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Covering Spaces in Representation-Theory

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In 1979 Chr. Riedtmann introduced coverings of the Auslander-Reiten quiver Γ_Λ of a representation-finite algebra Λ ([15]; see also 1.3 and 2.2 below). Our main results are that Γ_Λ admits many finite coverings in general, and that each of these is the Auslander-Reiten quiver Γ_M of some representation-finite M (2.9). In order to prove the first statement we show in § 1 that the finite coverings of Γ_Λ are classified by the actions of the fundamental group Π (1.2) of Γ_Λ on finite sets; in general, there are many such actions because Π is a free (non-commutative) group (4.2). We obtain our second main statement by considering the algebra E which is defined by the mesh relations of a finite covering Δ of Γ_Λ ([16], 1.4; see 2.5 below); such an E satisfies the Auslander conditions characterizing the algebras of the form $\mathrm{End}(\bigoplus_i M_i)$, where M_i ranges through chosen representatives of the indecomposable modules over some representation-finite algebra (2.3). In case $\Delta = \Gamma_\Lambda$, the relations between E and Λ are studied in § 5.

The theoretical notions developed in this paper give rise to concrete algorithms (and computer programs) which enable us to construct the Auslander-Reiten quivers for plenty of algebras. We enter upon these algorithms in § 6 tackling the special case $\Pi = 1$. The general case will be examined in a subsequent publication, from which we borrow the Auslander-Reiten quivers of the 14 "maximal" algebras listed at the end of our paper (each basic connected representation-finite algebra with two simple modules is isomorphic to a quotient of a "maximal" algebra or to its opposite). The list of these maximal algebras has also been obtained by A.V. Nikulin and C.A. Panasiuk [14] as an application of the methods developed in Kiev.

The present paper is intimately related to the results of Chr. Riedtmann ([15], [16]). Her unpublished collection of Auslander-Reiten quivers was a decisive help in proving that the fundamental group is free. Unfortunately, her own work on selfinjective algebras of class D_n and the distance between Boston and Zürich finally prevented us from carrying through the original plan of a common publication. We take pleasure in thanking her for encouragements and remarks.

The second author gave a series of lectures on this publication at the Ukrainian Academy of Sciences (Kiev, October 1980). The results (except §6, 7) were announced by him at the Conference on Representations of Algebras in Puebla (August 1980). His preliminary version was finally cleaned and improved by the first author.

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1. The Fundamental Group of a Translation-Quiver

1.1 Consider a quiver Γ together with a bijection τ, whose domain and range are both subsets of Γ_0 (= the set of vertices of Γ; see Fig. 1.1).

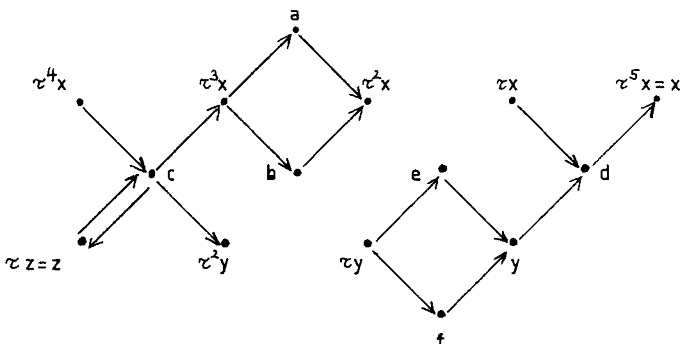
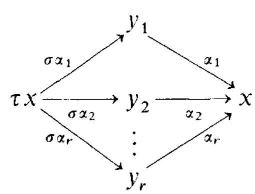


Fig. 1.1

The pair (Γ, τ) is called a *translation-quiver* (= Darstellungsköcher in the sense of Riedtmann [15]) if the following conditions a) and b) are satisfied:

a) Γ has no loop $\circlearrowleft$ and no multiple arrow $\cdot\rightrightarrows$.

b) Whenever τ is defined at some point $x \in \Gamma_0$, the set x^- of predecessors of x in Γ_0 coincides with the set $(\tau x)^+$ of successors of τx:



As usual, we often write Γ instead of (Γ, τ). The bijection τ is called the *translation* of (Γ, τ). The vertices of Γ where τ is not defined are called *projective*; those where τ^{-1} is not defined are called *injective*. The full subquiver of Γ formed by a non-projective x, by its non-injective translate τx and by the set $(\tau x)^+ = x^-$ is called the *mesh* starting at τx and stopping at x; for each $\alpha \in \Gamma_1$ (= the set of arrows of Γ) with non-projective head x and tail y we denote by $\sigma\alpha$ the unique arrow with *tail* τx and *head* y.

For a *geometric interpretation* we refer to §4 below.

1.2 Let Γ be a translation-quiver. In order to define the fundamental group of Γ at some point, we introduce "new" arrows $\tau x \xrightarrow{\gamma_x} x$, one for each non-projective vertex $x \in \Gamma_0$. We represent these new arrows by broken oriented line segments and say that they have *degree* 2 in contrast with the old arrows of Γ whose degree is defined as 1. The vertex-set Γ_0, the old and the new arrows give rise to some quiver $\hat{\Gamma}$, which may have loops (see Fig. 1.2 which represents $\hat{\Gamma}$, when Γ is the translation-quiver of Fig. 1.1).

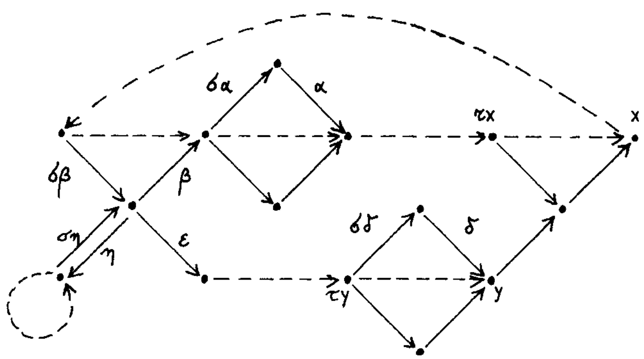


Fig. 1.2

Let $x, y \in \Gamma_0$. A *walk* of Γ is a path of the quiver formed by $\hat{\Gamma}$ and the formal inverses α^{-1} of the arrows $\alpha \in \hat{\Gamma}_1$ ([7], 4.1), i.e. a sequence $w = (y | \alpha_m, \ldots, \alpha_1 | x)$, where $\alpha_m \ldots \alpha_2 \alpha_1$ is a formal composition of arrows of $\hat{\Gamma}$ or of formal inverses of such arrows; this formal composition is supposed to start at x, to stop at y. In the case of Fig. 1.2 for instance, the sequence

$$(y | \delta, \sigma\delta, \gamma_{\tau y}, \varepsilon, \sigma\beta, (\sigma\beta)^{-1}, \beta^{-1}, (\sigma\alpha)^{-1}, \alpha^{-1}, \gamma_{\tau x}^{-1}, \gamma_x^{-1} | x)$$

is a walk from x to y. A walk from x to y may be composed with a walk from y to z according to the formula

$$(z | \beta_n, \ldots, \beta_1 | y)(y | \alpha_m, \ldots, \alpha_1 | x) = (z | \beta_n, \ldots, \beta_1, \alpha_m, \ldots, \alpha_1 | x).$$

On the set of all walks of Γ we define the *homotopy* relation as the smallest equivalence relation H satisfying the conditions a), b) and c):

a) $(x | \alpha, \alpha^{-1} | x) \underset{H}{\sim} (x || x) \underset{H}{\sim} (x | \beta^{-1}, \beta | x)$ for each arrow $\alpha \in \hat{\Gamma}_1$ with head x and each arrow $\beta \in \hat{\Gamma}_1$ with tail x.

b) $(x | \alpha, \sigma\alpha | \tau x) \underset{H}{\sim} (x | \gamma_x | \tau x)$ and $(\tau x | (\sigma\alpha)^{-1}, \alpha^{-1} | x) \underset{H}{\sim} (\tau x | \gamma_x^{-1} | x)$ for each arrow α of degree 1 with non-projective head x.

c) The relation $v \underset{H}{\sim} v'$ implies $wv \underset{H}{\sim} wv'$ and $vu \underset{H}{\sim} v'u$ whenever this makes sense.

Clearly, the composition of walks induces a (partially defined) composition of homotopy classes: if $\bar{w}$ denotes the homotopy class of a walk w, then $\bar{w}\bar{v}$ is defined whenever wv is, and we have $\bar{w}\bar{v} = \overline{wv}$. In particular, for any given $x \in \Gamma_0$, the composition is everywhere defined in the set $\Pi(\Gamma, x)$ of homotopy classes of walks from x to x. For this composition $\Pi(\Gamma, x)$ is a group: the *fundamental group* of Γ at x.

1.3 The *universal cover* $\tilde{\Gamma}$ of Γ at the point $x \in \Gamma_0$ is by definition the following translation-quiver: the points of $\tilde{\Gamma}$ are the homotopy classes $\bar{w}$ of walks w of Γ which start at the given fixed point $x = w^{\cdot}$ and stop at some (variable) point ${}^{\cdot}w \in \Gamma_0$. The arrows of $\tilde{\Gamma}$ are the pairs $(\bar{w}, \alpha)$ formed by a homotopy class $\bar{w} \in \tilde{\Gamma}_0$ and an arrow ${}^{\cdot}w \xrightarrow{\alpha} z$ of Γ; tail and head of $(\bar{w}, \alpha)$ are the classes $\bar{w}$ and $\overline{(z|\alpha|{}^{\cdot}w)\,w}$ respectively. Finally, the translation of $\tilde{\Gamma}$ is defined by the formula $\tau\bar{w} = \overline{(\tau^{\cdot}w|\gamma_{{}^{\cdot}w}^{-1}|{}^{\cdot}w)\,w}$, which makes sense whenever ${}^{\cdot}w$ is not projective; otherwise, $\bar{w}$ is a projective point of $\tilde{\Gamma}$.

Obviously, there is a unique translation-quiver-morphism $\pi: \tilde{\Gamma} \to \Gamma$ such that $\pi(\bar{w}) = {}^{\cdot}w$. This is a covering morphism in the following sense ([15]).

Definition. A translation-quiver morphism $f: \Delta \to \Gamma$ is called a covering if for each point $p \in \Delta_0$ the induced maps $p^- \to (fp)^-$ and $p^+ \to (fp)^+$ are bijective. Furthermore, τp and $\tau^{-1} q$ should be defined if τfp and $\tau^{-1} fq$ are respectively so (of course, since f is a translation-quiver-morphism, we have $f\tau p = \tau f p$ whenever τp is defined) (see Fig. 1.3a).

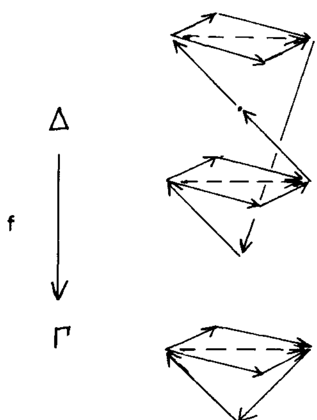


Fig. 1.3a

In Fig. 1.3b) we give a simple example of a universal covering. Only arrows of degree 1 are represented; p and p_n are projective points, whereas $\tau a = a$, $\tau b = b$ and $\tau c = c$. The quiver Γ is drawn on a cylinder (see §4), whose "universal covering" is a serrate strip.

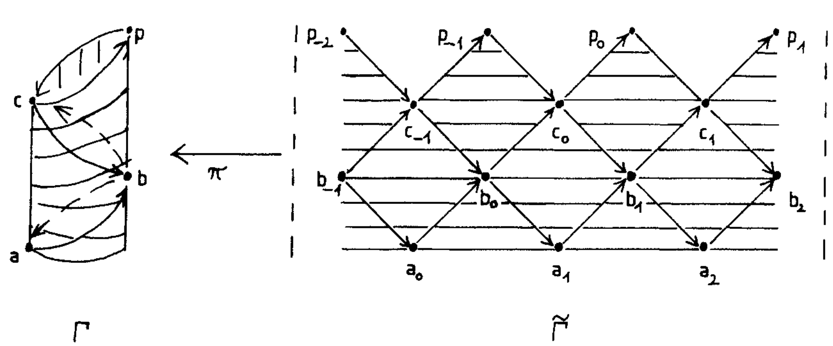


Fig. 1.3b

1.4 Let $f\colon \Delta \to \Gamma$ be a covering and $f^{-1}(x)$ the fibre of $x \in \Gamma_0$ in Δ_0. For each walk w of Γ starting at x and each point $y \in f^{-1}(x)$, there is a unique walk v of Δ, which starts at y and "lies over" w. The terminus ${}^{\cdot}v$ of v depends only on y and $\bar{w}$. In particular, if ${}^{\cdot}w = x$, ${}^{\cdot}v$ lies in $f^{-1}(x)$. In this way we get an operation of the fundamental group $\Pi(\Gamma, x)$ on $f^{-1}(x)$: $\bar{w}y = {}^{\cdot}v$.

Proposition. *If Γ is a connected translation-quiver and $x \in \Gamma_0$, the functor $f \mapsto f^{-1}(x)$ is an equivalence between the category of Γ-coverings and the category of $\Pi(\Gamma, x)$-sets.*

Clearly, the category of Γ-coverings has as objects the coverings $f\colon \Delta \to \Gamma$, as morphisms the commutative triangles

$$\begin{tikzcd} \Delta \arrow[rr, dashed, "d"] \arrow[dr, "f"'] & & \Delta' \arrow[dl, "f'"] \\ & \Gamma & \end{tikzcd}$$

where f and f' are coverings and d is a translation-quiver-morphism (this implies in fact that d is also a covering). A $\Pi(\Gamma, x)$-set is a set M together with an action of $\Pi(\Gamma, x)$ on M from the left.

Proof. Let us just produce a "quasi-inverse" functor. Starting with a $\Pi(\Gamma, x)$-set M we first construct a translation-quiver $\tilde{\Gamma} \times M$ having $\tilde{\Gamma}_0 \times M$ as set of vertices and $\tilde{\Gamma}_1 \times M$ as set of arrows: If $x \xrightarrow{\alpha} y$ is an arrow of $\tilde{\Gamma}$ and $m \in M$, the arrow $(\alpha, m) \in \tilde{\Gamma}_1 \times M$ starts at (x, m) and stops at (y, m). Furthermore, we set $\tau(x, m) = (\tau x, m)$ whenever x is non-projective. The translation-quiver $\tilde{\Gamma} \times M$ is the direct sum of some copies of $\tilde{\Gamma}$ indexed by M. It admits $\Pi(\Gamma, x)$ as a group of automorphisms: indeed, for each $\bar{\gamma} \in \Pi(\Gamma, x)$, the permutation $(\bar{w}, m) \mapsto (\bar{w}\bar{\gamma}^{-1}, \bar{\gamma}m)$ of $\tilde{\Gamma}_0 \times M$ yields an automorphism of $\tilde{\Gamma} \times M$. We denote by $\tilde{\Gamma} \overset{\Pi}{\times} M$ the quotient of $\tilde{\Gamma} \times M$ under this group-action. This is a translation-quiver having as points the orbits of $\Pi = \Pi(\Gamma, x)$ in $\tilde{\Gamma}_0 \times M$, as arrows the orbits of Π in $\tilde{\Gamma}_1 \times M$. It is related to Γ by means of a covering morphism $f\colon \tilde{\Gamma} \overset{\Pi}{\times} M \to \Gamma$, which is deduced from the projection $\tilde{\Gamma} \times M \to \Gamma$, $(\bar{w}, m) \mapsto {}^{\cdot}w = \pi\bar{w}$ by passing to the quotient. The construction $M \mapsto f$ supplies us with the wanted quasi-inverse functor.

1.5 In the particular case of the universal covering $\pi\colon \tilde{\Gamma} \to \Gamma$, the fibre $\pi^{-1}(x)$ is the fundamental group itself equipped with the action by left translations. An automorphism of the Π-set $\pi^{-1}(x)$ is just a right translation $\bar{\delta} \mapsto \bar{\delta}\bar{\gamma}$ of Π. The corresponding automorphism of the universal covering assigns to the homotopy class $\bar{w} \in \tilde{\Gamma}_0$ the composed class $\bar{w}\bar{\gamma} \in \tilde{\Gamma}_0$. Since each Π-set is a disjoint sum of Π-sets of the form Π/P, where P is a subgroup of Π, we deduce from Proposition 1.4 that *each covering of Γ is a "disjoint sum" of coverings of the form* $\kappa\colon \tilde{\Gamma}/P \to \Gamma$, where κ is deduced from π by passing to the quotient.

1.6 Of course, a translation-quiver Γ is called *simply connected* if it is connected and if $\Pi(\Gamma, x) = \{1\}$ for some $x \in \Gamma_0$. This implies $\Pi(\Gamma, y) = \{1\}$ for all $y \in \Gamma_0$ and is equivalent to saying that each connected covering $\kappa\colon \Delta \to \Gamma$ is an isomorphism.

Proposition. *Let Γ be a simply connected translation-quiver and $x \in \Gamma_0$. Then there is one and only one (translation-)quiver-morphism κ from Γ into the translation-quiver $\mathbb{Z}A_2$ of Fig.* 1.6 *such that* $\kappa(x)=0$.

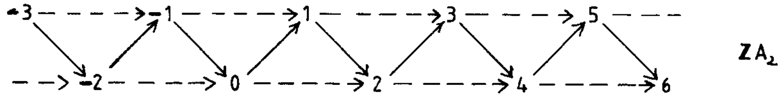


Fig. 1.6

Proof. Define the length $\lambda(w) \in \mathbb{Z}$ of a walk using the following formulae:

$$\lambda(x_n|\alpha_n, \dots, \alpha_1|x_0) = \lambda(x_n|\alpha_n|x_{n-1}) + \dots + \lambda(x_1|\alpha_1|x_0), \qquad \lambda(y|\alpha|x) = 1$$

and $\lambda(x|\alpha^{-1}|y) = -1$ if $\alpha \in \Gamma_1$, $\lambda(x|\gamma_x|\tau x) = 2$, $\lambda(\tau x|\gamma_x^{-1}|x) = -2$. By the definition of homotopy, λ is constant on each homotopy class. Now, since Γ is simply connected, the walks from x to any given $y \in \Gamma_0$ are homotopic to each other. So we may set $\kappa(y) = \lambda(w)$, where w is any walk from x to y. This yields the wanted quiver-morphism.

1.7 Following Riedtmann [15], a translation-quiver Γ is called *stable* if its translation is everywhere defined. The *simply connected stable translation-quivers* can be constructed in the following way: Start with any oriented tree T and set $(\mathbb{Z}T)_0 = \mathbb{Z} \times T_0$, $(\mathbb{Z}T)_1 = \{-1, 1\} \times \mathbb{Z} \times T_1$; if $x \xrightarrow{\alpha} y$ belongs to T_1, define the tails and the heads of the arrows $(-1, n, \alpha)$ and $(1, n, \alpha)$ as in the diagrams

$$(n, x) \xrightarrow{(1, n, \alpha)} (n, y) \quad \text{and} \quad (n-1, y) \xrightarrow{(-1, n, \alpha)} (n, x);$$

finally, set $\tau(n, y) = (n-1, y)$. This construction yields a simply-connected stable translation-quiver $\mathbb{Z}T$. Two translation-quivers $\mathbb{Z}T$ and $\mathbb{Z}T'$ are isomorphic iff the non-oriented trees $\bar{T}$ and $\bar{T}'$ underlying T and T' are isomorphic.

2. Auslander-Categories and Riedtmann-Quivers

In the sequel, k denotes a field, which we suppose to be *algebraically closed* for the sake of simplicity.

2.1 A *k-category* Λ is a category whose morphism-sets $\Lambda(x, y)$ are endowed with k-vectorspace structures such that the composition maps are k-bilinear.

Definition. A *locally finite-dimensional* (resp. a *locally bounded*) category is a k-category Λ satisfying the conditions a), b) and c) (resp. a), b) and c')) below:

a) For each $x \in \Lambda$, the endomorphism algebra $\Lambda(x, x)$ is local.
b) Distinct objects of Λ are not isomorphic.
c) $\forall x, y \in \Lambda$, $[\Lambda(x, y): k] < \infty$.
c') $\forall x \in \Lambda$, $\sum_{y \in \Lambda} [\Lambda(x, y): k] < \infty$ and $\sum_{y \in \Lambda} [\Lambda(y, x): k] < \infty$.

Locally bounded categories can be constructed in the following way: Start with a quiver Q, which may be infinite. The *path-category* kQ of Q has as objects the vertices of Q; if $x, y \in Q_0$, the morphism-space $kQ(x, y)$ consists of the formal linear combinations of paths from x to y. In the k-category kQ which we get in this way we distinguish two ideals kQ^+ and kQ^{+2}, whose values at some pair of objects (x, y) are the subspaces $kQ^+(x, y)$ and $kQ^{+2}(x, y)$ of $kQ(x, y)$ spanned by the paths of lengths $\geqq 1$ and $\geqq 2$ respectively. Given an ideal $I \subset kQ^{+2}$, it is easy to see, that the residue-category kQ/I is *locally bounded iff the conditions* d) *and* e) below *are satisfied:*

d) Q is *locally finite*, i.e. the number of arrows starting or stopping at any vertex is finite.

e) For each vertex $x \in Q_0$, there is a natural number N_x such that I contains each path of length $\geqq N_x$ which starts or stops at x.

Conversely, *each locally bounded category is isomorphic to such a kQ/I.* We recall the argumentation: Start with any *locally finite-dimensional category Λ.* The *radical* $\mathscr{R}\Lambda$ of Λ is the ideal assigning to a pair of objects (x, y) the subspace $\mathscr{R}\Lambda(x, y)$ of $\Lambda(x, y)$ formed by the non-invertible morphisms. The *radical-square* $\mathscr{R}^2\Lambda$ is defined by $\mathscr{R}^2\Lambda(x, y) = \sum_{z \in \Lambda} \mathscr{R}\Lambda(z, y)\,\mathscr{R}\Lambda(x, z)$. The *quiver* Q_Λ of Λ has as vertices the objects of Λ; if x, y are two such objects, we join x to y with a sequence of n arrows $x \to y$, where $n = [\mathscr{R}\Lambda(x, y)/\mathscr{R}^2\Lambda(x, y) : k]$. Now, if Λ is locally bounded, Q_Λ is locally finite. We get an isomorphism $kQ_\Lambda/I \xrightarrow{\sim} \Lambda$ for some ideal $I \subset kQ_\Lambda^{+2}$ by sending the different arrows $x \to y$ onto respresentatives in $\mathscr{R}\Lambda(x, y)$ of chosen basis vectors of $\mathscr{R}\Lambda(x, y)/\mathscr{R}^2\Lambda(x, y)$.

2.2 If Λ is a k-category, a *Λ-module* is a k-linear functor $\ell : \Lambda^{\mathrm{op}} \to \mathrm{Mod}\,k$, where $\mathrm{Mod}\,k$ is the category of k-vector-spaces. The Λ-module ℓ is *finitely generated* if it is a quotient of a finite direct sum of representable functors. We denote by $\mathrm{mod}\,\Lambda$ the category of all finitely generated Λ-modules, by $\mathrm{ind}\,\Lambda$ (resp. $\mathrm{proj}\,\Lambda$) the full subcategory formed by chosen representatives of the indecomposable modules (resp. by the projective modules).

Let Λ be *locally bounded:* a Λ-module ℓ is then finitely generated iff $\sum_{x \in \Lambda} [\ell(x) : k] < \infty$; it is indecomposable projective (resp. injective) iff it is isomorphic to some $\Lambda(?, x)$ (resp. to $D\Lambda(x, ?)$, where DV denotes the dual of a vector space V); the category $\mathrm{mod}\,\Lambda$ admits *Auslander-Reiten sequences.* In fact, the existence-proof given in [7] extends easily to the locally bounded situation. So does the construction of the *Auslander-Reiten quiver* Γ_Λ; this is a translation-quiver whose underlying quiver is obtained from $Q_{\mathrm{ind}\,\Lambda}$ by contracting possible multiple arrows to simple ones.

Definition. A *locally representation-finite* category is a locally bounded category Λ such that the number of $\ell \in \mathrm{ind}\,\Lambda$ satisfying $\ell(x) \neq 0$ is finite for each x.

It is easy to see that the last condition is equivalent to saying that $\mathrm{ind}\,\Lambda$ *is locally bounded.*

2.3 **Proposition** (M. Auslander [1]). *The following statements are equivalent:*

(i) *M is isomorphic to* $\mathrm{ind}\,\Lambda$ *for some locally representation-finite Λ.*

(ii) *M is locally bounded and satisfies the following conditions* a), b):

a) $g\ell \dim M \leqq 2$, *i.e.* $\mathrm{Ext}^3_M(m, n) = 0$ *for all* $m, n \in \mathrm{mod}\,M$.

b) *For each projective $p \in \mathrm{mod}\,M$, there is an exact sequence $0 \to p \to i_0 \to i_1$, where $i_0, i_1 \in \mathrm{mod}\,M$ are both injective and projective.*

Proof. (i)⇒(ii): Assume that $M = \operatorname{ind} \Lambda$, where Λ is locally representation-finite. As we noticed in 2.2, M is locally bounded. Obviously, the functor $\ell \mapsto \operatorname{Hom}_\Lambda(?, \ell)$ yields an equivalence $\operatorname{mod} \Lambda \xrightarrow{\sim} \operatorname{proj} M$. Accordingly, a morphism $g: m_0 \to m_1$ of $\operatorname{proj} M$ is isomorphic to $\operatorname{Hom}_\Lambda(?, f): \operatorname{Hom}_\Lambda(?, \ell_0) \to \operatorname{Hom}_\Lambda(?, \ell_1)$ for some morphism $f: \ell_0 \to \ell_1$ of $\operatorname{mod} \Lambda$. It follows that the kernel of g in $\operatorname{mod} M$ is isomorphic to $\operatorname{Hom}_\Lambda(?, \operatorname{Ker} f)$, which is projective. This proves a).

In order to prove b) we may assume that $p = \operatorname{Hom}_\Lambda(?, \ell)$. Let $0 \to \ell \to j_0 \to j_1$ be an injective resolution of ℓ in $\operatorname{mod} \Lambda$. This yields an exact sequence $0 \to p \to i_0 \to i_1$, where $i_0 = \operatorname{Hom}_\Lambda(?, j_0)$ and $i_1 = \operatorname{Hom}_\Lambda(?, j_1)$ are projective by construction. Statement b) now follows from the fact that $\operatorname{Hom}_\Lambda(?, j)$ is injective in $\operatorname{mod} M$, if j is so in $\operatorname{mod} \Lambda$: indeed, for each $\ell \in M = \operatorname{ind} \Lambda$ and each $\lambda \in \Lambda$, we have canonical isomorphisms

$$\operatorname{Hom}_\Lambda(\ell, D\Lambda(\lambda, ?)) \xrightarrow{\sim} \operatorname{Hom}_\Lambda(\Lambda(\lambda, ?), D\ell) \xrightarrow{\sim} D\ell(\lambda) \xrightarrow{\sim} D \operatorname{Hom}_\Lambda(\Lambda(?, \lambda), \ell),$$

which show that $\operatorname{Hom}_\Lambda(?, j)$ is identified with the injective M-module $DM(\Lambda(?, \lambda), ?)$ if $j = D\Lambda(\lambda, ?)$. Accordingly, $\operatorname{Hom}_\Lambda(?, j)$ is an indecomposable injective M-module, if j is an indecomposable injective Λ-module.

(ii)⇒(i): Let us first recall a classical result. With each additive category I we can associate a new additive category $\tilde{I}$ which looks as follows. The objects of $\tilde{I}$ are the morphisms of I; a morphism from $i_0 \xrightarrow{f} i_1$ to $j_0 \xrightarrow{g} j_1$ is an equivalence class of commutative squares

$$(h_0, h_1):\ \begin{tikzcd} i_0 \arrow[r, "h_0"] \arrow[d, "f"'] & j_0 \arrow[d, "g"] \\ i_1 \arrow[r, "h_1"] & j_1 \end{tikzcd}$$

two such squares (h_0, h_1) and $h'_0, h'_1)$ being equivalent if $h'_0 - h_0$ factors through f. Now, the classical result is the following: if I is a full subcategory of an abelian category C, and if I consists of injective objects of C, then the kernel-functor Ker: $\tilde{I} \to C$, which maps $i_0 \xrightarrow{f} i_i$ onto $\operatorname{Ker} f$, is fully faithful.

In the situation of our proof, we choose for I the full subcategory of $\operatorname{mod} M$ consisting of the modules which are projective and injective. The conditions a) and b) mean that the kernel-functor yields an equivalence $\tilde{I} \xrightarrow{\sim} \operatorname{proj} M$. On the other hand, choose Λ to be the full subcategory of I formed by chosen representatives of the indecomposable modules of I. The functor $j \mapsto D(j, ?)$ yields an equivalence from I onto the full subcategory of $\operatorname{mod} \Lambda$ formed by the injective modules; accordingly, the induced functor $\tilde{I} \to \operatorname{mod} \Lambda$, which maps $j_0 \xrightarrow{g} j_1$ onto $\operatorname{Ker} D(g, ?)$, is an equivalence. By composition of the obtained equivalences $\operatorname{proj} M \xleftarrow{\sim} \tilde{I} \xrightarrow{\sim} \operatorname{mod} \Lambda$, we get an isomorphism $M \xrightarrow{\sim} \operatorname{ind} \Lambda$, since M is identified with a full subcategory of $\operatorname{proj} M$ by means of the embedding $\mu \mapsto M(?, \mu)$.

2.4 *Definition.* A k-category M which satisfies the equivalent conditions of the preceding proposition is called an *Auslander-category.*

Assume that $M = \operatorname{ind} \Lambda$, where Λ is locally representation-finite. In 2.3 we used the embedding $\Lambda \to M$, $\lambda \to D\Lambda(\lambda, ?)$. In fact, it is often more convenient to use the embedding $\Lambda \to M$, $\lambda \mapsto \Lambda(?, \lambda)$. For this sake we need an M-internal characterization of the objects of M of the form $\Lambda(?, \lambda)$: let us say that an object μ of a locally finite-dimensional category M is *top-torsionfree* if there is a non-zero morphism $\alpha \in M(\mu, v)$ such that $\alpha\beta = 0$ for each non-invertible morphism β with range μ. In case $M = \operatorname{ind} \Lambda$, an object $\mu \in M$ is top-torsionfree iff it is isomorphic to some $\Lambda(?, \lambda)$ (the equivalence of these statements is obvious, as it simply provides us with a characterization of the projective indecomposable Λ-modules among all the indecomposable ones).

The terminology "top-torsionfree" can be justified as follows: the non-invertible morphisms $\beta \in M(\kappa, \mu)$ are the elements of the radical $\mathscr{R}M(\kappa, \mu) = (\mathscr{R}M(?, \mu))(\kappa)$; accordingly, a morphism $\alpha \in M(\mu, v)$ satisfies $\alpha\beta = 0$ for each non-invertible β iff $M(?, \alpha)$: $M(?, \mu) \to M(?, v)$ factors through the simple *top* $k_\mu = M(?, \mu)/\mathscr{R}M(?, \mu)$ of $M(?, \mu)$ (notice that $k_\mu(\mu) = k$ and $k_\mu(v) = 0$ for $v \neq \mu$). So μ is top-torsionfree iff the top k_μ is "torsionfree", i.e. can be embedded into some projective M-module $M(?, v)$.

There is another useful characterization, saying that *an object μ of an Auslander-category M is top-torsionfree iff k_μ has projective dimension $\leqq 1$ in* $\operatorname{mod} M$: indeed, if $M(?, \alpha)$ factors through k_μ, the sequence $0 \to \operatorname{Ker} M(?, \alpha) \to M(?, \mu) \to k_\mu \to 0$ is a projective resolution of k_μ ($\operatorname{Ker} M(?, \alpha)$ is projective, since $g\ell \dim M \leqq 2$). Conversely, if we have a projective resolution $0 \to p \to M(?, \mu) \to k_\mu \to 0$, then either $p = 0$ and μ is obviously top-torsionfree, or the resolution yields a non-zero element ε in $\operatorname{Ext}^1(k_\mu, p)$; now, if $0 \to p \to i_0 \to i_1$ is a minimal injective resolution, ε is associated with a non-zero morphism $k_\mu \to i_1$. Since i_1 is projective, k_μ is torsion-free.

In the sequel we need the preceding (trivial) developments in form of the following

Proposition. *Let P be the full subcategory formed by the top-torsionfree objects of an Auslander-category M. Then P is locally representation-finite and the functor which* maps *$\mu \in M$ onto the P-module $M(?, \mu)|P$ yields an isomorphism* $M \xrightarrow{\sim} \operatorname{ind} P$.

2.5 Let Γ be a *locally finite* translation-quiver. The *mesh-ideal* is the ideal I_Γ of the path-category $k\Gamma$ which is generated by the elements

$$\mu_x = \sum_\alpha \alpha(\sigma\alpha) \in k\Gamma(\tau x, x), \qquad x \text{ non-projective},$$

where α ranges over all arrows of Γ heading for x. The *mesh-category* of Γ is the residue-category $k(\Gamma) = k\Gamma/I_\Gamma$. We say that Γ is *locally bounded* if $k(\Gamma)$ is so.

Proposition (Riedtmann [15]). a) *If Γ is locally bounded, so is every full sub-translation-quiver.*

b) *Γ is locally bounded iff the universal covers of its connected components are so.*

c) *Γ is locally bounded, simply connected and stable iff it is isomorphic to a Dynkin-translation-quiver $\mathbb{Z}A_n$, $\mathbb{Z}D_p$ or $\mathbb{Z}E_q$ ($p, q, n \in \mathbb{N}$, $n \geqq 1$, $p \geqq 4$, $8 \geqq q \geqq 6$; see* [6], [15]).

We recall that a *full sub-translation-quiver* Δ of Γ is determined by a subset Δ_0 of Γ_0; an arrow $x \xrightarrow{\alpha} y$ of Γ belongs to Δ iff x and y belong to Δ_0; furthermore, a vertex $x \in \Delta_0$ is projective in Δ if it is so in Γ or if $\tau x \notin \Delta_0$.

Proof. If $x, y \in \Delta_0$, the morphism-space $k(\Delta)(x, y)$ is identified with the quotient of $k(\Gamma)(x, y)$ obtained by annihilating the paths which factor through a point of $\Gamma_0 \setminus \Delta_0$. This proves a).

If $\pi: \Delta \to \Gamma$ is a covering of translation-quivers, it is clear that the induced functor $k(\pi): k(\Delta) \to k(\Gamma)$ yields isomorphisms

$$\bigoplus_z k(\Delta)(x, z) \xrightarrow{\sim} k(\Gamma)(\pi x, \pi y) \quad \text{and} \quad \bigoplus_t k(\Delta)(t, y) \xrightarrow{\sim} k(\Gamma)(\pi x, \pi y),$$

where z and t range through the vertices of Δ lying over πy and πx respectively. In case $\pi_0: \Delta_0 \to \Gamma_0$ is surjective, Γ is locally bounded iff Δ is so.

For c) we refer to [15].

2.6 **Lemma.** *Let Γ be a locally bounded translation-quiver and $y_i \xrightarrow{\alpha_i} x$, $1 \leqq i \leqq r$, the arrows stopping at some vertex $x \in \Gamma_0$. If x is projective, the α_i induce a minimal projective resolution*

$$0 \to \bigoplus_{i=1}^{r} k(\Gamma)(?, y_i) \to k(\Gamma)(?, x) \to k_x \to 0$$

of the simple $k(\Gamma)$-module k_x. If x is not projective, the α_i and $\sigma\alpha_i$ induce a minimal projective resolution of length 2

$$k(\Gamma)(?, \tau x) \xrightarrow{[k(\Gamma)(?, \sigma\alpha_i)]} \bigoplus_{i=1}^{r} k(\Gamma)(?, y_i) \xrightarrow{[k(\Gamma)(?, \alpha_i)]} k(\Gamma)(?, x) \to k_x \to 0.$$

Proof. Obviously, the α_i produce bijections $\bigoplus_{i=1}^{r} k\Gamma(t, y_i) \xrightarrow{\sim} k\Gamma^+(t, x)$ (2.1). If x is projective, the induced maps $\bigoplus_{i=1}^{r} I_\Gamma(t, y_i) \to I_\Gamma(t, x)$ are bijective too. Passing to the quotients, we get the first sequence. If x is not projective, we have

$$I_\Gamma(t, x) = \sum_{i=1}^{r} \alpha_i I_\Gamma(t, y_i) + \mu_x k\Gamma(t, \tau x) \quad (2.5).$$

This yields the second sequence. It is clear that both sequences are minimal.

Remarks. a) The lemma shows that we can go back from the mesh-category $k(\Gamma)$ to the translation-quiver (Γ, τ). Namely, Γ is the quiver of $k(\Gamma)$ (2.1). A vertex x is projective iff the projective dimension of k_x is $\leqq 1$; if x is not projective, τx is determined by the fact that $k(\Gamma)(?, \tau x)$ is the component of degree 2 in the minimal projective resolution of k_x in $\operatorname{mod} k(\Gamma)$.

b) In general, the morphism $[k(\Gamma)(?, \sigma\alpha_i)]$ of our lemma is not mono. For instance, if $\Gamma = \mathbb{Z}A_2$ (1.6), the projective resolution of k_0 is $\dots p_{-3} \to p_{-2} \to p_{-1} \to p_0 \to k_0 \to 0$, where $p_i = k(\mathbb{Z}A_2)(?, i)$.

c) *If x is a non-projective vertex of Γ, we have* $\operatorname{Ext}^1(k_x, p)=0$ *for each projective $k(\Gamma)$-module p.* Indeed, we can compute $\operatorname{Ext}^1(k_x, p)$ by applying the functor $\operatorname{Hom}_{k(\Gamma)}(?, p)$ to the projective resolution of k_x given in the lemma. If $p=k(\Gamma)(?, z)$, this yields

$$0 \to \operatorname{Hom}_{k(\Gamma)}(k_x, p) \to k(\Gamma)(x, z) \to \bigoplus_{i=1}^{r} k(\Gamma)(y_i, z) \to k(\Gamma)(\tau x, z)$$

since $\operatorname{Hom}_{k(\Gamma)}(k(\Gamma)(?, y), k(\Gamma)(?, z)) \xrightarrow{\sim} k(\Gamma)(y, z)$. Applying our lemma to the translation-quiver Γ^{op}, we see that the preceding sequence is exact at the point $\bigoplus_{i=1}^{r} k(\Gamma)(y_i, z)$.

2.7 **Lemma.** *If Γ is a locally bounded translation-quiver, the following statements are equivalent:*

(i) $g\ell \dim k(\Gamma) \leqq 2$
(ii) *Each top-torsionfree* (2.4) *object of $k(\Gamma)$ is a projective vertex of Γ.*
(iii) *Each top-torsionfree object of $k(\Gamma^{\mathrm{op}})$ is an injective vertex of Γ.*

Proof. Clearly, the morphism $[k(\Gamma)(?, \sigma\alpha_i)]$ of Lemma 2.6 is mono for each non-projective x iff no τx is top-torsionfree in $k(\Gamma^{\mathrm{op}})$ (2.5). This yields the equivalence (i)$\Leftrightarrow$(iii). Similarly, we have (i)$\Leftrightarrow$(i$^{\mathrm{op}}$)$\Leftrightarrow$(ii), where (i$^{\mathrm{op}}$) is the statement saying that $k(\Gamma^{\mathrm{op}}) \xrightarrow{\sim} k(\Gamma)^{\mathrm{op}}$ has global dimension $\leqq 2$.

2.8 **Proposition.** *If Γ is a locally finite translation-quiver, the mesh-category $k(\Gamma)$ is an Auslander-category iff Γ satisfies the conditions* a), b) *and* c) *below:*

a) *$k(\Gamma)$ is locally bounded.*

b) *If x is a non-projective vertex of Γ and $\mu \in k(\Gamma)(x, y)$ a non-zero morphism, there is an arrow $x' \xrightarrow{\alpha} x$ of Γ such that $0 \neq \mu\bar{\alpha} \in k(\Gamma)(x', y)$.*

c) *For each projective vertex $p \in \Gamma_0$, there is a vertex $j \in \Gamma_0$ and a linear form ε: $k(\Gamma)(p, j) \to k$, such that the composition*

$$k(\Gamma)(p, x) \times k(\Gamma)(x, j) \to k(\Gamma)(p, j) \xrightarrow{\varepsilon} k$$

yields a vectorspace duality between $k(\Gamma)(p, x)$ and $k(\Gamma)(x, j)$ for any $x \in \Gamma_0$.

In statement b), $\bar{\alpha}$ denotes the residue class of α modulo the mesh-ideal I_Γ.

Proof. Assume that $k(\Gamma)$ is an Auslander-category. Then it is locally bounded and has global dimension $\leqq 2$ (2.3). So it follows from 2.7 that a non-projective vertex has "top-torsion". Accordingly, Γ satisfies a) and b). Now identify $k(\Gamma)$ with $\operatorname{ind} \Lambda$ for some locally-representation-finite Λ. If $p \in \Gamma_0$ is projective, k_p has projective dimension $\leqq 1$ (2.6). By 2.4, p is identified with some $\Lambda(?, \lambda)$, $\lambda \in \Lambda$. Now, $Dk(\Gamma)(p, ?)$ is isomorphic to $k(\Gamma)(?, j)$, where $j = D\Lambda(\lambda, ?)$ (see part (i)$\Rightarrow$(ii) of the proof of proposition 2.3). Condition c) expresses the well-known fact that an isomorphism $k(\Gamma)(?, j) \xrightarrow{\sim} Dk(\Gamma)(p, ?)$ is determined by some appropriate linear form $\varepsilon \in k(\Gamma)(p, j)$.

Conversely, assume that Γ satisfies the conditions a), b), c). Then we have $g\ell \dim k(\Gamma) \leqq 2$ by 2.7. Take a minimal injective resolution $0 \to k(\Gamma)(?, y) \to i_0 \to i_1$, $y \in \Gamma_0$, in $\operatorname{mod} k(\Gamma)$. A simple $k(\Gamma)$-module k_p occurs as a submodule of i_0 iff it occurs as a submodule of $k(\Gamma)(?, y)$; if this is so, p is top-torsionfree (2.4). Accordingly, i_0 is the direct sum of the injective hulls of simple modules of the form k_p, where p is a projective vertex (2.4 and 2.6). Now the injective hull of k_p is $Dk(\Gamma)(p,?)$, and this is isomorphic to some $k(\Gamma)(?,j)$ by condition c); so i_0 is projective. Similarly, if k_p occurs as a submodule of i_1, we have

$$0 \neq \operatorname{Hom}_{k(\Gamma)}(k_p, i_1) = \operatorname{Ext}^1_{k(\Gamma)}(k_p, k(\Gamma)(?, y)).$$

By 2.6c) it follows that p is a projective vertex. As we did for i_0, we infer that i_1 is projective. So $k(\Gamma)$ is an Auslander-category by 2.3.

Remarks. a) Let j have the property required in condition c) above. Denote by $\eta \in Dk(\Gamma)(j,j)$ the unique algebra-homomorphism of the local algebra $k(\Gamma)(j,j)$ onto k. The map $Dk(\Gamma)(\bar\beta, j)$: $Dk(\Gamma)(j,j) \to Dk(\Gamma)(y,j)$ which is induced by an arrow $j \xrightarrow{\beta} y$ of Γ assigns to η the zero-form $\xi \mapsto \eta(\xi\bar\beta) = 0$. Now $Dk(\Gamma)(?,j)$ is isomorphic to $k(\Gamma)(p,?)$. The morphism $\eta' \in k(\Gamma)(p,j)$ associated with η is non-zero and satisfies $\bar\beta\eta' = 0$ for each β. This means that j is top-torsionfree in $k(\Gamma^{\mathrm{op}})$ (2.4), i.e. that *j is an injective vertex of Γ*. In fact, this result also follows from the first part of the proof of our proposition.

b) *If Γ has no oriented cycle*, for instance if it is simply connected, we have $k \xrightarrow{\sim} k(\Gamma)(j,j) \xrightarrow{\sim} Dk(\Gamma)(p,j)$. In this case, *condition* c) just *means that the composition $k(\Gamma)(p,x) \times k(\Gamma)(x,j) \to k(p,j)$ is a duality.*

c) Figure 2.8 gives a simple example of a translation-quiver Γ satisfying a), b) and c). By proposition 2.5, we have $k(\Gamma) \xrightarrow{\sim} \operatorname{ind} \Lambda$, where Λ is the full subcategory of $k(\Gamma)$ formed by the projective vertices, i.e. the path-category of the quiver $\cdot \rightrightarrows \cdot$ (see § 6 and [7] § 6 for other examples of this kind).

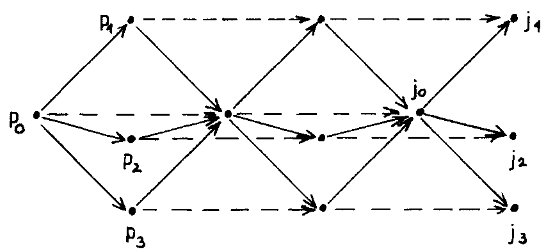


Fig. 2.8

Other examples are given by Fig. 1.3b, where $k(\Gamma) \xrightarrow{\sim} \operatorname{ind} k[Q]/I$, Q being the quiver $\cdot \circlearrowleft T$ and I the ideal generated by T^4. Similarly, we have $k(\tilde\Gamma) \xrightarrow{\sim} \operatorname{ind} k[\tilde Q]/\tilde I$, where $\tilde Q$ is the infinite quiver

$$\ldots n-2 \xrightarrow[T_{n-2}]{} n-1 \xrightarrow[T_{n-1}]{} n \xrightarrow[T_n]{} n+1 \xrightarrow[T_{n+1}]{} n+2 \ldots$$

and $\tilde I$ the ideal generated by the elements $T_{n+1} T_n T_{n-1} T_{n-2}$, $n \in \mathbb{Z}$.

For an application of proposition 2.8 we refer to [16], where it was proved that translation-quivers associated with Brauer-relations satisfy the conditions a), b), c).

2.9 *Definition.* A translation-quiver Γ satisfying the conditions a), b), c) of proposition 2.8 will be called a *Riedtmann-quiver*.

Theorem. *Let Γ be a connected translation-quiver, $\tilde{\Gamma}$ its universal cover. The following statements are equivalent:*

(i) *Γ is the Auslander-Reiten quiver of some locally representation-finite k-category.*

(ii) *Γ is a Riedtmann-quiver.*

(iii) *$\tilde{\Gamma}$ is a Riedtmann-quiver.*

We shall prove this theorem in § 3 below. It yields the wanted justification for the introduction of coverings. It will yield a construction of many representation-finite algebras out of one, as soon as we shall know that most Riedtmann-quivers admit plenty of finite coverings (see § 4).

3. Covering Functors

3.1 *Definition.* Let $F: M \to N$ be a k-linear functor between two k-categories. F is called a *covering functor* if the maps

$$\coprod_{z/b} M(x, z) \to N(a, b) \quad \text{and} \quad \coprod_{t/a} M(t, y) \to N(a, b),$$

which are induced by F, are bijective for any two objects a and b of N. Here t and z range over all objects of M such that $Ft = a$ and $Fz = b$ respectively; the maps are supposed to be bijective for all x and y chosen among the t and z respectively.

Examples. a) If $\pi: \Delta \to \Gamma$ is a covering of translation-quivers, we know that the induced functor $k(\pi): k(\Delta) \to k(\Gamma)$ is a covering functor (see 2.5).

b) Assume that Λ is *locally representation-finite* and *connected* (i.e. Λ is neither empty nor the disjoint sum of two non-empty subcategories). Then the Auslander-Reiten quiver Γ_Λ is connected. Denote by $\pi: \tilde{\Gamma}_\Lambda \to \Gamma_\Lambda$ its universal covering at the point m.

Under these assumptions, proposition 1.6 implies the existence of a *well-behaved* functor $F: k(\tilde{\Gamma}_\Lambda) \to \operatorname{ind} \Lambda$ in the sense of Riedtmann ([15], 2.2; [16], 1.5), i.e. of a k-linear functor which maps an object y of $\tilde{\Gamma}_\Lambda$ onto πy and the morphism $\bar{\beta}$ associated with an arrow β of $\tilde{\Gamma}_\Lambda$ onto an irreducible morphism of $\operatorname{mod} \Lambda$: *Indeed,* choose some point x of $\tilde{\Gamma}_\Lambda$ lying over m and consider the length-function κ as defined in 1.6. We use a first induction in order to define F on the arrows $y \xrightarrow{\beta} z$ such that $\kappa(z) \geqq 1$: If $\kappa(z) = 1$ or if z is projective, we choose for $F\bar{\beta} \in \operatorname{Hom}_\Lambda(\pi y, \pi z)$ any irreducible morphism; if z is not projective and $\kappa(z) \geqq 2$, consider the mesh stopping at z (Fig. 3.1):

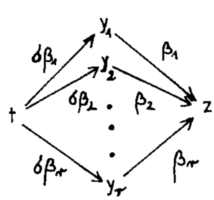


Fig. 3.1

We may suppose that $\overline{F\sigma\beta_i}$ is already constructed. We know then that there is an Auslander-Reiten sequence of mod Λ having the form

$$0 \to \pi t \xrightarrow{[F\sigma\overline{\beta_i}]} \bigoplus \pi y_i \xrightarrow{[\phi_i]} \pi z \to 0$$

(see for instance [7], 1.6). So we can set $F\bar{\beta}_i = \phi_i$. A second induction, resting on dual arguments, is used in order to define F on arrows $y \xrightarrow{\beta} z$ such that $\kappa(y) < 0$.

Now proposition 2.3 of [15] extends to the present situation. Using this proposition together with its dual, we see that F is a *covering functor.*

c) Let $F\colon M \to N$ be a covering functor and N' a full subcategory of N. Let $M' = F^{-1}(N')$ be the full subcategory of M formed by the objects mapped into N'. Clearly, the induced functor $F'\colon M' \to N'$ is also a covering functor.

Furthermore, consider the ideal of N generated by the morphisms of N which are factorized through N'. Factoring out that ideal and restricting to the objects of N not lying in N' we get a category which will be denoted by N/N' (compare with [2]). The functor $M/M' \to N/N'$ induced by F is again a covering functor.

3.2 Let $F\colon M \to N$ be a covering functor between k-categories. With each additive functor $m\colon M^{\mathrm{op}} \to Ab$ we associate its *push-down* $F_\lambda m\colon N^{\mathrm{op}} \to Ab$, which is constructed as follows: For each object $a \in N$, we set

$$(F_\lambda m)(a) = \coprod_{x/a} m(x),$$

where x ranges over all objects of M such that $F(x) = a$; if $b \xrightarrow{\alpha} a$ is a morphism of N, the map $(F_\lambda m)(\alpha)\colon (F_\lambda m)(a) \to (F_\lambda m)(b)$ to be defined assigns to $(\mu_x) \in \coprod_{x/a} m(x)$ the family $(\sum_x m({}_x^{\cdot}\alpha_y)(\mu_x)) \in \coprod_{y/b} m(y)$, where ${}_x^{\cdot}\alpha_y$ is determined by $\sum_{y/b} F({}_x^{\cdot}\alpha_y) = \alpha$.

People in search of an abstract justification will prove that pushing down $m \mapsto F_\lambda m$ is left adjoint to "*pulling up*" $n \mapsto n \circ F^{\mathrm{op}}$. More relevant for us is the fact that the second bijection in definition 3.1 yields a canonical isomorphism $F_\lambda M(?, y) \xrightarrow{\sim} N(?, Fy)$, that F_λ is exact, and that it maps k-finite-dimensional M-modules onto finite-dimensional N-modules.

Proposition. *Let $F\colon M \to N$ be a covering functor between locally bounded k-categories, and let $\mathscr{R}m$ denote the radical of $m \in$* mod M. *Then we have $F_\lambda \mathscr{R}m \xrightarrow{\sim} \mathscr{R}F_\lambda m$, and m is projective in* mod M *iff $F_\lambda m$ is so in* mod N.

Proof. Clearly, F_λ preserves dimension. So it maps one-dimensional M-modules onto one-dimenisonal N-modules, i.e. simple modules onto simple ones, and semi-simple modules onto semi-simple ones. Since $m/\mathscr{R}m$ is semi-simple, we infer that $F_\lambda m/F_\lambda \mathscr{R}m \xrightarrow{\sim} F_\lambda(m/\mathscr{R}m)$ is semi-simple, hence that $\mathscr{R}F_\lambda m \subset F_\lambda \mathscr{R}m$. On the other hand, if $p = M(?, x)$ is projective indecomposable, we know that $F_\lambda p \xrightarrow{\sim} N(?, Fx)$. In this case, $\mathscr{R}F_\lambda p$ and $F_\lambda \mathscr{R}p$ both have codimension 1 in $F_\lambda p$. We infer that $\mathscr{R}F_\lambda p = F_\lambda \mathscr{R}p$ if p is projective indecomposable, and more generally if p is projective. In the case of an arbitrary $m \in \operatorname{mod} M$, consider an epimorphism $p \xrightarrow{\pi} m \to 0$, where p is projective. In the induced square

$$\begin{tikzcd} \mathscr{R}F_\lambda p \arrow[r] \arrow[d, "\alpha"'] & \mathscr{R}F_\lambda m \arrow[d] \\ F_\lambda \mathscr{R} p \arrow[r, "\beta"'] & F_\lambda \mathscr{R} m \end{tikzcd}$$

α and β are epi. Accordingly, $\mathscr{R}F_\lambda m = F_\lambda \mathscr{R} m$.

Since $F_\lambda M(?, y) \xrightarrow{\sim} N(?, Fy)$, the image of a projective module is projective. Conversely, suppose that $F_\lambda m$ is projective, and let $p \xrightarrow{f} m$ be a projective cover of m. Then f induces an isomorphism of the tops $p/\mathscr{R}p \xrightarrow{\sim} m/\mathscr{R}m$. As F_λ preserves the radical, we get

$$F_\lambda p/\mathscr{R}F_\lambda p \xrightarrow{\sim} F_\lambda(p/\mathscr{R}p) \xrightarrow{\sim} F_\lambda(m/\mathscr{R}m) \xrightarrow{\sim} F_\lambda m/\mathscr{R}F_\lambda m,$$

so that $F_\lambda p \xrightarrow{F_\lambda f} F_\lambda m$ is a projective cover of $F_\lambda m$. Since $F_\lambda m$ is projective, $F_\lambda f$ is invertible, and so is obviously f.

3.3 Let $F\colon M \to N$ be a *covering functor* between *locally bounded* k-categories. Each object $x \in M$ yields canonical isomorphisms

$$F_\lambda M(?, x) \xrightarrow{\sim} N(?, Fx)$$

$$F_\lambda \mathscr{R} M(?, x) \xrightarrow{\sim} \mathscr{R}F_\lambda M(?, x) \xrightarrow{\sim} \mathscr{R}N(?, Fx)$$

$$F_\lambda \mathscr{R}^2 M(?, x) \xrightarrow{\sim} \mathscr{R}F_\lambda \mathscr{R} M(?, x) \xrightarrow{\sim} \mathscr{R}^2 N(?, Fx)$$

and

$$\begin{aligned} F_\lambda(\mathscr{R}M(?, x)/\mathscr{R}^2 M(?, x)) &\xrightarrow{\sim} F_\lambda \mathscr{R} M(?, x)/F_\lambda \mathscr{R}^2 M(?, x) \\ &\xrightarrow{\sim} \mathscr{R}N(?, Fx)/\mathscr{R}^2 N(?, Fx) \end{aligned}$$

i.e.

$$\coprod_{z/b} \mathscr{R}M(z, x)/\mathscr{R}^2 M(z, x) \xrightarrow{\sim} \mathscr{R}N(b, Fx)/\mathscr{R}^2(b, Fx), \qquad \forall b \in N.$$

Accordingly, if the quiver Q_M of M contains an arrow $y \to x$, then Q_N contains an arrow $Fy \to Fx$ (2.9). In other words, F *induces a quiver-morphism* $Q_F\colon Q_M \to Q_N$.

Definition. A locally finite-dimensional category N is called *square-free* if the spaces $\mathscr{R}N(b, a)/\mathscr{R}^2 N(b, a)$ have dimension ≤ 1 over k for all a, b.

In the foregoing situation, *M is obviously square-free if N is.* Furthermore, for each arrow $b \xrightarrow{\alpha} a$ of Q_N and each point x of Q_M lying over a, there is exactly one arrow $y \xrightarrow{\xi} x$ of Q_M lying over α. Taking into account that the definition of covering functors is self-dual, we deduce the dual statement saying that each y over b is the starting point of a unique arrow lying over α. In other words, we have the

Proposition. *A covering functor between square-free locally bounded k-categories induces a covering map between the associated ordinary quivers.*

3.4 **Proposition.** *Let $F: M \to N$ be a covering functor between square-free locally bounded k-categories, and let $\mathscr{S}m$ denote the socle of $m \in \operatorname{mod} M$. Then we have $F_\lambda \mathscr{S}m \xrightarrow{\sim} \mathscr{S}F_\lambda m$, and m is injective in* mod M *iff $F_\lambda m$ is so in* mod N.

Proof. First we prove the second part using the relation $F_\lambda \mathscr{S}m \xrightarrow{\sim} \mathscr{S}F_\lambda m$. If $x \in M$, the injective hull of the simple module k_x is identified with $DM(x, ?)$. Using $F_\lambda \mathscr{S}m \xrightarrow{\sim} \mathscr{S}F_\lambda m$ we infer that $F_\lambda DM(x, ?)$ has k_{Fx} as socle; so it is contained in the injective hull $DN(Fx, ?)$ of k_{Fx}. On the other hand, the first bijection of definition 3.1 tells us that $F_\lambda DM(x, ?)$ and $DN(Fx, ?)$ have the same dimension. Hence we have $F_\lambda DM(x, ?) \xrightarrow{\sim} DN(Fx, ?)$, and $F_\lambda m$ is injective if m is so. The converse is proved as in 3.2.

Now we come to the socle of $F_\lambda m$. Clearly, $F_\lambda \mathscr{S}m$ is semi-simple, and therefore we have $F_\lambda \mathscr{S}m \subset \mathscr{S}F_\lambda m$. We prove the equality by induction on the height ($=$Loewy length) h of m. The statement is clear if $h=1$. In order to tackle the case $h=2$, we first consider the second socle $\mathscr{S}^2 I_x$ of the indecomposable injective M-module $I_x = DM(x, ?)$. The socle k_x of $\mathscr{S}^2 I_x$ yields an exact sequence

$$e\colon 0 \to k_x \to \mathscr{S}^2 I_x \to \bigoplus_{i=1}^{r} k_{y_i} \to 0,$$

where y_i ranges over the heads of the arrows of Q_M starting at x. Suppose that the socle of $F_\lambda \mathscr{S}^2 I_x$ is not simple. Then there is a 2-dimensional semi-simple N-module n and a commutative diagram with exact rows

$$\begin{tikzcd}
F_\lambda e\colon 0 \arrow[r] & k_{Fx} \arrow[r] & F_\lambda \mathscr{S}^2 I_x \arrow[r] & \bigoplus_{i=1}^{r} k_{Fy_i} \arrow[r] & 0 \\
0 \arrow[r] & k_{Fx} \arrow[r] \arrow[u, equal] & n \arrow[r] \arrow[u] & k_{Fy_j} \arrow[r] \arrow[u, "\varepsilon"'] & 0.
\end{tikzcd}$$

By assumption, the k_{Fy_i} are pairwise non-isomorphic. Therefore, ε has the form $\varepsilon = F_\lambda \varepsilon'$, where $\varepsilon'\colon k_{y_j} \to \bigoplus_{i=1}^{r} k_{y_i}$. Now the pullback of e under ε' does not split, nor does its image under F_λ, since F_λ preserves the radical. This contradicts the semi-simplicity of n.

We infer that $F_\lambda \mathscr{S}\ell = \mathscr{S}F_\lambda \ell$ if $\ell = \mathscr{S}^2 I_x$, and more generally if ℓ is the second socle of an injective M-module. For an arbitrary M-module m of height 2 there is a socle-preserving embedding of m into such an ℓ. We infer that $\mathscr{S}F_\lambda m \subset \mathscr{S}F_\lambda \ell = F_\lambda \mathscr{S}\ell = F_\lambda \mathscr{S}m$, hence that $\mathscr{S}F_\lambda m = F_\lambda \mathscr{S}m$.

Finally, consider the case of height $h > 2$ and let $(\mu_t) \in (\mathscr{S}F_\lambda m)(a)$. If $(\mu_t) \in (F_\lambda \mathscr{S}^2 m)(a)$, we are reduced to the case $h=2$. Otherwise, the image $(\bar{\mu}_t)$ of (μ_t) in $F_\lambda(m/\mathscr{S}m)(a)$ lies in the socle of $F_\lambda(m/\mathscr{S}m)$ without lying in $F_\lambda(\mathscr{S}^2 m/\mathscr{S}m) \xrightarrow{\sim} F_\lambda \mathscr{S}(m/\mathscr{S}m)$, a contradiction by induction on h!

Remark. The propositions 3.2 and 3.4 are by no way dual, since the "dual" of the left adjoint functor F_λ is a right adjoint functor. In fact, elementary examples show that proposition 3.4 gets wrong if we drop the assumption that N is square-free.

3.5 **Proposition.** *Let $F: M \to N$ be a covering functor between connected square-free locally finite dimensional categories. Then M is an Auslander-category iff N is so.*

Proof. Clearly, M is locally bounded iff N is so. Suppose that M is an Auslander-category. Then each simple M-module k_x admits a projective resolution of the form $0 \to P_2 \to P_1 \to M(?, x) \to k_x \to 0$. The push down of this is a projective resolution of k_{Fx} in $\operatorname{mod} N$ (3.2). This shows that $g\ell \dim N \leqq 2$. Similarly, let $0 \to M(?, x) \to i_0 \to i_1 \ldots$ be a minimal injective resolution of the projective $M(?, x)$ in $\operatorname{mod} M$. Then i_0 and i_1 are also projective. The push down $0 \to F_\lambda M(?, x) \to F_\lambda i_0 \to F_\lambda i_1 \ldots$ is a minimal injective resolution of $N(?, Fx)$ by 3.4. As $F_\lambda i_0$ and $F_\lambda i_1$ are projective, we deduce from 2.3 that N is an Auslander-category.

Conversely, let N be an Auslander-category. The push down

$$\ldots F_\lambda P_2 \xrightarrow{F_\lambda f} F_\lambda P_1 \to N(?, Fx) \to k_{Fx} \to 0$$

of the minimal projective resolution of k_x, $x \in M$, is a minimal projective resolution by 3.2. Since $g\ell \dim N \leqq 2$, $F_\lambda f$ is a monomorphism, and so is f. Hence $g\ell \dim M \leqq 2$. Similarly, the push-down of a minimal injective resolution $0 \to M(?, x) \to i_0 \to i_1 \ldots$ is a minimal injective resolution of $N(?, Fx)$ by 3.4. Therefore $F_\lambda i_0$ and $F_\lambda i_1$ are projective, and so are i_0 and i_1 by 3.2.

3.6 *Proof of Theorem* 2.9. (i)$\Rightarrow$(iii): Suppose that $\Gamma = \Gamma_\Lambda$, where Λ is locally representation-finite. Consider a well-behaved functor $F: k(\tilde{\Gamma}_\Lambda) \to \operatorname{ind} \Lambda$ (3.1b). Since $\operatorname{ind} \Lambda$ satisfies the conditions a) and c) of 2.1, so does $k(\tilde{\Gamma}_\Lambda)$ (F is a covering functor!). Hence $k(\tilde{\Gamma}_\Lambda)$ is locally finite-dimensional (condition b) of 2.1 follows from the definition of the mesh category). On the other hand, $\operatorname{ind} \Lambda$ is square-free by a result of Bautista ([3], [15] 3.5, [17] 2.5). By proposition 3.5 $k(\tilde{\Gamma}_\Lambda)$ is an Auslander-category.

(iii)$\Rightarrow$(ii): Let $\pi: \tilde{\Gamma} \to \Gamma$ be the canonical projection and $k(\pi): k(\tilde{\Gamma}) \to k(\Gamma)$ the induced covering functor. By Proposition 2.5b) $k(\Gamma)$ is locally bounded. By construction $k(\Gamma)$ is square-free. As $k(\tilde{\Gamma})$ is an Auslander-category, $k(\Gamma)$ is one by 3.5.

(ii)$\Rightarrow$(i): Let P be the full subcategory of $k(\Gamma)$ whose objects are the projective points of Γ. Then $k(\Gamma) \xrightarrow{\sim} \operatorname{ind} P$ and Γ is identified with Γ_P.

4. The Fundamental Group of a Riedtmann-Quiver is Free

The result stated in the title can be proved by a combinatorial version of van Kampen's theorem. However, since our intuition is geometric, we shall side with topology, accepting to struggle with technical details beside the combinatorial point in order to borrow from topological attainments.

4.1 *The Geometric Realization of a Translation-Quiver.* Let Q be a *quiver* with vertex-set Q_0 and arrow-set Q_1. Associate with each arrow $x \xrightarrow{\alpha} y$ a copy I_α of the unit interval $I = [0, 1]$. Denote by $\dot{I}_\alpha = \{0, 1\}$ its "boundary", by $\partial_\alpha: \dot{I}_\alpha \to Q_0$ the map such that $\partial_\alpha(0) = x$, $\partial_\alpha(1) = y$. The *geometric realization* $|Q|$

of Q is the amalgamated sum attached to be following diagram of topological spaces

$$Q_0 \xleftarrow{[\partial_\alpha]} \coprod_{\alpha\in Q_1} \dot{I}_\alpha \xrightarrow{i} \coprod_{\alpha\in Q_1} I_\alpha,$$

where i is the inclusion and Q_0 carries the discrete topology. The canonical map $j_\alpha\colon I_\alpha \to |Q|$ yields a homeomorphism of I_α onto its image $\bar{I}_\alpha = j_\alpha(I_\alpha)$ if $x \neq y$; otherwise, $\bar{I}_\alpha$ is a circle. The topology of $|Q|$ is the weak Kelley-topology: a subset F is closed iff $F \cap \bar{I}_\alpha$ is closed for each α.

If G is a (non-oriented) *graph*, its *geometric realization* $|G|$ is by definition the geometric realization of the quiver $\vec{G}$ obtained from G by orienting each edge in some chosen direction. The choice made is of no consequence for us, since different orientations lead to canonically isomorphic geometric realizations.

Let us now turn to a *translation-quiver* Γ. The geometric realization of the associated quiver $\hat{\Gamma}$ (1.2) is the one dimensional skeleton of the space to be defined. In fact, the *geometric realization* $|\Gamma|$ of Γ is obtained by attaching triangles to $|\hat{\Gamma}|$, one along each $|\hat{\Delta}_\beta|$, where β ranges over all arrows of grade 1 with non-projective head and $\hat{\Delta}_\beta$ denotes the subquiver of $\hat{\Gamma}$ illustrated in Fig. 4.1.

case $\tau x \neq x$ case $\tau x = x$

Fig. 4.1

More precisely, denote by Δ_β a copy of the triangle $\Delta = \{x \in \mathbb{R}^3\colon 0 \leqq x_1, 0 \leqq x_2, 0 \leqq x_3, x_1 + x_2 + x_3 = 1\}$, by $\dot{\Delta}_\beta$ its "boundary", by $g_\beta\colon \dot{\Delta}_\beta \to |\hat{\Gamma}|$ the map such that $g_\beta(0, 1-t, t) = j_\beta(t)$, $g_\beta(1-t, 0, t) = j_{\gamma_x}(t)$ and $g_\beta(1-t, t, 0) = j_{\sigma\beta}(t)$. By definition, the geometric realization $|\Gamma|$ of the *translation*-quiver Γ is the topological amalgamated sum of the diagram

$$|\hat{\Gamma}| \xleftarrow{[g_\beta]} \coprod_\beta \dot{\Delta}_\beta \xrightarrow{j} \coprod_\beta \Delta_\beta,$$

where j is the inclusion-map. We identify $|\hat{\Gamma}|$ with is canonical image in $|\Gamma|$ and denote by $\bar{\Delta}_\beta$ the canonical image of Δ_β. All these canonical images are closed in $|\Gamma|$. Furthermore, a subset F of $|\Gamma|$ is closed iff all the intersections $|\hat{\Gamma}| \cap F$, $\bar{\Delta}_\beta \cap F$ are closed.

Proposition. *Let x be a vertex of the translation-quiver Γ. The fundamental groups $\Pi(\Gamma, x)$ and $\Pi(|\Gamma|, x)$ are naturally isomorphic.*

Proof. Denote by $K\Gamma$ the simplicial set of dimension $\leqq 2$ which has the vertices of Γ as 0-simplices, the arrows of $\hat{\Gamma}$ as non-degenerated 1-simplices, the diagrams

$$\begin{tikzcd} & y \arrow[dr, "\beta"] & \\ \tau x \arrow[ur, "\sigma\beta"] \arrow[rr, dashed] & & x \end{tikzcd}$$

of Γ as non-degenerated 2-simplices ([8], [15]). The groups $\Pi(\Gamma, x)$ and $\Pi(K\Gamma, x)$ are naturally isomorphic, since they admit the same description ([8], II, 7.1). The groups $\Pi(K\Gamma, x)$ and $\Pi(|\Gamma|, x)$ are naturally isomorphic by [8], Ap. 1, § 3 (notice that $|\Gamma|$ coincides with the geometric realization of $K\Gamma$ by [8], III, § 1).

4.2 Suppose from now on that the translation-quiver Γ is *locally finite*. Our objective is to compare the fundamental group of Γ with that of a graph.

Let x be a vertex of Γ. The set of all $n \in \mathbb{Z}$, such that $\tau^n x$ is defined, is clearly an interval D of $\mathbb{Z}$. We call the set $x^\tau = \{\tau^n x \colon n \in D\}$ the *τ-orbit* of x. The vertex x is *stable* if $D = \mathbb{Z}$ ([15]), it is *periodic* if it is stable and has finite τ-orbit; the cardinality of x^τ is then called the *period* of x. In a similar way, if $x \xrightarrow{\alpha} y$ is an arrow of Γ, we shall consider its *σ-orbit* α^σ, which is the set of all arrows of Γ of the form $\sigma^m \alpha$.

Whenever an arrow $x \xrightarrow{\alpha} y$ of Γ connects a periodic vertex x with a stable vertex y, then y is also periodic: otherwise, there would be infinitely many arrows starting at x. Accordingly, the τ-orbits of a connected component E of the stable part ${}_s\Gamma$ ([15], 1.4) of Γ are either all infinite or all finite. In the second case we call E a *periodic component* of Γ.

By Riedtmann's result ([15], 1.5) a periodic component E has the form $\mathbb{Z}T/\Pi$, where T is an oriented tree and Π an admissible automorphism group of $\mathbb{Z}T$. We call E *tree-finite* if the graph $\bar{T}$ underlying T is finite (notice that $\bar{T}$ is uniquely determined by E). The translation-quiver Γ itself will be called *tree-finite* if it is locally finite and all its periodic components are tree-finite.

The *graph* G_Γ associated with Γ has as vertices the periodic components and the non-periodic τ-orbits of Γ. To each periodic component, considered as a vertex of G_Γ, we attach a loop of G_Γ. The remaining edges of G_Γ are associated with the non-periodic σ-orbits of Γ. More precisely, let α^σ be a σ-orbit connecting the τ-orbits x^τ and y^τ of Γ. If both x and y are non-periodic, we associate with α^σ an edge of G_Γ connecting the vertices x^τ and y^τ. If y is not periodic and x belongs to the periodic component E, we associate with α^σ an edge of G_Γ connecting E and y^τ. For examples we refer to the lists at the end of the present paper.

Theorem. *If Γ is a tree-finite translation-quiver, the geometric realizations of Γ and G_Γ are homotopy-equivalent.*

Corollary. *If Γ is a Riedtmann-quiver and x a vertex of Γ, the fundamental group $\Pi(\Gamma, x)$ is free.*

By Riedtmann's result (see 2.5) we know that a Riedtmann-quiver Γ is tree-finite. On the other hand it is well-known that the fundamental group of a graph is free. Accordingly, the corollary follows easily from the foregoing theorem which will be proved in 4.3

4.3 **Proof of Theorem 4.2.** Consider an arrow $x \xrightarrow{\alpha} y$ of Γ and denote by Γ_α the sub-translation-quiver of Γ formed by x^τ, y^τ and α^σ. This Γ_α has only one σ-orbit; from the classification of the translation-quivers with one σ-orbit (a good exercise!) we deduce the following: Suppose that α is not periodic, i.e. that x and y are not both periodic. Denote by $\bar{\Gamma}_\alpha$ the subquiver of $\hat{\Gamma}_\alpha$ formed by

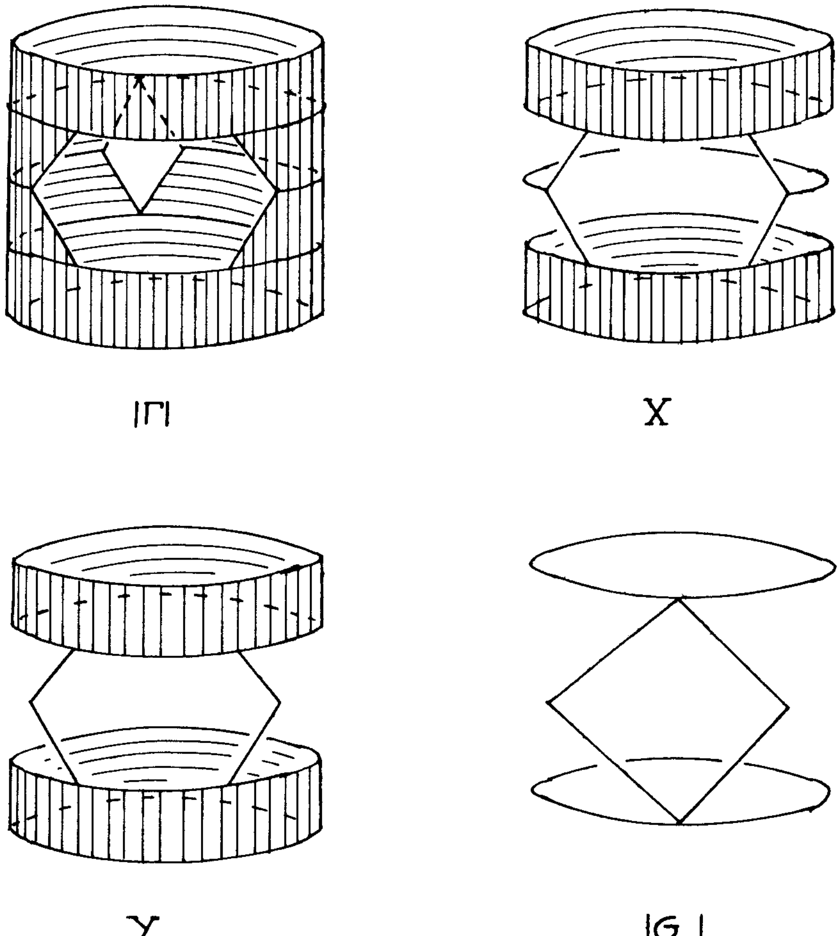


Fig. 4.3

α and all arrows of grade 2. Then $|\bar{\Gamma}_\alpha|$ is identified with a *strong deformation retract* of $|\Gamma_\alpha|$ ([20], Chap. 1 Sect. 4).

Now we choose a representative α of each non-periodic σ-orbit. This yields a family of subspaces $|\bar{\Gamma}_\alpha|$ of $|\Gamma|$. We denote by X the union of these subspaces and of the geometric realizations of the periodic components. Matching together strong deformation retractions f_α: $|\Gamma_\alpha| \times I \to |\Gamma_\alpha|$ of $|\Gamma_\alpha|$ onto $|\bar{\Gamma}_\alpha|$, we get a strong deformation retraction of $|\Gamma|$ onto X. Now X contains the geometric realization $|x^\tau|$ of each τ-orbit x^τ, considered as a sub-translation-quiver of $|\Gamma|$. If x is not periodic, $|x^\tau|$ is homeomorphic to $\mathbb{R}$, $[0, 1]$ or $[0, 1[$. Therefore, the space Y obtained from X by contracting each "non-periodic" $|x^\tau|$ to one point is easily seen to be homotopy equivalent to X (see Fig. 4.3).

The space Y is already quite near to $|G_\Gamma|$. But it still contains sub-spaces identified with the geometric realizations $|E|$ of the periodic components E of Γ. It remains for us to shrink each such $|E|$ to a loop. By [15], 4.2, E is identified with some $\mathbb{Z}B/\rho^{\mathbb{Z}}$, where B is an oriented tree and ρ an admissible automorphism of $\mathbb{Z}B$. Choose a vertex b in B and denote by $\kappa: \mathbb{Z}B \to \mathbb{Z}A_2$ the quiver-morphism such that $\kappa(0, b)=0$ (see 1.6). We identify $|\mathbb{Z}B|$ with $\mathbb{R} \times |B|$ by means of the "natural" homeomorphism which maps a vertex (n, c) onto $(\kappa(n, c), c)$ and is "affine" on the "triangles".

The automorphism $\bar{\rho}$ of the graph $\bar{B}$ underlying B, which is induced by ρ, either has a fixed point, or it exchanges two neighbours of $\bar{B}$. In any case, the induced automorphism $|\bar{\rho}|$ of $|\bar{B}|=|B|$ has a fixed point ω. Moreover, there is a strong deformation retraction $h: |\bar{B}| \times I \to |\bar{B}|$ of $|\bar{B}|$ onto ω, which is compatible with $|\bar{\rho}|$ (i.e. $h(|\bar{\rho}|\, x, t)=(h(x, t), \ \forall x \in |\bar{B}|, \ \forall t \in I)$. The induced strong deformation retraction $\mathbb{R} \times h$: $\mathbb{R} \times |\bar{B}| \times I \to \mathbb{R} \times |\bar{B}|$ is compatible with the action of $|\rho|$ on $\mathbb{R} \times |\bar{B}| = |\mathbb{Z}B|$. So it induces a strong deformation retraction of the residue space $|\mathbb{Z}B|/|\rho| = |E|$ onto a circle S. Under this retraction all the vertices fo E are mapped into some contractible closed arc of S. Shrinking this arc to one point, we get by composition a homotopy equivalence f_E of $|E|$ onto some circle S_E, which maps all vertices of E onto one single point.

The preceding construction is done for each periodic component E. The topological amalgamated sum of the resulting diagram

$$\coprod_E S_E \xleftarrow{\coprod f_E} \coprod_E |E| \longrightarrow Y$$

may then be identified with $|G_\Gamma|$. A simple classical homotopy extension argument shows that the induced map f: $Y \to |G_\Gamma|$ is a homotopy equivalence: Indeed, for each E there is a homotopy h_E: $|E| \times I \to |E|$ such that $h_E(x, 0)=x$, $h_E(x, 1)=h_E(y, 1)$ and $f_E(h_E(x, t))=f_E(h_E(y, t))$ for all t, whenever $f_E(x)=f_E(y)$. Construct a continuous extension h: $Y \times I \to Y$ of $\coprod_E h_E$, using intuition or the general homotopy extension property ([20]). Then $h(?, 1)$ factors through $|G_\Gamma|$, i.e. we have $h(y, 1)=s(f(y))$ for some continuous s: $|G_\Gamma| \to Y$ and all $y \in Y$. Moreover, h is a homotopy between $\mathbb{1}_Y$ and sf. The map $\bar{h}$: $|G_\Gamma| \times I \to |G_\Gamma|$, such that $\bar{h}(f(y), t)=f(h(y, t))$ for all (y, t), is a homotopy between $\mathbb{1}_{|G_\Gamma|}$ and fs.

5. Standard Representation-Finite Algebras

5.1 *Definition.* A locally representation-finite k-category Λ is said to be *standard* if $\operatorname{ind} \Lambda$ is isomorphic to a mesh-category $k(\Gamma)$ (2.2). A Riedtmann-quiver Γ is called *standard* if each locally representation-finite k-category whose Auslander-Reiten quiver is a cover of Γ is standard.

Clearly, if Λ is standard, there is an isomorphism $k(\Gamma_\Lambda) \xrightarrow{\sim} \operatorname{ind} \Lambda$ which is the identity on the objects. Most of the known examples are standard. The first known non-standard example is due to Chr. Riedtmann: see number 14 bis) in our list of the maximal representation-finite k-categories with 2 objects.

Our purpose in this paragraph is to relate non-standard algebras to standard ones. In order to do so, we first consider a locally finite-dimensional k-category M with radical $\mathscr{R}M$ (2.1). The *powers* $\mathscr{R}^n M$ are the ideals of M which are defined inductively by the formulae: $\mathscr{R}^0 M(x, y) = M(x, y)$ and

$$\mathscr{R}^{n+1} M(x, y) = \sum_z \mathscr{R}M(z, y)\, \mathscr{R}^n M(x, z).$$

The *associated graded category* GrM has the same objects as M; its morphism-spaces are the direct sums

$$(GrM)(x,y)=\coprod_{n\in\mathbb{N}} \mathscr{R}^n M(x,y)/\mathscr{R}^{n+1}M(x,y);$$

the composition of GrM is induced in the usual way by that of M.

Proposition. *If Λ is a locally representation-finite category, there is an isomorphism $k(\Gamma_\Lambda)\xrightarrow{\sim} Gr(\operatorname{ind}\Lambda)$ which is the identity on the objects.*

Proof. Set $I=\operatorname{ind}\Lambda$. First we associate an irreducible morphism $\underline{\beta}\in\mathscr{R}I(y,x)\setminus\mathscr{R}^2I(y,x)$ with each arrow $y\xrightarrow{\beta}x$ of Γ_Λ. Then we choose an Auslander-Reiten sequence of the form

$$0\to\tau x\xrightarrow{[\underline{\beta}]}\bigoplus_\beta y_\beta\xrightarrow{[\underline{\beta}]}x\to 0$$

for each non-projective x; here $y_\beta\xrightarrow{\beta}x$ ranges over all arrows of Γ_Λ heading for x. Since $\mathscr{R}^2I(\tau x,y_\beta)$ has codimension 1 in $\mathscr{R}I(\tau x,y_\beta)$ ([3], [15] 3.5, [17] 2.5), we have $\underline{\beta}-z_\beta\,\underline{\sigma\beta}\in\mathscr{R}^2I(\tau x,y_\beta)$ for some $z_\beta\in k^*=k\setminus\{0\}$. Together with the equation $\sum\limits_\beta\underline{\beta}\underline{\beta}=0$ this yields $\sum\limits_\beta z_\beta\,\underline{\beta}\underline{\sigma\beta}\in\mathscr{R}^3I(\tau x,x)$.

Now, by the lemma stated below we can attach a scalar $b_\gamma\in k^*$ to each arrow γ of $\hat{\Gamma}_\Lambda$ (4.1) in such a way that $z_\beta=b_\beta b_{\gamma_x}^{-1}b_{\sigma\beta}$. Hence we get $\sum\limits_\beta(b_\beta\underline{\beta})(b_{\sigma\beta}\underline{\sigma\beta})=b_{\gamma_x}\sum\limits_\beta z_\beta\underline{\beta}\,\underline{\sigma\beta}\in\mathscr{R}^3I(\tau x,\tau)$. In other words, the map $\alpha\mapsto b_\alpha\underline{\alpha}$, where α ranges through the arrow-set of Γ, induces a k-linear functor $F\colon k(\Gamma_\Lambda)\to\operatorname{Gr}I$ which is the identity on the objects and is surjective on the morphisms (F hits the generating morphisms of $\operatorname{Gr}I$). We infer that F is bijective on the morphisms, since we have

$$\dim(\operatorname{Gr}I)(x,y)=\dim I(x,y)=\sum_{\tilde{z}/x}\dim k(\tilde{\Gamma}_\Lambda)(\tilde{z},\tilde{y})=\dim(\Gamma_\Lambda)(x,y).$$

Here $\tilde{\Gamma}_\Lambda$ is the universal cover of Γ_Λ, which we may assume connected; $\tilde{y}$ is a point of $\tilde{\Gamma}_\Lambda$ over y, and $\tilde{z}$ ranges over all points of $\tilde{\Gamma}_\Lambda$ over x. Of course, we use the existence of covering functors $k(\tilde{\Gamma}_\Lambda)\to I$ and $k(\tilde{\Gamma}_\Lambda)\to k(\Gamma_\Lambda)$.

Lemma. *Let Γ be a tree-finite translation-quiver and (z_β) a family of non-zero scalars indexed by the arrows β of Γ with non-projective head. There is a family (b_γ) of non-zero scalars indexed by the arrows of $\hat{\Gamma}$ (1.2) such that $z_\beta=b_\beta b_{\gamma_x}^{-1}b_{\sigma\beta}$, $\forall\beta$.*

$$\begin{tikzcd} & y_\beta \arrow[dr, "\beta"] & \\ \tau x \arrow[ur, "\sigma\beta"] \arrow[rr, dashed, "\gamma_x"'] & & x \end{tikzcd}$$

We shall produce the proof of this lemma in 5.4 below.

5.2 **Corollary.** *Let A be a representation-finite algebra. The standard representation-finite algebra $\bar{A}$ with Auslander-Reiten quiver Γ_A is a degeneration of A.*

Proof. Set $M = \operatorname{ind} A$. Choose a supplementary subspace $S^n(x,y)$ of $\mathscr{R}^{n+1}M(x,y)$ in $\mathscr{R}^n M(x,y)$ for all $x,y \in M$ and each $n \in \mathbb{N}$. This yields finite direct sum decompositions $M(x,y) = S^0(x,y) \oplus S^1(x,y) \oplus S^2(x,y) \oplus \dots$. Denote by $\phi_t\colon M(x,y) \xrightarrow{\sim} M(x,y)$ the vector-space automorphism such that $\phi_t(f_0 + f_1 + f_2 + \dots) = f_0 + t f_1 + t^2 f_2 + \dots$ if $f_n \in S^n(x,y)$ $(t \in k, t \neq 0)$. Using these automorphisms we construct a new category M_t having the same objects and the same morphism spaces as M. The composition $g \underset{t}{\circ} f$ of two morphisms of M_t is expressed in terms of the composition $g \circ f$ of M by means of the formula $g \underset{t}{\circ} f = \phi_t(\phi_t^{-1}(g) \circ \phi_t^{-1}(f))$. Clearly, $g \underset{t}{\circ} f$ is a polynomial in t, whose value for $t=0$ is the composition of g and f in $\operatorname{Gr} M$ (identify $\mathscr{R}^n M(x,y)/\mathscr{R}^{n+1} M(x,y)$ with $S^n(x,y)$). Accordingly, the algebraic family $(M_t)_{t \in k}$ yields a degeneration of M into $\operatorname{Gr} M$, or equivalently a *deformation* of $\operatorname{Gr} M$ into M.

The algebra A, which we may suppose to be basic, is identified with $\bigoplus_{p,q} M(p,q)$, where p and q range through the projective points of Γ_A. Similarly, $\bar{A}$ is identified with $\bigoplus_{p,q} (\operatorname{Gr} M)(p,q)$. We infer that the algebraic family $\bigoplus_{p,q} M_t(p,q)$ yields a degeneration of A into $\bar{A}$.

5.3 **Corollary.** *Every finite Riedtmann-quiver has a finite covering which is standard.*

Proof. Let Γ be a finite connected Riedtmann-quiver, Π its fundamental group, $\pi\colon \tilde{\Gamma} \to \Gamma$ its universal covering. For each vertex x of Γ we choose a vertex $\tilde{x}$ of $\tilde{\Gamma}$ such that $\pi(\tilde{x}) = x$, and we denote by R_x the set of vertices y of $\tilde{\Gamma}$ such that $k(\tilde{\Gamma})(\tilde{x}, y) \neq 0$. The elements $\gamma \in \Pi$ such that $\gamma \neq 1$ and $\gamma(R_x) \cap R_x \neq \emptyset$ for some x form a finite subset S of Π. As Π is free, it has an invariant subgroup P of finite index such that $P \cap S = \emptyset$. The finite cover $\Delta = \tilde{\Gamma}/P$ of Γ is our candidate.

Indeed, let A be a representation-finite algebra with Auslander-Reiten quiver Δ, and let $F\colon k(\tilde{\Gamma}) \to M = \operatorname{ind} A$ be a well-behaved functor. For any two $s, t \in M$ such that $M(s,t) \neq 0$ and for each $\tilde{s} \in \tilde{\Gamma}_0$ lying over s, there is exactly one $\tilde{t} \in \tilde{\Gamma}_0$ lying over t and such that $k(\tilde{\Gamma})(\tilde{s}, \tilde{t}) \neq 0$: In fact, we have $\gamma \tilde{s} = \tilde{x}$ for some $x \in \Gamma_0$ and some $\gamma \in \Pi$; hence $\gamma \tilde{t} \in R_x$; the relation $k(\tilde{\Gamma})(\tilde{s}, \delta \tilde{t}) \neq 0$, $1 \neq \delta \in P$, would imply $\gamma \delta \tilde{t} \in R_x$ and $\gamma \delta \gamma^{-1} \in S$, a contradiction to our assumption $P \cap S = \emptyset$.

Being a covering functor, F induces an isomorphism $k(\tilde{\Gamma})(\tilde{s}, \tilde{t}) \xrightarrow{\sim} M(s,t)$, where $s, t \dots$ are as above. On the other hand, we clearly have $k(\tilde{\Gamma})(\tilde{s}, \tilde{t}) \subset \mathscr{R}^n k(\tilde{\Gamma})(\tilde{s}, \tilde{t})$ and $\mathscr{R}^{n+1} k(\tilde{\Gamma})(\tilde{s}, \tilde{t}) = 0$, where $n = \kappa(\tilde{t}) - \kappa(\tilde{s})$ is determined by the grading morphism κ introduced in 1.6. Applying proposition 3.2 we infer that $M(s,t) \subset \mathscr{R}^n M(s,t)$ and $\mathscr{R}^{n+1} M(s,t) = 0$. In other words, only one grade really occurs in $(\operatorname{Gr} M)(s,t)$. So we can deduce $M = \operatorname{Gr} M$ from $k(\tilde{\Gamma}) = \operatorname{Gr} k(\tilde{\Gamma})$ and apply proposition 5.1.

A similar argument applies to any cover of Δ.

5.4 **Proof of Lemma 5.1.** Let us assume that Γ is a connected translation-quiver, or equivalently that $\hat{\Gamma}$ is a connnected quiver. Consider the following differential complex

$$S_2(\Gamma) \xrightarrow{\delta_2} S_1(\Gamma) \xrightarrow{\delta_1} S_0(\Gamma),$$

where $S_0(\Gamma)$, $S_1(\Gamma)$ and $S_2(\Gamma)$ are the free abelian groups generated by the vertices of Γ, the arrows of $\hat{\Gamma}$ and the arrows of Γ with non-projective heads respectively. For basis elements $a \xrightarrow{\alpha} z$ and $y \xrightarrow{\beta} x$ of $S_1(\Gamma)$ and $S_2(\Gamma)$ we set $\delta_1 \alpha = z - a$ and $\delta_2 \beta = \beta - \gamma_x + \sigma\beta$ respectively. Clearly, Coker δ_1 is identified with $\mathbb{Z}$.

We claim that $\operatorname{Ker}\delta_2 = 0$ *if Γ is simply connected:* indeed, assume that $n = \sum_\beta n_\beta \beta \in \operatorname{Ker}\delta_2$. In order to show that $n_\beta = 0$ for each β, consider the sub-translation-quiver Γ_β of Γ, which is formed by the σ-orbit of β and the τ-orbits of its extremities (4.3). Let $z \xrightarrow{\sigma^r\beta} t$ be a σ-translate of β and m_r the coordinate of $\delta_2 n$ with respect to the basis-element $\sigma^r\beta$ of $S_1(\Gamma)$: if t is projective, we have $m_r = n_{\sigma^{r-1}\beta}$; if z is injective, $m_r = n_{\sigma^r\beta}$; in all other cases $m_r = n_{\sigma^{r-1}\beta} + n_{\sigma^r\beta}$. Now, since Γ is simply connected, it has no periodic component (7.2), so that Γ_β is neither periodic nor semi-periodic. A glance at the list of the translation-quivers with one σ-orbit tells us that, either z is injective for some r, or the arrows $\sigma^r\beta$ are all defined and distinct for small values of $r \in \mathbb{Z}$. On the other hand, we have $m_r = 0$ for each r and $n_{\sigma^r\beta} = 0$ if r is small enough. We infer that $n_{\sigma^r\beta} = 0$ by induction on r.

If Γ is simply-connected, we also have $\operatorname{Im}\delta_2 = \operatorname{Ker}\delta_1$. Indeed, we can show that each homomorphism of abelian groups $f\colon S_1(\Gamma) \to M$ such that $f\delta_2 = 0$ has the form $f = g\delta_1$: Indeed, given f, we set

$$\ell(w) = \pm f(\alpha_m) \pm \dots \pm f(\alpha_1)$$

for any walk $w = (y|\alpha_m, \dots, \alpha_1|x)$ (1.2), where $f(\alpha_i)$ is endowed with the sign $+$ or $-$ according as α_i is oriented from x to y or not. The relation $f\delta_2 = 0$ means that ℓ is constant on the homotopy classes. If Γ is simply connected, we construct $g\colon S_0(\Gamma) \to M$ by setting $g(y) = \ell(w)$, where w is an arbitrary walk from a chosen fixed vertex x to the vertex y. The result is independent of w and provides us with a g such that $f = g\delta_1$.

If Γ is not simply connected, denote by $\pi\colon \tilde{\Gamma} \to \Gamma$ is universal covering, by Π its fundamental group. The sequence

$$0 \to S_2(\tilde{\Gamma}) \xrightarrow{d_2} S_1(\tilde{\Gamma}) \xrightarrow{d_1} S_0(\tilde{\Gamma}) \to \mathbb{Z} \to 0$$

is exact and provides us with a free resolution of the trivial Π-module $\mathbb{Z}$. Applying to this resolution the functor $\operatorname{Hom}_\Pi(?, k^*)$, where $k^* = k \setminus \{0\}$ is endowed with the trivial Π-structure, we obtain the differential complex

$$\operatorname{Hom}_{\mathbb{Z}}(S_0(\Gamma), k^*) \to \operatorname{Hom}_{\mathbb{Z}}(S_1(\Gamma), k^*) \to \operatorname{Hom}_{\mathbb{Z}}(S_2(\Gamma), k^*) \to 0$$

whose second cohomology group is identified with $H^2(\Pi, k^*)$. Since Π is free, we have $H^2(\Pi, k^*) = \{1\}$. So the second cohomology group is trivial. This is the statement of our lemma.

Remark. The preceding proof can be interpreted as follows: the differential complex $S_.(\Gamma)$ is a subcomplex of the singular complex of the simplical set $K\Gamma$ associated with Γ (4.1). It is the subcomplex generated by the non-degenerated singular simplices. It is well-known that this subcomplex $S_.(\Gamma)$ is homotopy-equivalent to the singular complex. Therefore, the n-th cohomology group of $\mathrm{Hom}_{\mathbb{Z}}(S.(\Gamma), k^*)$ is identified with the singular cohomology group $H^n(|\Gamma|, k^*)$ of the topological space $|\Gamma|$. Now, the universal cover $|\tilde{\Gamma}|$ is acyclic by theorem 4.2. It follows that $H^n(|\Gamma|, k^*) \xrightarrow{\sim} H^n(\Pi, k^*)$ by [12], IV 7.3.

6. Simply Connected Algebras

Up to the end of this paragraph we denote by A an algebra over k which is *simply connected*, i.e. representation-finite, connected, basic, finite-dimensional and *having a simply connected Auslander-Reiten quiver* Γ_A. We denote by G_A the associated graph, which is a *tree* by theorem 4.2. Like G_A, all *trees* considered here are supposed to be *finite*.

6.1 Among the known classes of representation-finite algebras the following ones turn out simply-connected: the algebras (with commutativity relations) associated with connected partially ordered sets ([11], [19]); the tree-algebras of Bongartz-Ringel [5]; the tilted algebras of Happel-Ringel produced by a hereditary tree-algebra ([4], [9]). We shall revert to these examples in a subsequent publication.

Since the simply-connected algebra A admits a well-behaved isomorphism $k(\Gamma_A) \xrightarrow{\sim} \mathrm{ind}\, A$ (3.1 b)), it is *standard* and isomorphic to $\bigoplus_{p,q} k(\Gamma_A)(p, q)$, where p, q range over all projective vertices of Γ_A. Accordingly, the classification of the simply connected algebras is equivalent to the classification of the simply connected finite Riedtmann-quivers. In this paragraph we try for a first approach to this problem by demonstrating the existence of an inductive construction of the involved Riedtmann-quivers. Among other things our construction will yield the

Theorem. *For each tree T, the number n_T of isomorphism classes of simply connected algebras A such that $G_A \xrightarrow{\sim} T$ is finite.*

6.2 Since Γ_A is simply connected and finite, there is a unique quiver-morphism $\kappa: \Gamma_A \to \mathbb{Z}A_2$ such that $0 = \operatorname{Min}_x \kappa(x)$, where the minimum is taken over all vertices x of Γ_A; we denote this quiver-morphism by κ_A. Since G_A has no loop, each τ-orbit t of Γ_A contains exactly one projective vertex p_t. We set $g_A(t) = \kappa_A(p_t) \in \mathbb{N}$. The function g_A thus defined is a grading of G_A in the following sense.

Definition. A grading of a tree T is a function $g\colon T_0 \to \mathbb{N}$ satisfying the conditions a) and b) below. A *graded tree* is a pair (T, g) formed by a tree T and a grading g.

a) $g(x) - g(y) \in 1 + 2\mathbb{Z}$, whenever x and y are neighbours in T.
b) $g^{-1}(0) \neq \emptyset$.

At the end of this paper we include the list of the gradings of some chosen small trees which arise from simply connected algebras.

Our purpose is to show that A is completely determined by (G_A, g_A). In order to do so, we first attach a translation-quiver Q_T to each graded tree

$T=(T,g)$: the vertices of Q_T are the pairs $(n,t) \in \mathbb{N} \times T_0$ such that $n-g(t) \in 2\mathbb{N}$; two such vertices (m,s) and (n,t) are joined by an arrow $(m,s) \to (n,t)$ if s,t are neighbours in T and $n=m+1$; the projective vertices are the pairs $(g(t),t)$; the translate of a non-projective vertex is defined by $\tau(n,t)=(n-2,t)$ (see Fig. 6.2).

6.3 Let us examine the case of the graded tree (G_A, g_A) attached to the simply connected algebra A. The map $x \to (\kappa_A(x), x^\tau)$, where x^τ denotes the τ-orbit of x (4.2), *identifies* Γ_A with a full sub-translation-quiver of $Q_{(G_A, g_A)}$ and yields a *dimension map* d_A: by definition, this map associates with each vertex x of $Q_{(G_A, g_A)}$ a function $d_A(x)\colon (G_A)_0 \to \mathbb{N}$, which is 0 if x lies outside Γ_A and equals $t \mapsto [k(\Gamma_A)(p_t, x): k]$ if x lies in Γ_A.

The support of d_A is by construction the set of vertices of Γ_A. The point now is that we can describe d_A in a purely combinatorial way in terms of g_A. More precisely, for each graded tree $T=(T,g)$, there is a unique map $d\colon (Q_T)_0 \to \mathbb{N}^{T_0}$ satisfying the conditions a), b) and c) below. This d equals d_A if $T=(G_A, g_A)$.

a) We have $d(g(t),t)=\delta_t+\sum_s d(g(t)-1,s)$, whenever t is such that $d(g(t)-1,s)>0$ for each neighbour s of t in T satisfying $g(s)<g(t)$. In the preceding sum s ranges over the neighbours s of t such that $g(s)<g(t)$; the Kronecker function δ_t takes at r the value 1 or 0 according as $r=t$ or $r \neq t$; a function is >0 if all its values are $\geqq 0$ and one of them at least is >0.

b) We have $d(n,t)=\sum_s d(n-1,s)-d(n-2,t)$, whenever (n,t) is a non-projective vertex of Q_T for which the functions $d(n-2,t)$ and $\sum_s d(n-1,s)-d(n-2,t)$ are both >0. Here s ranges over the neighbours of t in T such that $g(s)<n$.

c) For any other vertex (n,t) of Q_T we have $d(n,t)=0$.

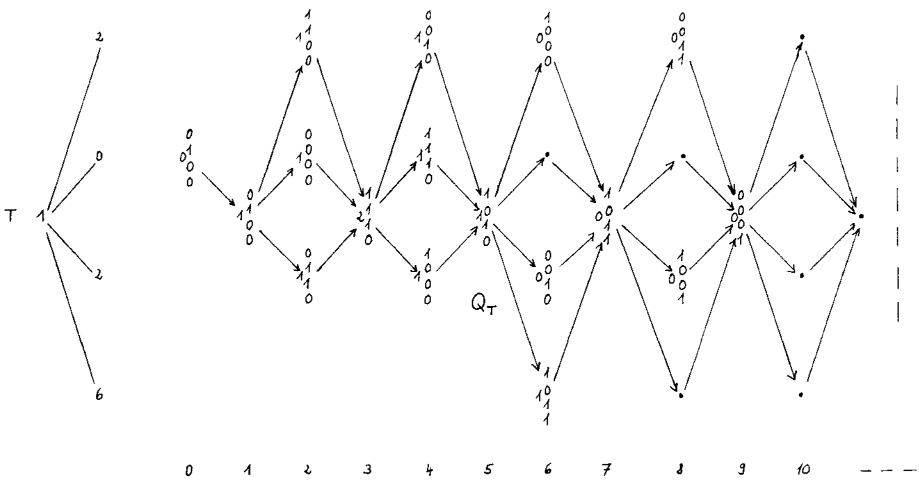


Fig. 6.2. The grading g takes the values 0, 1, 2 (twice) and 6. A vertex x of Q_T, such that $d(x)>0$, is represented by the values of the map $d(x)\colon T_0 \to \mathbb{N}$

The preceding conditions allow us to compute the functions $d(n,t)$ by induction on n, starting with $n=g(t)$. On the analogy of case $T=G_A$ we call d the *dimension map* of Q_T. We denote by R_T the full sub-translation-quiver of Q_T formed by the vertices (n,t) such that $d(n,t)>0$. The grading g is called *admissible*, if R_T contains all the projective vertices $(g(t),t)$ of Q_T; it is called *representation-finite* if it is admissible and R_T is finite. Accordingly, (T,g) is called an *admissible* or a *representation-finite graded tree*. For instance, (G_A, g_A) is representation-finite and R_{G_A} is identified with Γ_A.

6.4 Let T be an *admissible graded tree*. Our next step is to examine the finite-dimensional algebra $A^T=\bigoplus_{p,q} k(R_T)(q,p)$, where $k(R_T)$ is the mesh category of R_T and p,q range over all *projective* vertices of R_T:

a) Each vertex x of R_T is associated with an A^T-module $M(x) =\bigoplus_p k(R_T)(p,x)$, where p ranges over all projective vertices of R_T. The map $x \mapsto M(x)$ yields a functor $M\colon k(R_T)\to \operatorname{mod} A^T$, whose restriction to the projective vertices is fully faithful by construction. Accordingly, the algebra of endomorphisms of $M(p)$ has dimension 1 if p is projective. The formula $A^T=\bigoplus_p M(p)$ further shows that the $M(p)$ furnish a complete list of indecomposable projective A^T-modules.

b) Let $x \xrightarrow{\alpha} p$ range over the arrows of R_T heading for some projective vertex $p=(g(t),t)$. Denote by $\bar\alpha$ the morphism of $k(R_T)$ associated with α. It follows from Lemma 2.6 that the induced morphisms $M(\bar\alpha)$ yield an isomorphism $\bigoplus_\alpha M(x) \xrightarrow{\sim} \mathscr{R}M(p)$ (= radical of $M(p)$).

c) Let $n\in\mathbb{Z}$. We want to show that the following statements hold:

α) For each vertex (n,t) of R_T, the A^T-module $M(n,t)$ *is indecomposable and its dimension-vector is* $d(n,t)$ (6.3). In other words, the value of $d(n,t)$ at $s\in T_0$ is the multiplicity of the top of $M(g(s),s)$ as a Jordan-Hölder factor of $M(n,t)$; equivalently, it is the dimension of $k(R_T)(M(g(s),s), M(n,t))$.

β) *For each non-projective vertex* (n,t) of R_T, *the sequence*

$$M(n-2,t) \xrightarrow{[M(\bar\sigma\bar\alpha)]} \bigoplus_\alpha M(n-1,s) \xrightarrow{[M(\bar\alpha)]} M(n,t),$$

which is induced by the arrows of R_T of the form $(n-1,s)\xrightarrow{\alpha}(n,t)$, is *Auslander-Reiten.*

γ) *For each injective vertex of* R_T *of the form* $(n-2,s)$, $M(n-2,s)$ *is injective;* moreover, in the quotient of $M(n-2,s)$ by its socle each direct summand occurs with multiplicity 1 and is isomorphic to $M(n-1,r)$ for some arrow $(n-2,s)\xrightarrow{\beta}(n-1,r)$ of R_T.

Our proof proceeds as follows: Let m be a natural number and denote by H_m the hypothesis claiming that the statements α), β), γ) hold for each $n\leqq m$. We shall prove that H_m is true by induction on m and on the cardinality $|T_0|$ of T_0.

H_m is obviously true if $|T_0|=1$. So we shall suppose that $|T_0|>1$ and that H_m is true for each m and each tree having cardinality strictly less than T. The hypothesis H_m is also trivially satisfied for $m=0$. So we shall suppose that $m>0$ and that H_{m-1} holds.

The proof of the induction step is given in d) and e) below. It uses the following trivial implication of H_{m-1}: if $n<m$, *$M(n,t)$ is isomorphic to the Auslander-Reiten translate* $\mathscr{A}^{-r}M(g(t),t)$, $r=\frac{1}{2}(n-g(t))$, *of the projective module* $M(g(t),t)$. Accordingly, *we have* $M(n,t) \not\to M(\ell,s)$ *if* $(n,t)\neq(\ell,s)$ and $\ell, n<m$.

d) Let $(m-2,t)$ be a vertex of R_T and $N=M(m-2,t)$. Denoting by $\mathscr{R}$ the radical of the category $\operatorname{mod} A^T$, we first show that $[\mathscr{R}(N,L)/\mathscr{R}^2(N,L): k]\leqq 1$ for any $L\in\operatorname{ind} A^T$. Moreover, *in the case* $[\mathscr{R}(N,L)/\mathscr{R}^2(N,L): k]=1$, *$L$ is isomorphic to $M(m-1,s)$ for some arrow $(m-2,t)\xrightarrow{\delta}(m-1,s)$ of R_T.*

Indeed, let $\mu\colon N\to L$ be an irreducible map, $L\in\operatorname{ind} A$. *If L is not projective*, its Auslander-Reiten translate $\mathscr{A}L$ is the domain of an irreducible map $\nu\colon \mathscr{A}L\to N$. If $(m-2,t)$ is not projective, part β) of H_{m-1} gives us the structure of the Auslander-Reiten sequence stopping at N (statement β) of Sect. c)). We infer that $\mathscr{R}(\mathscr{A}L,N)/\mathscr{R}^2(\mathscr{A}L,N)$ has dimension 1, and that $\mathscr{A}L\xrightarrow{\sim}M(m-3,s)$ for some arrow $(m-3,s)\xrightarrow{\varepsilon}(m-2,t)$ of R_T. The same conclusion holds if $(m-2,t)$ is projective by section b) above. Since $\mathscr{A}L$ is not injective, $(m-3,s)$ is not an injective vertex of R_T by part γ) of H_{m-1}. Hence $(m-1,s)$ belongs to R_T and L is isomorphic to $\mathscr{A}^{-1}M(m-3,s)\xrightarrow{\sim}M(m-1,s)$ by part β) of H_{m-1}. Our claim follows for $\delta=\sigma^{-1}\varepsilon$, since $[\mathscr{R}(N,L)/\mathscr{R}^2(N,L): k] =[\mathscr{R}(\mathscr{A}L,N)/\mathscr{R}^2(\mathscr{A}L,N): k]$.

Suppose now that $L=M(g(s),s)$ is projective. By section b) we have $M(m-2,t)=N\xrightarrow{\sim}M(g(s)-1,r)$ for some arrow $(g(s)-1,r)\xrightarrow{\delta}(g(s),s)$ of R_T. Now, if $g(s)-1<m$, the last assertion of section c) tells us that $(m-2,t)=(g(s)-1,r)$. Accordingly, our claim follows from section b), if we can exclude the possibility $g(s)-1\geqq m$: in fact, if $g(s)>m$, we consider the full subgraph of T formed by the vertices v such that $g(v)<g(s)$. This subgraph is a disjoint union of trees T^i, which we grade with $g_i=g\,|\,T_0^i-\mu_i$, where $\mu_i=\operatorname{Min}\{g(x)\colon x\in T_0^i\}$. Clearly, we have $(g(s)-1-\mu_i,r)\in R_{T^i}$ and $(m-2-\mu_j,t)\in R_{T^j}$ for some i,j. As $M(g(s)-1,r)$ and $M(m-2,t)$ are isomorphic, they must have the same "support", i.e. the same Jordan-Hölder factors. Hence $i=j$. Since T^i has less vertices than T, we already know that $M(m-2-\mu_i,t)\xrightarrow{\sim}M(g(s)-1-\mu_j,r)$ implies $t=r$ and $m-2-\mu_i=g(s)-1-\mu_j$, a contradiction to $g(s)>m$.

e) *Proof of H_m*: Let $(m-2,t)$ be a vertex of R_T. Each arrow $(m-2,t)\xrightarrow{\delta}(m-1,s)$ induces an irreducible map $M(\bar\delta)$. This follows from section b) if $(m-1,s)$ is projective, from part β) of H_{m-1} otherwise. We infer that the induced map $M(m-2,t)\xrightarrow{[M(\delta)]}\bigoplus_\delta M(m-1,s)$, where δ ranges over all the arrows of R_T with tail $(m-2,t)$, is irreducible. It is maximal irreducible by section d).

If $(m-2,t)$ is an injective vertex of R_T, we have $d(m-2,t)\geqq\sum_\delta d(m-1,s)$ by 6.3b), c). Since $d(m-2,t)$ and $d(m-1,s)$ are the dimension-vectors of $M(m-2,t)$ and $M(m-1,s)$ by part α) of H_{m-1}, $M(m-2,t)$ is injective. This and section d) prove part γ) of H_m.

If $(m-2,t)$ is not an injective vertex of R_T, the dimension-vector of $M(m-2,t)$ is strictly smaller than that of $\bigoplus_\delta M(m-1,s)$. Accordingly, $M(m-2,t)$ is not injective. The maximal irreducible map $[M(\delta)]$ yields an Auslander-Reiten sequence

$$0 \to M(m-2,t) \xrightarrow{[M(\delta)]} \bigoplus_{\delta} M(m-1,s) \to \operatorname{Coker}[M(\bar{\delta})] \to 0.$$

Now, $\operatorname{Coker}[M(\bar{\delta})]$ is identified with $M(m,t)$ by Lemma 2.6. This proves part β) of H_m, and implies that $M(m,t)$ is indecomposable. The exactness of the Auslander-Reiten sequence and part α) of H_{m-1} imply that $M(m,t)$ has the dimension-vector $\sum_{\delta} d(m-1,s) - d(m-2,t) = d(m,t)$ (6.3b)). This proves part α) of H_m.

We summarize our findings in the following proposition.

Proposition. *If* (T,g) *is an admissible graded tree, the functor* $M: k(R_T) \to \operatorname{mod} A^T$ *yields an equivalence between* $k(R_T)$ *and a full subcategory of* $\operatorname{ind} A^T$*; it induces a translation-quiver-isomorphism of* R_T *onto a connected component of the Auslander-Reiten-quiver* Γ_{A^T}.

Proof. The proof that a well-behaved functor is a covering functor ([15], 2.3) extends to the present non-representation-finite case. It yields that M is fully faithful. The rest of the proposition has already been proved.

6.5 **Corollary.** *The* map $(T,g) \mapsto A^T$ *yields a bijection between the isomorphism classes of representation-finite graded trees and the isomorphism-classes of simply connected algebras.*

Proof. Consider the map $A \mapsto (G_A, g_A)$ in the reverse direction. By 6.1 and 6.3 we know that $A \xrightarrow{\sim} A^{G_A}$. By 6.4 we know that $(G_{A^T}, g_{A^T}) \xrightarrow{\sim} (T,g)$.

6.6 At last we give the promised inductive recipe to construct all representation-finite graded trees.

Let $T=(T,g)$ be a graded tree and x a vertex of R_T. The *starting function* $s_x = s_x^T: (R_T)_0 \to \mathbb{N}$ at x is defined by $s_x^T(y) = [k(R_T)(x,y): k]$. Its *support* $s_x^{-1}(\mathbb{N} \setminus \{0\})$ is denoted by S_x^T. The full subquiver of R_T formed by S_x^T is the Hasse-diagram of a *partial order*, with which we endow S_x^T.

In the sequel we denote by m a *vertex* of T *with maximal grade* ($g(m) \geqq g(t)$ for all $t \in T_0$). We denote by $t_1, \ldots, t_r$ the neighbouring vertices of m in T, by $T^1, \ldots, T^r$ the corresponding connected components of $T \setminus \{m\}$, by μ_i the minimum of g on T_0^i, by g_i the grading $g \mid T_0^i - \mu_i$ of T^i.

Proposition. *With the above notations the following statements are true:*

a) (T,g) *is admissible iff each* (T^i, g_i) *is admissible and each* R_{T^i} *contains* $x_i = (g(m) - 1 - \mu_i, t_i)$, $1 \leqq i \leqq r$.

b) (T,g) *is representation-finite iff the conditions* α) *and* β) *below are satisfied:*

α) *Each* (T^i, g_i) *is representation-finite, each* R_{T^i} *contains* $x_i = (g(m) - 1 - \mu_i, t_i)$*, the values of each* $s_{x_i}^{T^i}$ *are* $\leqq 1$.

β) *The partially ordered set* $S_{x_1}^{T^1} \coprod \ldots \coprod S_{x_r}^{T^r}$ *is representation-finite in the sense of Nazarova-Roiter* ([13]).

Let S be a partially ordered set. A *S-space* is by definition a vector-space M together with a family of subspaces M_s, $s \in S$, such that $M_s \subset M_t$ if $s \leqq t$. The S-space M is the direct sum of two S-spaces M' and M'' if $M = M' \oplus M''$ and $M_s = M'_s \oplus M''_s$ for all $s \in S$. The partially ordered set S is called *representation-finite* in the sense of Nazarova-Roiter if there are only finitely many finite-dimensional S-spaces admitting no proper direct sum decomposition. It has been shown by Kleiner, Nazarova and Roiter that this is equivalent to saying that S is finite and contains no subset whose Hasse-diagram for the induced order has one of the 5 forms given in Fig. 6.6 [13].

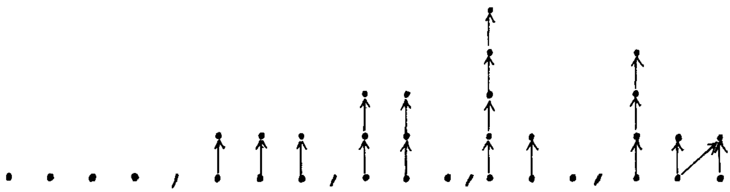


Fig. 6.6.

We postpone the proof of the proposition to Sect. 6.10.

6.7 **Corollary.** *Each tree T admits only a finite number of representation-finite gradings.*

Proof. Suppose that the statement is already proved for the graded trees S having strictly less vertices than T. Then there is a natural number N such that

$$(R_S)_0 \subset \{(n, s) : s \in S_0, n \leqq N\}$$

for each such S. As a consequence, each representation-finite grading g of T satisfies the relation $g(t) \leqq N+1$, $\forall t \in T_0$. This proves our statement.

6.8 We will use the following well-known facts in 6.10: Let A be a basic-finite-dimensional algebra and $A_A = P_1 \oplus \ldots \oplus P_m$ a decomposition of A_A into indecomposable projectives such that $\mathrm{Hom}_A(P_m, P_m) = k$ and $\mathrm{Hom}_A(P_m, P_i) = 0$ for all $i \neq m$. The decomposition yields an isomorphism

$$A = \mathrm{End}\, A_A \xrightarrow{\sim} \begin{bmatrix} B & 0 \\ R & k \end{bmatrix},$$

where $B = \mathrm{End}_A\left(\bigoplus_{i=1}^{m-1} P_i\right)$ and $R = \mathrm{Hom}_A\left(\bigoplus_{i=1}^{m-1} P_i, P_m\right)$. Accordingly, each A-module can be interpreted as a triple (M_1, M_2, ϕ), where M_1 is a B-module, M_2 a k-vectorspace and ϕ an element of $\mathrm{Hom}_B(M_2 \otimes_k R, M_1) \xrightarrow{\sim} \mathrm{Hom}_k(M_2, \mathrm{Hom}_B(R, M_1))$.

Let S be the support of the functor $\mathrm{Hom}_B(R, ?)$ in $\mathrm{ind}\, B$ and $U \in S$. If $f_1, f_2 \in \mathrm{Hom}_B(R, U)$ are linearly independent over k and $\mathrm{End}_B U = k$, we have a one-parameter family $(U, k, f_1 + \lambda f_2)$ of non-isomorphic indecomposable triples, and A is representation-infinite.

If $[\mathrm{Hom}_B(R, U) : k] = 1$ for all $U \in S$, we endow S with the partial order such that $U \geqq V$ iff $\mathrm{Hom}_B(R, f) \neq 0$ for some $f \in \mathrm{Hom}_B(U, V)$. With this definition, the second component M_2 of each triple (M_1, M_2, ϕ) carries a natural S-space structure (6.6): set $M_{2U} = \phi^{-1}(H_U)$, where H_U is the image of the composition-map

$$\mathrm{Hom}_B(R, U) \otimes_k \mathrm{Hom}_B(U, M_1) \to \mathrm{Hom}_B(R, M_1).$$

The functor $(M_1, M_2, \phi) \mapsto M_2$ thus defined induces a bijection between the isomorphism classes of indecomposable triples (M_1, M_2, ϕ) such that $\phi \neq 0$ and the isomorphism classes of indecomposable S-spaces M such that $M_s \neq M$ for some $s \in S$.

For details see for instance [18].

6.9 **Lemma.** *Assume that (T, g) is admissible and that $y, z \in S_x = s_x^{-1}(1)$ (6.6). Then we have $y \geqq z$ iff $k(R_T)(x, f) \neq 0$ for some $f \in k(R_T)(y, z)$.*

Proof. Let u, v be two vertices of S_x, $u \xrightarrow{\alpha} v$ an arrow of R_T and ψ a path from x to u inducing a non-zero-morphism $\bar{\psi} \in k(R_T)(x, u)$. It is clearly enough to show that $\bar{\alpha}\bar{\psi} \neq 0$. We proceed by induction on $\kappa(v)$, where κ is the first projection: $\kappa(n, t) = n$.

If v is projective, $k(R_T)(x, \bar{\alpha})$ is injective; so $\bar{\alpha}\bar{\psi} \neq 0$. If v is not projective, consider the mesh stopping at v (Fig. 6.9) and the associated exact squence

$$0 \to k(R_T)(x, \tau v) \to \bigoplus_{i=1}^{r} k(R_T)(x, v_i) \to k(R_T)(x, v) \quad (2.6).$$

Suppose that $\alpha = \beta_1$ and $\bar{\alpha}\bar{\psi} = 0$; then $\bar{\psi} = (\overline{\sigma\beta_1})f$ for some $f \in k(R_T)(x, \tau v)$ such that $(\overline{\sigma\beta_i})f = 0$ for $i \neq 1$. We infer that $\tau v \in S_x$ and that $f = \lambda\bar{\psi}'$ for some path ψ' and some $\lambda \neq 0$. This implies $(\overline{\sigma\beta_i})\bar{\psi}' = \lambda^{-1}(\overline{\sigma\beta_i})f = 0$ for $i \neq 1$, hence $v_i \notin S_x$ by our induction hypothesis $(\kappa(v_i) < \kappa(n))$. On the other hand, $v \in S_x$; so there is a path χ from x to some v_j such that $\overline{\beta_j \chi} \neq 0$. Accordingly, we have $j = 1$, $\bar{\chi} = \mu\bar{\psi}$ for some $\mu \in k$, and $\overline{\beta_1 \chi} = \mu\overline{\beta_1 \psi} = 0$, a contradiction.

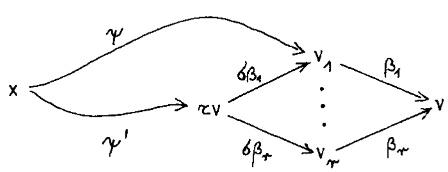


Fig. 6.9

6.10 **Proof of Proposition 6.6.** a) The inductive definition of R_T implies that

$$\{(n, t) \in (R_T)_0 : n \leqq g(m), t \neq m\}$$
$$= \coprod_{i=1}^{r} \{(n + \mu_i, t) : (n, t) \in (R_{T^i})_0, n + \mu_i \leqq g(m)\}.$$

Furthermore $(g(m), m)$ belongs to $(R_T)_0$ iff $x_i \in (R_{T^i})_0$ for all i. This proves a).

b) Suppose (T, g) representation-finite. Then all (T^i, g_i) are so, since the corresponding algebras A^{T^i} are residue-algebras of A^T. Applying 6.8 to the case $A = A^T$ and $P_m = M(g(m), m)$, we have $B = \prod_{i=1}^{r} A^{T^i}$ and $R = \prod_{i=1}^{r} M(x_i)$. Since all A^{T^i} are simply connected, each indecomposable B-module has k as ring of endomorphisms. According to 6.8, this implies α). Moreover, the partially ordered set S considered in 6.8 is identified with $S_{x_1}^{T^1}\coprod\ldots\coprod S_{x_r}^{T^r}$ by 6.9. It is representation-finite, because A^T is so (6.8). This proves that the conditions α) and β) are necessary. The sufficiency proof is similar.

7. The Representation-Finite Gradings of A_n

Our purpose in this section is to describe the representation-finite gradings of the tree A_n illustrated below. They coincide with the admissible ones. We skip the proofs.

$$A_n \qquad 1—2—3—\ldots—n-1—n.$$

7.1 With each grading g of A_n we associate a *bounden quiver* K_g which is defined as follows. The vertices of K_g are the projective vertices $\underline{s}=(g(s), s)$ of $Q_{(A_n, g)}$ (6.2). Two such vertices $\underline{s}$, $\underline{t}$ are connected by an arrow $\underline{s}\xrightarrow{\phi}\underline{t}$ if one of the two following conditions holds: either $s<t$, $g(s)-s=g(t)-t$ and $g(x)-x<g(t)-t$ whenever $s<x<t$; or $s>t$, $g(s)+s=g(t)+t$ and $g(x)+x<g(t)+t$ whenever $t<x<s$. We call ϕ an *α-arrow* in the first case, a *β-arrow* in the second (see Fig. 7.1). We require that the composition of any α-arrow with any β-arrow be zero (symbolically: $\alpha\beta=0=\beta\alpha$).

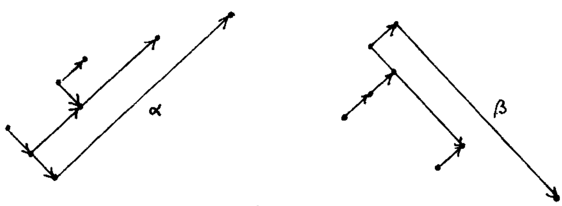
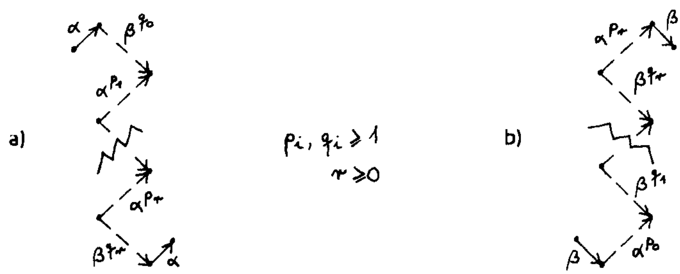


Fig. 7.1

Proposition. *A grading g of A_n is representation-finite iff K_g is connected and contains no subquiver of the form* a) *or* b) *below. If these conditions hold, the algebra $A^{(A_n, g)}$ (6.4) is defined by the quiver K_g and the relations $\alpha\beta=0=\beta\alpha$.*



7.2 The quiver $K=K_g$ attached to a representation-finite grading g of A_n satisfies the following conditions:

A) K is a tree.

B) The arrows of K can be divided into an α-class and a β-class which satisfy B_1, B_2 and B_3:

B_1) The composition of two arrows belonging to different classes is zero, whereas any composition of arrows of the same class is not.

B_2) Each vertex x of K is the head of one α-arrow and one β-arrow at most; similarly, at most one α-arrow and one β-arrow start at x.

B_3) K contains no subquiver of the form a) or b) (7.1).

Conversely, *let A be the algebra of a bounden quiver K which has n vertices and satisfies the conditions* A) *and* B). *Then A is simply connected, and the associated tree G_A* (4.2) *is isomorphic to A_n*. In order to describe the grading g_A and an isomorphism $G_A \xrightarrow{\sim} A_n$, we divide the arrows of K into two classes α, β and consider the map $K_0 \to \mathbb{Z}^2$ whose value $(g(x), \ell(x))$ at $x \in K_0$ is constructed as follows by induction on the distance from x to a chosen origin $\sigma \in K_0$: at the origin we set $(g(\sigma), \ell(\sigma)) = (0, 1)$; if $c \to d$ is an α-arrow of K, we require that $(g(d), \ell(d)) = (g(c)+w, \ell(c)+w)$, where $w-1$ is the number of vertices x such that the shortest walk from c to x has the form illustrated in Fig. 7.2(*); similarly, if $c \to d$ is a β-arrow, we require that $(g(d), \ell(d)) = (g(c)+w, \ell(c) - g(w))$, where $w-1$ is the number of vertices x such that the shortest walk from c to x has the form illustrated in Fig. 7.2(**). The construction of the injection $K_0 \to \mathbb{Z}^2$ is illustrated in the Figs. 7.1, 7.3.2, 7.3.3 and 7.4.1.

Now set $\lambda = \operatorname*{Min}_{x \in K_0} \ell(x)$ and $\gamma = \operatorname*{Min}_{x \in K_0} g(x)$. Denote by $P_x \in \operatorname{mod} A$ the projective cover of the simple A-module with support x, by $x^\tau \in G_A$ the τ-orbit of P_x. Then: *the map $\bar{\ell}: x \mapsto \ell(x) - \lambda + 1$ induces a bijection $K_0 \xrightarrow{\sim} \{1, \dots, n\}$; the τ-orbits x^τ* and *y^τ are neighbours in G_A iff $|\bar{\ell}(y) - \bar{\ell}(x)| = |\ell(x) - \ell(y)| = 1$; the grade $g_A(x^\tau)$* of *$x^\tau \in G_A$ equals $g(x) - \gamma$.*

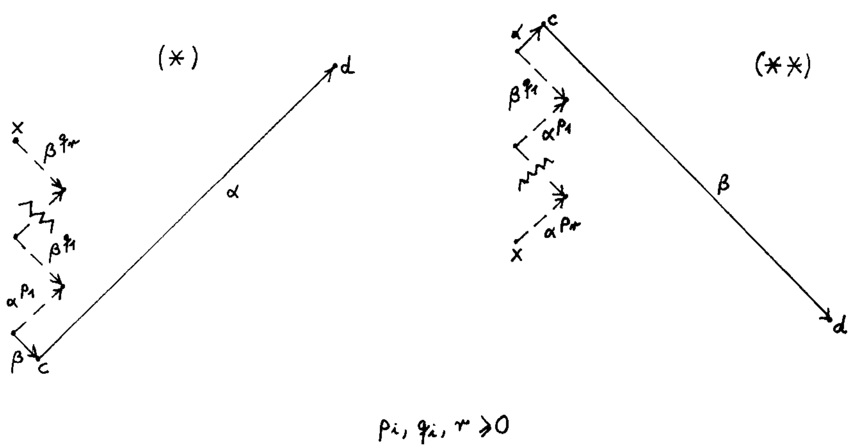


Fig. 7.2

7.3 *Definition. We call $\mathbb{A}$-quiver a bounden oriented tree whose arrows are divided into an* α-class *and a* β-class *such that the conditions* B_1, B_2 *and* B_3 *of* 7.2 *are satisfied.*

Figure 7.3.1 proposes two examples (compare with Happel-Ringel [23]). In $\tilde{S}^N$ the vertices are words formed with the letters a and b^{-1}. We order them

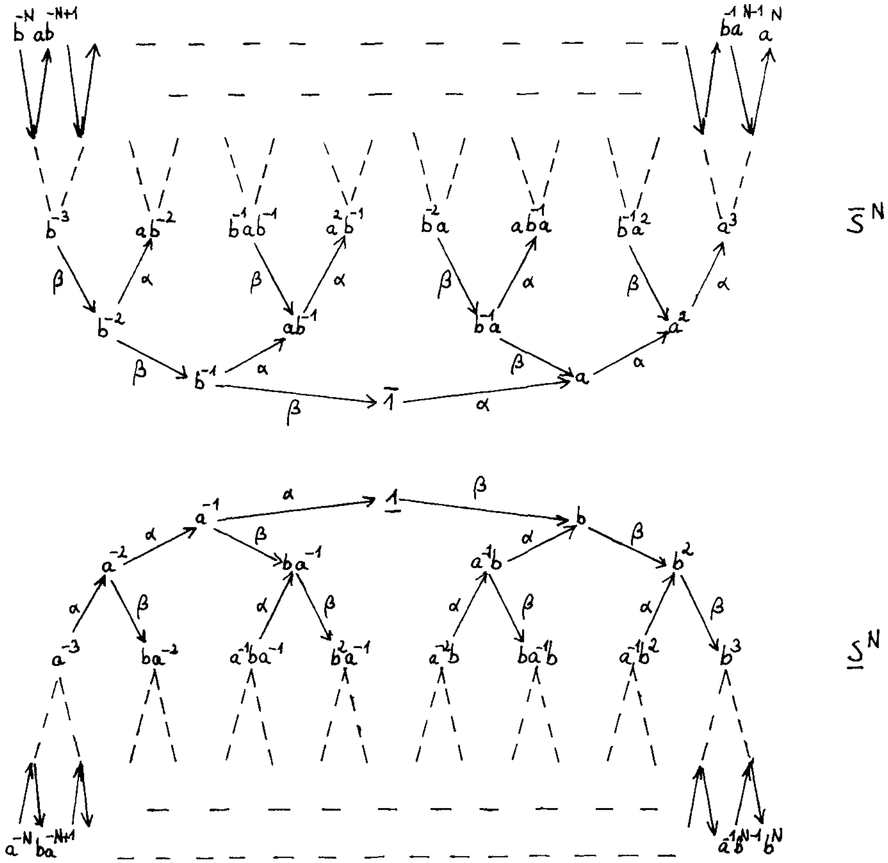


Fig. 7.3.1

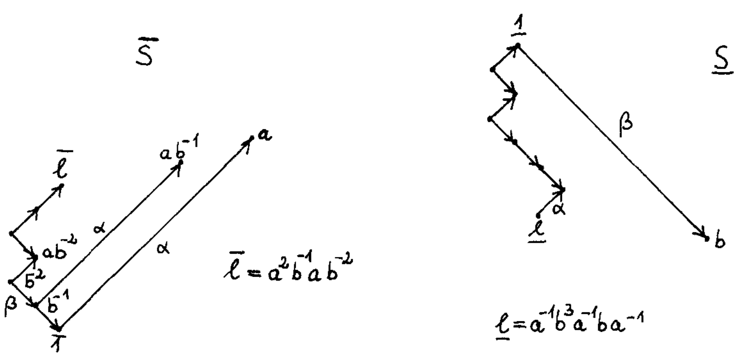


Fig. 7.3.2

lexicographically writing them from the right to the left and setting $b^{-1}<a$; this yields for instance $1<b^{-1}<b^{-2}<ab^{-1}<a<b^{-1}a<a^2$. We call *atom of class* α any connected full bounden subquiver $\bar{S}$ of $\bar{S}^N$ which has $\bar{1}$ as *smallest* and a as *biggest vertex* (see Fig. 7.3.2). The number c_p of atoms of class α with $p \leqq N$ arrows is given by the formula

$$\begin{aligned} c(t) &= \sum_{p \geqq 1} c_p t^p = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4t} = \sum_{p \geqq 1} \frac{1}{p}\binom{2p-2}{p-1} t^p \\ &= t + t^2 + 2t^3 + 5t^4 + 14t^5 + 42t^6 \ldots. \end{aligned}$$

If $\bar{1}$ is chosen as origin and $(g, \ell)\colon \bar{S}_0 \to \mathbb{Z}^2$ is the injection constructed in 7.2, $\ell(x)$ coincides with the ordinal of x in the lexicographic ordering of $\bar{S}_0$; so $\ell(\bar{1})$ is 1, and $\ell(a)$ is the number of vertices of $\bar{S}$. On the other hand, $\frac{1}{2}(\ell(x) - g(x) - 1)$ is the number of letters equal to b^{-1} in the word x. A word x is a *tip* of $\bar{S}$ if x belongs to $\bar{S}$, whereas $b^{-1}x$ and ax do not. By construction a is the biggest tip. If $\bar{\ell}$ denotes the smallest tip, the interval $\{x \in \bar{S}_0 \colon \bar{1} \leqq x \leqq \bar{\ell}\}$ is called the left *lineage* of $\bar{S}$.

Similarly, the vertices of $\underline{S}^N$ are words in b and a^{-1}. We endow them with the opposite of the lexicographic order such that $a^{-1}<b$; so we have $\underline{1}>a^{-1}>a^{-2}>ba^{-1}>b>a^{-1}b>b^2$. We call *atom of class* β any connected full bounden subquiver $\underline{S}$ of some $\underline{S}^N$ which has $\underline{1}$ as biggest and b as smallest vertex (see Fig. 7.3.2). A word x is a tip of $\underline{S}$ if it belongs to $\underline{S}$, whereas $a^{-1}x$ and bx do not. If $\underline{\ell}$ is the biggest tip of $\underline{S}$, the interval $\{x \in \underline{S}_0 \colon \underline{1} \geqq x \geqq \underline{\ell}\}$ is called the *left lineage of* $\underline{S}$.

Now suppose that m belongs to the left lineage of some $\bar{S}$ and m^{-1} to the left lineage of some $\underline{S}$ (for instance, set $m = ab^{-1}ab^{-2}$ and $m^{-1} = b^2a^{-1}ba^{-1}$ in Fig. 7.3.2). In case $\bar{1}<m<a$ and $\underline{1}>m^{-1}>b$ we match $\bar{S}$ and $\underline{S}$ together along the intervalls $\{x \in \bar{S}_0 \colon \bar{1} \leqq x \leqq m\}$ and $\{y \in \underline{S}_0 \colon \underline{1} \geqq y \geqq m^{-1}\}$ by identifying y with $x = ym$. The resulting $\mathbb{A}$-quiver $\bar{\underline{S}}$ will be called an *atom of class* $\alpha\beta$ (see Fig. 7.3.3). We endow it with the total order which extends the orders of $\underline{S}$ and $\bar{S}$; for this order b is minimal and a maximal. The number b_p of atoms of class $\alpha\beta$ with p arrows is given by the formula

$$\begin{aligned} b(t) &= \sum_{p \geqq 1} b_p t^p = -\tfrac{1}{2} + t + \tfrac{1}{2}\sqrt{1-4t} + \frac{t^2}{\sqrt{1-4t}} \\ &= \sum_{p \geqq 4} \binom{2p-4}{p-4} t^p = t^4 + 6t^5 + 28t^6 + \ldots. \end{aligned}$$

7.4 Let $S^1, \ldots, S^m$ be a sequence of atoms of class α, β or $\alpha\beta$. Then we can amalgamate $S^1, \ldots, S^m$ by identifying the biggest vertex of S^i with the smallest vertex of S^{i+1}, $1 \leqq i < m$. *The resulting amalgamation* $S^m \ldots S^1$ *is an* $\mathbb{A}$-*quiver* (see Fig. 7.4). In case $m=0$ we agree that the amalgamation consists of one vertex only. With this convention, *each* $\mathbb{A}$-*quiver can be written in a unique way as an amalgamation.*

It follows that *the number* g_p *of representation-finite gradings of* A_{p+1} *is given by* the formula

$$g(t)=\sum_{p\geqq 0} g_p t^p=\frac{1}{1-2t+\sqrt{1-4t}}+\frac{1}{-1+2t+3\sqrt{1-4t}}$$
$$=1+2t+6t^2+20t^3+71t^4+262t^5+992t^6$$
$$+3824t^7+14934t^8+58892t^9+233974t^{10}+\dots.$$

Accordingly, *the number a_p of isomorphism classes of simply connected algebras A such that $G_A \xrightarrow{\sim} A_{p+1}$ is given by*

$$a(t)=\sum_{p\geqq 0} a_p t^p=\tfrac{1}{2}g(t)+\tfrac{1}{4}\left(1+\frac{1-2t^2}{\sqrt{1-4t^2}}\right) g(t^2)$$
$$=1+t+4t^2+10t^3+39t^4+131t^5+509t^6$$
$$+1912t^7+7517t^8+29446t^9+117183t^{10}+\dots.$$

We infer that

$$\tfrac{1}{8}(\tfrac{3}{2}\sqrt{2}+2)^p/g_p\to 1 \quad\text{and}\quad \tfrac{1}{16}(\tfrac{3}{2}\sqrt{2}+2)^p/a_p\to 1$$

when p tends to ∞.

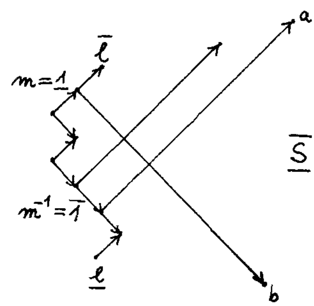


Fig. 7.3.3

7.5 Let g be a representation-finite grading of A_n, $K=K_g$ the associated $\mathbb{A}$-quiver (7.1), $R=R_{(A_n,g)}$ the associated Riedtmann-quiver (6.3). For each $s\in\{1,\dots,n\}$ we denote by u_s (resp. by ℓ_s) the number of vertices $x\neq \underline{s}$ of K such that the shortest walk from $\underline{s}$ to x in K has the form illustrated in Fig. 7.5.1 (resp. 7.5.2). With these notations, *the vertices y of R satisfying $k(R)(\underline{s}, y)\neq 0$ are the pairs $(p,q)\in\mathbb{N}\times\{1,\dots,n\}$ such that $(p-q)-(g(s)-s)\in\{0,2,4,\dots,2\ell_s\}$ and* $(p+q)-(g(s)+s)\in\{0,2,4,\dots,2u_s\}$. The set of these pairs is called the *rectangle starting at* $\underline{s}$ (see Fig. 7.5.3). It "stops" at the *injective vertex* $\bar{s}=(g(s)+u_s+\ell_s,$ $s+u_s-\ell_s)$ (compare with Prop. 2.8c). *For each point y of the rectangle starting at s we have $[k(R)(\underline{s},y): k]=1$. The vertex-set of R is the union of the rectangles starting at the different projective vertices* (see Fig. 7.4).

As a corollary we infer that *the simply connected algebras A such that $G_A \xrightarrow{\sim} A_n$ coincide with the tilted algebras produced by hereditary algebras of*

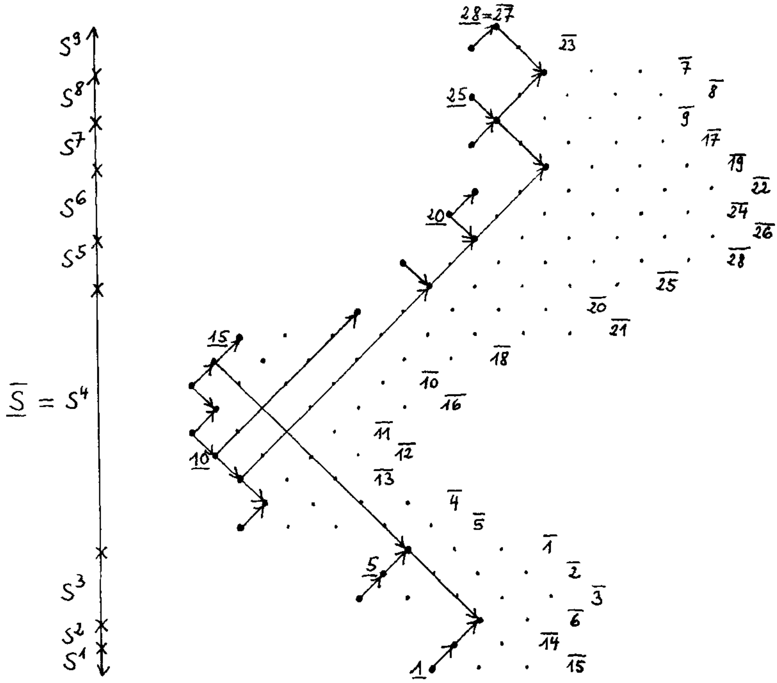


Fig. 7.4

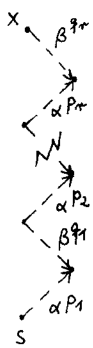


Fig. 7.5.1

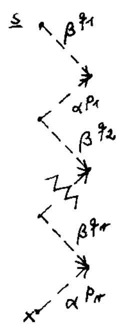


Fig. 7.5.2

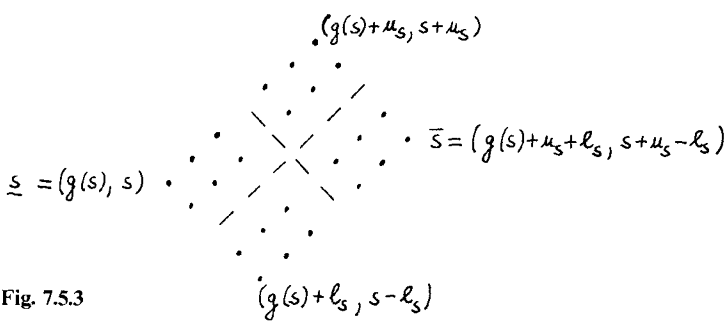


Fig. 7.5.3

class A_n (it follows from 7.4 and 7.5 that the Auslander-Reiten quiver of A has a section [4]; see Fig. 7.4).

7.6 *Remarks.* a) After the completion of our results we received an article by I. Assem and D. Happel on Generalized Tilted Algebras of Type A_n [21]. They prove that the algebras of the bounden quivers satisfying condition A and part B_1, B_2 of condition B (7.2) coincide with the algebras obtained from hereditary algebras of class A_n by a finite sequence of tilts.

b) The numbers $c_{n+1}=\frac{1}{n+1}\binom{2n}{n}$ occuring in 7.3 are well-known in combinatorics as *Catalan numbers.* Three different combinatorial interpretations of them can be found in L. Comtet [22]. In our case we use a fourth interpretation of c_{n+1} as the number of subtrees S of $\bar{S}^n$ which have n vertices and contain $\bar{1}$. In the terminology of Happel-Ringel [23], there is a natural bijection between these S and the isomorphism classes of multiplicity-free tilting modules over the algebra A of the Dynkin-quiver $n \to n-1 \to \ldots \to 2 \to 1$.

c) The numbers a_p and g_p of 7.4 can be computed by means of the following induction-formulae:

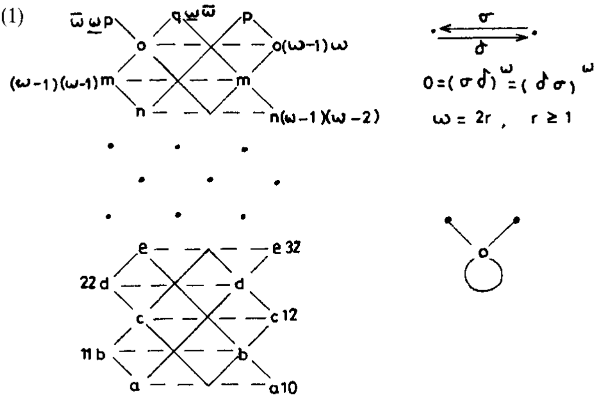
$$g_{p+1}=2g_p+2g_{p-1}+4g_{p-2}+\sum_{i=4}^{p+1}\binom{2i-4}{i-4}\frac{i^2+3i-6}{(i-2)(i-3)}\,g_{p+1-i}$$

$$2a_{2p}=g_{2p}+g_p+\sum_{i=1}^{p}\frac{i-1}{i}\binom{2i-2}{i-1}\,g_{p-i}.$$

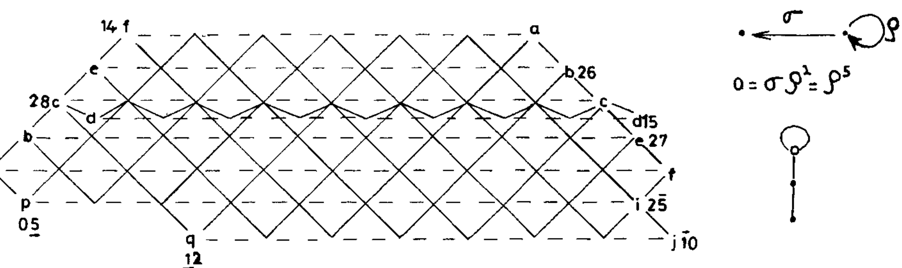
Maximal Algebras with 2 Simple Modules

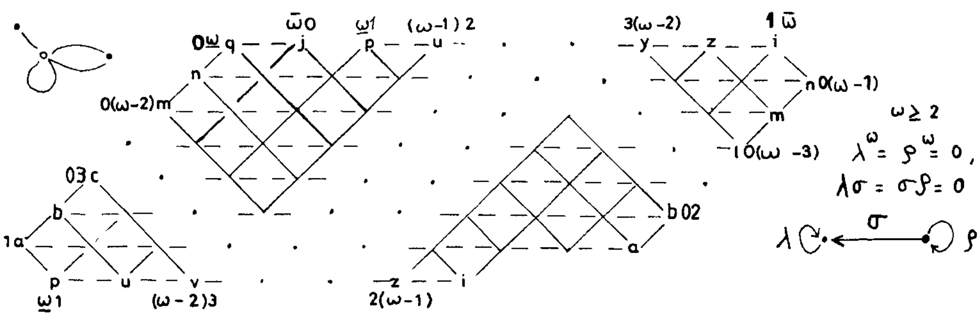
Each representation-finite basic connected finite-dimensional algebra over k (=algebraically closed) whose radical has codimension 2 is isomorphic or anti-isomorphic to a quotient of an algebra of the following list. The listed algebras are defined by quivers and relations. With each algebra A we produce its Auslander-Reiten quiver Γ_A and the associated graph G_A (4.2). In Γ_A we have omitted the tips of the arrows, which are directed from the left to the right. We have to identitify two vertices denoted by the same letter, as well as the arrows connecting such vertices. Although the dimension-vector of an indecomposable module is completely determined by its position in Γ_A, we indicate it in some cases: for instance, the notation e32 means that [32] is the dimension-vector of the module represented by the vertex e; similarly, $1\underline{3}p$ (resp. $j\bar{2}2$) denotes a projective (resp. injective) indecomposable with dimension-vector [13] (resp. [22]) and top-dimension-vector [01] (resp. socle-dimension-vector [10]); the letters $A, B, C\ldots$ stand for the numbers 10, 11, 12.... Using the given dimension-vectors and the additivity of the dimension occuring in a mesh, it is easy to compute all dimension-vectors. With the exception of algebra 14 bis, which is not isomorphic to algebra 14 in characteristic 2, all listed algebras are standard.

(1)



(2)





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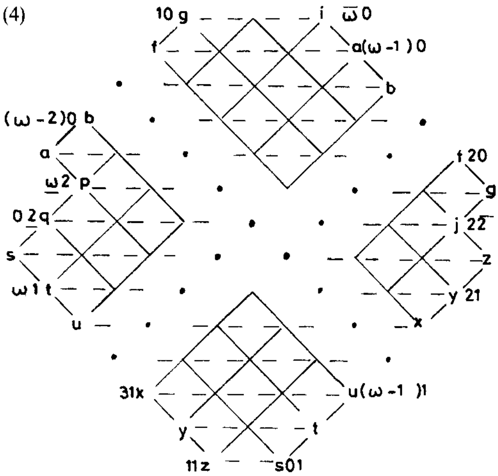
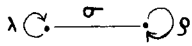
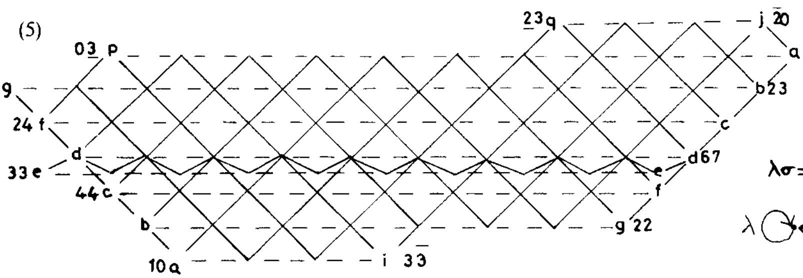


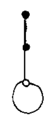


$$0 = \lambda\sigma = \lambda^{\omega} = \rho^2$$

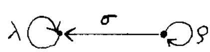
$$\omega \geq 2$$

(5)

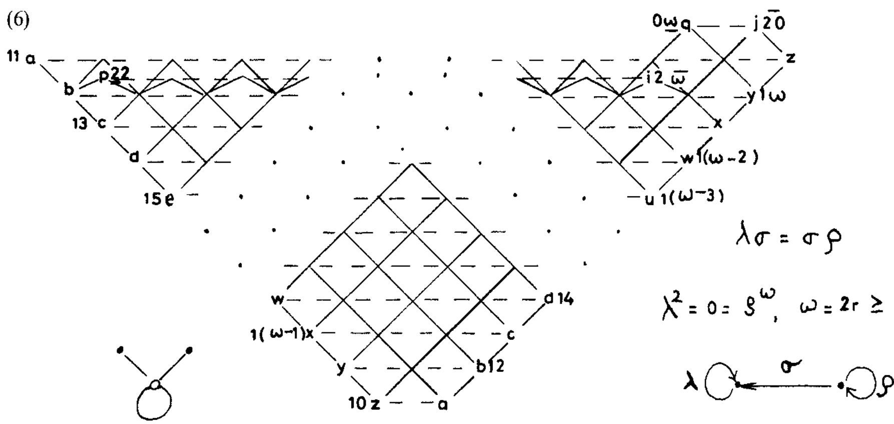




$$\lambda\sigma = \sigma\rho^2, \quad 0 = \lambda^2 = \rho^3$$

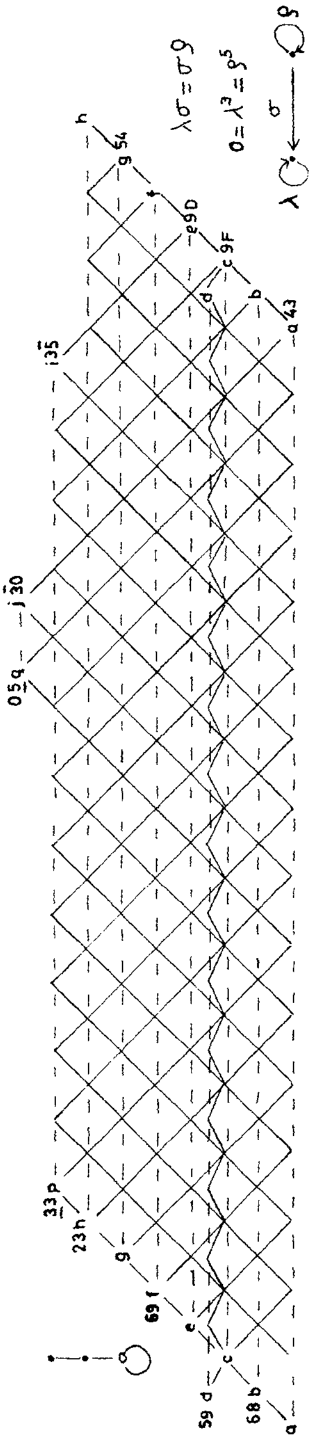


(6)



$$\lambda\sigma = \sigma\rho$$

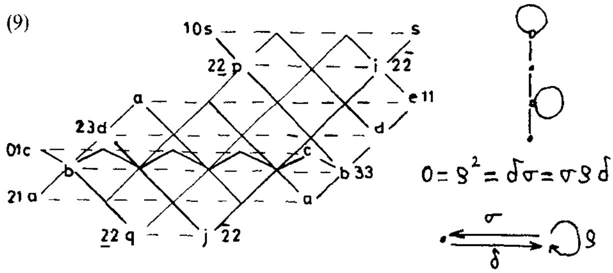
$$\lambda^2 = 0 = \rho^{\omega}, \quad \omega = 2r \geq 2$$


$\lambda\sigma = \sigma\rho$
$0 = \lambda^3 = \rho^5$

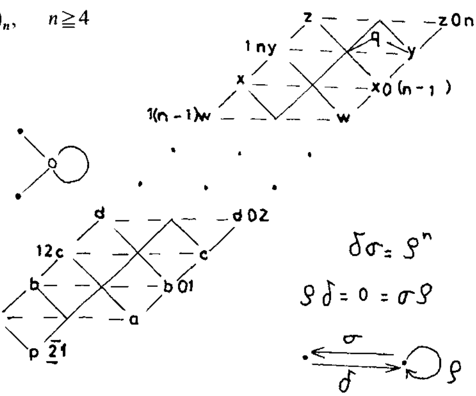
(8)


$0 = \rho^3 = \rho\delta = \sigma\delta = \delta\sigma$

(9)



$(10)_n$, $n\geqq 4$



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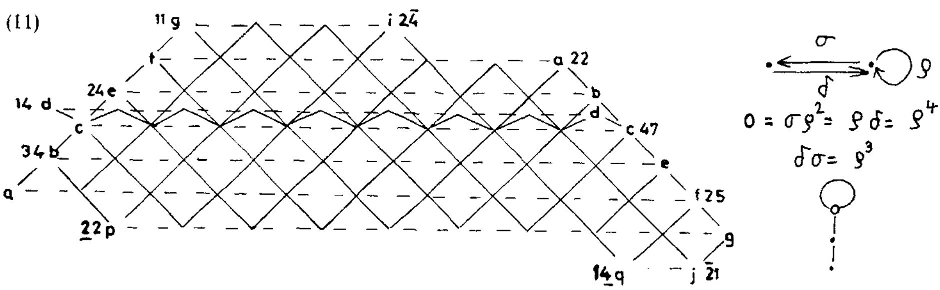
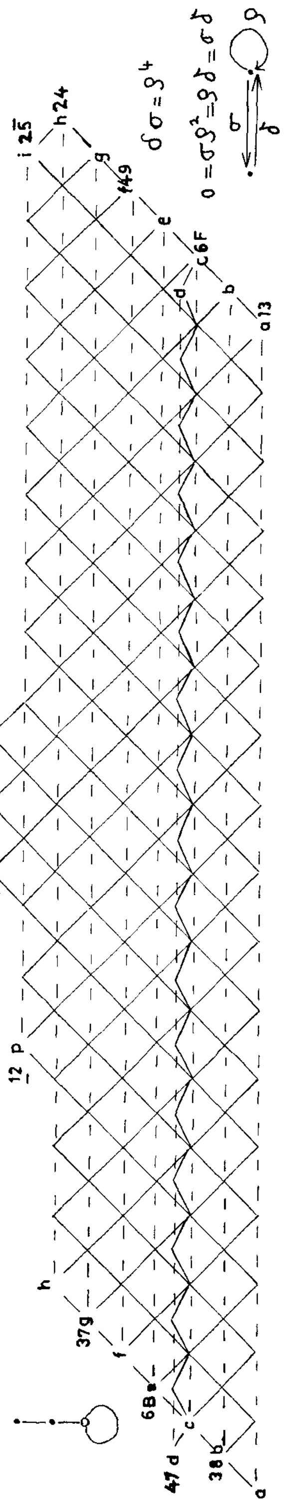
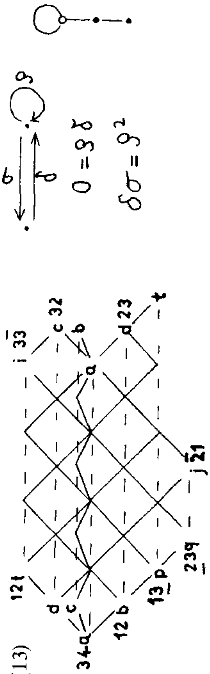


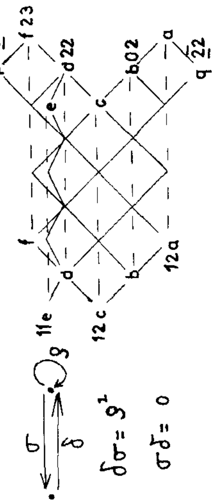
(12)



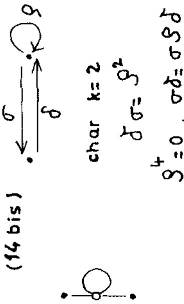
(13)



(14)



(14 bis)

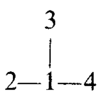


Representation-Finite Gradings of Small Trees

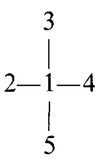
The vertices of a tree are labelled from 1 to n according to the pictures given adhead of each list. In the symbol

$$\begin{matrix} g_1 g_2 g_3 \cdots \\ a_1 a_2 a_3 \ldots, \end{matrix}$$

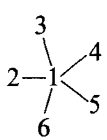
g_i is the grade of the vertex p_i with label i, a_i the number of vertices in the τ-orbit of p_i; as a consequence, $a_1+a_2+a_3\ldots$ is the number of indecomposable representations of the associated algebra. The gradings are ordered lexicographically. We only list the first grading in each orbit under the automorphism group of the considered tree. The letters A and B stand for the numbers 10 and 11 respectively.



0 1 1 1	0 1 1 3	0 1 3 3	0 1 3 5	1 0 0 0	1 0 0 2	1 0 0 4	1 0 2 2	1 0 2 4
3 3 3 3	4 3 3 1	3 1 2 2	4 1 1 1	3 3 3 3	3 3 3 3	3 2 2 1	3 3 3 3	3 4 1 1



0 1 1 1 5	0 1 1 3 5	0 1 1 3 7	0 1 3 3 3	0 1 3 3 5	0 1 3 5 5	0 1 3 5 7
6 5 5 5 1	5 4 4 1 3	5 3 3 1 1	4 1 3 3 3	5 1 3 3 1	4 1 1 2 2	5 1 1 1 1
1 0 0 0 6	1 0 0 2 6	1 0 0 4 4	1 0 0 4 6	1 0 2 2 6	1 0 2 4 4	1 0 2 4 6
4 3 3 3 1	4 3 3 4 1	3 2 2 2 2	4 2 2 1 1	5 3 4 4 1	4 4 1 3 3	4 5 1 1 1



0 1 1 1 5 B	0 1 1 3 5 9	0 1 1 3 7 7	0 1 1 3 7 9	0 1 3 3 3 7	0 1 3 3 5 7
7 5 5 5 1 1	6 4 4 1 4 1	5 3 3 1 2 2	6 3 3 1 1 1	7 1 5 5 5 1	6 1 4 4 1 3
0 1 3 3 5 9	0 1 3 5 5 5	0 1 3 5 5 7	0 1 3 5 7 7	0 1 3 5 7 9	1 0 0 0 6 6
6 1 3 3 1 1	5 1 1 3 3 3	6 1 1 3 3 1	5 1 1 1 2 2	6 1 1 1 1 1	4 3 3 3 2 2
1 0 0 0 6 8	1 0 0 2 6 6	1 0 0 2 6 8	1 0 0 4 4 4	1 0 0 4 4 6	1 0 0 4 6 6
5 3 3 3 1 1	5 3 3 4 3 3	5 3 3 5 1 1	4 2 2 3 3 3	5 2 2 3 3 1	4 2 2 1 2 2
1 0 0 4 6 8	1 0 2 2 6 8	1 0 2 2 6 A	1 0 2 4 4 8	1 0 2 4 6 6	1 0 2 4 6 8
5 2 2 1 1 1	6 3 5 5 1 3	6 3 4 4 1 1	6 4 1 4 4 1	5 5 1 1 3 3	5 6 1 1 1 1

$$\begin{array}{ccccccc} & & 5 & & \\ & & | & & \\ 1 & — 2 — & 3 & — 4 \end{array}$$

0 1 0 1 1	0 1 0 1 3	0 1 0 1 5	0 1 0 3 3	0 1 0 3 5	0 1 2 1 1	0 1 2 1 3
4 4 4 4 4	4 4 5 3 3	3 2 4 2 1	2 3 4 3 3	2 4 4 1 1	4 4 4 4 4	4 4 4 5 3
0 1 2 1 5	0 1 2 3 3	0 1 2 3 5	1 0 1 0 0	1 0 1 0 2	1 0 1 0 4	1 0 1 2 2
3 4 3 2 1	4 4 4 4 4	5 4 3 1 1	4 4 4 4 4	4 4 4 5 3	2 4 4 4 1	4 4 4 4 4
1 0 1 2 4	1 0 1 4 4	1 0 1 4 6	1 0 3 2 2	1 0 3 2 4	1 0 3 2 6	1 0 3 4 4
4 5 4 3 1	2 2 3 2 2	2 2 4 1 1	1 4 3 3 3	1 4 3 3 3	1 3 3 2 1	1 4 3 3 3
1 0 3 4 6	1 2 1 0 0	1 2 1 0 2	1 2 1 0 4	1 2 1 0 6	2 1 0 1 1	2 1 0 1 3
1 5 3 1 1	4 4 4 4 4	4 4 4 5 3	2 3 4 4 3	2 2 4 3 1	4 4 4 4 4	4 4 5 3 3
2 1 0 1 5	2 1 0 3 3	2 1 0 3 5	2 1 0 5 5	2 1 0 5 7	2 3 0 1 1	2 3 0 1 3
2 2 4 4 1	1 4 4 3 3	1 4 5 3 1	1 2 4 2 2	1 2 5 1 1	2 3 5 4 4	3 2 4 1 2
2 3 0 1 5	3 0 1 0 0	3 0 1 0 2	3 0 1 0 4	3 0 1 0 6	3 0 1 2 2	3 0 1 2 4
2 3 4 1 1	2 5 4 4 4	2 5 4 5 3	1 4 3 2 2	1 3 4 2 1	2 5 4 4 4	3 4 3 1 2
3 0 1 2 6	3 2 1 0 0	3 2 1 0 2	3 2 1 0 4	3 2 1 0 6	3 4 1 0 0	3 4 1 0 2
2 3 4 1 1	4 4 4 4 4	4 4 4 5 3	1 4 4 4 3	1 2 4 5 1	2 2 3 2 2	2 2 4 4 1
4 1 0 1 1	4 1 0 1 3	4 3 0 1 1	4 3 0 1 3	4 3 0 1 5	4 3 0 1 7	4 5 0 1 3
2 3 5 4 4	2 4 4 4 1	1 4 5 4 4	2 2 4 1 3	1 3 4 1 2	1 2 5 1 1	2 2 4 1 1
5 0 1 0 0	5 0 1 0 2	5 0 1 2 2	5 0 1 2 4	5 2 1 0 0	5 2 1 0 2	5 4 1 0 0
1 4 4 3 3	1 4 4 3 4	1 4 5 4 4	2 4 4 1 1	2 3 4 3 3	2 3 4 3 4	1 2 4 2 2
5 4 1 0 2	6 1 0 1 1	6 1 0 1 3	6 3 0 1 3	6 5 0 1 3	7 0 1 2 4	7 2 1 0 0
1 2 4 5 1	1 4 3 3 3	1 4 5 3 1	1 3 3 1 2	1 2 5 1 1	1 5 3 1 1	1 4 3 3 3
7 2 1 0 2						
1 4 3 3 3						

$$\begin{array}{ccccc} & & 6 & & \\ & & | & & \\ 1 & — 2 — & 3 & — 4 — 5 \end{array}$$

0 1 0 1 0 1	0 1 0 1 0 3	0 1 0 1 0 5	0 1 0 1 2 1	0 1 0 1 2 3	0 1 0 1 2 5
6 6 6 6 6 6	6 6 7 6 6 4	3 4 5 4 3 1	7 6 6 6 5 6	7 6 7 6 5 4	5 5 5 4 2 1
0 1 0 1 4 1	0 1 0 1 4 3	0 1 0 1 4 5	0 1 0 1 4 7	0 1 0 1 6 1	0 1 0 1 6 3
6 5 7 6 4 6	6 6 6 5 2 5	3 2 4 4 1 2	3 2 5 3 1 1	4 5 6 4 2 5	4 5 6 6 1 4
0 1 0 1 8 1	0 1 0 3 2 1	0 1 0 3 2 3	0 1 0 3 2 5	0 1 0 3 2 7	0 1 0 3 4 1
4 4 4 5 1 4	5 6 7 5 5 6	2 5 5 4 4 3	2 4 5 3 2 3	2 3 5 2 2 1	5 7 7 5 4 6
0 1 0 3 4 3	0 1 0 3 4 5	0 1 0 3 4 7	0 1 0 3 6 1	0 1 0 3 6 3	0 1 0 3 8 1
2 5 5 4 4 3	2 4 5 4 1 3	2 5 5 2 1 1	4 5 5 4 2 3	2 3 5 3 2 4	4 4 6 4 1 3
0 1 0 3 8 3	0 1 0 5 4 1	0 1 0 5 4 3	0 1 0 5 6 1	0 1 0 5 6 3	0 1 2 1 0 1
2 3 4 4 1 3	3 2 4 2 2 2	2 4 5 2 2 1	3 2 5 2 1 2	2 5 5 2 1 1	6 6 6 6 6 6
0 1 2 1 0 3	0 1 2 1 0 5	0 1 2 1 0 7	0 1 2 1 2 1	0 1 2 1 2 3	0 1 2 1 2 5
6 6 6 6 6 6	3 4 4 4 3 3	3 3 4 3 3 1	7 6 6 6 5 6	7 6 6 6 5 6	5 5 5 4 2 3

012127	012141	012143	012145	012147	012161
434321	676646	676646	344513	354311	455625
012163	012181	012183	0121A3	012321	012323
455644	445514	445523	444613	776556	776556
012325	012327	012341	012343	012345	012347
555523	454221	876546	876546	555514	654211
012361	012363	012381	012383	0123A3	012541
465525	465544	445416	445425	444514	344222
012543	012561	012563	101010	101012	101014
554221	354212	654211	666666	666666	466644
101016	101030	101032	101034	101036	101050
334331	576656	576656	245543	254321	566625
101052	101054	101070	101072	101074	101092
566744	254521	445514	445523	245511	444613
101210	101212	101214	101216	101230	101232
676566	676566	466664	354231	776556	776556
101234	101236	101250	101252	101254	101270
476654	554221	666525	666644	555421	445416
101272	101274	101292	101430	101432	101434
445425	465411	444514	255325	566425	224232
101436	101450	101452	101454	101456	101458
224321	255415	566416	224223	224312	225211
101474	101654	101674	103012	103014	103016
223312	224221	225211	143413	143413	133311
103212	103214	103216	103232	103234	103236
154444	164443	133431	154444	164443	154421
103252	103254	103256	103258	103272	103274
154524	164523	133412	134311	144413	144414
103432	103434	103436	103438	103452	103454
154444	164443	154323	144221	154444	164443
103456	103458	103472	103474	103492	103494
154413	164211	144323	144324	143413	143413
103652	103654	103672	103674	121030	121032
133222	154221	134212	164211	566756	566756
121034	121036	121050	121052	121054	121056
255543	245521	556627	557746	255623	224512
121058	121070	121072	121074	121092	121210
225411	455514	455523	235514	444613	666666
121230	121250	121270	121430	121450	121470
766656	656527	456416	245445	245445	235324

121490	121650	121670	210121	210123	210125
234414	224223	225213	666666	667664	456541
210141	210143	210145	210147	210161	210163
557656	566625	224432	225321	566425	567614
210165	210181	210185	210321	210323	210325
225421	444514	224511	467566	155444	146443
210327	210341	210343	210345	210347	210361
135231	477556	155444	146443	155221	555423
210363	210365	210381	210383	210385	210541
156324	155421	446413	144413	146411	225224
210543	210545	210547	210561	210563	210565
156324	125232	125321	225215	156414	125223
210567	210569	210585	210765	210785	230141
125312	126211	124312	125221	126211	246545
230161	230181	230321	230341	230361	230541
246625	236514	334331	434321	325311	235221
230561	301030	301032	301034	301036	301050
245211	476746	476746	154513	155511	255626
301052	301070	301072	301092	301230	301232
255645	265514	265523	254613	676646	676646
301234	301236	301250	301252	301270	301272
455514	465411	265526	265545	266416	266425
301292	301430	301432	301450	301452	301470
254614	144232	354231	154222	454221	143312
301472	301650	301652	301670	301672	321050
343311	134222	234221	135212	235211	456727
321052	321054	321056	321058	321070	321072
457756	155624	124532	125421	555514	555523
321074	321076	321092	321096	321230	321250
156514	125521	444613	124611	666666	556627
321270	321430	321450	321470	321490	321650
566416	145446	145446	155324	144414	125225
321670	341052	341072	410141	410143	410161
125216	234521	225511	466644	345431	235525
410163	410181	410183	410341	430161	430181
255421	236514	244511	456416	146626	156514
430341	430361	430381	430541	430561	430581
334331	225321	224411	135231	145221	134311
430761	430781	501050	501052	501250	501252
125221	126211	154513	165614	154433	166544

5 0 1 2 7 0	5 0 1 2 7 2	5 0 1 2 9 0	5 0 1 2 9 2	5 0 1 4 5 2	5 2 1 2 5 0
1 4 5 4 2 3	1 4 5 5 2 5	1 4 4 5 1 3	1 4 6 5 1 4	3 5 4 3 1 1	3 4 4 4 3 3
5 2 1 2 7 0	5 2 1 2 9 0	5 4 1 0 7 2	5 4 1 0 9 2	6 1 0 1 6 1	6 1 0 1 6 3
2 3 5 4 2 3	2 3 4 5 1 3	1 2 5 5 2 1	1 2 4 6 1 1	1 4 3 4 1 3	1 5 5 5 1 1
6 3 0 3 6 1	7 2 1 2 7 0				
1 3 3 3 1 1	1 4 3 4 1 3				

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