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Finite Dimensional Approximation of Nonlinear Problems

Part. II: Limit Points

F. Brezzi[1]*, J. Rappaz[2]**, and P.A. Raviart[3]

[1] Istituto di Mathematica Applicata and Istituto di Analisi Numerica del C.N.R., Università di Pavia - 27100 Pavia, Italy
[2] Centre de Mathématiques Appliquées (ERA/CNRS 747), Ecole Polytechnique 91128 Palaiseau Cedex, France
[3] Analyse Numérique, Université Pierre et Marie Curie, 4, Place Jussieu, 75230 Paris Cedex 05, France

Summary. We continue here the study of a general method of approximation of nonlinear equations in a Banach space yet considered in [2]. In this paper, we give fairly general approximation results for the solutions in a neighborhood of a simple limit point. We then apply the previous analysis to the study of Galerkin approximations for a class of variationally posed nonlinear problems and to a mixed finite element method for the Navier-Stokes equations.

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1. Introduction

Consider nonlinear problems of the form:

$$F(\lambda, u)=0, \tag{1.1}$$

where F is a sufficiently smooth function from $\mathbb{R}\times V$ into V for some Banach space V. In the first paper of this series [2], we have studied the numerical approximation of branches $\{(\lambda, u(\lambda));\ \lambda\in\Lambda\}$ of nonsingular solutions of problem (1.1), where $\Lambda\subset\mathbb{R}$ is a compact interval. We now turn to the approximation of singular solutions such as limit points and bifurcation points.

In this paper, we shall be concerned with the approximation of the solutions of (1.1) in a neighborhood of a simple limit point (λ_0, u_0) of F, i.e. a point

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$(\lambda_0, u_0) \in \mathbb{R} \times V$ which satisfies the following properties:

$$F(\lambda_0, u_0) = 0; \tag{1.2}$$

$$D_u F(\lambda_0, u_0) \quad \text{is singular and} \quad \dim \operatorname{Ker}(D_u F(\lambda_0, u_0)) = \operatorname{codim} \operatorname{Range}(D_u F(\lambda_0, u_0)) = 1; \tag{1.3}$$

$$D_\lambda F(\lambda_0, u_0) \notin \operatorname{Range}(D_u F(\lambda_0, u_0)). \tag{1.4}$$

The third paper of this series will be devoted to the study of simple bifurcation points in the general case and in the presence of symmetry properties.

As in [2], we shall give a fairly general analysis in order to include various approximation schemes such as conforming finite element methods or mixed finite element methods. Moreover, our results can be extended so as to cover the cases of finite element methods with numerical integration and finite difference methods; in that direction, we refer to [11] which improves and generalizes the difference results of [13].

An outline of the paper is as follows. In Sect. 2, we consider a simple singular point (λ_0, u_0) of F, i.e. a point which satisfies the conditions (1.2) and (1.3), and we derive the corresponding bifurcation equation. Then we introduce a general method of approximation of problem (1.1) and we establish various results concerning the approximation of this bifurcation equation. These results will be constantly used in the subsequent sections and the part 3 of this series of papers. In Sect. 3, we assume that (λ_0, u_0) is a simple limit point of F so that there exists a unique branch $\{(\lambda(\alpha), u(\alpha)); |\alpha| \leqq \alpha_0\}$ of solutions of (1.1) passing through the point (λ_0, u_0). We then show that the approximate problem has a unique branch $\{(\lambda_h(\alpha), u_h(\alpha)); |\alpha| \leqq \alpha_0\}$ of solutions in a neighborhood of (λ_0, u_0) and we give estimates of $|\lambda_h(\alpha) - \lambda(\alpha)| + \|u(\alpha) - u_h(\alpha)\|_V$ which are uniform in the parameter α. In Sect. 4, we suppose that (λ_0, u_0) is indeed a nondegenerate turning point; we prove that the approximate problem has indeed a unique turning point (λ_h^0, u_h^0) in a neighborhood of (λ_0, u_0) and we derive estimates of $|\lambda_h^0 - \lambda_0|$ and $\|u_h^0 - u_0\|_V$. In Sect. 5, we apply the above results to the Galerkin approximations of nonlinear variational problems. We then obtain generalizations of the results of [7–9]; see also [3]. Finally, Sect 6 is concerned with the analysis of a mixed finite element method for the Navier-Stokes equations in a stream function-vorticity formulation yet considered in [2]; for a related work, see [12]. Let us point out that the techniques developed in Sect. 6 may be adapted for analyzing various mixed finite element approximations of other nonlinear problems.

For the numerical computation of the turning points of the discretized problems, see [6] and the references therein.

2. General Analysis of Simple Singular Points

2.1. A Preliminary Result

Let us first state an useful version of the abstract results of [2] concerning the approximation of branches of nonsingular solutions of nonlinear problems. This preliminary result will be of constant use in all the sequel of this paper.

Let X, Y, Z be three (real) Banach spaces and Φ be a C^r mapping ($r \geq 2$) from $B \times Y$ into Z where B is a bounded open subset of X. We shall denote by $D\Phi(x, y) \in \mathscr{L}(X \times Y; Z)$ the total derivative of Φ at the point (x, y) and by $D_x\Phi(x, y) \in \mathscr{L}(X; Z)$ and $D_y\Phi(x, y) \in \mathscr{L}(Y; Z)$ the corresponding partial derivatives. We shall also denote by $D^l\Phi(x, y) \in \mathscr{L}_l(X \times Y; Z)$, $2 \leq l \leq r$, the l-th total derivative of Φ where $\mathscr{L}_l(X \times Y; Z)$ is the space of all continuous l-linear mappings from $X \times Y$ into Z.

Theorem 1. *We assume that the mapping $D^r\Phi$ is bounded on all bounded subsets of $B \times Y$. Let g be a bounded C^r function from B into Y such that, for all $x \in B$, the two following properties hold:*

$$\Phi(x, g(x)) = 0; \tag{2.1}$$

$$\begin{gathered} D_y\Phi(x, g(x)) \quad \text{is an isomorphism from } Y \text{ onto } Z \text{ with} \\ \|D_y\Phi(x, g(x))^{-1}\|_{\mathscr{L}(Z;Y)} \leq c. \end{gathered} \tag{2.2}$$

For each value of a parameter $h > 0$, let Φ_h be a C^r mapping from $B \times Y$ into Z such that

$$\text{(i)} \quad \lim_{h \to 0} \sup_{(x,y) \in \mathscr{B}} \|D^l\Phi(x, y) - D^l\Phi_h(x, y)\|_{\mathscr{L}_l(X \times Y; Z)} = 0, \; 0 \leq l \leq r-1, \tag{2.3}$$

$$\text{(ii)} \quad \sup_{(x,y) \in \mathscr{B}} \|D^r\Phi_h\|_{\mathscr{L}_r(X \times Y; Z} \leq \tilde{c} (\tilde{c} \text{ independent of } h) \tag{2.3}$$

for all bounded subset $\mathscr{B} \subset B \times Y$.

Then there exist two constants a and $h_0 > 0$ and, for $h \leq h_0$, a unique C^r mapping g_h from B into Y such that we have for all $x \in B$

$$\begin{gathered} \Phi_h(x, g_h(x)) = 0, \\ \|g_h(x) - g(x)\|_y \leq a. \end{gathered} \tag{2.4}$$

Moreover, we have for all $x, x^* \in B$ and all integer m with $0 \leq m \leq r-1$ the following error bound

$$\begin{aligned} \text{(i)} \quad & \|D^m g_h(x^*) - D^m g(x)\|_{\mathscr{L}_m(X;Y)} \\ & \leq K\left\{\|x^* - x\|_X + \sum_{l=0}^{m} \left\|\frac{d^l}{dx^l}(\Phi(x, g(x)) - \Phi_h(x, g(x)))\right\|_{\mathscr{L}_l(X;Z)}\right\}, \end{aligned} \tag{2.5}$$

$$\text{(ii)} \quad \sup_{x \in B} \|D^r g_h\|_{\mathscr{L}_r(X;Y)} \leq K, \tag{2.5}$$

where $D^m g_h$ and $D^m g$ are the m-th derivatives of g_h and g respectively and $K > 0$ is a constant independent of h.

Proof. Let us first check that, for $h \leq h_0$ small enough and $x \in B$, $D_y\Phi_h(x, g(x))$ is an isomorphism from Y into Z. In fact, we may write

$$D_y\Phi_h(x, g(x)) = D_y\Phi(x, g(x))(I + A_h(x)),$$

where

$$A_h(x) = D_y\Phi(x, g(x))^{-1}(D_y\Phi_h(x, g(x)) - D_y\Phi(x, g(x))).$$

Using (2.2) and (2.3), we get for $h \leqq h_0$

$$\sup_{x \in B} \|A_h(x)\|_{\mathscr{L}(Y;Y)} \leqq \tfrac{1}{2}.$$

Hence, $I + A_h(x)$ is an isomorphism of Y and we have for all $x \in B$

$$\|D_y \Phi_h(x, g(x))^{-1}\|_{\mathscr{L}(Z;Y)} \leqq \frac{1}{1 - \|A_h(x)\|_{\mathscr{L}(Y;Y)}} \|D_y \Phi(x, g(x, g(x))^{-1}\|_{\mathscr{L}(Z;Y)} \leqq 2c^1.$$

Next, the mapping $D^l \Phi$ is bounded on all bounded subsets of $B \times Y$, $0 \leqq l \leqq r$. Using (2.3), we obtain:

(i) $\sup_{x \in B} \|D_x \Phi_h(x, g(x)\|_{\mathscr{L}(X;Z)} \leqq c_1$

(ii) each mapping $D^l \Phi_h$, $0 \leqq l \leqq r-1$, is Lipschitz continuous on all bounded subsets of $B \times Y$ *uniformly in h*.

Since by (2.1) and (2.3)

$$\sup_{x \in B} \|\Phi_h(x, g(x))\|_Z = \sup_{x \in B} \|\Phi_h(x, g(x)) - \Phi(x, g(x))\|_Z \to 0$$

as $h \to 0$, we may apply Theorems 1 and 2 of [2]: there exists a unique C^r function g_h from B into Y such that (2.4) holds. Moreover, we have for all $x, x^* \in B$ and all integer m with $0 \leqq m \leqq r-1$

$$\|D^m g_h(x) - D^m g(x)\|_{\mathscr{L}_m(X;Y)} \leqq K \left\{ \|x^* - x\|_X + \sum_{l=0}^{m} \|\Phi_h^{(l)}(x, g(x), Dg(x), \ldots, D^l g(x))\|_{\mathscr{L}_l(X;Z)} \right\},$$

where the mappings $\Phi_h^{(l)}$: $(x, y, y^{(1)}, \ldots, y^{(l)}) \in B \times Y \times \mathscr{L}(X;Y) \times \ldots \times \mathscr{L}_l(X;Y) \to \Phi_h^{(l)}(x, y, y^{(1)}, \ldots, y^{(l)}) \in \mathscr{L}_l(X;Z)$ are defined by induction

$$\Phi_h^{(0)}(x, y) = \Phi_h(x, y),$$

$$\Phi_h^{(l+1)}(x, y, y^{(1)}, \ldots, y^{(l+1)}) = D_x \Phi_h^{(l)}(x, y, y^{(1)}, \ldots, y^{(l)}) + \sum_{i=0}^{l} D_{y^{(i)}} \Phi_h^{(l)}(x, y, y^{(1)}, \ldots, y^{(l)}), y^{(i+1)}; \; y^{(0)} = y.$$

Now, we notice that

$$\frac{d^l}{dx^l} \Phi_h(x, g(x)) = \Phi_h^{(l)}(x, g(x), Dg(x), \ldots, D^l g(x))$$

and

$$\frac{d^l}{dx^l} \Phi(x, g(x)) = 0,$$

so that the estimate (2.5) (i) follows immediately. The property (2.5) (ii) is obvious. ∎

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[1] Here and in all the sequel, $c, c_1, c_2, \ldots, c_i, \ldots$, will denote various positive constants independent of h

2.2. *The Continuous Case*

Let V and W be two (real) Banach spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively. We introduce a C^p mapping ($p \geqq 1$) $G: \mathbb{R} \times V \to W$ and a linear *compact* operator $T \in \mathscr{L}(W; V)$. We set:

$$F(\lambda, u) = u + TG(\lambda, u). \tag{2.6}$$

We assume that $(\lambda_0, u_0) \in \mathbb{R} \times V$ is a *simple singular point* of F in the sense that

$$\begin{aligned} &\text{(i) } F^0 \equiv F(\lambda_0, u_0) = 0,\\ &\text{(ii) } D_u F^0 \equiv D_u F(\lambda_0, u_0) = I + TD_u G(\lambda_0, u_0) \in \mathscr{L}(V; V)\\ &\quad \text{is singular and } -1 \text{ is an eigenvalue of the compact}\\ &\quad \text{operator } TD_u G(\lambda_0, u_0) \text{ with algebraic multiplicity 1.} \end{aligned} \tag{2.7}$$

The problem is to solve the equation

$$F(\lambda, u) = 0 \tag{2.8}$$

in the neighborhood of the singular point (λ_0, u_0).

Let us denote by V' the dual space of V and by $\langle \cdot, \cdot \rangle$ the duality pairing between the spaces V and V'. Then, as a consequence of (2.7) (ii) and the classical theory on compact operators, we have

Lemma 1. *There exist $\varphi_0 \in V$ and $\varphi_0^* \in V'$ such that on the one hand*

$$\begin{aligned} &D_u F^0 \cdot \varphi_0 = 0, \ \|\varphi_0\|_V = 1,\\ &V_1 \equiv \operatorname{Ker}(D_u F^0) = \mathbb{R}\varphi_0, \end{aligned} \tag{2.9}$$

and on the other hand

$$\begin{aligned} &(D_u F^0)^* \cdot \varphi_0^* = 0, \ \langle \varphi, \varphi_0^* \rangle = 1,\\ &V_2 \equiv \operatorname{Range}(D_u F^0) = \{v \in V; \langle v, \varphi_0^* \rangle = 0\}. \end{aligned} \tag{2.10}$$

Moreover, we have

$$V = V_1 \oplus V_2$$

and $D_u F^0$ is an isomorphism of V_2. ■

We shall denote by $L = (D_u F^0|_{V_2})^{-1}$ the inverse isomorphism of $D_u F^0|_{V_2}$. Let us now define the projection operator $Q: V \to V_2$ by

$$Qv = v - \langle v, \varphi_0^* \rangle \varphi_0, \ v \in V. \tag{2.11}$$

Then the Eq. (2.8) is equivalent to the system

$$\begin{aligned} &QF(\lambda, u) = 0,\\ &(I - Q) F(\lambda, u) = 0. \end{aligned} \tag{2.12}$$

Given $u \in V$, there exists a unique decomposition of the form

$$u = u_0 + \alpha\varphi_0 + v, \ \alpha \in \mathbb{R}, \ v \in V_2. \tag{2.13}$$

Setting:

$$\lambda = \lambda_0 + \xi, \tag{2.14}$$

the first equation of (2.12) becomes

$$\mathscr{F}(\xi, \alpha, v) = 0, \tag{2.15}$$

where the C^p function $\mathscr{F}: \mathbb{R}^2 \times V_2 \to V_2$ is defined by

$$\mathscr{F}(\xi, \alpha, v) = QF(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v). \tag{2.16}$$

By using (2.7) (i) and Lemma 1, we find that $\mathscr{F}(0,0,0) = 0$ and $D_v\mathscr{F}(0,0,0) = D_u F^0|_{V_2}$ is an isomorphism of V_2. Hence, by the implicit function theorem, we get

Lemma 2. *Assume the hypothesis* (2.7). *Then there exist two positive constants* ξ_0, α_0 *and a unique* C^p *mapping* $v: [-\xi_0, \xi_0] \times [-\alpha_0, \alpha_0] \to V_2$ *such that*

$$\begin{aligned} &\mathscr{F}(\xi, \alpha, v(\xi, \alpha)) = 0, \\ &v(0,0) = 0. \end{aligned} \tag{2.17}$$

Hence, solving the Eq. (2.3) in a neighborhood of the singular point (λ_0, u_0) amounts to solve the *bifurcation equation* (see [1] for instance for a similar approach)

$$(I - Q)\,F(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v(\xi, \alpha)) = 0,$$

i.e. the equation

$$f(\xi, \alpha) \equiv \langle F(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v(\xi, \alpha)), \varphi_0^* \rangle = 0, \tag{2.18}$$

in a neighborhood of the origin.

Elementary calculations show that:

$$f(0,0) = \frac{\partial f}{\partial \alpha}(0,0) = 0. \tag{2.19}$$

In Sect. 3, we shall discuss the case where $\dfrac{\partial f}{\partial \xi}(0,0) \neq 0$.

2.3. The Approximation

Let us next study the finite-dimensional approximation of Eq. (2.8) in the neighborhood of the simple singular point (λ_0, u_0). For each value of a real parameter $h > 0$ which will tend to zero, we introduce a finite-dimensional subspace V_h of the space V and an operator $T_h \in \mathscr{L}(W; V_h)$. We set:

$$F_h(\lambda, u) = u + T_h G(\lambda, u), \; \lambda \in \mathbb{R}, \; u \in V. \tag{2.20}$$

The approximate problem consists in solving the equation:

$$F_h(\lambda, u_h) = 0, \tag{2.21}$$

i.e. in finding pairs $(\lambda, u_h)\in \mathbb{R} \times V_h$ solutions of (2.21). Let us notice that we can equivalently solve the equation (2.21) in $\mathbb{R} \times V$.

As in the previous subsection, the equation (2.21) is equivalent to the system

$$\begin{aligned} &QF_h(\lambda, u_h)=0,\\ &(I-Q)\,F_h(\lambda, u_h)=\langle F_h(\lambda, u_h), \varphi_0^*\rangle\, \varphi_0=0. \end{aligned} \tag{2.22}$$

Setting

$$\begin{aligned} &\lambda=\lambda_0+\xi,\\ &u_h=u_0+\alpha\varphi_0+v_h,\ \alpha\in\mathbb{R},\ v_h\in V_2, \end{aligned} \tag{2.23}$$

the first equation of (2.22) becomes

$$\mathscr{F}_h(\xi, \alpha, v_h)=0, \tag{2.24}$$

where the C^p function $\mathscr{F}_h\colon \mathbb{R}^2 \times V_2 \to V_2$ is defined by

$$\mathscr{F}_h(\xi, \alpha, v)=QF_h(\lambda_0+\xi, u_0+\alpha\varphi_0+v). \tag{2.25}$$

We introduce the C^p mapping $J\colon [-\xi_0, \xi_0]\times[-\alpha_0, \alpha_0]\to W$ defined by

$$J(\xi, \alpha)=G(\lambda_0+\xi, u_0+\alpha\varphi_0+v(\xi, \alpha)). \tag{2.26}$$

Theorem 2. *Assume the hypothesis* (2.7). *Assume in addition that* G *is a* C^p *mapping* ($p\geqq 2$) *and the mapping* D^pG *is bounded on all bounded subsets of* $\mathbb{R}\times V$ *and*

$$\lim_{h\to 0}\|T-T_h\|_{\mathscr{L}(W;V)}=0. \tag{2.27}$$

Then, there exist three positive constants ξ_0, α_0, a *and, for* $h\leqq h_0$ *small enough, a unique* C^p *mapping* $v_h\colon [-\xi_0, \xi_0]\times[-\alpha_0, \alpha_0]\to V_2$ *such that*

$$\begin{aligned} &\mathscr{F}_h(\xi, \alpha, v_h(\xi, \alpha))=0,\\ &\|v_h(\xi, \alpha)-v(\xi, \alpha)\|_V\leqq a,\ |\xi|\leqq\xi_0,\ |\alpha|\leqq\alpha_0 \end{aligned} \tag{2.28}$$

Moreover, there exists a constant $K>0$ *independent of* h *such that, for all* $\xi, \xi^*\in[-\xi_0, \xi_0]$, *all* α, $\alpha^*\in[-\alpha_0, \alpha_0]$ *and all integer* m *with* $0\leqq m\leqq p-1$, *the following error estimate holds:*

$$\begin{aligned} \text{(i)}\quad &\|D^m v_h(\xi^*, \alpha^*)-D^m v(\xi, \alpha)\|_{\mathscr{L}_m(\mathbb{R}^2;V)}\\ &\leqq K\left\{|\xi^*-\xi|+|\alpha^*-\alpha|+\sum_{l=0}^{m}\|(T-T_h)\,D^l J(\xi, \alpha)\|_{\mathscr{L}_l(\mathbb{R}^2;V)}\right\}, \end{aligned} \tag{2.29}$$

$$\text{(ii)}\quad \|D^p v_h(\xi^*, \alpha^*)\|_{\mathscr{L}_p(\mathbb{R}^2;V)}\leqq K. \tag{2.29}$$

Proof. Since $D_v\mathscr{F}(0,0,0)$ is an isomorphism of V_2, we may suppose that in Lemma 2 the constants ξ_0 and α_0 are chosen in such a way that

$$\|D_v\mathscr{F}(\xi, \alpha, v(\xi, \alpha))^{-1}\|_{\mathscr{L}(V_2;V_2)}\leqq c,\ |\xi|\leqq\xi_0,\ |\alpha|\leqq\alpha_0.$$

On the other hand, we have

$$\mathscr{F}(\xi,\alpha,v)-\mathscr{F}_h(\xi,\alpha,v)=Q(T-T_h)\,G(\lambda_0+\xi,u_0+\alpha\varphi_0+v). \tag{2.30}$$

Hence, it follows from (2.27) that $D^l\mathscr{F}_h\to D^l\mathscr{F}$ in $\mathscr{L}_l(\mathbb{R}^2\times V_2;V_2)$, $0\leqq l\leqq p$, uniformly on every bounded subset of $[-\xi_0,\xi_0]\times[-\alpha_0,\alpha_0]\times V_2$. Therefore, we may apply Theorem 1: there exists a unique C^p function $v_h\colon[-\xi_0,\xi_0]\times[-\alpha_0,\alpha_0]\to V_2$ which satisfies (2.28).

Moreover, using (2.26) and (2.30), we obtain for $\xi,\xi^*\in[-\xi_0,\xi_0]$, $\alpha,\alpha^*\in[-\alpha_0,\alpha_0]$ and $0\leqq m\leqq p-1$

$$\begin{aligned}&\|D^m v_h(\xi^*,\alpha^*)-D^m v(\xi,\alpha)\|_{\mathscr{L}_m(\mathbb{R}^2;V)}\\&\quad\leqq K\left\{|\xi^*-\xi|+|\alpha^*-\alpha|+\sum_{l=0}^{m}\|Q(T-T_h)\,D^l J(\xi,\alpha)\|_{\mathscr{L}_l(\mathbb{R}^2;V)}\right\},\end{aligned}$$

from which (2.29) follows. ■

In fact, we shall also need a more specific version of the previous result. Let $t\to(\xi(t),\alpha(t))$ be a pair of C^p real functions defined for $|t|\leqq t_0$ and let $t\to(\xi_h^*(t),\alpha_h^*(t))$ be a family of pairs of C^p real functions defined for $|t|\leqq t_0$ which satisfy for $h\leqq h_0$

$$\sup_{|t|\leqq t_0}|\xi(t)|\leqq\xi_0,\qquad \sup_{|t|\leqq t_0}|\xi_h^*(t)|\leqq\xi_0,$$
$$\sup_{|t|\leqq t_0}|\alpha(t)|\leqq\alpha_0,\qquad \sup_{|t|\leqq t_0}|\alpha_h^*(t)|\leqq\alpha_0.$$

Lemma 3. *Assume the hypotheses of Theorem 2. Assume in addition that for* $0\leqq m\leqq p-1$

(i) $$\lim_{h\to 0}\sup_{|t|\leqq t_0}\left(\left|\frac{d^m}{dt^m}(\xi_h^*(t)-\xi(t))\right|+\left|\frac{d^m}{dt^m}(\alpha_h^*(t)-\alpha(t))\right|\right)=0 \tag{2.31}$$

(ii) $$\sup_{|t|\leqq t_0}\left(\left|\frac{d^p}{dt^p}\xi_h^*(t)\right|+\left|\frac{d^p}{dt^p}\alpha_h^*(t)\right|\right)=c\quad\text{independent of } h. \tag{2.31}$$

Then, we get for all $t\in[-t_0,t_0]$ *and all integer* m *with* $0\leqq m\leqq p-1$

$$\begin{aligned}&\left\|\frac{d^m}{dt^m}\big(v_h(\xi_h^*(t),\alpha_h^*(t))-v(\xi(t),\alpha(t))\big)\right\|_V\\&\quad\leqq K\sum_{l=0}^{m}\left\{\left|\frac{d^l}{dt^l}(\xi_h^*(t)-\xi(t))\right|+\left|\frac{d^l}{dt^l}(\alpha_h^*(t)-\alpha(t))\right|\right.\\&\qquad\left.+\left\|(T-T_h)\frac{d^l}{dt^l}J(\xi(t),\alpha(t))\right\|_V\right\}.\end{aligned} \tag{2.32}$$

Proof. Let us introduce the functions ψ and $\psi_h\colon[-t_0,t_0]\times V_2\to V_2$ defined by

$$\psi(t,w)=\mathscr{F}(\xi(t),\alpha(t),w),$$
$$\psi_h(t,w)=\mathscr{F}_h(\xi_h^*(t),\alpha_h^*(t),w).$$

Setting

$$w(t)=v(\xi(t),\alpha(t)),\ w_h(t)=v_h(\xi_h^*(t),\alpha_h^*(t)),$$

we obtain

$$\psi(t,w(t))=\psi_h(t,w_h(t))=0,\ |t|\leqq t_0.$$

On the other hand, the constants ξ_0 and α_0 being choosen as in the proof of Theorem 2,

$$D_w\psi(t,w(t))=D_v\mathscr{F}(\xi(t),\alpha(t),v(\xi(t),\alpha(t)))$$

is an isomorphism of V_2 for $|t|\leqq t_0$ with

$$\sup_{|t|\leqq t_0}\|D_w\psi(t,w(t))^{-1}\|_{\mathscr{L}(V_2;V_2)}\leqq c.$$

Furthermore, we have

$$\begin{aligned}\psi(t,w)-\psi_h(t,w)&=Q(T-T_h)G(\lambda_0+\xi(t),u_0+\alpha(t)\varphi_0+w)\\&+QT_h[G(\lambda_0+\xi(t),u_0+\alpha(t)\varphi_0+w)\\&-G(\lambda_0+\xi_h^*(t),u_0+\alpha_h^*(t)\varphi_0+w)].\end{aligned}\tag{2.33}$$

Then it follows from (2.27) and (2.31) that $D^l\psi_h\to D^l\psi$ in $\mathscr{L}_l(\mathbb{R}\times V_2;V_2)$, $0\leqq l\leqq p-1$, and $D^p\psi_h$ is bounded, uniformly on every bounded subset of $[-t_0,t_0]\times V_2$.

Now, applying Theorem 1 to the function ψ_h gives for all $t\in[-t_0,t_0]$ and all integer m with $0\leqq m\leqq p-1$

$$\left\|\frac{d^m}{dt^m}(w_h(t)-w(t))\right\|_V\leqq c\sum_{l=0}^{m}\left\|\frac{d^l}{dt^l}(\psi(t,w(t))-\psi_h(t,w(t)))\right\|_V.$$

The estimate (2.32) is a consequence of (2.33) and the previous inequality. ∎

By Theorem 2, we see that solving the Eq. (2.21) in a neighbourhood of the singular point (λ_0,u_0) amounts to solve the approximate bifurcation equation

$$f_h(\xi,\alpha)\equiv\langle F_h(\lambda_0+\xi,u_0+\alpha\varphi_0+v_h(\xi,\alpha)),\varphi_0^*\rangle=0\tag{2.34}$$

in a neighbourhood of the origin. Now, in order to analyze the approximation of the solutions of problem (2.8) by those of problem (2.21), it remains to compare the solutions of the bifurcation equation (2.18) with those of (2.34). This will be done in the next section in the case where $\frac{\partial f}{\partial\xi}(0,0)\neq 0$. The case $\frac{\partial f}{\partial\xi}(0,0)=0$ will be analyzed in the 3rd paper of this series.

We shall need in the sequel estimates of $D^m f_h(\xi,\alpha)$, $0\leqq m\leqq p$.

Lemma 4. *Assume the hypotheses of Theorem* 2. *Then we have for all* $\xi, \xi^* \in [-\xi_0, \xi_0]$, $\alpha, \alpha^* \in [-\alpha_0, \alpha_0]$ *and all integer* m *with* $0 \leqq m \leqq p-1$

$$\text{(i)} \quad \|D^m f_h(\xi^*, \alpha^*) - D^m f(\xi, \alpha)\|_{\mathscr{L}_m(\mathbb{R}^2; \mathbb{R})} \leqq K\left\{|\xi^* - \xi| + |\alpha^* - \alpha| + \sum_{l=0}^{m} \|(T - T_h) D^l J(\xi, \alpha)\|_{\mathscr{L}_l(\mathbb{R}^2; V)}\right\}, \tag{2.35}$$

$$\text{(ii)} \quad \|D^p f_h(\xi^*, \alpha^*)\|_{\mathscr{L}_p(\mathbb{R}^2; \mathbb{R})} \leqq K.$$

Proof: We first set

$$J_h(\xi, \alpha) = G(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v_h(\xi, \alpha)). \tag{2.36}$$

By using the definitions (2.6), (2.18), (2.20) and (2.34) of F, f, F_h and f_h respectively, it is easy to check that

$$f_h(\xi^*, \alpha^*) - f(\xi, \alpha) = \alpha^* - \alpha + \langle (T_h - T) J(\xi, \alpha), \varphi_0^* \rangle + \langle T_h(J_h(\xi^*, \alpha^*) - J(\xi, \alpha)), \varphi_0^* \rangle, \tag{2.37}$$

and

$$D^m f_h(\xi^*, \alpha^*) - D^m f(\xi, \alpha) = \langle (T_h - T) D^m J(\xi, \alpha), \varphi_0^* \rangle + \langle T_h(D^m J_h(\xi^*, \alpha^*) - D^m J(\xi, \alpha)), \varphi_0^* \rangle, \quad 1 \leqq m \leqq p-1.$$

By (2.27) and the boundedness of the mapping $D^l G$, $1 \leqq l \leqq m+1$, we obtain for $0 \leqq m \leqq p-1$

$$\|D^m f_h(\xi^*, \alpha^*) - D^m f(\xi, \alpha)\|_{\mathscr{L}_m(\mathbb{R}^2; \mathbb{R})} \leqq c\left\{|\xi^* - \xi| + |\alpha^* - \alpha| + \sum_{l=0}^{m} |D^l v_h(\xi^*, \alpha^*) - D^l v(\xi, \alpha)\|_{\mathscr{L}_l(\mathbb{R}^2; V)} + \|(T - T_h) D^m J(\xi, \alpha)\|_{\mathscr{L}_m(\mathbb{R}^2; V)}\right\}.$$

so that the estimate (2.35) follows from Theorem 2. ■

Again, we shall need a more specific estimate. Introducing as in Lemma 3 the pairs of functions $t \to (\xi(t), \alpha(t))$ and $t \to (\xi_h^*(t), \alpha_h^*(t))$, we have

Lemma 5. *Assume the hypotheses of Theorem* 2 *together with* (2.31). *Then, we get for all* $t \in [-t_0, t_0]$ *and all integer* m *with* $0 \leqq m \leqq p-1$

$$\left|\frac{d^m}{dt^m}(f_h(\xi_h^*(t), \alpha_h^*(t)) - f(\xi(t), \alpha(t)))\right| \leqq K \sum_{l=0}^{m} \left\{\left|\frac{d^l}{dt^l}(\xi(t) - \xi_h^*(t))\right| + \left|\frac{d^l}{dt^l}(\alpha(t) - \alpha_h^*(t))\right| + \left\|(T - T_h)\frac{d^l}{dt^l} J(\xi(t), \alpha(t))\right\|_V\right\}. \tag{2.38}$$

Proof. It follows from (2.37) that

$$\begin{aligned} & f_h(\xi_h^*(t), \alpha_h^*(t)) - f(\xi(t), \alpha(t)) \\ & \quad = \alpha_h^* t) - \alpha(t) + \langle (T_h - T) J(\xi(t), \alpha(t)), \varphi_0^* \rangle \\ & \qquad + \langle T_h(J_h(\xi_h^*(t), \alpha_h^*(t)) - J(\xi(t), \alpha(t)), \varphi_0^* \rangle. \end{aligned}$$

Differentiating m times the above expression and using (2.27) together with the boundedness of $D^l G$, $1 \leqq l \leqq m+1$, gives

$$\begin{aligned} & \left| \frac{d^m}{dt^m} (f_h(\xi_h^*(t), \alpha_h^*(t)) - f(\xi(t), \alpha(t))) \right| \\ & \quad \leqq c \sum_{l=0}^{m} \left\{ \left| \frac{d^l}{dt^l} (\xi_h^*(t) - \xi(t)) \right| + \left| \frac{d^l}{dt^l} (\alpha_h^*(t) - \alpha(t)) \right| \right. \\ & \qquad + \left\| \frac{d^l}{dt^l} (v_h(\xi_h^*(t), \alpha_h^*(t)) - v(\xi(t), \alpha(t))) \right\|_V \\ & \qquad \left. + \left\| (T - T_h) \frac{d^m}{dt^m} J(\xi(t), \alpha(t)) \right\|_V \right\}. \end{aligned}$$

Hence the estimate (2.38) follows from Lemma 3. ■

3. Simple Limit Points

3.1. The Continuous Case

Let us consider again problem (2.8). From now on, we shall assume that (λ_0, u_0) is a *simple limit point* of F, i.e. a simple singular point of F which satisfies in addition

$$D_\lambda F^0 \equiv D_\lambda F(\lambda_0, u_0) \notin \text{Range}(D_u F^0). \tag{3.1}$$

Let us state the following classical result

Lemma 6. *Assume the hypotheses* (2.7) *and* (3.1). *Then there exist* $\alpha_0 > 0$ *and a unique* C^p *mapping* $\alpha \in [-\alpha_0, \alpha_0] \to \xi(\alpha) \in \mathbb{R}$ *such that*

$$\begin{aligned} & f(\xi(\alpha), \alpha) = 0, \qquad |\alpha| \leqq \alpha_0, \\ & \xi(0) = 0. \end{aligned} \tag{3.2}$$

Hence, there exists a unique branch $\{(\lambda(\alpha), u(\alpha)); |\alpha| \leqq \alpha_0\}$ *of solutions of* (2.8) *in a neighborhood of the simple limit point* (λ_0, u_0), *where* $\alpha \to \lambda(\alpha)$ *and* $\alpha \to u(\alpha)$ *are* C^p *functions given by*

$$\begin{aligned} & \lambda(\alpha) = \lambda_0 + \xi(\alpha), \\ & u(\alpha) = u_0 + \alpha \varphi_0 + v(\xi(\alpha), \alpha). \end{aligned} \tag{3.3}$$

Proof. It follows from Lemmata 1 and 2 that the condition (3.1) can be equivalently stated in the form

$$\frac{\partial f}{\partial \xi}(0, 0) = \langle D_\lambda F^0, \varphi_0^* \rangle \neq 0. \tag{3.4}$$

Therefore, using (2.19) and applying the implicit function theorem to the function f give the desired result. ■

Let us compute the first and second derivatives of the function $\alpha \to \xi(\alpha)$ at the origin. First, differentiating (3.2) and using (2.19), we obtain

$$\frac{d\xi}{d\alpha}(0)=0. \tag{3.5}$$

On the other hand, by differentiating (2.17) with respect to ξ and α at the point (0, 0), we get

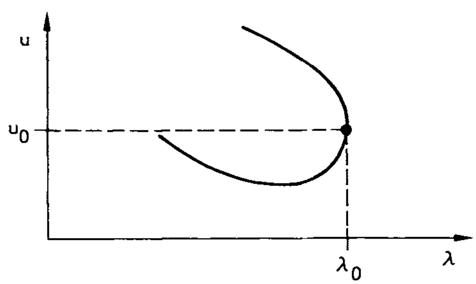
$$\begin{aligned} &\frac{\partial v}{\partial \xi}(0,0)=-LQD_\lambda F^0, \\ &\frac{\partial v}{\partial \alpha}(0,0)=0, \end{aligned} \tag{3.6}$$

where $L=(D_u F^0|_{V_2})^{-1}$. Then, differentiating (3.2) twice and using (3.5) and (3.6), we find after straightforward calculations

$$\frac{d^2\xi}{d\alpha^2}(0)=-\langle D_\lambda F^0, \varphi_0^*\rangle^{-1}\langle D_{uu}^2 F^0\cdot(\varphi_0,\varphi_0),\varphi_0^*\rangle, \tag{3.7}$$

where $D_{uu}^2 F^0 \equiv D_{uu}^2 F(\lambda_0, u_0)\in \mathscr{L}_2(V; V)$ denotes the second partial derivative of F with respect to u at the point (λ_0, u_0).

When $\langle D_{uu}^2 F^0\cdot(\varphi_0,\varphi_0),\varphi_0^*\rangle$ is $\neq 0$, the point (λ_0, u_0) is called a *nondegenerate turning point* or *a normal limit point* of F. In that case, we have the following diagram for the branch of solutions of (2.8)



3.2. The Approximation

We now want to establish the existence of a branch of solutions of the equation (2,21) in a neighbourhood of the branch $\{(\lambda(\alpha), u(\alpha));\, |\alpha|\leq \alpha_0\}$ of solutions of (2.8), at least for $h\leq h_0$ sufficiently small. To do that, we begin by considering the approximation bifurcation equation (2.34).

Lemma 7. *Assume the hypotheses of Theorem 2 together with the condition* (3.1). *Then there exist two positive constants α_0, b and, for $h\leq h_0$ small enough, a unique*

C^p mapping $\alpha\in[-\alpha_0,\alpha_0]\to\xi_h(\alpha)\in\mathbb{R}$ such that, for $|\alpha|\leqq\alpha_0$

$$\begin{aligned} &f_h(\xi_h(\alpha),\alpha)=0,\\ &|\xi_h(\alpha)-\xi(\alpha)|\leqq b. \end{aligned} \tag{3.8}$$

Moreover, there exists a constant $K>0$ independent of h such that, for all $\alpha\in[-\alpha_0,\alpha_0]$ and all integer m with $0\leqq m\leqq p-1$, the following error estimates hold:

$$\begin{aligned} &\text{(i)} && \left|\frac{d^m}{d\alpha^m}(\xi_h(\alpha)-\xi(\alpha))\right|\leqq K\sum_{l=0}^{m}\left\|(T-T_h)\frac{d^l}{d\alpha^l}G(\lambda(\alpha),u(\alpha))\right\|_V,\\ &\text{(ii)} && \left|\frac{d^p}{d\alpha^p}\xi_h(\alpha)\right|\leqq K. \end{aligned} \tag{3.9}$$

Proof. Lemma 4 together with (2.27) and Lemma 6 enables us to apply Theorem 1: there exists a unique C^p function ξ_h: $[-\alpha_0,\alpha_0]\to\mathbb{R}$ which satisfies (3.8). Moreover, we obtain for $|\alpha|\leqq\alpha_0$ and $0\leqq m\leqq p-1$

$$\left|\frac{d^m}{d\alpha^m}(\xi_h(\alpha)-\xi(\alpha))\right|\leqq c\sum_{l=0}^{m}\left|\frac{d^l}{d\alpha^l}(f(\xi(\alpha),\alpha)-f_h(\xi(\alpha),\alpha))\right|. \tag{3.10}$$

The estimate (3.9)(i) follows from (3.10) and Lemma 5 used with $t=\alpha(t)=\alpha_h^*(t)=\alpha$ and $\xi(t)=\xi_h^*(t)=\xi(\alpha)$. The estimate (3.9)(ii) follows from Theorem 1. ■

We define the pair of C^p functions $\alpha\in[-\alpha_0,\alpha_0]\to(\lambda_h(\alpha),u_h(\alpha))\in\mathbb{R}\times V$ by

$$\begin{aligned} &\lambda_h(\alpha)=\lambda_0+\xi_h(\alpha),\\ &u_h(\alpha)=u_0+\alpha\varphi_0+v_h(\xi_h(\alpha),\alpha). \end{aligned} \tag{3.11}$$

We have

$$F_h(\lambda_h(\alpha),u_h(\alpha))=0,\qquad |\alpha|\leqq\alpha_0,$$

so that $\{(\lambda_h(\alpha),u_h(\alpha));\ |\alpha|\leqq\alpha_0\}$ is a branch of solutions of problem (2.21). Let us denote by $\lambda^{(m)}(\alpha)$, $u^{(m)}(\alpha)$, $\lambda_h^{(m)}(\alpha)$, $u_h^{(m)}(\alpha)$ the m-th derivatives of the functions $\lambda(\alpha)$, $u(\alpha)$, $\lambda_h(\alpha)$, $u_h(\alpha)$.

We can now state our main result.

Theorem 3. *Assume the hypotheses of Theorem 2. Assume in addition that the condition* (3.1) *holds. Then the approximate problem* (2.21) *has a unique branch of solutions $\{(\lambda_h(\alpha),u_h(\alpha));|\alpha|\leqq\alpha_0\}$ in a neighbourhood of the branch of solutions $\{(\lambda(\alpha),u(\alpha));|\alpha|\leqq\alpha_0\}$ of the continuous problem* (2.8).

Moreover, these branches of solutions are of class C^p and we obtain for all $\alpha\in[-\alpha_0,\alpha_0]$ and all integer m with $0\leqq m\leqq p-1$ the error estimate

$$\begin{aligned} &|\lambda_h^{(m)}(\alpha)-\lambda^{(m)}(\alpha)|+\|u_h^{(m)}(\alpha)-u^{(m)}(\alpha)\|_V\\ &\leqq K\sum_{l=0}^{m}\left\|(T-T_h)\frac{d^l}{d\alpha^l}G(\lambda(\alpha),u(\alpha))\right\|_V. \end{aligned} \tag{3.12}$$

Proof. The first part of the theorem follows from Lemma 7. The estimate (3.12) will follow from (3.9)(i) if we prove that we have for $0\leqq m\leqq p-1$

$$\left\| \frac{d^m}{d\alpha^m} (v_h(\xi_h(\alpha), \alpha) - v(\xi(\alpha), \alpha)) \right\|_V$$
$$\leq c \sum_{l=0}^{m} \left\| (T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha)) \right\|_V.$$

But this a direct consequence of the estimate (3.9) and Lemma 3 used with $t = \alpha(t) = \alpha_h^*(t) = \alpha$, $\xi(t) = \xi(\alpha)$, $\xi_h^*(t) = \xi_h(\alpha)$. ∎

In order to get practical bounds for the error $|\lambda_h^{(m)}(\alpha) - \lambda^{(m)}(\alpha)| + \|u_h^{(m)}(\alpha) - u^{(m)}(\alpha)\|_V$, it remains to estimate the right-hand side member of (3.12). We observe that

$$u(\alpha) + G(\lambda(\alpha), u(\alpha)) = 0$$

and therefore

$$T \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha)) = -u^{(l)}(\alpha). \tag{3.13}$$

In Sects. 5 and 6 where we consider Galerkin or finite element approximations of nonlinear boundary value problems, using (3.13), we shall obtain such estimates by only assuming that the solution $u(\alpha)$ is "sufficiently smooth" together with its derivatives $u^{(l)}(\alpha)$, $1 \leq l \leq m$.

4. Nondegenerate Turning Points

In this section, we assume that the simple limit point (λ_0, u_0) is a nondegenerate turning point, i.e. (λ_0, u_0) satisfies the conditions (2.7), (3.1), and

$$\langle D_{uu}^2 F^0 \cdot (\varphi_0, \varphi_0), \varphi_0^* \rangle \neq 0. \tag{4.1}$$

In this case, using (3.5) and (3.7), we have

$$\frac{d\xi}{d\alpha}(0) = 0, \qquad \frac{d^2\xi}{d\alpha^2}(0) \neq 0,$$

so that the function $\alpha \to \xi(\alpha)$ has a local maximum or minimum at the point $\alpha = 0$.

Assume that G is a C^p mapping with $p \geq 3$. Then it follows from the estimate (3.9) used with $m = 0, 1, 2$ that there exists an interval $[-\alpha_1, \alpha_1]$ and, for $h \leq h_0$ small enough, a unique value $\alpha_h^0 \in [-\alpha_1, \alpha_1]$ such that

$$\lim_{h \to 0} \alpha_h^0 = 0,$$
$$\frac{d}{d\alpha} \xi_h(\alpha_h^0) = 0,$$
$$\left| \frac{d^2}{d\alpha^2} \xi_h(\alpha) \right| \geq \varepsilon > 0 \qquad \text{for all } \alpha \in [-\alpha_1, \alpha_1], \tag{4.2}$$

where the constant ε is independent of h.

We set:

$$\lambda_h^0 = \lambda_h(\alpha_h^0), \qquad u_h^0 = u_h(\alpha_h^0). \tag{4.3}$$

We can easily check that the point (λ_h^0, u_h^0) is indeed a nondegenerate turning point of F_h.

Theorem 4. *Assume the hypotheses of Theorem 3 with $p \geqq 3$. Assume in addition that the condition (4.1) holds. Then for $h \leqq h_0$ small enough, we have the error estimate*

$$|\lambda_h^0 - \lambda^0| + \|u_h^0 - u_0\|_V \leqq K \sum_{l=0}^{1} \left\| (T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha))|_{\alpha=0} \right\|_V, \tag{4.4}$$

where K is a positive constant independent of h.

Proof. We get from (4.2)

$$|\alpha_h^0| \leqq \varepsilon^{-1} \left| \frac{d}{d\alpha} \xi_h(0) \right| = \varepsilon^{-1} \left| \frac{d}{d\alpha} (\xi_h(0) - \xi(0)) \right|.$$

Using Lemma 7, we obtain

$$|\alpha_h^0| \leqq c_1 \sum_{l=0}^{1} \left\| (T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha))|_{\alpha=0} \right\|_V. \tag{4.5}$$

Next, we write

$$|\lambda_h^0 - \lambda_0| = |\xi_h(\alpha_h^0)| \leqq |\xi_h(\alpha_h^0) - \xi_h(0)| + |\xi_h(0)|$$

and, using again Lemma 7 and the estimate (4.5) together with the uniform boundedness of $d\xi_h/d\alpha$, we find

$$|\lambda_h^0 - \lambda_0| \leqq c_2 \sum_{l=0}^{1} \left\| (T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha))|_{\alpha=0} \right\|_V.$$

Similarly, the function $du_h/d\alpha$ is uniformly bounded and

$$\begin{aligned} \|u_h^0 - u_0\|_V &\leqq \|u_h(\alpha_h^0) - u_h(0)\|_V + \|u_h(0) - u(0)\|_V \\ &\leqq c_3 |\alpha_h^0| + \|u_h(0) - u(0)\|_V. \end{aligned}$$

Hence, using (4.5) and Theorem 3, we obtain

$$\|u_h^0 - u_0\|_V \leqq c_4 \sum_{l=0}^{1} \left\| (T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha))|_{\alpha=0} \right\|_V.$$

This proves the theorem. ■

In many cases the above error bound for $|\lambda_h^0 - \lambda_0|$ can be improved. We set $G^0 = G(\lambda_0, u_0)$, $D_u G^0 = D_u G(\lambda_0, u_0)$ and we denote by $[(T - T_h) D_u G^0]^* \in \mathscr{L}(V'; V')$ the adjoint operator of $(T - T_h) D_u G^0 \in \mathscr{L}(V; V)$.

Theorem 5. *Assume the hypotheses of Theorem 4. Then, for $h \leqq h_0$ small enough, we have the error estimate*

$$|\lambda_h^0 - \lambda_0| \leqq K\{|\langle (T - T_h) G^0, \varphi_0^* \rangle| + \|(T - T_h) G^0\|_V \cdot \|[(T - T_h) D_u G^0]^* \varphi_0^*\|_{V'} + \sum_{l=0}^{1} \left\| (T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha))|_{\alpha=0} \right\|_V^2. \tag{4.6}$$

Proof. By the uniform boundedness of the function $d^2\xi_h/d\alpha^2$, we have

$$|\xi_h(\alpha_h^0)| \leqq |\xi_h(0)| + \left|\frac{d\xi_h}{d\alpha}(0)\right| (\alpha_h^0| + c_0 |\alpha_h^0|^2$$

so that by Lemma 7 and (4.5)

$$|\xi_h(\alpha_h^0)| \leqq |\xi_h(0)| + c_1 \sum_{l=0}^{1} \left\| (T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha))|_{\alpha=0} \right\|_V^2. \tag{4.7}$$

Let us next give an estimate for $|\xi_h(0)|$. The bound (3.10) with $m = 0$ gives

$$|\xi_h(0)| \leqq c_2 |f_h(0, 0)|. \tag{4.8}$$

On the other hand, using (2.37) with $\xi = \xi^* = \alpha = \alpha^* = 0$, we get

$$f_h(0, 0) = \langle (T_h - T) G^0, \varphi_0^* \rangle + \langle T_h(G(\lambda_0, u_0 + v_h(0, 0)) - G^0), \varphi_0^* \rangle \tag{4.9}$$

The error estimate (4.6) will follow from (4.7), (4.8) and (4.9) if we check that

$$|\langle T_h(G(\lambda_0, u_0 + v_h(0, 0)) - G^0), \varphi_0^* \rangle| \leqq c_3 \|(T - T_h) G^0\|_V \{\|(T - T_h) G^0\|_V + \|[(T - T_h) D_u G^0]^* \varphi_0^*\|_{V'}\}. \tag{4.10}$$

In fact, we have

$$\begin{aligned} &|\langle T_h(G(\lambda_0, u_0 + v_h(0, 0)) - G^0), \varphi_0^* \rangle \\ &\leqq |\langle T_h D_u G^0 \cdot v_h(0, 0), \varphi_0^* \rangle| + c_4 \|v_h(0, 0)\|_V^2 \\ &\leqq |\langle T D_u G^0 \cdot v_h(0, 0), \varphi_0^* \rangle| + |\langle (T_h - T) D_u G^0 \cdot v_h(0, 0), \varphi_0^* \rangle| \\ &\qquad + c_4 \|v_h(0, 0)\|_V^2. \end{aligned}$$

Thus, we obtain

$$|\langle T_h(G(\lambda_0, u_0 + v_h(0, 0)) - G^0), \varphi_0^* \rangle| \leqq |\langle T D_u G^0 \cdot v_h(0, 0), \varphi_0^* \rangle| + c_5 \|v_h(0, 0)\|_V (\|v_h(0, 0)\|_V + \|[(T_h - T) D_u G^0]^* \cdot \varphi_0^*\|_{V'}).$$

Since $v_h(0, 0) \in V_2$, we have by Lemma 1

$$\langle T D_u G^0 \cdot v_h(0, 0), \varphi_0^* \rangle = \langle D_u F^0 \cdot v_h(0, 0), \varphi_0^* \rangle = 0.$$

Moreover, we have by Theorem 2 used with $m = \xi^* = \xi = \alpha^* = \alpha = 0$

$$\|v_h(0, 0)\|_V \leqq c_6 \|(T - T_h) G^0\|_V.$$

Hence (4.10) is proved. ■

Remark. It is worthwhile to notice that the results of Theorem 3 can also be obtained in a more direct way. In fact, let us consider the functions $\mathscr{G}$ and $\mathscr{G}_h$: $\mathbb{R} \times \mathbb{R} \times V_2 \to \mathbb{R} \times V_2$ defined by

$$\mathscr{G}(\alpha, \xi, v) = (QF(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v), \quad \langle F(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v), \varphi_0^* \rangle)$$

and

$$\mathscr{G}_h(\alpha, \xi, v) = (QF_h(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v), \langle F(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v), \varphi_0^* \rangle).$$

Clearly, problems (2.8) and (2.21) are respectively equivalent to

$$\mathscr{G}(\alpha, \xi, v) = 0 \tag{4.11}$$

and

$$\mathscr{G}_h(\alpha, \xi, v) = 0. \tag{4.12}$$

Now, it is easy to check that $\mathscr{G}(0,0,0) = 0$ and $D_{(\xi,v)}\mathscr{G}(0,0,0)$ is an isomorphism of $\mathbb{R} + V_2$ so that $(0,0,0)$ is now a nonsingular point of $\mathscr{G}$ (for a related approach, see [6]). Moreover $D^l\mathscr{G}_h \to D^l\mathscr{G}$ uniformly in a neighborhood of $(0,0,0)$, $0 \leq l \leq p$. Hence, applying Theorem 1 to (4.11), (4.12) gives (3.12).

However, using this approach, the proof of Theorem 5 appears to be more complicated. Furthermore, the results of Lemmata 3-5 will be constantly used in the third paper of this series devoted to the study of bifurcation points. ■

5. Application I: Galerkin Approximation of Nonlinear Problems

In this section, we want to apply the above results to a class of *conforming* approximations of variationally posed nonlinear problems.

Let V and H be two (real) Hilbert spaces with scalar products $((.,.))$, $(.,.)$ and norms $\|.\|$, $|.|$ respectively. We suppose that $V \subset H$ with continuous imbedding and V is dense in H. If we identify H with its dual space H', we have $V \subset H \subset V'$ with densely continuous imbeddings and the scalar product $(.,.)$ may also represent the duality pairing between the spaces V and V'.

Let W be a reflexive Banach space such that $H \subset W \subset V'$ with continuous imbeddings. We assume that the canonical injection of W into V' is *compact*[2]. We introduce a continuous bilinear form $a: V \times V \to \mathbb{R}$ and a C^p mapping $(p \geq 2)\, G: \mathbb{R} \times V \to W$. Then we consider the nonlinear problem: Find pairs $(\lambda, u) \in \mathbb{R} \times V$ solutions of

$$a(u, v) + (G(\lambda, u), v) = 0, \qquad \forall v \in V. \tag{5.1}$$

We further assume that the bilinear form a is *V-elliptic* in the sense that there exists a positive constant γ such that

$$a(v, v) \geq \gamma \|v\|^2, \qquad \forall v \in V. \tag{5.2}$$

We can now define the operators T, $T^* \in \mathscr{L}(V'; V)$ by

$$a(Tf, v) = a(v, T^*f) = (f, v), \qquad \forall v \in V,\ \forall f \in V'. \tag{5.3}$$

[2] This implies that the canonical injection of V into H is also compact

Then an equivalent form of problem (5.1) consists in finding pairs $(\lambda, u) \in \mathbb{R} \times V$ solutions of

$$F(\lambda, u) \equiv u + TG(\lambda, u) = 0. \tag{5.4}$$

Next, we are given a family $\{V_h\}$ of finite-dimensional subspaces of V and we consider the approximate problem: Find pairs $(\lambda, u_h) \in \mathbb{R} \times V_h$ such that

$$a(u_h, v_h) + (G(\lambda, u_h), v_h) = 0, \quad \forall v_h \in V_h. \tag{5.5}$$

Let us define the operators $\Pi_h \in \mathscr{L}(V; V_h)$ and $T_h \in \mathscr{L}(V'; V_h)$ by

$$a(\Pi_h u - u, v_h) = 0, \quad \forall v_h \in V_h, \ \forall u \in V, \tag{5.6}$$

and

$$a(T_h f, v_h) = (f, v_h), \quad \forall v_h \in V_h, \ \forall f \in V'. \tag{5.7}$$

Clearly, we have

$$T_h = \Pi_h T, \tag{5.8}$$

and problem (5.5) consists in finding pairs $(\lambda, u_h) \in \mathbb{R} \times V$ solutions of

$$F_h(\lambda, u_h) \equiv u_h + T_h G(\lambda, u_h) = 0. \tag{5.9}$$

Assume that for all $v \in V$

$$\lim_{h \to 0} \inf_{v_h \in V_h} \|v - v_h\| = 0. \tag{5.10}$$

Then, as an easy and classical consequence of (5.2) and (5.6), we have

$$\lim_{h \to 0} \|v - \Pi_h v\| = 0, \quad \forall v \in V.$$

Moreover, since $T \in \mathscr{L}(W; V)$ is compact, we obtain

$$\lim_{h \to 0} \|T - T_h\|_{\mathscr{L}(W;V)} = \lim_{h \to 0} \|(I - \Pi_h) T\|_{\mathscr{L}(W;V)} = 0.$$

Now, we suppose that $(\lambda_0, u_0) \in \mathbb{R} \times V$ is a simple singular point of F and we choose φ_0 and φ_0^* as in Lemma 1. It is an easy matter to check that $\varphi_0 \in V$ is an eigenvector corresponding to the eigenvalue $\mu = 0$ (with algebraic multiplicity 1) of the linearized variationally posed eigenproblem: Find $\mu \in \mathbb{R}$ and $\varphi \in V$, $\varphi \neq 0$ such that

$$a(\varphi, v) + (D_u G^0 \cdot \varphi, v) = \mu(\varphi, v), \quad \forall v \in V\,[3] \tag{5.11}$$

Setting

$$\psi_0^* = T^* \varphi_0^*, \tag{5.12}$$

we check that $\psi_0^* \in V$ is the eigenvector of the adjoint variationally posed eigenproblem

$$a(v, \psi_0^*) + (D_u G^0 \cdot v, \psi_0^*) = 0, \quad \forall v \in V \tag{5.13}$$

[3] Usually, one proceeds to the complexification of the spaces V, H and W and one looks for eigenvalues $\mu \in \mathbb{C}$. But this is not necessary here

such that

$$a(\varphi_0, \psi_0^*) = 1. \tag{5.14}$$

Suppose that the condition (3.1), or equivalently the condition

$$(D_\lambda G^0, \psi_0^*) \neq 0 \tag{5.15}$$

holds, i.e. (λ_0, u_0) is a simple limit point of F. Then, there exists a unique branch $\{(\lambda(\alpha), u(\alpha)); |\alpha| \leqq \alpha_0\}$ of solutions of (5.1) (or (5.4)) in a neighbourhood of (λ_0, u_0). This branch is of class C^p and may be parametrized as in (3.3) with $\lambda(0) = \lambda_0$, $u(0) = u_0$.

Theorem 6. *Assume that G is a C^p mapping $(p \geqq 2)$ and the mapping $D^p G$ is bounded on all bounded subsets of $\mathbb{R} \times V$. Assume in addition that (λ_0, u_0) is s simple limit point of F and that the approximation property* (5.10) *holds. Then, there exists a unique branch $\{(\lambda_h(\alpha), u_h(\alpha)); |\alpha| \leqq \alpha_0\}$ of solutions of* (5.5) (*or* (5.9)) *in the neighbourhood of the branch $\{(\lambda(\alpha), u(\alpha)); |\alpha| \leqq \alpha_0\}$. This branch is of class C^p and, for all $\alpha \in [-\alpha_0, \alpha_0]$ and all integer m with $0 \leqq m \leqq p-1$, we get the error estimate*

$$|\lambda_h^{(m)}(\alpha) - \lambda^{(m)}(\alpha)| + \|u_h^{(m)}(\alpha) - u^{(m)}(\alpha)\| \leqq K \sum_{l=0}^{m} \inf_{v_h \in V_h} \|u^{(l)}(\alpha) - v_h\|. \tag{5.16}$$

Proof. Let us show that this is a consequence of Theorem 3. In fact, we have only to check that for $0 \leqq l \leqq m$

$$\left\| (T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha)) \right\| \leqq c \inf_{v_h \in V_h} \|u^{(l)}(\alpha) - v_h\|.$$

But, using (3.13) and (5.8), we get

$$(T - T_h) \frac{d^l}{d\alpha^l} G(\lambda(\alpha), u(\alpha)) = \Pi_h u^{(l)}(\alpha) - u^{(l)}(\alpha).$$

Since, by (5.2), we have for all $v \in V$

$$\|v - \Pi_h v\| \leqq c \inf_{v_h \in V_h} \|v - v_h\|,$$

the result follows. ∎

Finally, we assume further the condition (4.1), or equivalently

$$(D_{uu}^2 G^0 \cdot (\varphi_0, \varphi_0), \psi_0^*) \neq 0, \tag{5.17}$$

i.e. (λ_0, u_0) is a nondegenerate turning point of F. By the results of Section 4, there exists a unique nondegenerate turning point (λ_h^0, u_h^0) of F_h in a sufficiently small neighbourhood of (λ_0, u_0).

Theorem 7. *Assume the hypotheses of Theorem 6 with $p \geqq 3$. Assume in addition that the condition* (5.17) *holds. Then we have the error estimates*

$$\|u_h^0 - u_0\| \leqq K\{\inf_{v_h \in V_h} \|u_0 - v_h\| + \inf_{v_h \in V_h} \|u_0' - v_h\|\} \tag{5.18}$$

and

$$|\lambda_h^0-\lambda_0|\leqq K\{(\inf_{v_h\in V_h}\|u_0-v_h\|)^2+(\inf_{v_h\in V_h}\|u_0'-v_h\|)^2$$
$$+(\inf_{v_h\in V_h}\|u_0-v_h\|)(\inf_{\psi_h\in V_h}\|\psi_0^*-\psi_h\|)\}, \tag{5.19}$$

where $u_0'=\frac{du}{d\alpha}(0)$.

Proof. The estimate (5.18) follows directly from Theorem 4, (3.13) and (5.8). On the other hand, (5.19) will be a consequence of Theorem 5 if we show that

$$|((T-T_h)G^0,\varphi_0^*)|\leqq c_1(\inf_{v_h\in V_h}\|u_0-v_h\|)(\inf_{\psi_h\in V_h}\|\psi_0^*-\psi_h\|) \tag{5.20}$$

and

$$\|[(T-T_h)G^0]^*\cdot\varphi_0^*\|_{V'}\leqq c_2\inf_{\psi_h\in V_h}\|\psi_0^*-\psi_h\|. \tag{5.21}$$

First, using (3.13), (5.3), (5.6), (5.8) and (5.12), we have

$$((T-T_h)G^0,\varphi_0^*)=a((I-\Pi_h)TG^0,T^*\varphi_0^*)=a(\Pi_h u_0-u_0,\psi_0^*)$$
$$=a(\Pi_h u_0-u_0,\psi_0^*-\psi_h),\qquad \forall\psi_h\in V_h,$$

from which (5.20) follows immediately.

Next, for proving (5.21), we write:

$$\|[(T-T_h)D_uG^0]^*\cdot\varphi_0^*\|_{V'}=\sup_{\substack{v\in V\\ \|v\|=1}}|((T-T_h)D_uG^0\cdot v,\varphi_0^*)|.$$

Then, given $v\in V$, we have

$$((T-T_h)D_uG^0\cdot v,\varphi_0^*)=a((I-\Pi_h)TD_uG^0\cdot v,T^*\varphi_0^*)$$
$$=a((I-\Pi_h)TD_uG^0\cdot v,\psi_0^*-\psi_h),\qquad \forall\psi_h\in V_h,$$

so that (5.21) holds. ∎

6. Application II. A Mixed Finite Element Approximation of the Navier-Stokes Equation

Let Ω be a bounded simply connected plane domain with boundary Γ; we consider the Navier-Stokes equations for an incompressible viscous fluid confined in Ω in the stream function formulation

$$\begin{aligned}&\nu\Delta^2\psi-\operatorname{curl}(\Delta\psi\,\mathbf{grad}\,\psi)=\mathrm{f}\quad \text{in }\Omega,\\ &\psi=\frac{\partial\psi}{\partial n}=0\quad\text{on }\Gamma,\end{aligned} \tag{6.1}$$

where f is given in $H^{-1}(\Omega)$, $\nu>0$ is the viscosity coefficient and $\partial/\partial n$ denotes the outer normal derivative along Γ. Problem (6.1) has at least one solution $\psi\in H_0^2(\Omega)$.

We introduce the linear operator $\mathcal{R} \in \mathcal{L}(H^{-2}(\Omega); H_0^2(\Omega))$: $g \in H^{-2}(\Omega) \to \psi = \mathcal{R} g \in H_0^2(\Omega)$ defined by

$$\begin{aligned} \Delta^2 \psi &= g \quad \text{in } \Omega, \\ \psi &= \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \Gamma. \end{aligned} \tag{6.2}$$

In addition, we consider the C^∞ mapping $\mathcal{G}: (\lambda, \psi) \in \mathbb{R} \times H_0^2(\Omega) \to \mathcal{G}(\lambda, \psi) \in H^{-2}(\Omega)$ defined by

$$\mathcal{G}(\lambda, \psi) = -\lambda(\operatorname{curl}(\Delta \psi \, \mathbf{grad}\, \psi) + f). \tag{6.3}$$

Clearly, solving problem (6.1) amounts to find $\psi \in H_0^2(\Omega)$ solution of

$$\mathcal{F}(\lambda, \psi) \equiv \psi + \mathcal{R}\mathcal{G}(\lambda, \psi) = 0, \qquad \lambda = \frac{1}{v}. \tag{6.4}$$

Notice that the operator $\mathcal{R} D_\psi \mathcal{G}(\lambda, \psi) \in \mathcal{L}(H_0^2(\Omega); H_0^2(\Omega))$ is compact.

Now, let $\psi_0 \in H_0^2(\Omega)$ be a *simple singular solution* of (6.1) corresponding to $v = v_0$. This means that the linearized Navier-Stokes operator

$$X \to v_0 \Delta^2 X - \operatorname{curl}(\Delta \psi_0 \, \mathbf{grad}\, X + \Delta X \, \mathbf{grad}\, \psi_0)$$

has an eigenfunction $X_0 \in H_0^2(\Omega)$ corresponding to a zero eigenvalue of algebraic multiplicity 1, or equivalently X_0 is an eigenvector of the compact operator $\mathcal{R} D_\psi \mathcal{G}(\lambda_0, \psi_0)$ corresponding to the eigenvalue -1 of algebraic multiplicity 1.

We denote by $X_0^* \in H_0^2(\Omega)$ an eigenfunction of the formal adjoint of the linearized Navier-Stokes operator

$$X \to v_0 \Delta^2 X + \operatorname{div}(\Delta \psi_0 \, \mathbf{curl}\, X) - \Delta \, \mathbf{grad}\, \psi_0 \cdot \mathbf{curl}\, X)$$

corresponding to the zero eigenvalue. If $D_\psi \mathcal{G}(\lambda_0, \psi_0)^* \in \mathcal{L}(H_0^2(\Omega); H^{-2}(\Omega))$ is the adjount operator of $D_\psi \mathcal{G}(\lambda_0, \psi_0)$, then X_0^* is an eigenvector of the operator $\mathcal{R} D_\psi \mathcal{G}(\lambda_0, \psi_0)^*$ corresponding to the eigenvalue -1. Note that $(\mathcal{R} D_\psi \mathcal{G}(\lambda_0, \psi_0))^* = D_\psi \mathcal{G}(\lambda_0, \psi_0)^* \mathcal{R} \in \mathcal{L}(H^{-2}(\Omega); H^{-2}(\Omega))$ is the adjoint operator of $\mathcal{R} D_\psi \mathcal{G}(\lambda_0, \psi_0)$ and $\mathcal{R}^{-1} X_0^*$ is an eigenvector of $(\mathcal{R} D_\psi \mathcal{G}(\lambda_0, \psi_0))^*$ corresponding to the eigenvalue -1. According to Lemma 1, we may choose the functions X_0 and X_0^* in such a way that

$$\|X_0\|_{H^2(\Omega)} = 1, \qquad \langle X_0, \mathcal{R}^{-1} X_0^* \rangle = \int_\Omega \Delta X_0 \Delta X_0^* \, dx = 1. \tag{6.5}$$

We further assume that ψ_0 is a *simple limit solution* of the Navier-Stokes equations (6.1) with $v = v_0$ in the sense that (λ_0, ψ_0) is a simple limit point of $\mathcal{F}$; i.e. ψ_0 is a simple singular solution of (6.1) which satisfies

$$\langle D_\lambda \mathcal{F}(\lambda_0, \psi_0), \mathcal{R}^{-1} X_0^* \rangle \neq 0$$

or equivalently

$$\int_\Omega \Delta\psi_0 \,\mathbf{grad}\,\psi_0 \cdot \mathbf{curl}\, X_0^* \,dx + \langle f, X_0^* \rangle \neq 0^4 \tag{6.6}$$

This limit point (λ_0, ψ_0) will be a *nondegenerate turning point* of the Navier-Stokes problem if in addition we have

$$\langle D^2_{\psi\psi} \mathscr{F}(\lambda_0, \psi_0) \cdot (X_0, X_0), \quad \mathscr{R}^{-1} X_0^* \rangle \neq 0$$

or equivalently

$$\int_\Omega \Delta X_0 \,\mathbf{grad}\, X_0 \cdot \mathbf{curl}\, X_0^* \,dx \neq 0. \tag{6.7}$$

For approximation purposes, we need to introduce another formulation of the Navier-Stokes problem. As in [2, Sect. 4], we are looking for a pair $u = (\psi, \omega)$ where $\omega = -\Delta\psi$ is the vorticity. We set (with standard notations for the Sobolev spaces):

$$V = W_0^{1,4}(\Omega) \times L^2(\Omega), \qquad W = W^{-1,\frac{4}{3}}(\Omega). \tag{6.8}$$

We introduce the linear operator $T\colon H^{-2}(\Omega) \to H_0^2(\Omega) \times L^2(\Omega)$ defined by

$$Tg = (\mathscr{R}g, -\Delta\mathscr{R}g). \tag{6.9}$$

By the Sobolev imbedding theorem, T belongs also to the space $\mathscr{L}(W; V)$. We next define the C^∞ mapping $G\colon (\lambda, u = (\psi, \omega)) \in \mathbb{R} \times V \to G(\lambda, u) \in W$ by

$$G(\lambda, u) = \lambda(\mathrm{curl}(\omega\, \mathbf{grad}\, \psi) - f). \tag{6.10}$$

Then a pair $u = (\psi, \omega) \in V$ satisfies the equation

$$F(\lambda, u) \equiv u + TG(\lambda, u) = 0 \tag{6.11}$$

if and only if the function ψ is a solution of the Navier-Stokes problem (6.1) corresponding to $\nu = 1/\lambda$ and $\omega = -\Delta\psi$.

Now, we assume very weak regularity hypotheses on the domain Ω so that $\mathscr{R} \in \mathscr{L}(W^{-1,\frac{4}{3}}(\Omega); H^{2+s}(\Omega))$ for some $s > 0$ and therefore the operator $T \in \mathscr{L}(W; V)$ is compact.

Let ψ_0 be a solution of problem (6.1) corresponding to $\nu = \nu_0$ and let (λ_0, u_0), $\lambda_0 = 1/\nu_0$, $u_0 = (\psi_0, \omega_0 = -\Delta\psi_0)$, be the associate solution of problem (6.11). Then, we need the following natural result whose proof is left to the reader.

Lemma 8. *A function $\psi_0 \in H_0^2(\Omega)$ is a simple singular solution of* (6.1) (*resp. a simple limit solution of* (6.1)) *corresponding to $\nu = \nu_0$ if and only if (λ_0, u_0) is a simple singular point of F (resp. a simple limit point of F). In that case, setting*

$$\tilde{\varphi}_0 = (X_0, -\Delta X_0) \in V, \qquad \tilde{\varphi}_0^* = (\xi_0^*, \eta_0^*) \in V' \tag{6.12}$$

---

[4] $\langle \cdot, \cdot \rangle$ denotes the duality pairing between any space and its dual

with

$$\xi_0^* = -\lambda_0 \operatorname{div}(\Delta \psi_0 \,\mathbf{curl}\, X_0^*),$$
$$\eta_0^* = -\lambda_0(\mathbf{grad}\, \psi_0 \cdot \mathbf{curl}\, X_0^*), \tag{6.13}$$

we have

$$D_u F^0 \cdot \tilde{\varphi}_0 = 0 \qquad (D_u F^0)^* \tilde{\varphi}_0^* = 0. \tag{6.14}$$

Moreover, a simple limit point (λ_0, u_0) is a nondegenerate turning point of the Navier-Stokes problem if and only if (λ_0, u_0) is a nondegenerate turning point of F. ■

Note that, by normalizing the eigenfunctions $\tilde{\varphi}_0$ and $\tilde{\varphi}_0^*$, we obtain the eigenvectors φ_0 and φ_0^* of the abstract theory. On the other hand, since

$$X_0^* + \lambda_0 \mathscr{R}(\operatorname{div}(\Delta \psi_0 \,\mathbf{curl}\, X_0^*) - \Delta\,(\mathbf{grad}\, \psi_0 \cdot \mathbf{curl}\, X_0^*)) = 0,$$

it follows from (6.13) that

$$X_0^* = \mathscr{R}(\xi_0^* - \Delta \eta_0^*). \tag{6.15}$$

Assume that ψ_0 is a simple limit solution of (6.1) corresponding to $v = v_0$. Then, by the results of Sects. 2 and 3, there exists a unique branch $\{(\lambda(\alpha), \psi(\alpha));\ |\alpha| \leqq \alpha_0\}$ of solutions of (6.1) in a neighbourhood of (λ_0, ψ_0) in $\mathbb{R} \times H_0^2(\Omega)$ with $\lambda(0) = \lambda_0$, $\psi(0) = \lambda_0$; or equivalently, there exists a unique branch $\{(\lambda(\alpha), u(\alpha) = (\psi(\alpha), -\Delta \psi(\alpha)));\ |\alpha| \leqq \alpha_0\}$ of solutions of (6.11) in a neighbourhood of (λ_0, u_0) with $\lambda(0) = \lambda_0$, $u(0) = u_0$. This branch is of class C^∞.

Let us introduce a mixed finite element method yet considered in [4], [2, Section 4]. *For simplicity*, we suppose that Ω is a *polygonal domain* which is assumed to be *convex* so that the linear operator T is continuous from $W^{-1,\frac{4}{3}}(\Omega)$ into $W^{3,\frac{4}{3}}(\Omega) \times W^{1,\frac{4}{3}}(\Omega)$ (cf. [10], [5]).

Let $(\mathscr{T}_h)$ be a family of triangulations of $\bar{\Omega}$ made with triangles K whose diameters are $\leqq h$. We assume that $(\mathscr{T}_h)$ is uniformly regular in the sense that there exist two constants $\sigma, \tau > 0$ independent of h such that

$$h_K \leqq \sigma \rho_K, \qquad \tau h \leqq h_K \leqq h,$$

where h_K is the diameter of K and ρ_K is the diameter of the inscribed circle in K. Then, we define for each integer $l \geqq 1$ the finite-dimensional spaces

$$\Theta_h = \Theta_h^{(l)} = \{\theta \in C^0(\bar{\Omega});\ \theta|_K \in P_l \quad \text{for all} \quad K \in \mathscr{T}_h\},$$
$$\Phi_h = \Phi_h^{(l)} = \{\varphi \in \Theta_h;\ \varphi|_\Gamma = 0\}, \tag{6.16}$$
$$V_h = V_h^{(l)} = \Theta_h \times \Phi_h \subset V,$$

where P_l denotes the space of all polynomials of degree $\leqq l$ in the two variables x_1, x_2.

Let us define the operator T_h: $g \in W \to u_h = (\psi_h, \omega_h) = T_h g \in V_h$ by

$$\int_\Omega \nabla \omega_h \cdot \nabla \varphi \, dx = \langle g, \varphi \rangle \qquad \text{for all } \varphi \in \Phi_h,$$
$$\int_\Omega (\omega_h \theta - \nabla \psi_h \cdot \nabla \theta)\, dx = 0 \qquad \text{for all } \theta \in \Theta_h, \tag{6.17}$$

where ∇ stands for the operator **grad**. Then a mixed finite element approximation of the Navier-Stokes problem consists in finding a pair $u_h=(\psi_h, \omega_h)\in V_h$ solution of

$$F_h(\lambda, u_h)\equiv u_h + T_h G(\lambda, u_h)=0, \qquad \lambda=\frac{1}{v}. \tag{6.16}$$

Let us recall the approximation properties of the operator T_h (cf. [4], [2, Lemma 6]).

Lemma 9. *Assume that the polygonal domain is convex. Then, we have*

$$\lim_{h\to 0} \| T-T_h\|_{\mathscr{L}(W;V)}=0. \tag{6.19}$$

If $g\in W$ is chosen in such a way that $u=(\psi,\omega)=Tg$ satisfies the smoothness property $\psi\in H^{k+\frac{3}{2}}(\Omega)\cap W^{k+1,\infty}(\Omega)$ for some $k\in\mathbb{R}$ with $1\leqq k\leqq l$, we have

$$\|(T-T_h) g\|_V \leqq C h^{k-\frac{1}{2}} |\ln h|^{\beta} (\|\psi\|_{H^{k+\frac{3}{2}}(\Omega)} + \|\psi\|_{W^{k+1,\infty}(\Omega)}), \tag{6.20}$$

where $\beta=0$ if $l\geqq 2$ and $\beta=1$ if $l=1$.

We are now able to prove

Theorem 8. *Let ψ_0 be a simple limit solution of the Navier-Stokes problem* (6.1) *corresponding to $\lambda_0=1/v_0$ and let $\{(\lambda(\alpha), \psi(\alpha)); |\alpha|\leqq\alpha_0\}$ be the branch of solutions of* (6.1) *such that $\lambda(0)=\lambda_0$, $\psi(0)=\psi_0$. Then, there exists a neighbourhood $\mathcal{O}$ of the origin in $\mathbb{R}\times W_0^{1,4}(\Omega)\times L^2(\Omega)$ and, for $h\leqq h_0$ small enough, a unique branch $\{(\lambda_h(\alpha), u_h(\alpha)=(\psi_h(\alpha), \omega_h(\alpha))); |\alpha|\leqq\alpha_0\}$ of solutions of* (6.18) *such that $(\lambda_h(\alpha)-\lambda(\alpha), u_h(\alpha)-u(\alpha))\in\mathcal{O}$ for all $|\alpha|\leqq\alpha_0$ where $u(\alpha)=(\psi(\alpha), \omega(\alpha)=-\Delta\psi(\alpha))$. Moreover, we have*

$$\alpha\to(\lambda_h(\alpha), u_h(\alpha)) \quad \text{is a } C^\infty \text{ functions from } [-\alpha_0, +\alpha_0] \text{ into } V_h; \tag{6.21}$$

$$\begin{aligned}\lim_{h\to 0} \sup_{|\alpha|\leqq\alpha_0} \{&|\lambda_h^{(m)}(\alpha)-\lambda^{(m)}(\alpha)| + \|\psi_h^{(m)}(\alpha)-\psi^{(m)}(\alpha)\|_{W_0^{1,4}(\Omega)} \\ &+ \|\omega_h^{(m)}(\alpha)-\omega^{(m)}(\alpha)\|_{L^2(\Omega)}\}=0,\end{aligned} \tag{6.22}$$

for all integer $m\geqq 0$.

If, in addition, $\alpha\to\psi(\alpha)$ is a C^m function from $[-\alpha_0, +\alpha_0]$ into $H^{k+\frac{3}{2}}(\Omega)\cap W^{k+1,\infty}(\Omega)$ for some $k\in\mathbb{R}$ with $1\leqq k\leqq l$, we get the estimate

$$\begin{aligned}|\lambda_h^{(m)}(\alpha)-\lambda^{(m)}(\alpha)| &+ \|\psi_h^{(m)}(\alpha)-\psi^{(m)}(\alpha)\|_{W_0^{1,4}(\Omega)} \\ &+ \|\omega_h^{(m)}(\alpha)-\omega^{(m)}(\alpha)\|_{L^2(\Omega)}\leqq C h^{k-\frac{1}{2}}|\ln h|^{\beta},\end{aligned} \tag{6.23}$$

where $\beta=0$ if $l\leqq 2$ and $\beta=1$ if $l=1$ and C is a constant independent of h and $\alpha\in[-\alpha_0, \alpha_0]$.

Proof. Let us check the hypotheses of Theorem 3. First, the properties (2.7) and (3.1) follow from Lemma 8. Next, we may write $G(\lambda, u)=\lambda H(u)$ where H is a C^∞ quadratic mapping from V into W and D^2H is bounded on all bounded subsets of V. Moreover, (2.27) holds by Lemma 9. Hence, we may apply Theorem 3: the desired results are therefore consequence of Lemma 9 again and the fact that $u(\alpha) = -TG(\lambda(\alpha), u(\alpha))$. ■

We conclude this section by considering the case of a nondegenerate turning point of the Navier-Stokes problem. We begin by a preliminary result. Let $g, g^* \in W$; we set

$$Tg = (\psi, \omega), \qquad Tg^* = (\psi^*, \omega^*), \qquad T_h g = (\psi_h, \omega_h), \qquad T_h g^* = (\psi_h^*, \omega_h^*).$$

Lemma 10. *We have for all $\theta, \theta^* \in \Theta_h$*

$$\langle \psi - \psi_h, g^* \rangle = \int_\Omega \{ \nabla(\psi - \psi_h) \cdot \nabla(\omega^* - \theta^*) + \nabla(\psi^* - \psi_h^*) \cdot \nabla(\omega - \theta) + (\omega - \omega_h)(\theta^* - \omega^*) + (\omega^* - \omega_h^*)(\theta - \omega) + (\omega - \omega_h)(\omega^* - \omega_h^*) \} \, dx. \tag{6.24}$$

Proof. Using (6.2), (6.9) and (6.17), it is an easy matter to check that the following properties hold:

$$\begin{aligned}
&\text{(i)} && \int_\Omega \nabla(\omega - \omega_h) \cdot \nabla\varphi \, dx = 0 && \\
&\text{(ii)} && \int_\Omega \nabla(\omega^* - \omega_h^*) \cdot \nabla\varphi \, dx = 0 && \text{for all } \varphi \in \Phi_h, \\
&\text{(iii)} && \int_\Omega \nabla(\psi - \psi_h) \cdot \nabla\theta \, dx = \int_\Omega (\omega - \omega_h)\, \theta \, dx && \\
&\text{(iv)} && \int_\Omega \nabla(\psi^* - \psi_h^*) \cdot \nabla\theta \, dx = \int_\Omega (\omega^* - \omega_h^*)\, \theta \, dx && \text{for all } \theta \in \Theta_h.
\end{aligned} \tag{6.25}$$

We have

$$\langle \psi - \psi_h, g^* \rangle = -\langle \psi - \psi_h, \Delta\omega^* \rangle = \int_\Omega \nabla(\psi - \psi_h) \cdot \nabla\omega^* \, dx$$

so that we obtain for all $\theta^* \in \Theta_h$

$$\langle \psi - \psi_h, g^* \rangle = \int_\Omega \nabla(\psi - \psi_h) \cdot \nabla(\omega^* - \theta^*) \, dx + \int_\Omega \nabla(\psi - \psi_h) \cdot \nabla\theta^* \, dx. \tag{6.26}$$

Next, using (6.25) (iii) with $\theta = \theta^*$, we get

$$\begin{aligned}
\int_\Omega \nabla(\psi - \psi_h) \cdot \nabla\theta^* \, dx &= \int_\Omega (\omega - \omega_h)\, \theta^* \, dx \\
&= \int_\Omega (\omega - \omega_h)(\theta^* - \omega^*) \, dx + \int_\Omega (\omega - \omega_h)\, \omega^* \, dx.
\end{aligned} \tag{6.27}$$

Using (6.25) (i) with $\varphi = \psi_h^*$ gives

$$\int_\Omega (\omega - \omega_h)\, \omega^* \, dx = \int_\Omega \nabla(\omega - \omega_h) \cdot \nabla\psi^* \, dx = \int_\Omega \nabla(\omega - \omega_h) \cdot \nabla(\psi^* - \psi_h^*) \, dx.$$

Hence, by (6.25) (iv), we obtain for all $\theta \in \Theta_h$

$$\int_\Omega (\omega - \omega_h)\, \omega^* \, dx = \int_\Omega \nabla(\omega - \theta) \cdot \nabla(\psi^* - \psi_h^*) \, dx + \int_\Omega (\theta - \omega_h)(\omega^* - \omega_h^*) \, dx. \tag{6.28}$$

Now, (6.24) follows trivially from (6.26), (6.27) and (6.28). ■

We can now state

Theorem 9. *Assume the hypotheses of Theorem 8. Assume in addition that* (λ_0, ψ_0) *is a nondegenerate turning point of the Navier-Stokes problem. Then, the approximate problem* (6.18) *has a unique nondegenerate turning point* $(\lambda_h^0, u_h^0 = (\psi_h^0, \omega_h^0)) \in \mathbb{R} \times V_h$ *in a suitable neighbourhood of* $(\lambda_0, u_0) = (\lambda_0, (\psi_0, \omega_0 = -\Delta\psi_0))$ *in* $\mathbb{R} \times V$. *Moreover, we have*

$$\lim_{h\to 0} \{|\lambda_h^0 - \lambda_0| + \|\psi_h^0 - \psi_0\|_{W_0^{1,4}(\Omega)} + \|\omega_h^0 - \omega_0\|_{L^2(\Omega)}\} = 0. \tag{6.29}$$

If, in addition, $\alpha \to \psi(\alpha)$ *is a* C^1 *function from* $[-\alpha_0, \alpha_0]$ *into* $H^{k+\frac{3}{2}}(\Omega) \cap W^{k+1,\infty}(\Omega)$ *for some* $k \in \mathbb{R}$ *with* $1 \leq k \leq l$, *we get the estimate*

$$|\lambda_h^0 - \lambda_0| + \|\psi_h^0 - \psi_0\|_{W_0^{1,4}(\Omega)} + \|\omega_h^0 - \omega_0\|_{L^2(\Omega)} \leq C h^{k-\frac{1}{2}} |\ln h|^\beta, \tag{6.30}$$

where β *is defined as in Theorem 8. Furthermore, we obtain if the functions* ψ_0, X_0^* *belong to* $H^{k+2}(\Omega) \cap W^{k+1,\infty}(\Omega)$

$$|\lambda_h^0 - \lambda_0| \leq \begin{cases} C h^{2k-1} & \text{if } l \geq 2 \\ C h^{1-\varepsilon} & \text{if } l = 1 \end{cases}. \tag{6.31}$$

Proof. The first part of the theorem together with the bound (6.30) follow immediately from Theorem 4 and Lemmata 8 and 9. It remains only to check the bound (6.31).

For the sake of simplicity, we restrict ourselves to the case $l = 2$.

Then (6.31) will be a consequence of Theorem 5 and Lemma 9 if we show that the two following estimates hold:

$$|\langle (T - T_h) G^0, \varphi_0^* \rangle| \leq C h^{2k-1}, \tag{6.32}$$

$$\|[(T - T_h) D_u G^0]^* \varphi_0^*\|_{V'} \leq C h^{k-\frac{1}{2}}, \tag{6.33}$$

where $\varphi_0^* = \mu \tilde{\varphi}_0^*$, $\tilde{\varphi}_0^*$ being given by (6.12) and (6.13) and μ being a normalizing factor. We first notice that

$$\|[(T - T_h) D_u G^0]^* \varphi_0^*\|_{V'} = \sup_{\substack{v \in V \\ \|v\|_V = 1}} |\langle (T - T_h) D_u G^0 \cdot v, \varphi_0^* \rangle|.$$

Hence, in order to prove (6.32) and (6.33), we need to estimate expressions of the form

$$\langle (T - T_h) g, \tilde{\varphi}_0^* \rangle, \qquad g \in W.$$

Given $g \in W$, we set:

$$Tg = (\psi, \omega), \qquad T_h g = (\psi_h, \omega_h)$$

Using (6.13) and the regularity hypotheses on the functions ψ_0 and X_0^*, we have $\eta_0^* \in H^k(\Omega)$. Thus, we may write

$$\langle (T - T_h) g, \tilde{\varphi}_0^* \rangle = \langle \psi - \psi_h, \xi_0^* - \Delta\eta_0^* \rangle + \langle \psi - \psi_h, \Delta\eta_0^* \rangle + \langle \omega - \omega_h, \eta_0^* \rangle. \tag{6.34}$$

Let us first derive a bound for $|\langle\psi-\psi_h, \xi_0^*-\Delta\eta_0^*\rangle|$. We set:

$$g^*=\xi^*-\Delta\eta_0^*, \qquad Tg^*=(\psi^*,\omega^*), \qquad T_h g^*=(\psi_h^*,\omega_h^*).$$

Using (6.15), we have $\psi^*=X_0^*$, $\omega^*=-\Delta X_0^*$. Moreover, by Lemma 10, we have, for all $\theta,\theta^*\in\Theta_h$

$$\begin{aligned}|\langle\psi-\psi_h, \xi_0^*-\Delta\eta_0^*\rangle| \leqq & \|\psi-\psi_h\|_{H^1(\Omega)}\|\omega^*-\theta^*\|_{H^1(\Omega)}+\|\psi^*-\psi_h^*\|_{H^1(\Omega)}\|\omega-\theta\|_{H^1(\Omega)}\\ &+\|\omega-\omega_h\|_{L^2(\Omega)}\|\omega^*-\theta^*\|_{L^2(\Omega)}+\|\omega^*-\omega_h^*\|_{L^2(\Omega)}\|\omega-\theta\|_{L^2(\Omega)}\\ &+\|\omega-\omega_h\|_{L^2(\Omega)}\|\omega^*-\omega_h^*\|_{L^2(\Omega)}. \end{aligned} \tag{6.35}$$

On the other hand, using (6.17) and $\omega=-\Delta\psi$, we may write for all $\zeta\in\Theta_h$

$$\begin{aligned}\langle\psi-\psi_h, \Delta\eta_0^*\rangle+\langle\omega-\omega_h,\eta_0^*\rangle = & -\int_\Omega \nabla(\psi-\psi_h)\cdot\nabla(\eta_0^*-\zeta)\,dx\\ &+\int_\Omega(\omega-\omega_h)(\eta_0^*-\zeta)\,dx,\end{aligned}$$

so that

$$\begin{aligned}|\langle\psi-\psi_h, \Delta\eta_0^*\rangle+\langle\omega-\omega_h,\eta_0^*\rangle| \leqq & \|\psi-\psi_h\|_{H^1(\Omega)}\|\eta_0^*-\zeta\|_{H^1(\Omega)}\\ &+\|\omega-\omega_h\|_{L^2(\Omega)}\|\eta_0^*-\zeta\|_{L^2(\Omega)}. \end{aligned} \tag{6.36}$$

Thus, combining (6.34)–(6.36), we get for all $\theta,\theta^*,\zeta\in\Theta_h$

$$\begin{aligned}|\langle(T-T_h)g,\tilde\varphi_0^*\rangle| \leqq & \|\psi-\psi_h\|_{H^1(\Omega)}(\|\omega^*-\theta^*\|_{H^1(\Omega)}+\|\eta_0^*-\zeta\|_{H^1(\Omega)})\\ &+\|\psi^*-\psi_h^*\|_{H^1(\Omega)}\|\omega-\theta\|_{H^1(\Omega)}+\|\omega-\omega_h\|_{L^2(\Omega)}\\ &+(\|\omega^*-\theta^*\|_{L^2(\Omega)}+\|\eta_0^*-\zeta\|_{L^2(\Omega)})\\ &+\|\omega^*-\omega_h^*\|_{L^2(\Omega)}\|\omega-\theta\|_{L^2(\Omega)}\\ &+\|\omega-\omega_h\|_{L^2(\Omega)}\|\omega^*-\omega_h^*\|_{L^2(\Omega)}. \end{aligned} \tag{6.37}$$

Were are now able to obtain (6.32) and (6.33). First, we choose in (6.37) $g=-G^0$ so that $\psi=\psi_0$, $\omega=-\Delta\psi_0$. Since the functions $\psi=\psi_0$ and $\psi^*=X_0^*$ belong to $H^k(\Omega)$, we have by [4, Remark 6.1]

$$\|\psi-\psi_h\|_{H^1(\Omega)}+\|\psi^*-\psi_h^*\|_{H^1(\Omega)}\leqq Ch^k.$$

Thus, it follows from (6.37) that we have for all $\theta,\theta^*,\zeta\in\Theta_h$

$$\begin{aligned}|\langle(T-T_h)G^0,\tilde\varphi_0^*\rangle| \leqq & Ch^{k-1}\{h\|\omega-\theta\|_{H^1(\Omega)}+\|\omega-\theta\|_{L^2(\Omega)}\\ &+h\|\omega^*-\theta^*\|_{H^1(\Omega)}+\|\omega^*-\theta^*\|_{L^2(\Omega)}+h\|\eta_0^*-\zeta\|_{H^1(\Omega)}+\|\eta_0^*-\zeta\|_{L^2(\Omega)}+h^k\}.\end{aligned}$$

Since $\omega,\omega^*,\eta_0^*\in H^k(\Omega)$, the bound (6.32) is a consequence of the previous inequality and standard approximation results.

In order to derive (6.33), we choose in (6.37) $g=D_uG^0\cdot v$, $v\in V$. We observe that $Tg\in H^3(\Omega)\times H^1(\Omega)$. Then using Lemma 9 and the same technique as above gives the desired estimate. ∎

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