, dy \end{aligned} \tag{1}$$

$$\int_s^t \iint_{R^2} |w-k|\,\phi_t + \operatorname{sgn}(w-k)(f_2(w)-f_2(k))\,\phi_y\,dx\,dy\,dt$$
$$\geqq \iint_{R^2} |w-k|\,\phi(x,y,t)\,dx\,dy - \iint_{R^2} |w-k|\,\phi(x,y,s)\,dx\,dy. \tag{2}$$

We remark that 'mutatis mutandi' the analogous result is valid in R^N. We use that fact in the appendix. Next, using the above two facts we complete the proof of Theorem 1. We postpone the proof of Lemma 3.1 until the end of this section.

We begin with our proof of convergence for the first splitting algorithm in (0.2). By the L^1 contraction estimates for $S(t)$, $S^x(t)$, $S^y(t)$ from (4) in Proposition 2.1, it is sufficient to prove the convergence for initial data, $u_0(x,y)$, of the form $u_0 \in L^1 \cap L^\infty \cap BV$ with the support of u_0 contained in $|x| \leqq R_0$. With such initial data, define $u^{\Delta t} \equiv S^{\Delta t}(t)\,u_0$ by the formula,

$$u^{\Delta t} \equiv S^{\Delta t}(t)\,u_0$$
$$\equiv \begin{cases} S^y(2(t-n\Delta t))(S^x(\Delta t)\,S^y(\Delta t))^n u_0, & n\,\Delta t \leqq t < (n+\tfrac{1}{2})\,\Delta t \\ S^x(2(t-n\,\Delta t))\,S^y(\Delta t)(S^x(\Delta t)\,S^y(\Delta t))^n u_0, & (n+\tfrac{1}{2})\,\Delta t \leqq t < (n+1)\,\Delta t. \end{cases} \tag{3.3}$$

From the properties in (3), (4), and (5) of Proposition 2.1, it follows that

$$\|S^{\Delta t}(t)\,u_0\|_{BV \cap L^1 \cap L^\infty} \leqq \|u_0\|_{BV \cap L^1 \cap L^\infty} \tag{3.4}$$

and for any fixed T, by (7) of Proposition 2.1,

$$\operatorname{supp} S^{\Delta t}(t)\,u_0 \subseteq \{x \mid |x| \leqq R_0 + c\,T\}$$
$$\text{for any } 0 \leqq t \leqq T. \tag{3.5}$$

Furthermore, from (6) of Proposition 2.1, we deduce that

$$|S^{\Delta t}(t_1)\,u_0 - S^{\Delta t}(t_2)\,u_0|_{L^1} \leqq C\,|t_1 - t_2|\,|u_0|_{BV}. \tag{3.6}$$

Thus, by the compactness property in (3.1), (3.2) and the estimates in (3.4), (3.5), and (3.6), we repeat the proof of the Arzela-Aseoli theorem (see Sect. 4 of [8] for similar details) to conclude the following: for every sequence $\{\Delta t_l\}$ tending to zero, there is a subsequence (still denoted by Δt_l) and a function $\tilde{u} \in C([0,\infty), L^1(R^2))$ with $|u|_{L^\infty} \leqq |u_0|_{L^\infty}$, $\tilde{u}(x,y,0) = u_0(x,y)$, and

$$S^{\Delta t_l}(t)\,u_0 \text{ converging boundedly a.e. to } \tilde{u}(x,y,t) \text{ as } l \to \infty. \tag{3.7}$$

Once we deduce that $\tilde{u}(x,y) \equiv S(t)\,u_0$, the proof of the convergence of the first splitting algorithm in (0.3) is complete. (We apply the elementary fact that if every subsequence has a sub-subsequence converging to a unique limit - the original sequence must converge to the same limit.)

We claim that $\tilde{u}(x,y,t)$ satisfies

$$\iiint (\phi_t\,|\tilde{u}-k| + \phi_x \operatorname{sgn}(\tilde{u}-k)(f_1(\tilde{u}) - f_1(k))$$
$$+ \phi_y \operatorname{sgn}(\tilde{u}-k)(f_2(\tilde{u}) - f_2(k)))\,dx\,dy\,dt \geqq 0$$
$$\text{for all } \phi \in C_0^1(R^+ \times R^2) \text{ with } \phi \geqq 0. \tag{3.8}$$

Assuming this for the moment, we choose $k = \pm 2\|u_0\|_{L^\infty}$ and use $u(x,y,0) = u_0(x,y)$ to deduce that u_0 is a weak solution of (0.1) and furthermore from (3.8) we see that u satisfies all the entropy inequalities in (2.2) - since $\tilde{u} \in C([0,\infty), L^1) \cap L^\infty$, by Kruzkov's uniqueness theorem [13] mentioned in Sect. 2, $\tilde{u}(x,y,t) \equiv S(t)\,u_0$.

To establish the inequality in (3.8), we consider the new test function $\tilde{\phi} \in C_0^1(R^+ \times R^2)$ defined by $\tilde{\phi}(x,y,t) = \phi(x,y,2t)$ and apply parts (1) and (2) of Lemma 3.1 respectively to $v_n(t) \equiv S^y(t)(S^x(\Delta t)\,S^y(\Delta t))^n u_0$, $w_n(t) \equiv S^x(t)\,S^y(\Delta t) \cdot (S^x(\Delta t)\,S^y(\Delta t))^n u$ to obtain

$$\begin{aligned}
&\int_{n\Delta t}^{(n+\frac{1}{2})\Delta t} \iint_{R^2} (\tfrac{1}{2}\phi_t\,|u^{\Delta t} - k| + \phi_y \operatorname{sgn}(u^{\Delta t} - k)(f_2(u^{\Delta t}) - f_2(k))\\
&\quad \equiv \tfrac{1}{2}\int_0^{\Delta t} \iint_{R^2} (\tilde{\phi}_t(\tau + 2n(|t|))\,|v_n(t) - k|\\
&\qquad + \tilde{\phi}_y(\tau + 2n(\Delta t))\operatorname{sgn}(v_n(t) - k)(f_2(v_n(t) - f_2(k)))\,dx\,dy\,d\tau\\
&\quad \geqq \tfrac{1}{2}\iint_{R^2} \phi((n+\tfrac{1}{2})\Delta t)\,|u^{\Delta t}((n+\tfrac{1}{2})\Delta t) - k|\,dx\,dy\\
&\qquad - \tfrac{1}{2}\iint_{R^2} \phi(n\Delta t)\,|u^{\Delta t}(n\Delta t) - K|\,dx\,dy
\end{aligned} \tag{3.9}$$

and similarly,

$$\begin{aligned}
&\int_{(n+\frac{1}{2})\Delta t}^{(n+1)\Delta t} \iint_{R^2} (\tfrac{1}{2}\phi_t\,|u^{\Delta t} - k| + \phi_x \operatorname{sgn}(u^{\Delta t} - k)(f_1(u^{\Delta t}) - f_1(k))\\
&\quad \geqq \tfrac{1}{2}\iint_{R^2} \phi((n+1)\Delta t)\,|u^{\Delta t}((n+1)\Delta t) - k|\,dx\,dy\\
&\qquad - \tfrac{1}{2}\iint_{R^2} \phi((n+\tfrac{1}{2})\Delta t)\,|u^{\Delta t}((n+\tfrac{1}{2})\Delta t) - k|\,dx\,dy.
\end{aligned} \tag{3.10}$$

We add the inequalities in (3.9) and (3.10) and sum over n - we observe that the right hand side of the resulting inequality is a telescoping sum and collapses to zero. Let χ_n denote the characteristic function of $\{(x,y,t) \mid n\Delta t \leqq t \leqq (n+\frac{1}{2})\Delta t\}$, then this resulting inequality has the form,

$$\begin{aligned}
&\int_0^\infty \iint_{R^2} \tfrac{1}{2}\phi_t\,|u^{\Delta t} - k| + \int_0^\infty \iint_{R^2} \chi_n\,\phi_y \operatorname{sgn}(u^{\Delta t} - k)(f_2(u) - f_2(k))\\
&\qquad + \int_0^\infty \iint_{R^2} (1 - \chi_n)\,\phi_x \operatorname{sgn}(u^{\Delta t} - k)(f_1(u) - f_1(k))\\
&\quad \geqq 0.
\end{aligned} \tag{3.11}$$

It is a simple exercise (see the appendix) to check that

$$\left.\begin{aligned}
\chi_n &\xrightarrow{\text{weakly}} \tfrac{1}{2}\\
1 - \chi_n &\xrightarrow{\text{weakly}} \tfrac{1}{2}
\end{aligned}\right\} \text{ in } L^2_{\text{loc}}([0,\infty) \times R^2). \tag{3.12}$$

From (3.7) it follows that as $l \to \infty$

$$\begin{aligned}&\phi_x \operatorname{sgn}(u^{\Delta t_l}-k)(f_1(u^{\Delta t_l})-f_1(k)) \xrightarrow{\text{strongly}} \phi_x \operatorname{sgn}(\tilde{u}-k)(f_1(\tilde{u})-f_1(k))\\ &\phi_y \operatorname{sgn}(u^{\Delta t_l}-k)(f_2(u^{\Delta t_l})-f_1(k)) \xrightarrow{\text{strongly}} \phi_y \operatorname{sgn}(\tilde{u}-k)(f_2(\tilde{u})-f_2(k))\end{aligned} \tag{3.13}$$

where the strong convergence takes place in $L^2_{\text{comp}}(R^+ \times R^2)$. Thus, by using (3.12) and (3.13) in (3.11) and passing to the limit, we obtain the required inequality in (3.8). Thus, we have proved that with $n\Delta t = T$

$$\max_{0 \leqq T \leqq T_0} \|(S^x(\Delta t) S^y(\Delta t))^n u_0 - S(T) u_0\|_{L^1} \to 0 \tag{3.14}$$

as $\Delta t \to 0$.

It is a simple matter to deduce from (3.14) that with $n\Delta t = T$

$$\max_{0 \leqq T \leqq T_0} \left\| \left(S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t) S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0 - S(T) u_0 \right\|_1 \to 0$$

as $\Delta t \to 0$. (See the appendix for a direct proof.) From the semigroup property,

$$\begin{aligned}
&\left(S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t) S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0 - (S^x(\Delta t) S^y(\Delta t))^n u_0\\
&= \left[S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t)(S^x(\Delta t) S^y(\Delta t))^{n-1} S^x\left(\frac{\Delta t}{2}\right) u_0\right.\\
&\quad \left. - S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t)(S^x(\Delta t) S^y(\Delta t))^{n-1} u_0\right]\\
&= \left[S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t)(S^x(\Delta t) S^y(\Delta t))^{n-1} S^x\left(\frac{\Delta t}{2}\right) u_0\right.\\
&\quad \left. - S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t)(S^x(\Delta t) S^y(\Delta t))^{n-1} u_0\right]\\
&\quad + \left[S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t)(S^x(\Delta t) S^y(\Delta t))^{n-1} u_0 - (S^x(\Delta t) S^y(\Delta t))^n u_0\right].
\end{aligned} \tag{3.15}$$

For $w \in BV \cap L^\infty$, it follows from (6) in Proposition 2.1 that

$$\left\| S\left(\frac{\Delta t}{2}\right) w - w \right\|_{L^1} \leqq C \Delta t \, |w|_{BV}. \tag{3.16}$$

By combining (3.15), (3.16), and the BV stability estimates in (5) of Proposition 2.1, we obtain for $u_0 \in L^\infty \cap BV$,

$$\left\| \left(S^x\left(\frac{\Delta t}{2}\right) S^y(\Delta t) S^x\left(\frac{\Delta t}{2}\right)\right)^n u_0 - (S^x(\Delta t) S^y(\Delta t))^n u_0 \right\|_{L^1} \leqq 2 C \Delta t \, |u_0|_{BV} \tag{3.17}$$

and this estimate with (3.14) implies the convergence of the Strang iterates of (0.4). Since analogous arguments hold with the roles of x and y reversed, the proof of Theorem 1 is complete.

Proof of Lemma 3.1. Our proof is based on the main convergence result of [8] on general monotone schemes – this lemma could also be proved by the techniques of [13] by using the viscosity method and approximating $|u-k|$ by smooth convex functions. To fix our ideas, to prove (1) we consider the solution of the one-dimensional Lax-Friedrich's scheme, given by

$$V_{j,l}^{n+1} = V_{j,l}^{n} - \lambda^x \Delta_x^+ g_1(V_{j,l}^n, V_{j-1,l}^n) \tag{3.18}$$

with discrete initial data given by $I^{\Delta x} I^{\Delta y} v_0$. Here we assume that $\Delta x \equiv \Delta y$,

$$\lambda^x \max_{-\|v_0\|_{L^\infty} \leqq u \leqq \|v_0\|_{L^\infty}} \left|\frac{\partial f_1}{\partial u}\right| \leqq I \tag{3.19}$$

and g_1 is the explicit numerical flux function,

$$g_1(V_j, V_{j-1}) \equiv \frac{f(V_j)+f(V_{j-1})}{2} - \frac{1}{2\lambda^x}(V_j - V_{j-1}). \tag{3.20}$$

Under the conditions in (3.19) and (3.20), the scheme in (3.18) is monotone (see Sect. 4 for definitions) when applied to the discrete initial data $I^{\Delta x} I^{\Delta y} v_0$ and since $\|v_0\|_{L^1 \cap L^\infty \cap BV} \leqq C$, it follows from the main theorem of [8] that as $\Delta t \to 0$ with λ^x fixed and $\Delta x \equiv \Delta y$,

$$\begin{gathered} V^n \text{ converges boundedly a.e. to} \\ v(x,y,t) = S^x(t)\, v_0(\cdot, y, t), \qquad n\Delta t = t. \end{gathered} \tag{3.21}$$

Set $v \vee w = \max(v,w)$ and $v \wedge w = \min(v,w)$. Define $g^k(V_1, V_2)$ by

$$g^k(V_1, V_2) \equiv g_1(V_1 \vee k, V_2 \vee k) - g_1(V_1 \wedge k, V_2 \wedge k).$$

It follows from an explicit calculation using (3.19) (3.20) (and more generally from Proposition 4.1 of [8]) that

$$\begin{aligned} &\text{(a)}\quad \Delta_+^t |V_{j,l}^n - k| + \lambda^x \Delta_+^x (g^k(V_{j,l}^n, V_{j-1,l}^n)) \leqq 0 \\ &\text{(b)}\quad g^k(V,V) = \operatorname{sgn}(V-k)(f_1(V) - f_1(k)). \end{aligned} \tag{3.22}$$

We set $\phi_{jl}^n = \phi(j\Delta x, l\Delta y, n\Delta t)$, multiply (3.22)(a) by $\phi_{j,l}^n$, and sum (3.22) over j, l, and n with $m_0 \leqq n \leqq n_0 - 1$ where $n_0 \Delta t \leqq t \leqq (n_0+1)\Delta t$, $m_0 \Delta t \leqq s \leqq (m_0+1)\Delta t$ to arrive at

$$\sum_{n=m_0}^{n_0-1} \Delta t \sum_{j,l} \Delta x \Delta y\, \phi_{jl}^n (\Delta_+^t |V_{j,l}^n - k| + \lambda^x \Delta_+^x g^k(V_{j,l}^n, V_{j-1,l}^n)) \leqq 0. \tag{3.23}$$

We sum by parts in x and t on the left hand side of the identity in (3.23) to obtain

$$\begin{aligned} &\sum_{j,l} \phi_{j,l}^{n_0-1} |V_{j,l}^{n_0} - k| \Delta x \Delta y - \sum_{j,l} \phi_{j,l}^{m_0} |V_{j,l}^{m_0} - k| \Delta x \Delta y \\ &\quad \leqq \sum_{n=m_0+1}^{(n_0-1)} \Delta t \left(\sum_{j,l} \frac{\Delta_+^t \phi_{j,l}^n}{\Delta t} |V_{j,l}^n - k| \Delta x \Delta y \right) \\ &\qquad + \sum_{n=m_0}^{(n_0-1)} \Delta t \left(\sum_{j,l} \frac{\Delta_+^x \phi_{j,l}^n}{\Delta x} g^k(V_{j,l}^n, V_{j-1,l}^n) \Delta x \Delta y \right). \end{aligned} \tag{3.24}$$

By using the bounded a.e. convergence of V^n to $S^x(t)v_0$ from (3.21), the fact that $S^x(t)v_0 \in C([0,\infty), L^1(R^2)) \cap L^\infty(R^+ \times R^2)$, the estimate in (3.24), and finally (3.22)(b), we obtain (1) of Lemma 3.1 by applying the dominated convergence theorem to (3.24). The proof of (2) is similar.

Section 4. Fractional Steps for Monotone Schemes

A difference scheme, $U_j^{n+1} = G(U_{j-p}^n, \ldots, U_{j+q+1}^n)$ is locally *monotone* on the interval, $[a,b]$, if G is an increasing (i.e., non-decreasing) function of all arguments as they vary over $[a,b]$. In this section, we assume that the difference schemes, $V^{n+1} = \vec{G}^x(V^n)$, $W^{n+1} = \vec{G}^y(W^n)$, defined in (2.4), not only have conservation form and are consistent but also are monotone on $[a,b]$ with respective numerical flux functions, g_1, g_2, continuous functions of all arguments. The upwind scheme (differenced through stagnation points), Godunov's scheme, and the Lax-Friedrich's scheme are all familiar examples of one-dimensional monotone schemes [8]. First, we make the elementary remark:

A consistent one-dimensional conservation-form monotone scheme with an a priori continuous numerical flux, $g_1(v_{-p}, \ldots, v_q)$, automatically has a Lipschitz continuous flux, $g_1(v_{-p}, \ldots, v_q)$. (4.1)

As proved in [8], when g_1 is only continuous, such schemes define L^1-contractions; thus for $a \leqq v_j$, $\tilde{v}_j \leqq b$ $j = -p, \ldots, q+1$,

$$|G(v_{-p}, \ldots, v_{q+1}) - G(v_{-p}, \ldots, v_{q+1})| \leqq \sum_{l=-p}^{q+1} |v_l - v_l| \tag{4.2}$$

so that G is a Lipschitz function of all $p+q+1$ arguments. Therefore, $\Delta^+ g_1$ is a Lipschitz function of all arguments, in particular, of the $q+1$st argument; i.e. g_1 is Lipschitz in v_q and a trivial induction completes the proof. Next, we make the observation that if the schemes $V^{n+1} = \vec{G}^x(V^n)$, $W^{n+1} = \vec{G}^y(W^n)$ are both monotone on $[a,b]$, then the scheme defined in (2.6) is the composition of two increasing maps and therefore is monotone when all arguments vary over $[a,b]$. From this fact, (4.1), and the remark below (2.6), we conclude that the difference scheme in (2.6), formed from $\vec{G}^x$, $\vec{G}^y$ via the discrete splitting algorithm, has the following properties:

(1) Conservation-form and consistent with with (0.1).
(2) Lipschitz continuous numerical fluxes, g_1, g_2. (4.3)
(3) Monotone on $[a,b]$.

The basic result proved in [8] was that difference schemes with the structure in (4.3) always converge to the unique weak solution of (0.1) satisfying the entropy inequalities in (2.2). As a corollary to this fact, we immediately obtain the following:

Theorem 2 (Fractional steps for Monotone schemes). *Given* $u_0 \in L^1(R^2) \cap L^\infty(R^2)$ *with* ess $\sup u_0 \leqq \cdot b$, ess $\inf u_0 \geqq \cdot a$, *we define discrete initial data,* U^0, *by*

$U^0 \equiv I^{\Delta y} I^{\Delta x} u_0$. Compute U^{n+1} iteratively from U^0 via the discrete splitting algorithm in (2.6) where it is assumed that the component schemes, $\vec{G}^x$, $\vec{G}^y$ are monotone on $[a,b]$, have conservation-form, are consistent with the one-dimensional operators in (0.2), and have continuous numerical fluxes. Then as $\Delta t \to 0$ with λ^x, λ^y fixed, U^n converges to $S(T)u_0$. More precisely, if $n\Delta t = T$ and T_0 is any given number,

$$\max_{0 \leqq T \leqq T_0} \|U^n - S(T)u_0\|_{L^1(R^2)} \to 0 \quad \text{as} \quad \Delta t \to 0$$

The same results are valid for the other discrete splitting algorithms in (0.3), (0.4).

Remark 1. The above theorem is false if the assumption of monotonicity is dropped (see [12] and Section 5).

Remark 2. Weaker hypotheses than the differentiability of the numerical fluxes are of more than purely mathematical interest - Godunov's scheme has a non-differentiable numerical flux (see [8]).

Remark 3. From a practical point of view, it is a moot point that Theorem 1 applies to the discrete Strang splittings in (0.4) since monotone schemes are only first order accurate on smooth solutions (see [12]).

Finally, we conclude this section by mentioning that it is possible to give another proof of Theorem 1 by approximating the x and y sweeps by monotone difference schemes and applying Theorem 2. For this purpose we need to introduce a monotone, conservation-form, $2m+1$ point generalization of Godunov's method to approximate the x and y sweeps and prove certain uniform estimates on these approximations as $m \to \infty$ which guarantee the convergence in Theorem 1. In fact, this argument was the first proof of Theorem 1 discovered by the authors.

Section 5. Entropy Production for Fractional Step Algorithms Using the Lax-Wendroff Scheme

In a single space dimension, the *standard Lax-Wendroff* [16] difference schemes approximating the equations in (0.2) are three-point, conservation-form, second-order difference methods with the specific numerical flux functions from (2.4) given by

$$\begin{aligned} g_1(V_{j-1}, V_j) &\equiv \frac{f_1(V_j)+f_1(V_{j-1})}{2} + \frac{\lambda_n^x}{2} a_1\left(\frac{V_j+V_{j-1}}{2}\right)(f_1(V_j)-f_1(V_{j-1})) \\ g_2(W_{k-1}, W_k) &\equiv \frac{f_2(W_k)+f_2(W_{k-1})}{2} + \frac{\lambda_n^x}{2} a_2\left(\frac{W_k+W_{k-1}}{2}\right)(f_2(W_k)-f_2(W_{k-1})) \end{aligned} \tag{5.1}$$

where $a_1(u) = \frac{\partial f_1}{\partial u}$, $a_2(u) = \frac{\partial f_2}{\partial u}$. We denote the corresponding standard Lax-Wendroff schemes by $L^x_{\Delta t^n}$, $L^y_{\Delta t^n}$. When Strang introduced the algorithm in (0.4) into inviscid computational dynamics, he proposed using $S^x(\Delta t^n) \cong L^x_{\Delta t^n}$,

$S^y(\Delta t^n) \cong L^y_{\Delta t^n}$ to construct the *discrete Lax-Wendroff splitting algorithms* via

$$\prod_{r=1}^{n} L^x_{\frac{\Delta t^r}{2}} L^y_{\Delta t^r} L^x_{\frac{\Delta t^r}{2}} U^0 \quad \text{or} \quad \prod_{r=1}^{n} L^y_{\frac{\Delta t^r}{2}} L^x_{\Delta t^r} L^y_{\frac{\Delta t^r}{2}} U^0. \tag{5.2}$$

We note that one needs a priori varying time steps because the linearized stability conditions, $\frac{\Delta t^r}{\Delta x} |a_1(U^r_{j,k})| \leqq \varepsilon_0 \leqq 1$, which dictate the time step size, Δt^r, vary from time level to time level.

Here, we analyze the question, when are limits of the discrete Lax-Wendroff splitting algorithm (and appropriate modifications) the unique weak solution of (0.1) obeying the entropy inequalities in (2.2)? We begin with the following simple remark:

Example 5.1. *Entropy violating solutions can be computed by* (5.2).

Let U^0 have discrete initial data of the form

$$(U^0)_{j,k} = \begin{cases} u_1, & j \leqq 0 \\ u_2, & j > 0 \end{cases} \tag{5.3}$$

where the data on the right hand side define initial data for a one-dimensional steady entropy violating shock for the standard Lax-Wendroff scheme. Such data always exist even when f_1 is convex as long as $f_1(u_1) = f_1(u_2)$ (see [12, 20], for many examples).

A trivial calculation establishes that the initial datum in (5.3) is a steady entropy violating shock computed by the split Lax-Wendroff algorithm defined in (5.2). The same remarks apply for systems of conservation laws [20] and for splitting algorithms when $\vec{G}^y$ is any conservation-form difference method. Furthermore, the examples in [18] also can be generalized to prove that the algorithms in (5.2) can be nonlinearly unstable. The Lax-Wendroff scheme is not a monotone scheme and the above example indicates that Theorem 2 in Sect. 4 can be false when one of the steps is not monotone.

In [19], Majda and Osher studied a *modified Lax-Wendroff* difference scheme in a single space dimension with numerical flux functions from (2.6) given by

$$\begin{aligned} \tilde{g}_1(V_{j-1}, V_j) \equiv & \frac{f_1(V_j) + f_1(V_{j-1})}{2} \\ & + \frac{\lambda_n}{2} \frac{(f_1(V_j) - f_1(V_{j-1}))^2}{V_j - V_{j-1}} + C|a_1(V_j) - a_1(V_{j-1})|(V_j - V_{j-1}) \\ \tilde{g}_2(W_{k-1}, W_k) \equiv & \frac{f_2(W_k) + f_2(W_{k-1})}{2} \\ & + \frac{\lambda_n}{2} \frac{(f_2(W_k) - f_2(W_{k-1}))^2}{W_k - W_{k-1}} + C|a_2(W_k) - a_2(W_{k-1})|(W_k - W_{k-1}). \end{aligned} \tag{5.4}$$

These schemes, denoted by $\tilde{L}^x_{\Delta t^n}$, $\tilde{L}^y_{\Delta t^n}$ below, retain the desirable computational features of the standard Lax-Wendroff schemes. They are three-point, conservation-form, and second-order accurate - in fact, $\tilde{g}_1(V_{j-1}, V_j) = g_1(V_{j-1}, V_j) + O(|V_{j-1} - V_j|^2)$, etc. Furthermore, it was proved in [15] that under appropriate technical restrictions on C and on the C.F.L. condition, when f_1 is strictly convex so that $|f_1''(u)| > \delta > 0$, the undesirable computational features of the standard Lax-Wendroff scheme alluded to in Example 5.1 are eliminated for the one dimensional modified Lax-Wendroff scheme, $\tilde{L}^x_{\Delta t^n}$.

Here, we study correct entropy production for the *modified Lax-Wendroff splitting algorithm* analogous to (5.2) and defined by

$$U^n \equiv \prod_{r=1}^{n} \tilde{L}^x_{\frac{\Delta t^r}{2}} \tilde{L}^y_{\Delta t^r} \tilde{L}^x_{\frac{\Delta t^r}{2}} U^0$$

or (5.5)

$$U^n \equiv \prod_{r=1}^{n} \tilde{L}^y_{\frac{\Delta t^r}{2}} \tilde{L}^x_{\Delta t^r} \tilde{L}^y_{\frac{\Delta t^r}{2}} U^0$$

and use the notation, $U^{n+\frac{1}{3}} = \tilde{L}^x_{\frac{\Delta t^r}{2}} U^n$, $U^{n+\frac{2}{3}} = \tilde{L}^y_{\Delta t^r} U^{n+\frac{1}{3}}$ when studying the first algorithm in (5.5) (symmetric arguments apply to the second algorithm in (5.5)).

We assume that the space steps satisfy $\Delta y = \gamma \Delta x$ for γ a fixed constant and also that the time steps Δt^n and the constant, C, in (5.5) are chosen so that the linearized C.F.L conditions, $\Delta t^n \leqq \Delta t^0$ and

$$\begin{aligned} &\frac{\Delta t^n}{2\Delta x} \max_{j,k} |a_1(U^n_{j,k})| \leqq \varepsilon_0 \\ &\frac{\Delta t^n}{\gamma \Delta x} \max_{j,k} |a_2(U^{n+\frac{1}{3}}_{j,k})| \leqq \varepsilon_0 \\ &\frac{\Delta t^n}{2\Delta x} \max_{j,k} |a_1(U^{n+\frac{2}{3}}_{j,k})| \leqq \varepsilon_0 \end{aligned} \tag{5.6}$$

are obeyed. Here ε_0 and C are chosen to satisfy the technical restrictions guaranteeing stability and correct entropy production for the corresponding one-dimensional difference operators, $\tilde{L}^x_{\frac{\Delta t^r}{2}}$, $\tilde{L}^y_{\Delta t^r}$ as listed in the main Theorem of [19] provided that additionally,

$$|f_1''(u)| \geqq \delta > 0, \quad |f_2''(u)| \geqq \delta > 0. \tag{5.7}$$

We remark that such strategies for the time steps, Δt^n, as given in (5.6) are always possible for the modified Lax-Wendroff splitting algorithms in (5.5) under the mild growth restrictions,

$$\begin{aligned} &|a_1(u)| \leqq C(1 + |u|^\rho), \quad \text{for some } \rho > 0 \\ &|a_2(u) \leqq C(1 + |u|^\rho), \end{aligned} \tag{5.8}$$

provided $\varepsilon_0 \geqq .14$ is sufficiently small. We omit the proof since it is a simple generalization of the argument in Proposition 4.1 of [18] but only state the conclusion of that argument. Such a strategy meeting the conditions in (5.6) is always possible under the restrictions in (5.7), (5.8) with the time steps, Δt^n, satisfying

$$\Delta t^0 \geqq \Delta t^n \geqq \frac{\varepsilon_0 \Delta x}{C_\gamma}\left(1+\gamma^{-\frac{\rho}{2}}(\|U_0\|_2)^\rho(\Delta x)^{-\rho}\right)^{-1} \tag{5.9}$$

with C_γ a fixed positive constant.

The *entropy inequality* which we verify below for limits of (5.5) (see [14, 19] for the motivation) has the form

$$\iiint \left(\frac{\partial \rho}{\partial t}\frac{1}{2}u^2+\frac{\partial \rho}{\partial x}F_1(u)+\frac{\partial \rho}{\partial y}F_2(u)\right)dx\,dy\,dt \geqq 0 \tag{5.10}$$

$$\text{for all } \rho \geqq 0,\ \rho \in C_0^1(R^+\times R^2)$$

where $F_j(u)=\int_0^u s\,a_j(s)\,ds$, $j=1,2$,

Given a grid function, U^n, computed by (5.5), we define the interpolant of U^n by

$$U^{\Delta x}(t)=\begin{cases} U^n, & \sum_{l=0}^{n-1}\Delta t^l \leqq t < \sum_{l=0}^{n-1}\Delta t^l+\frac{1}{4}\Delta t^n \\ U^{n+\frac{1}{4}}, & \sum_{l=0}^{n-1}\Delta t^l+\frac{1}{4}\Delta t^n \leqq t < \sum_{l=0}^{n-1}\Delta t^l+\frac{3}{4}\Delta t^n \\ U^{n+\frac{3}{4}}, & \sum_{l=0}^{n-1}\Delta t^l+\frac{3}{4}\Delta t^n \leqq t < \sum_{l=0}^{n}\Delta t^2. \end{cases} \tag{5.11}$$

We have the following result:

Theorem 3. *Assume that* $\|U^0\|_{L^2(R^2)} \leqq C$ *and that* U^n *is computed from* U^0 *via the modified Lax-Wendroff splitting algorithm in* (5.5) *with a strategy satisfying the restrictions in* (5.6) *and that* f_1, f_2 *satisfy the nondegeneracy conditions in* (5.7). *Also assume that* $U^{\Delta x}(t)$ *converges boundedly a.e. to* $u(x, y, t)$ *as* $\Delta x \to 0$. Then,

(1) $u(x, y, t)$ is a weak solution of (0.1) which satisfies the entropy inequality in (5.10).

(2) If additionally, $u(x, y, t) \in BV(R^2 \times R^+)$ and $f_2(u) \equiv a f_1(u) + b(u) + c$, then $u(x, y, t)$ is the unique weak solution which satisfies the entropy inequalities in (2.2). In particular, correct entropy production is valid for the two dimensional inviscid Burger's equations in (0.7).

For the moment, we assume the conclusion of (1) in Theorem 3 and indicate, via examples, that we expect this result to be sharp. Assuming (1), we also indicate how the conclusion in (2) follows.

Our first example indicates that the modified split Lax-Wendroff scheme can still compute entropy violating solutions when the nondegeneracy conditions in (5.7) are not assumed.

Example 5.2. *Consider discrete initial data,* U^0, *with*

$$(U^0)_{j,k}=\begin{cases} u_1, & j \leqq 0 \\ u_2, & j > 0 \end{cases}$$

where f_1 *is any smooth function satisfying* $f_1(u_1)=f_1(u_2)$, $a_1(u_1)=a_1(u_2)$. *Also* *assume that there exists u^*, u_* belonging to the interval (u_1, u_2) with $f_1(u^*) > f_1(u_1)$, $f_1(u_*) < f_1(u_1)$. The function, $f_2(u)$, can be arbitrary.* Then,

$$\prod_{r=1}^{n} \tilde{L}^x_{\frac{\Delta t^r}{2}} \tilde{L}^y_{\Delta t^r} \tilde{L}^x_{\frac{\Delta t^r}{2}} U^0 \equiv U^0$$

and the limit of U^0 as $\Delta x \to 0$ is a steady weak solution which violates some of the entropy inequalities in (2.2).

Next, we discuss the implications of (1) when the nondegeneracy conditions in (5.7) are enforced so that the entropy inequality from (5.10) is satisfied by the limit. Also suppose, as in (2) above, that the limit $u(x, y, t)$, has the mild regularity $u \in BV(R^2 \times R^+) \cap L^\infty(R^2 \times R^+)$ (we get the L^∞ bound from the assumed bounded a.e. convergence). Because of this regularity, it follows from the work of Volpert ([24]) that u can be treated formally as if this solution were piecewise smooth. Thus, as in [14], across shock surfaces, the inequalities,

$$\begin{gathered} S\tfrac{1}{2}(u_L^2 - u_R^2) - (n_1(F_1(u_L) - F_1(u_R)) + n_2(F_2(u_L) - F_2(u_R))) \leqq 0, \\ S = \frac{n_1(f_1(u_L) - f_1(u_R)) + n_2(f_2(u_L) - f_2(u_R))}{u_L - u_R} \end{gathered} \tag{5.12}$$

are implied by the entropy condition in (5.10).

For general, $f_1(u), f_2(u)$ satisfying the conditions in (5.7), the inequalities in (5.12) are not powerful enough to characterize the unique weak solution satisfying the entropy inequalities in (2.2). We present the following example which illustrates this fact:

Example 5.3. *Consider solutions of* (0.1) *with* $f_1(u) \equiv u^2 + \cos(u)$, $f_2(u) \equiv u^2$, *then* $f_1''(u) \geqq 1$, $f_2''(u) = 2$ *so that the conditions in* (5.7) *are satisfied. A steady weak solution of* (0.1) *violating the entropy conditions in* (2.2) *is given by the data,*

$$u = \begin{cases} (2p + \tfrac{1}{2})\pi, & x > y \\ (2q + \tfrac{1}{2})\pi, & x \leqq y \end{cases}$$

where p and q are integers with $p < q$. On the other hand, the inequalities in (5.12) are satisfied since for the above data, they are

$$S = 0,$$

$$\frac{S}{2}(u_L^2 - u_R^2) - \frac{1}{\sqrt{2}}((F_1(u_L) - F_1(u_R) - (F_2(u_2) - F_2(u_R))) \equiv -\frac{1}{\sqrt{2}}(2q - 2p) < 0.$$

Thus, the entropy inequality in (5.10) is obeyed by this weak solution but (some of) those in (2.2) are violated. In fact to rule out such entropy violating solutions, one needs at least $2(p-q)$ of the entropy inequalities in (2.2) satisfied. It is unlikely that limits of the split (modified) Lax-Wendroff scheme conserve more than a few additional functionals and, of course, $(p-q)$ can be arbitrarily large. This example is 'evidence' to support the claim that we expect Theorem 3, although restrictive, to be sharp in general.

Next, assuming (1) and the additional regularity that u belongs to $BV(R^2 \times R^+)$, we claim that the conclusion in (2) of Theorem 3 is valid. It is a special case of results of Mock [21] that the entropy inequality in (5.12) implies the infinite number of inequalities in (2.2) provided that, for fixed $\vec{n}$, $n_1 f_1(u) + n_2 f_2(u)$ is either genuinely non-linear or identically linear. But as mentioned in the introduction, these conditions are equivalent to (5.7) and $f_2(u) \equiv a f_1(u) + bu + c$. Once the inequalities in (2.2) are satisfied, it follows from Kruzkov's uniqueness theorem that u is the unique 'physical' weak solution.

Proof of (1) *in Theorem 3.* Below, the constant K denotes a number satisfying $|U^{\Delta x}(t)| \leqq K$ - the existence of such a K follows from the assumed bounded a.e. convergence of $U^{\Delta x}$.

Under the restrictions in (5.6) on the time steps, Δt^n, it follows from Lemma 2.1 in [19] (with the switch, $\theta \equiv 1$) by utilizing the choice, $\rho \equiv 1$ and a trivial summation over the other variable that the following inequalities are valid: for a fixed constant $\beta > 0$,

$$\begin{aligned}
\|U^{n+\frac14}\|^2_{L^2(R^2)} + \beta \left\| \frac{|\Delta^x_+ U^n|^2 \, |\Delta^x_+ a_1(U^n)|}{\Delta x} \right\|_{L^1(R^2)} &\leqq \|U^n\|^2_{L^2(R^2)} \\
\|U^{n+\frac34}\|^2_{L^2(R^2)} + \beta \left\| \frac{|\Delta^y_+ U^{n+\frac14}|^2 \, |\Delta^y_+ a_2(U^{n+\frac14})|}{\Delta y} \right\|_{L^1(R^2)} &\leqq \|U^{n+\frac14}\|^2_{L^2(R^2)} \qquad (5.13) \\
\|U^{n+1}\|^2_{L^2(R^2)} + \beta \left\| \frac{|\Delta^x_+ U^{n+\frac34}|^2 \, |\Delta^x_+ a_1(U^{n+\frac34})|}{\Delta x} \right\|_{L^1(R^2)} &\leqq \|U^{n+\frac34}\|^2_{L^2(R^2)}.
\end{aligned}$$

Similarly, by using $\rho \geqq 0$, $\rho \in C^1_0$ in Lemma 2.1 from [15], we obtain

$$\begin{aligned}
&\sum_{j,k} \Delta x \, \Delta y \left[\frac{\rho^{n+\frac14}_{j,k} - \rho^n_{j,k}}{\frac{\Delta t^n}{2}} \, \frac12 ((U^n)_{j,k})^2 + \frac{\Delta^x_0 \rho^n_{j,k}}{2\Delta x} F_1((U^n)_{j,k}) \right] \\
&\quad \geqq -K \sum_{j,k} \Delta x \, \Delta y \left(\sum_{-v_0}^{v_0} \frac{|\Delta^x_+ \rho^n_{j+v,k}|}{\Delta x} |\Delta^x_+ (U^n)_{j,k}| \right), \\
&\sum_{j,k} \Delta x \, \Delta y \left[\frac{\rho^{n+\frac34}_{j,k} - \rho^{n+\frac14}_{j,k}}{\Delta t^n} \, \frac12 ((U^{n+\frac14})_{j,k})^2 + \frac{\Delta^y_0 \rho^n_{j,k}}{2\Delta y} F_2((U^{n+\frac14})_{j,k}) \right] \\
&\quad \geqq -K \sum_{j,k} \Delta x \, \Delta y \left(\sum_{-v_0}^{v_0} \frac{|\Delta^y_+ \rho^n_{j,k+v}|}{\Delta y} |\Delta^y_+ (U^{n+\frac14})_{j,k}| \right), \qquad (5.14) \\
&\sum_{j,k} \Delta x \, \Delta y \left[\frac{\rho^{n+1}_{j,k} - \rho^{n+\frac34}_{j,k}}{\frac{\Delta t^n}{2}} \, \frac12 ((U^{n+\frac34})_{j,k})^2 + \frac{\Delta^x_0 \rho^n_{j,k}}{2\Delta x} F_1((U^{n+\frac34})_{j,k}) \right] \\
&\quad \geqq -K \sum_{j,k} \Delta x \, \Delta y \left(\sum_{-v_0}^{v_0} \frac{|\Delta^x_+ \rho^n_{j+v,k}|}{\Delta x} |\Delta^x_+ (U^n)_{j,k}| \right)
\end{aligned}$$

where V_0 is a fixed integer. We assume that $\operatorname{supp} \rho \subseteq R^2 \times [0, T_0 - 1]$; by summing the first and third inequalities in (5.13) over Δt^n with

$T_0 \geqq \sum_{n=0}^{N} \Delta t^n > T_0 - 1$, we obtain

$$\sum_{\substack{n=0\\ l=0,3}}^{N} \left\| \Delta t^n \frac{|\Delta_+^x U^{n+\frac{l}{4}}|^2 \, |\Delta_+^x a_1(U^{n+\frac{l}{4}})|}{\Delta x} \right\|_{L^1} \leqq \frac{T_0}{\beta} \|U^0\|_{L^2(R^2)}^2 . \tag{5.15}$$

As in [19], it follows from this estimate, the genuine nonlinearity condition in (5.7), and Hölder's inequality (with $\frac{1}{3}+\frac{2}{3}=1$) in time that

$$\sum_{\substack{n=0\\ l=0,3}}^{N} \Delta t^n \, \| \, |\Delta_+^x U^{n+\frac{l}{4}}| \, \|_{L^1(R^2)} \leqq \frac{C T_0^{\frac{2}{3}} (\Delta x)^{\frac{1}{3}}}{\gamma} \|U^0\|_{L^2(R^2)}^{\frac{2}{3}} \tag{5.15a}$$

and similarly, that

$$\sum_{n=0}^{N} \Delta t^n \, \| \, |\Delta_+^y U^{n+\frac{1}{4}}| \, \|_{L^1(R^2)} \leqq \frac{C T_0^{\frac{2}{3}} (\Delta x)^{\frac{1}{3}}}{\gamma} \|U^0\|_{L^2(R^2)}^{\frac{2}{3}} \tag{5.16}$$

where C is a fixed constant depending only on γ.

We define $\chi^{\Delta x}(t)$ to be the characteristic function of the set,

$$\bigcup_{n=0}^{\infty} \left\{ t \,\middle|\, \sum_{r=0}^{n} \Delta t^n + \tfrac{1}{4} \Delta t^{n+1} \leqq t < \Delta t^n + \tfrac{3}{4} \Delta t^{n+1} \right\}.$$

Following a remark in Sect. 3, we observe that since (5.6) is satisfied for the time steps,

$$\begin{aligned} \chi^{\Delta x}(t) &\xrightarrow{\text{weakly}} \tfrac{1}{2} \quad \text{in } L^2_{\mathrm{loc}}(R^2 \times R^+) \\ 1-\chi^{\Delta x}(t) &\xrightarrow{\text{weakly}} \tfrac{1}{2} \quad \text{in } L^2_{\mathrm{loc}}(R^2 \times R^+) \end{aligned} \tag{5.17}$$

as $\Delta x \to 0$. Given $q=0, 1, 2, 3$, we define l_q by $l_0=0$, $l_1=1$, $l_2=3$, $l_3=4$. Next we sum the inequalities in (5.14) over time using the weights $\Delta t^{n,q}$ where $\Delta t^{n,0} = \frac{\Delta t^n}{4}$, $\Delta t^{n,1} = \frac{\Delta t^n}{2}$, $\Delta t^{n,2} = \frac{\Delta t^n}{4}$ and apply the estimates in (5.15), (5.16) to obtain

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{n=0\\ q=0,1,2}}^{N} \Delta t^{n,q} \sum_{j,k} \Delta x \, \Delta y \left(\frac{\rho^{n+\frac{l_{q+1}}{4}} - \rho^{n+\frac{q}{4}}}{\Delta t^{n,q}} \right) \tfrac{1}{2} (U^{n+\frac{l_q}{4}})_{j,k}^2 \\ &\quad + \sum_{n=0}^{N} \sum_{j,k} \frac{\Delta t^n}{2} \Delta x \, \Delta y \left(\frac{\Delta_0^x \rho_{j,k}^n + \Delta_0^x \rho_{j,k}^{n+\frac{3}{4}}}{4 \Delta x} \right) (1-\chi^{\Delta x}(t)) \, F_1(U_{j,k}^{\Delta x}(t)) \\ &\quad + \sum_{n=0}^{N} \sum_{j,k} \frac{\Delta t^n}{2} \Delta x \, \Delta y \left(\frac{\Delta_0^y \rho_{j,k}^{n+\frac{1}{4}}}{2 \Delta y} \right) (\chi^{\Delta x}(t)) \, F_2(U_{j,k}^{\Delta x}(t)) \\ &\geqq -K C |\rho|_{C_0^1} (\Delta x)^{\frac{1}{3}} \end{aligned} \tag{5.18}$$

where C is a fixed constant. In the above, we have symmetrized the contributions from the first and third terms of (5.14) by using

$$\frac{\Delta_0^x \rho_{j,k}^n}{\Delta x} = \tfrac{1}{2}\left(\frac{\Delta_0^x \rho_{j,k}^n + \Delta_0^x \rho_{j,k}^{n+\frac{3}{4}}}{\Delta x}\right) + O(\Delta t^n)$$

which results in errors of the same form as implied by (5.15), (5.16).

By assumption, $U^{\Delta x}(t)$ converges boundedly a.e. to $u(x, y, t)$; thus, by the dominated convergence theorem, the first term in (5.18) converges to

$$\tfrac{1}{2}\iiint \rho_t(\tfrac{1}{2}u^2)\,dx\,dy\,dt \quad \text{as } \Delta x \to 0.$$

By the same assumption, we obtain (using the obvious piecewise constant interpolation for $\Delta_0^y \rho_{j,k}^{n+\frac{1}{4}}$) that

$$\frac{\Delta_0^y \rho_{j,k}^{n+\frac{1}{4}}}{2\Delta y} F_2(U_{j,k}^{\Delta x}(t)) \xrightarrow{\text{strongly}} \frac{\partial \rho}{\partial y} F_2(u) \tag{5.19}$$

where the strong convergence takes place in $L^2_{\mathrm{comp}}(R^2 \times R^+)$. From (5.17) and (5.19) it follows that the third term in (5.18) which is a discrete L^2 inner product of the term on the left in (5.19) and $\chi^{\Delta x}(t)$ converges to

$$\tfrac{1}{2}\iiint \frac{\partial \rho}{\partial y} F_2(u)\,dx\,dy\,dt$$

as $\Delta x \to 0$.

Since a similar argument applies to the second term in (5.18), as $\Delta x \to 0$, these estimates imply the required entropy inequality from (5.10). The fact that u is a weak solution follows directly from the remark below (2.6) and a standard result of Lax and Wendroff [16].

The entropy inequality in (5.10) can characterize physical weak solutions with *sufficiently weak shock strengths* (in contrast to *Example* 5.3) under appropriate additional assumptions but this matter is too special and uninteresting to discuss here.

Appendix

A General Theorem on Dimensional Splitting for Conservation Laws

Consider the initial value problem

$$\begin{aligned} &\text{(i)} \qquad u_t + \sum_{i=1}^{N} f_i(u)_{x_i} = 0 \qquad \text{for } t>0,\ x \in \mathbb{R}^N \\ &\text{(ii)} \qquad u(x,0) = u_0(x) \qquad \text{for } x \in \mathbb{R}^N \end{aligned} \tag{A.1}$$

where $(f_1, \ldots, f_N)$: $\mathbb{R} \to \mathbb{R}^N$ is locally Lipschitz continuous and $u_0 \in L^\infty(\mathbb{R}^N)$. It is known [13] that then there is a unique function $u(x,t)$, called the *entropy solution* of (A.1), which satisfies:

$$u \in L^\infty(\mathbb{R}^N \times (0, \infty)), \qquad t \to u(\cdot, t) \in C([0, \infty)\colon L^1_{\mathrm{loc}}(\mathbb{R}^N)),$$

$u(\cdot, 0) = u_0$, and the entropy condition

For each $\phi \in C_0^1(\mathbb{R}^+ \times \mathbb{R}^N)$, $\phi \geqq 0$, and $k \in \mathbb{R}$

$$\int_0^\infty \int_{\mathbb{R}^N} (|u(x,t)-k|\,\phi_t + \sum_{i=1}^N \operatorname{sign}(u(x,t)-k)(f_i(u(x,t)-f_i(k))\,\phi_{x_i}(x,t)\,dx\,dt \geqq 0. \tag{A.2}$$

Denote the entropy solution of (A.1) by $S^{(f_1,\ldots,f_N)}(t)\,u_0$. That is,

$$S^{(f_1,\ldots,f_N)}(t)\,u_0 = u(x_1,\ldots,x_N,t)$$

when u is the solution of (A.1). Set $S^{f_1}(t)=S^{(f_1,0,\ldots,0)}(t)$, $S^{f_2}(t)=S^{(0,f_2,0,\ldots,0)}(t)$, etc. We wish to recover S from the $S^{f_i}(t)$. The semigroup associated with the one space variable problem

$$u_t + h(u)_z = 0,$$
$$u(z,0) = u_0(z)$$

will also be written $S^h(t)$. This is consistent with the above notation in that, for example, if we solve

$$\frac{\partial}{\partial t} u(x_1,\ldots,x_N,t) + \frac{\partial}{\partial x_1} f_1(u(x_1,x_2,\ldots,x_N,t)) = 0$$
$$u(x_1,x_2,\ldots,x_N,0) = u_0(x_1,\ldots,x_N)$$

by regarding $(x_2,\ldots,x_N)$ as parametrizing the initial data in a one space variable problem we obtain the entropy solution of the N space variable problem. One just checks the entropy condition to see that this is so.

We define general dimensional splitting methods in terms of finite sequences $\alpha_1,\ldots,\alpha_M$ of numbers satisfying

$$\alpha_i > 0 \quad \text{for } i=1,\ldots,M \quad \text{and} \quad \alpha_1+\ldots+\alpha_M = 1 \tag{A.3}$$

and functions $\{h\colon 1,\ldots,M\} \to \{f_1,\ldots,f_N\}$, that is:

$$\text{for each } i \in \{1,\ldots,M\}, \qquad h(i) \in \{f_1,\ldots,f_N\}. \tag{A.4}$$

Given $(\Delta t)>0$, a candidate approximation $S^{\Delta t}$ to $S^{(f_1,\ldots,f_N)}$ is defined as follows: Set

(i) $S^{\Delta t}(t)\,u_0 = S^{h(1)}(t)\,u_0$ for $0 \leqq t \leqq \alpha_1 \Delta t$,

(ii) $S^{\Delta t}(t)\,u_0 = S^{h(j+1)}(t)\,S^{\Delta t}((\alpha_1+\ldots+\alpha_j)\,\Delta t)\,u_0$ for $j+1 \leqq M$ and $(\alpha_1+\ldots+\alpha_j)\,\Delta t \leqq t \leqq (\alpha_1+\ldots+\alpha_{j+1})\,\Delta t$, (A.5)

(iii) $S^{\Delta t}(n\,\Delta t+\xi)\,u_0 = S^{\Delta t}(\xi)\,S^{\Delta t}(\Delta t)^n\,u_0$ for $n=1,2,\ldots$ and $0 \leqq \xi \leqq \Delta t$.

Together, (A.5)(i) and (A.5)(ii) define $S^{\Delta t}(t)$ for $0 \leqq t \leqq (\alpha_1+\ldots+\alpha_M)\,\Delta t = \Delta t$ (by (A.3)) and then (iii) extends this definition to all $t \geqq 0$. The approximation $S^{\Delta t}$ is *consistent* with $S^{(f_1,\ldots,f_N)}$ if

$$\beta_j = \sum_{h(i)=f_j} \alpha_i \tag{A.6}$$

is independent of j. In this case we have:

Theorem A. *Let $\beta=\beta_1=\beta_2=\ldots=\beta_N$. Then for every $u_0\in L^\infty(\mathbb{R}^N)$*

$$\lim_{\Delta t\to 0} S^{\Delta t}\left(\frac{t}{\beta}\right)u_0 = S^{(f_1,f_2,\ldots,f_N)}(t)\,u_0$$

where the convergence takes place in $C([0,\infty): L^1_{\mathrm{LOC}}(\mathbb{R}^N))$.

By repeating the argument in (3.4)–(3.7) of Sect. 3, we establish Theorem A as a consequence of

Proposition A. *Let $\Delta t_l>0$ and $\Delta t_l\to 0$ as $l\to\infty$. Let $\lim_{l\to\infty} S^{\Delta t_l}(t)\,u_0 = v(x_1,\ldots,x_N,t)$* a.e. *Then for every $\gamma\equiv|u-k|$ and $\phi\in C^1(\mathbb{R}^N\times(0,\infty))$ with $\phi\geqq 0$.*

$$\int_0^\infty \int_{\mathbb{R}^N} \left(\gamma(v)\,\phi_t + \sum_{i=1}^N \beta_i F_i(v)\,\phi_{x_i}\right) dx\,dt \geqq 0 \tag{A.7}$$

where β_j is given by (A.8) *and $F_i=\operatorname{sgn}(v-k)(f_i(v)-f_i(k))$.*

One finds from (A.3) that v satisfies the entropy condition associated with $v_t+\sum_{i=1}^N \beta_i f_i(v)_{x_i}=0$ and, if $\beta=\beta_1=\ldots=\beta_N$, then $v(x,t/\beta)$ satisfies the entropy condition for (A.1).

It remains to prove Proposition A.

Proof of Proposition A. Let $t_{n,j}=n\Delta t+(\alpha_1+\ldots+\alpha_j)\Delta t$ for $n=0,1,2,\ldots$ and $j=1,\ldots,M$. By the definition of $S^{\Delta t}(t)$, $u^{\Delta t}=S^{\Delta t}(t)\,u_0$ satisfies (using the remark below Lemma 3.1 for $S^{h(j+1)}$ and setting $\mathscr{H}(j)=i$ if $h(j)=f_i$)

$$\begin{aligned}
&\int_{t_{n,j}}^{t_{n,j+1}} \int_{\mathbb{R}^N} \left(\gamma(u^{\Delta t})\,\phi_t + F_{\mathscr{H}(j+1)}(u^{\Delta t})\,\phi_{x_{\mathscr{H}(j+1)}}\right) dx\,dt \\
&\quad \geqq \int_{\mathbb{R}^N} \gamma(u^{\Delta t}(x,t_{n,j+1}))\,\phi(x,t_{n,j+1})\,dx \\
&\qquad - \int_{\mathbb{R}^N} \gamma(u^{\Delta t}(x,t_{n,j})\,\phi(x,t_{n,j})\,dx
\end{aligned} \tag{A.8}$$

where $F_{\mathscr{H}(j+1)}$ is given by $\operatorname{sgn}(u^{\Delta t}-k)(f_i(u^{\Delta t})-f_i(k))$ provided that $h(j)=f_i$. Summing the relations (A.8) over $j=0,\ldots,M-1$ and then $n=1,2,\ldots$, noticing that $t_{r,M}=t_{r+1,0}$ and using the collapsing nature of the sum on the right together with the definition of $\mathscr{H}(j)$ yields

$$\int_0^\infty \int_{\mathbb{R}^N} \left(\gamma(u^{\Delta t})\,\phi_t + \sum_{i=1}^N \chi_i^{\Delta t}(t)\,F_i(u^{\Delta t})\,\phi_{x_i}\right) dx\,dt \geqq 0, \tag{A.9}$$

where

$$\begin{aligned}
\chi_i^{\Delta t}(t)=1 \ \text{ if } t_{n,j}\leqq t\leqq t_{n,j+1} \text{ for some } n, j \text{ with } h(j+1)=f_i,\\
\text{and } \chi_i^{\Delta t}(t)=0 \quad \text{otherwise.}
\end{aligned} \tag{A.10}$$

Now we use the elementary facts that

$$\lim_{\Delta t\to 0}\int_0^\infty\int_{\mathbb{R}^N}\chi_i^{\Delta t}(t)\,\psi(x,t)\,dx\,dt=\beta_i\int_0^\infty\int_{\mathbb{R}^N}\psi(x,t)\,dx\,dt \tag{A.11}$$

for each $\psi\in L^1(\mathbb{R}^N\times(0,\infty))$ (see below), $|\chi_i^{\Delta t}|\leqq 1$ and $F_i(u^{\Delta t_l})\,\phi_{x_i}\to F_i(v)\,\phi_{x_i}$ in $L^1(\mathbb{R}^N\times(0,\infty))$ to pass to the limit in (A.9) as $\Delta t=\Delta t_l\to 0$ and find (A.7). A remark about (A.11) may be in order. If $\chi_i(t)=1$ for $\alpha_1+\ldots+\alpha_j\leqq t\leqq\alpha_1+\ldots+\alpha_{j+1}$ and $h(j+1)=f_i$, $\chi_i(t)=0$ for other t in $[0,1]$ and χ_i is 1-periodic, then $\chi_i^{\Delta t}(t)=\chi_i(t(\Delta t))$. Now

$$\int_0^1\chi_i(s)\,ds=\sum_{h(i)=f_j}\alpha_j=\beta_i$$

so

$$\int_{n\Delta t}^{(n+1)\Delta t}\chi_i^{\Delta t}(s)\,ds=\Delta t\,\beta_i.$$

If $\psi\in C_0^1((0,\infty)\times\mathbb{R}^N)$ and $\psi(x,t)=0$ for $t>T$ we thus have

$$\int_0^\infty\int_{\mathbb{R}^N}\chi_i^{\Delta t}(t)\,\psi(x,t)\,dt=\int_{\mathbb{R}^N}\sum_{n=0}^{[T/\Delta t]+1}\int_{n\Delta t}^{(n+1)\Delta t}\chi_i^{\Delta t}(t)\,\psi(x,t)\,dt\,dx$$

where $[r]$ is the greatest integer in r. Now $\psi(x,t)=\psi(x_1 n\Delta t)+O(\Delta t)$ for $n\Delta t\leqq t\leqq(n+1)\Delta t$ so the right-hand side above is

$$\begin{aligned}&\int_{\mathbb{R}^N}\sum_{n=0}^{[T/\Delta t]+1}\int_{n\Delta t}^{(n+1)\Delta t}\chi_i^{\Delta t}(t)(\psi(x,n\Delta t)+O(\Delta t))\,dx\,dt\\&=\sum_{n=0}^{[T/\Delta t]+1}\beta_i\Delta t\int_{n\Delta t}^{(n+1)\Delta t}\psi(x,n\Delta t)\,dx+O(\Delta t)\\&=\beta_i\int_0^\infty\int_{\mathbb{R}^N}\psi(x,t)\,dx\,dt+O(\Delta t).\end{aligned}$$

The general result (A.15) follows from approximation by elements of $C_0^1(\mathbb{R}^N\times(0,\infty))$. (This result is, by the way, well known.)

We remark that the structure of the arguments in this appendix allows one to apply them to the notion of 'integral solution' in nonlinear semigroup theory.

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