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Response of a Lorentzian Gas to an a. c. Pulse*

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The response of a "Lorentz gas" to a pulsed a.c. electric field and its relaxation after the cessation of this field has been studied by solving the Boltzmann's transfer equation. Explicit expressions for the electron distribution function and the current density are obtained under the assumption that the collision frequency is independent of the electron velocity.

Introduction

In this communication we have studied the response of a Lorentz gas** to an applied a.c. electric pulse by solving the Boltzmann's transfer equation under the assumption that the collision frequency is independent of velocity. After the cessation of the pulse, the discrete modes describing the relaxation of the velocity distribution to the equilibrium Maxwellian distribution are shown to be product of the Maxwellian distribution and associated Laguerre polynomials. The corresponding discrete relaxation constants have also been obtained.

Boltzmann's Transfer Equation and its Solutions

The Boltzmann's transfer equation for electrons in a homogeneous plasma may be written as

$$\frac{\partial f}{\partial t}+\boldsymbol{\alpha}\cdot\nabla_v f=\left(\frac{\partial f}{\partial t}\right)_C, \tag{1}$$

where $f(\boldsymbol{v}, t)$ is the distribution function of electron velocities, t is time, $\boldsymbol{\alpha}$ is the acceleration of electrons; ∇_v represents the gradient in the velocity space and $(\partial f/\partial t)_c$ is the rate of change of f due to collisions.

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** By a Lorentz gas we imply that the electric field is not disturbed by the plasma itself and electrons interact only with neutral gas molecules and not with electrons or ions.

The acceleration α of electrons in the presence of an electric field $\boldsymbol{E}(t)$ is given by

$$\alpha = -\frac{e\,E(t)}{m}, \tag{2}$$

where e is the electronic charge and m is the electronic mass. In the present investigation we will consider the electric field acting along the x-axis. Further, we will assume the time dependence of the electric field to be of the form[1]:

$$E(t) = A(t)\,\mathrm{Cos}\,\omega t, \tag{3}$$

where $A(t)$ is a slowly varying function of t (except for certain points where $A(t)$ may vary rapidly). In the simplest case of the "cut off sinusoid"

$$\begin{aligned} A(t) &= E_0 \quad \text{for } 0<t<\tau \\ &= 0 \quad\;\; \text{for } t<0 \;\text{ and }\; t>\tau, \end{aligned} \tag{4}$$

i.e. the electric field is $E_0 \exp(i\omega t)$ in the interval $0<t<\tau$ and zero outside the interval. The transfer equation thus becomes:

$$\frac{\partial f}{\partial t} - \frac{e\,E_0}{m}\,\mathrm{Cos}\,\omega t\,\frac{\partial f}{\partial v_x} = \left(\frac{\partial f}{\partial t}\right)_C, \quad 0<t<\tau \tag{5}$$

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_C, \quad t<0 \;\text{ and }\; t>\tau. \tag{6}$$

The distribution function is expanded as

$$f = f_0(v,t) + v_x\,g(v,t). \tag{7}$$

Further, for a slightly ionized gas consisting of neutral molecules and electrons, the rate of change of the electron velocity distribution function due to collisions can be shown to be of the form[2]

$$\left(\frac{\partial f}{\partial t}\right)_C = -\nu(f-f_0) + \frac{m}{M v^2}\frac{\partial}{\partial v}(\nu v^3 f_0) + \frac{kT}{M v^2}\frac{\partial}{\partial v}\left(\nu v^2 \frac{\partial f_0}{\partial v}\right), \tag{8}$$

where $v = |\boldsymbol{v}|$ denotes the speed of the electron, $\nu(v)$ is the electron collision frequency, M is the mass of molecules, k is the Boltzmann constant and T the temperature of the gas.

Let us first obtain the solution of the transfer equation for $0<t<\tau$ subject to the boundary condition that before the pulse the electron

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1. Ginzburg, V. L.: Propagation of electromagnetic waves in plasma, chap. IV. New York: Gordon and Breach 1961.
2. Desloge, E. A., and S. W. Mathysee: Am. J. Phys. **28**, 1 (1960).

distribution was in thermal equilibrium and therefore given by

$$f(\boldsymbol{v})=f_0(v)=C\exp\left(-\frac{mv^2}{2kT}\right),\qquad t\leqq 0.$$

The above represents a Maxwellian distribution; C being an arbitrary constant. Substituting Eqs. (7) and (8) in Eq. (5), and equating the terms having v_x and the remaining terms on both sides of the equation, we obtain

$$\frac{\partial g}{\partial t}-\frac{eE(t)}{mv}\frac{\partial f_0}{\partial v}=-\nu g \tag{9}$$

and

$$\frac{\partial f_0}{\partial t}-\frac{eE(t)}{mv}\left[g(v,t)+\frac{v}{3}\frac{\partial g(v,t)}{\partial v}\right] =\frac{m}{Mv^2}\frac{\partial}{\partial v}\left[\nu v^3 f_0\right]+\frac{kT}{Mv^2}\left[\nu v^2\frac{\partial f_0}{\partial v}\right], \tag{10}$$

where in Eq. (10), v_x^2 has been replaced by $v^2/3$. Since $g(v,t)=0$ at $t=0$ the solution of Eq. (9) is immediately obtained:

$$g(v,t)=\frac{e}{mv}e^{-\nu t}\int_0^t e^{\nu t}E(t)\frac{\partial f_0}{\partial v}dt. \tag{11}$$

From now on we will assume that the duration of the pulse is so short that the isotropic part of the distribution function, f_0, does not change appreciably during the interval $0<t<\tau$*. Thus we will take $\partial f_0/\partial v$ outside the integral in Eq. (11) and neglect $\partial f_0/\partial t$ in Eq. (10). Substituting the expression for $E(t)$ in Eq. (11) we obtain

$$g(v,t)=\frac{e}{mv}\frac{\partial f_0(v,t)}{\partial v}K(v,t),\qquad 0<t<\tau \tag{12}$$

where

$$K(v,t)=\frac{E_0}{\omega^2+\nu^2}\left[\omega\,\mathrm{Sin}\,\omega t+\nu\,\mathrm{Cos}\,\omega t-\nu e^{-\nu t}\right]. \tag{13}$$

Next, in order to calculate the isotropic part of the velocity distribution we neglect $\partial f_0/\partial t$ in Eq. (10) and rewrite it in the following form:

$$\frac{\partial}{\partial v}\left[\frac{m}{M}\nu v^3 f_0+\frac{kT}{M}\nu v^2\frac{\partial f_0}{\partial v}\right]=-\frac{\partial}{\partial v}\left[\frac{eE(t)}{3m}v^3 g(v,t)\right].$$

* As will be shown later, this will be true if

$$\left|\frac{Me^2E_0^2\cos\omega t}{3m^2kT\nu(\omega^2+\nu^2)}\left[\omega\sin\omega t+\nu\cos\omega t-\nu e^{-\nu t}\right]\right|\ll 1$$

for all values of t lying between 0 and τ.

Integrating, we obtain

$$\frac{\partial f_0}{\partial v}+\frac{m v}{kT}f_0=-\frac{eE(t)}{3\nu kT}\frac{M}{m}v\,g(v,t),$$

where we have put the constant of integration to be zero by using the boundary condition at $v=0$. The above equation can be written in the form

$$\frac{\partial f_0}{\partial \eta^2}[1+L(t,v)]+f_0=0,$$

where we have used Eq. (11) and

$$\eta=\left[\frac{m}{2kT}\right]^{\frac{1}{2}}v, \tag{14}$$

$$L(t,v)=\frac{Me^2}{3m^2kT}\frac{E(t)K(t,v)}{\nu}. \tag{15}$$

The solution of the above equation, subject to Eq. (8) is given by

$$f_0=C\exp\left[-\int_0^{\eta^2}\frac{d\eta^2}{1+L(t,\eta^2)}\right]. \tag{16}$$

The velocity dependence of the functions K and L come through the velocity dependence of the collision frequency ν. Therefore, if we assume the collision frequency to be velocity independent, the trivial integration in Eq. (16) gives

$$f_0=C\exp\left[-\frac{\eta^2}{1+L(t)}\right] \quad \text{for } 0<t<\tau. \tag{17}$$

This corresponds to a Maxwellian distribution with effective temperature, T_e, given by

$$T_e=T[1+L(t)]. \tag{18}$$

At $t=0$, obviously $L(t)=0$.

Thus, during the duration of the pulse the isotropic part of the velocity distribution is approximately a Maxwellian distribution with an effective temperature changing with time.

Next, let us consider the approach to the equilibrium distribution when the electric field is no more there, i.e. for $t>\tau$. For this we have to solve Eq. (6) subject to the continuity of the distribution function at $t=\tau$. Substituting Eq. (7) in Eq. (6) and equating the terms having

v_x and the remaining terms on both sides of the resulting equation we obtain:

$$\frac{\partial g}{\partial t} + \nu g = 0 \tag{19}$$

and

$$\frac{\partial f_0}{\partial t} = \frac{m}{M v^2} \frac{\partial}{\partial v} [\nu v^3 f_0] + \frac{kT}{M v^2} \frac{\partial}{\partial v} \left[\nu v^2 \frac{\partial f_0}{\partial v} \right]. \tag{20}$$

The solution of Eq. (19) subject to its continuity at $t=\tau$ is

$$g(v, t) = \frac{e}{m v} \frac{\partial f_0(v, \tau)}{\partial v} \frac{E_0}{\omega^2 + \nu^2} [\omega \operatorname{Sin} \omega \tau + \nu \operatorname{Cos} \omega \tau - \nu e^{-\nu \tau}] e^{-\nu (t-\tau)} \tag{21}$$

for $t \geqq \tau$,

showing the exponential decay of the nonisotropic component of the velocity distribution.

In order to solve Eq. (20) we try the method of separation of variables and therefore look for solutions of the form:

$$f_0(v, t) = F_0(v)\, T_0(t),$$

On substitution we obtain

$$\frac{1}{T_0} \frac{dT_0}{dt} = \frac{1}{F_0} \frac{m}{M v^2} \frac{d}{dv} [\nu v^3 F_0] + \frac{1}{F_0} \frac{kT}{M v^2} \frac{d}{dv} \left[\nu v^2 \frac{dF_0}{dv} \right] = -\lambda$$

indicating the variables have indeed separated out. Therefore

$$T_0(t) = \text{Constant} \times \exp(-\lambda t) \tag{22}$$

and

$$\frac{m}{M v^2} \frac{d}{dv} [\nu v^3 F_0] + \frac{kT}{M v^2} \frac{d}{dv} \left[\nu v^2 \frac{dF_0}{dv} \right] + \lambda F_0 = 0. \tag{23}$$

We introduce

$$\Psi(v) = F_0(v) \exp \left[\frac{m v^2}{2kT} \right],$$

which gives us

$$\frac{d}{dv} \left[\nu v^2 \exp\left(-\frac{m v^2}{2kT} \right) \frac{d\Psi}{dv} \right] + \frac{M \lambda}{kT} v^2 \exp\left(-\frac{m v^2}{2kT} \right) \Psi(v) = 0$$

Assuming ν to be independent of velocity and using the dimensionless variable η we obtain

$$\frac{d^2 \Psi}{d\eta^2} + \frac{1}{\eta} (2 - \eta^2) \frac{d\Psi}{d\eta} + 4\kappa \Psi = 0, \tag{24}$$

where

$$\kappa = \frac{M\lambda}{2\nu m}.$$

A further change of variable to $\varepsilon = \eta^2$ leads to the following confluent hypergeometric equation

$$\varepsilon \frac{d^2\Psi}{d\varepsilon^2} + (\tfrac{3}{2} - \varepsilon)\frac{d\Psi}{d\varepsilon} + \kappa\Psi = 0. \tag{25}$$

We will use the standard Frobenius method to solve the above equation. One of the solutions is singular at $v=0$. If we exclude such solutions, the solution of Eq. (25) is a confluent hypergeometric function

$$\Psi = F_1(-\kappa, \tfrac{3}{2}, \varepsilon).$$

The only solutions to Eq. (25) with a satisfactory behaviour as ε approaches infinity occur for κ equal to zero or positive integer. Thus, the corresponding values of λ are given by

$$\lambda_n = \frac{2\nu m}{M} n$$

where

$$n = 0, 1, 2, \ldots$$

For these values of κ, the function $F_1(-\kappa, \frac{3}{2}, \varepsilon)$ becomes a polynomial[3]. The solutions of Eq. (23) are thus

$$F_0(v) = \exp\left[-\frac{m v^2}{2kT}\right] L_n^{\frac{1}{2}}(\varepsilon). \tag{26}$$

The associated Laguerre polynomials are orthogonal in the sense that[3]

$$\int_0^\infty \varepsilon^{\frac{1}{2}} e^{-\varepsilon} L_n^{\frac{1}{2}}(\varepsilon) L_m^{\frac{1}{2}}(\varepsilon)\, d\varepsilon = \delta_{mn} \frac{[\Gamma(n+\frac{3}{2})]^3}{\Gamma(n+1)}. \tag{27}$$

The complete solution of Eq. (20) is therefore

$$f_0(v,t) = \Big[\sum_n A_n L_n^{\frac{1}{2}}(\varepsilon) e^{-\lambda_n t}\Big] e^{-m v^2/2kT}, \qquad t > \tau. \tag{28}$$

The constants, A_n, are to be determined from the initial conditions (i.e. from the value of f_0 at $t=\tau$).

We can rewrite Eq. (28) as

$$f_0(v,t) = \sum_{n=0}^{\infty} B_n L_n^{\frac{1}{2}}(\varepsilon) e^{-\varepsilon} e^{-\lambda_n (t-\tau)}$$

3. Morse, P. M., and H. Feshbach: Methods of theoretical physics, part. I. p. 784, London, New York: McGraw-Hill 1953.

where

$$B_n = A_n e^{-\lambda_n \tau}$$

From the above equation we get

$$f_0(v, \tau) = \sum B_n L_n^{\frac{1}{2}}(\varepsilon) e^{-\varepsilon}.$$

Using Eq. (27) we obtain

$$B_n = \frac{\Gamma(n+1)}{[\Gamma(n+\frac{3}{2})]^3} \int_0^\infty \varepsilon^{\frac{1}{2}} L_n^{\frac{1}{2}}(\varepsilon) f_0(v, \tau) \, d\varepsilon. \tag{29}$$

Substituting Eq. (17) we get

$$B_n = C \frac{\Gamma(n+1)}{[\Gamma(n+\frac{3}{2})]^3} \int_0^\infty \varepsilon^{\frac{1}{2}} L_n^{\frac{1}{2}}(\varepsilon) e^{-\frac{\varepsilon}{1+L(\tau)}} \, d\varepsilon. \tag{30}$$

Since $L_n^{\frac{1}{2}}(\varepsilon)$ are polynomials the evaluation of the integrals is quite straightforward. The result is in terms of the gamma functions. The explicit forms of the first few polynomials are

$$\begin{aligned} L_0^{\frac{1}{2}}(\varepsilon) &= \Gamma(\tfrac{3}{2}) \\ L_1^{\frac{1}{2}}(\varepsilon) &= \Gamma(\tfrac{5}{2}) [\tfrac{3}{2} - \varepsilon] \\ L_2^{\frac{1}{2}}(\varepsilon) &= \tfrac{1}{2} \Gamma(\tfrac{7}{2}) [\tfrac{15}{4} - 5\varepsilon + \varepsilon^2] \end{aligned} \tag{31}$$

in general

$$L_n^{\frac{1}{2}}(\varepsilon) = \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{e^\varepsilon}{\varepsilon^{\frac{1}{2}}} \frac{d^n}{d\varepsilon^n} [\varepsilon^{n+\frac{1}{2}} e^{-\varepsilon}].$$

Expression for Current Density

The current density in a plasma is given by:

$$\boldsymbol{J} = -e \iiint_{-\infty}^{+\infty} \boldsymbol{v} f(\boldsymbol{v}, t) \, dv_x \, dv_y \, dv_z.$$

Since

$$\begin{aligned} & f(\boldsymbol{v}, t) = f_0(v, t) + v_x g(v, t) \\ & J_y = J_z = 0 \quad \text{and} \quad \iiint_{-\infty}^{+\infty} v_x f_0(v, t) \, dv_x \, dv_y \, dv_z = 0 \\ & \therefore \; J_x = -e \iiint_{-\infty}^{+\infty} v_x^2 g(v, t) \, dv_x \, dv_y \, dv_z \\ & \quad = -\frac{4\pi e}{3} \int_0^\infty v^4 g(v, t) \, dv. \end{aligned} \tag{32}$$

In writing the above expression we have, by symmetry, replaced v_x^2 by $v^2/3$, $dv_x\, dv_y\, dv_z$ has been replaced by $4\pi\, v^2\, dv$ and the triple integral has been replaced by a single integral extending from 0 to ∞.

For $t<\tau$ we use Eq. (12) to obtain

$$J_x = -\frac{4\pi e^2}{3m} K(t) \int_0^\infty \frac{\partial f_0}{\partial v} v^3\, dv$$

$$= \frac{4\pi e^2}{m} K(t) \int_0^\infty f_0\, v^2\, dv \qquad [\because\ f_0 v^3 |_0^\infty = 0].$$

We use Eq. (17) for $f_0(v, t)$ and obtain after simple manipulations

$$J_x = C \frac{e^2}{m} K(t) \left[\frac{2\pi k T}{m}(1+L(t))\right]^{\frac{3}{2}} t<\tau. \tag{33}$$

For $t>\tau$ we use Eq. (21) for $g(v, t)$ and obtain in an identical manner

$$J_x = C \frac{e^2}{m} K(\tau) \left[\frac{2\pi k T}{m}(1+L(\tau))\right]^{\frac{3}{2}} e^{-\nu(t-\tau)}. \tag{34}$$

We now summarize our results

(i) For $t \leqq 0$ (i.e. before the pulse)

$$f(\boldsymbol{v}, t) = C \exp\left(-\frac{m v^2}{2kT}\right)$$

$$\boldsymbol{J} = 0.$$

(ii) For $0<t<\tau$

$$f_0(v, t) = C \exp\left[-\frac{m v^2}{2kT(1+L(t))}\right]$$

$$g(v, t) = \frac{e}{m v} \frac{\partial f_0(v, t)}{\partial v} K(t)$$

$$= -C \frac{e}{kT} \frac{K(t)}{1+L(t)} \exp\left[-\frac{m v^2}{2kT(1+L(t))}\right]$$

$$J_x = C \frac{e^2}{m} K(t) \left[\frac{2\pi k T}{m}(1+L(t))\right]^{\frac{3}{2}}.$$

(iii) For $t \geqq \tau$

$$f_0(v, t) = \sum B_n L_n^{\frac{1}{2}}(\varepsilon)\, e^{-\varepsilon}\, e^{-\lambda_n (t-\tau)}$$

$$g(v, t) = -C \frac{e}{kT} \frac{K(\tau)}{1+L(\tau)} \exp\left[-\frac{m v^2}{2kT(1+L(\tau))}\right] e^{-\nu(t-\tau)}$$

$$J_x = C \frac{e^2}{m} K(\tau) \left[\frac{2\pi k T}{m}(1+L(\tau))\right]^{\frac{3}{2}} \exp(-\nu(t-\tau)).$$

Now, the total electron density, N, is given by:

$$N = \iiint f(\mathbf{v}, t)\, d\mathbf{v} = \int_0^\infty f_0(v, t)\, 4\pi v^2\, dv. \tag{35}$$

For $0 < t < \tau$ a straightforward calculation gives

$$N = C\left[\frac{2\pi k T\{1 + L(t)\}}{m}\right]^{\frac{3}{2}},$$

which on substitution gives

$$J_x = \frac{\Omega_p^2}{4\pi} K(t)$$

where

$$\Omega_p^2 = \frac{4\pi N(t)\, e^2}{m}$$

is a slowly varying function of time.

For $t > \tau$, we have to use the orthogonality relation (Eq. (27)) to obtain

$$N = \frac{\pi^2}{2}\left(\frac{2kT}{m}\right)^{\frac{3}{2}} B_0.$$

Substituting the expression for B_0 (Eq. (30)) we obtain

$$N = C\left[\frac{2\pi k T(1 + L(\tau))}{m}\right]^{\frac{3}{2}}$$

showing that the total electron density remains constant for $t \geqq \tau$. Eliminating the constant c we get

$$J_x = \frac{\omega_p^2}{4\pi} K(\tau) \exp[-\nu(t-\tau)] \qquad (t \geqq \tau)$$

where the plasma frequency

$$\omega_p^2 = \frac{4\pi N e^2}{m}.$$

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