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Collocation Methods for Parabolic Partial Differential Equations in One Space Dimension*

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Summary. Collocation at Gaussian points for a scalar m-th order ordinary differential equation has been studied by C. de Boor and B. Swartz. J. Douglas, Jr. and T. Dupont, using collocation at Gaussian points, and a combination of "energy estimates" and approximation theory have given a comprehensive theory for parabolic problems in a single space variable. While the results of this report parallel those of Douglas and Dupont, the approach is basically different. The Laplace transform is used to "lift" the results of de Boor and Swartz to linear parabolic problems. This indicates a general procedure that may be used to "lift" schemes for elliptic problems to schemes for parabolic problems. Additionally there is a section on longtime integration and A-stability.

1. Introduction

Let $T>0$ be given and let

$$\mathbb{R}_T \equiv \{(x, t);\quad 0<x<1,\quad 0<t\leqq T\}, \tag{1.1}$$

and let $\overline{\mathbb{R}}_T$ denote its closure. Consider the mixed initial value-boundary value problem for a function $u(x, t)\in C(\overline{\mathbb{R}}_T)\cap C^2(\mathbb{R}_T)$ which satisfies

$$c(x)\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+b(x)\frac{\partial u}{\partial x}+q(x)\,u+f(x,t), \qquad (x,t)\in\mathbb{R}_T \tag{1.2a}$$

$$u(x,0)=u_0(x), \qquad 0\leqq x\leqq 1, \tag{1.2b}$$

$$u(0,t)=u(1,t)=0, \qquad 0\leqq t\leqq T. \tag{1.2c}$$

We assume that equation (1.2a) is parabolic and that

$$\begin{aligned} 0< \quad m \quad &\leqq c(x)\leqq M, \qquad 0\leqq x\leqq 1\\ |b(x)|&\leqq B\\ |q(x)|&\leqq Q \end{aligned} \tag{1.2d}$$

for appropriate positive constants m, M, B and Q.

Several authors, Douglas and Dupont [5-7], and Archer [1] in particular, have been concerned with numerical methods for this problem based on collocation in the space of piecewise polynomials of order $k+2$ in the x variable.

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Douglas and Dupont consider collocation at Gaussian points and using a combination of "Energy Estimates" and approximation theory arrive at a rather comprehensive study for non-linear problems.

At about the same time, de Boor and Swartz [2] developed an extensive theory for collocation at Gaussian points in general boundary value problems for a scalar, m-th order, ordinary differential equation. These results have been extended by Wittenbrink [13] to include nonlinear boundary conditions and by Russell [10] and Cerutti [4] for systems of equations.

In this report we consider collocation at the Gaussian points for the problem (1.2a)–(1.2d). While our results parallel the results of Dupont and Douglas, our approach is basically different. Our purpose is to "lift" the results of de Boor and Swartz [2] to this parabolic problem via the Laplace transform. This discussion indicates a general procedure for "lifting" results for elliptic problems to results for parabolic problems. While the Laplace transform has been used by Strang and Fix [11] in connection with Galerkin's method for parabolic problems, the extension to collocation methods is not completely straightforward. For one thing, the location of the spectrum of the discrete operators is not immediately apparent as it is in many Ritz-Galerkin methods. Moreover the results of de Boor and Swartz [2] and Dupont and Douglas are so strikingly similar we felt it particularly intriguing to "tie" them together in complete detail.

Finally, it should be mentioned that many of our results are of intrinsic interest, e.g. the results on A-stability of Section 6.

In Section 2 we develop some fundamental concepts. Section 3 is concerned with some basic estimates including the aforementioned results on the spectrum of the discrete operators. In Section 4 we obtain global error estimates and in Section 5 we obtain the "superconvergence" results at the knots. Section 6 is concerned with longtime integration, the approach to the "steady state" and A-stability.

2. Fundamental Concepts

In this section we describe the space S_Δ of piecewise polynomials and the numerical method used to find the approximate solution $V(x, t; \Delta) \in S_\Delta$. Our notation will be consistent with the notation of [2] and [6] wherever possible.

As in the discussion in [2], let $\Delta = \{x_i\}_0^N$ be a strict partition of the interval [0, 1],

$$0 = x_0 < x_1 < \cdots < x_N = 1$$

and set

$$I_j \equiv [x_{j-1}, x_j],$$
$$C_\Delta \equiv C(I_1) \times C(I_2) \times \cdots \times C(I_N).$$

An element $f \in C_\Delta$ consists of N pieces $f_1, f_2, \ldots, f_N$, with $f_j \in C(I_j)$, $j = 1, 2, \ldots, N$ and has two values at the interior breakpoints $\{x_i\}_1^{N-1}$.

C_Δ is a Banach space with respect to the norm

$$\|f\|_\infty \equiv \max_{1 \leq j \leq N} \max_{t \in I_j} |f_j(t)| = \max_{1 \leq j \leq N} \|f_j\|_\infty.$$

Let $\xi_1, \xi_2, \ldots, \xi_k$ be the Gaussian points on the interval [0, 1] with associated quadrature weights $w_1, w_2, \ldots, w_k$. That is

$$p_k = \prod_{j=1}^{k} (x-\xi_j) \tag{2.1}$$

satisfies

$$\int_0^1 p_k(x)\, q(x)\, dx = 0 \qquad \forall q(x) \in \mathbb{P}_k[0,1] \tag{2.2}$$

and

$$\int_0^1 q(x)\, dx = \sum_{j=1}^{k} w_j\, q(\xi_j) \qquad \forall q(x) \in \mathbb{P}_{2k}[0,1] \tag{2.3}$$

where $\mathbb{P}_k[0,1]$ denotes the set of polynomials of order k (degree less than k) on $[0,1]$.

With $\Delta_i = x_i - x_{i-1}$, $i=1,2,\ldots,N$ let

$$\begin{aligned} \xi_{ij} &= x_{i-1} + \Delta_i \xi_j, \\ w_{ij} &= \Delta_i w_j, \qquad i=1,2,\ldots,N, \qquad j=1,2,\ldots,k, \end{aligned} \tag{2.4}$$

thus

$$\int_{x_{i-1}}^{x_i} \prod_{j=1}^{k} (x-\xi_{ij})\, q(x) = 0 \qquad \forall q(x) \in \mathbb{P}_k[x_{i-1}, x_i], \qquad i=1,2,\ldots,N. \tag{2.5}$$

Let

$$S_\Delta \equiv \{f \in C^1[0,1]; \quad f|_{I_j} \in \mathbb{P}_{k+2},\ j=1,2,\ldots,N; \quad f(0)=f(1)=0\} \tag{2.6}$$

Lemma 2.1. Given real numbers Y_{ij} $i=1,2,\ldots,N$, $j=1,2,\ldots,k$ there exists a unique $f \in S_\Delta$ such that $f(\xi_{ij}) = Y_{ij}$.

Proof. This result follows from well known results on interpolation by piecewise polynomials. See e.g. [3]. □

We seek a function $U(x,t,\Delta)$ which is an element of S_Δ for each $t \geqq 0$ and satisfies the collocation equations

$$\begin{aligned} c(\xi_{ij}) \frac{\partial U}{\partial t}(\xi_{ij}, t, \Delta) &= \frac{\partial^2 U}{\partial x^2}(\xi_{ij}, t, \Delta) + b(\xi_{ij}) \frac{\partial U}{\partial x}(\xi_{ij}, t; \Delta) \\ &\quad + q(\xi_{ij})\, U(\xi_{ij}, t, \Delta) + f(\xi_{ij}, t) \\ &\quad 0 < t \leqq T, \quad i=1,\ldots,N, \quad j=1,\ldots,k, \\ U(x,0;\Delta) &= U_0(x;\Delta) \in S_\Delta, \end{aligned} \tag{2.7}$$

where the initial data $U_0(x;\Delta)$ are suitably chosen. There are several ways to choose $U_0(x;\Delta)$ to obtain optimum error estimates and we discuss this question in Sections 4 and 5.

Collocation is a spatial operator and hence commutes with the Laplace transform (with respect to t) of equation (1.2). However, for the sake of completeness we give a detailed discussion of this fact.

Let $\{\phi_n\}_{n=1}^{Nk}$ be a real basis for S_Δ. If we let

$$U(x,t;\Delta) = \sum_{n=1}^{Nk} \alpha_n(t)\, \phi_n(x) \tag{2.8}$$

then the collocation equations (2.7) become the following ordinary differential equations for the functions $\alpha_n(t)$,

$$\begin{aligned} c(\xi_{ij}) \sum_{n=1}^{Nk} \frac{d}{dt} \alpha_n(t)\, \phi_n(\xi_{ij}) &= \sum_{n=1}^{Nk} \alpha_n(t)\,[\phi_n''(\xi_{ij}) + b(\xi_{ij})\, \phi_n'(\xi_{ij}) + q(\xi_{ij})\, \phi_n(\xi_{ij})] \\ &\quad + f(\xi_{ij}, t); \quad i=1,2,\ldots,N; \quad j=1,2,\ldots,k; \\ \sum_{n=1}^{Nk} \alpha_n(0)\, \phi_n(x) &= U_0(x;\Delta). \end{aligned} \tag{2.9}$$

The Laplace transform of (2.9) with respect to t is

$$\begin{aligned} s\, c(\xi_{ij}) \sum_{n=1}^{Nk} \hat{\alpha}_n(s)\, \phi_n(\xi_{ij}) - c(\xi_{ij})\, U_0(\xi_{ij}) &= \sum_{n=1}^{Nk} \hat{\alpha}_n(s)\,[\phi_n''(\xi_{ij}) + b(\xi_{ij})\, \phi_n'(\xi_{ij}) + q(\xi_{ij})\, \phi_n(\xi_{ij})] \\ &\quad + \hat{f}(\xi_{ij}, s); \quad i=1,2,\ldots,N; \quad j=1,2,\ldots,k; \end{aligned} \tag{2.10}$$

where

$$\hat{\alpha}_n(s) = \int_0^\infty e^{-st}\, \alpha_n(t)\, dt$$

and

$$\hat{f}(\xi_{ij}, s) = \int_0^\infty e^{-st} f(\xi_{ij}, t)\, dt.$$

The Laplace transform of (1.2) is

$$\begin{aligned} s\, c(x)\, \hat{u}(x,s) - c(x)\, u_0(x) &= \hat{u}_{xx}(x,s) + b(x)\, \hat{u}_x(x,s) \\ &\quad + q(x)\, \hat{u}(x,s) + \hat{f}(x,s). \end{aligned} \tag{2.11}$$

The collocation equations for this ordinary differential equation (depending on the parameter s) are

$$\begin{aligned} s\, c(\xi_{ij})\, \hat{U}(\xi_{ij}, s) - c(\xi_{ij})\, U_0(\xi_{ij}) &= \hat{U}_{xx}(\xi_{ij}, s) + b(\xi_{ij})\, \hat{U}_x(\xi_{ij}, s) \\ &\quad + q(\xi_{ij})\, \hat{U}(\xi_{ij}, s) + \hat{f}(\xi_{ij}, s). \end{aligned} \tag{2.12}$$

By (2.8)

$$\hat{U}(x, s; \Delta) = \sum_{n=1}^{NK} \hat{\alpha}_n(s)\, \phi_n(x).$$

Thus (2.12) is the same as (2.9), hence collocation commutes with the Laplace transform.

3. Basic Estimates

We begin this section with two fundamental lemmas which are the basis of the "energy estimates" of Douglas and Dupont [5–7]. These estimates give relations between the discrete inner product described below and the usual L^2 inner product.

Let $(\cdot,\cdot)$ be the usual L^2 inner product;

$$(u, v) = \int_0^1 u(x)\,\bar{v}(x)\,dx, \qquad \|u\|^2 = (u, u), \tag{3.1}$$

and let $\langle\cdot,\cdot\rangle$ be the quadratic form

$$\langle f, g\rangle = \sum_{i=1}^{N} \sum_{j=1}^{k} w_{ij}\, f(\xi_{ij})\, \bar{g}(\xi_{ij}), \qquad \|f\|_d^2 = \langle f, f\rangle. \tag{3.2}$$

Lemma 3.1. For all $f, g \in S_\Delta$

$$\langle f, g''\rangle = -(f', g') - P_k \sum_{i=1}^{N} f_i^{k+1} \bar{g}_i^{\,k+1}\,(\Delta_i)^{2k+1} \tag{3.3}$$

where f_j^{k+1} is the constant $(k+1)$ st derivative of f_j on I_j and P_k is a positive constant depending only on k.

Proof. First consider $\langle\cdot,\cdot\rangle$ on an interval I_i i.e.

$$\langle f, g''\rangle_i = \sum_{j=1}^{k} w_{ij}\, f(\xi_{ij})\, \bar{g}''(\xi_{ij}).$$

Since on I_i $f, g \in \mathbb{P}_{k+2}$ we have

$$f_i = f\,|_{I_i} = \sum_{r=0}^{k+1} a_r\, x^r, \quad g_i = g\,|_{I_i} = \sum_{r=0}^{k+1} b_r\, x^r.$$

The error term in the Gaussian quadrature formula

$$\int_{I_i} f_i = \sum_{j=1}^{k} w_{ij}\, f_i(\xi_{ij}) + E_k(f)$$

is given by (see, e.g., Isaacson and Keller [9], pp. 330)

$$E_k(f) = \frac{f_i^{(2k)}(\xi)}{(2k)!} \int_{I_i} \left[\prod_{j=1}^{k} (x - \xi_{ij})\right]^2 dx.$$

Hence

$$\langle f, g''\rangle_i = \int_{I_i} f\,\bar{g}'' - \frac{(2k)!\,k(k+1)\,a_{k+1}\,\bar{b}_{k+1}}{(2k)!} \int_{x_{i-1}}^{x_i} \left[\prod_{j=1}^{k} (x - \xi_{ij})\right]^2 dx.$$

With the change of variable $x = x_{i-1} + t\Delta_i$ we see that

$$\int_{x_{i-1}}^{x_i} \left[\prod_{j=1}^{k} (x - \xi_{ij})\right]^2 dx = \int_0^1 \left[\prod_{j=1}^{k} (t - \xi_j)\,\Delta_i\right]^2 \Delta_i\,dt = \Delta_i^{2k+1} \int_0^1 \left[\prod_{i=1}^{k} (t - \xi_j)\right]^2 dt.$$

Letting $P_k = \dfrac{k(k+1)}{[(k+1)!]^2} \displaystyle\int_0^1 \left[\prod_{j=1}^{k} (t - \xi_j)\right]^2 dt$

and integrating once by parts we obtain

$$\langle f, g''\rangle_i = -\int_{I_i} f'\,\bar{g}' + f\,\bar{g}'\,\Big|_{x_{i-1}}^{x_i} - [(k+1)!]^2\, P_k\, a_{k+1}\,\bar{b}_{k+1}\,\Delta_i^{2k+1}.$$

Since $f, g \in C^1[0,1]$, and satisfy zero boundary conditions, the terms $f\bar{g}'|_{x_{i-1}}^{x_i}$ cancel upon summation and we have

$$\langle f, g''\rangle = \sum_{i-1}^{N} \langle f, g''\rangle_i = -(f', g') - [(k+1)!]^2 P_k \sum_{i=1}^{N} a_{k+1} \bar{b}_{k+1} \Delta_i^{2k+1}.$$

Since a_{k+1}, b_{k+1} depend on the interval I_i, (3.3) follows. □

Corollary 3.1. For all $f \in S_\Delta$

$$\langle f, f''\rangle = \langle f'', f\rangle = -(f', f') - P_k \sum_{i=1}^{N} |f_i^{(k+1)}|^2 \Delta_i^{2k+1} \tag{3.4}$$

and

$$\frac{d}{dt} \langle V_{xx}, V\rangle = \langle V_{xx}, V_t\rangle + \langle V_t, V_{xx}\rangle \tag{3.5}$$

where

$$V(x, t; \Delta) = \sum_{i=1}^{Nk} \alpha_i(t)\, \phi_i(x),\ \{\phi_i\}_1^{Nk}$$

is a basis for S_Δ and $\alpha_i(t) \in C^1[0, T]$.

Proof. (3.4) follows immediately from Lemma 3.1 with $g = f$. Moreover

$$\frac{d}{dt} \langle V_{xx}, V\rangle = \langle V_{xx}, V_t\rangle + \langle \overline{V, V_{txx}}\rangle$$

and by Lemma 3.1

$$= \langle V_{xx}, V_t\rangle - (V_{tx}, V_x) - P_k \sum_{i=1}^{N} (V_t)_i^{(k+1)} \bar{V}_i^{(k+1)}$$
$$= \langle V_{xx}, V_t\rangle + \langle V_t, V_{xx}\rangle$$

and we have (3.5). □

Lemma 3.2. There exists a constant Λ depending only on $k+2$ such that for $f \in \mathbb{P}_{k+2,\Delta} = \{f \mid f|_{I_j} \in \mathbb{P}_{k+2},\ j = 1, 2, \ldots, N\}$

$$\|f\|_\Delta \leq \Lambda \, \|f\|. \tag{3.6}$$

Proof. On $\mathbb{P}_{k+2,\Delta}$, a finite dimensional linear space, $\|\cdot\|$ is a norm. Since $\|\cdot\|_\Delta$ is a semi-norm on this space it follows that such a constant Λ exists, and by homogeneity in Δ is independent of the partition Δ. □

Since $\mathbb{P}_{k+1,\Delta} \supset S_\Delta$ the relation (3.6) holds for $f \in S_\Delta$. Moreover derivatives of functions in $\mathbb{P}_{k+2,\Delta}$ are in this same space so for $f \in \mathbb{P}_{k+2,\Delta}$

$$\|f'\|_\Delta \leq \Lambda \|f'\|. \tag{3.7}$$

Actually the inequality (3.7) holds for some constant smaller than Λ (depending on $k+1$), but we do not make use of this fact.

We now consider the discrete eigenvalue problem associated with

$$u'' + \lambda\, c(x)\, u = 0, \qquad u(0) = u(1) = 0, \tag{3.8}$$

where $0 < m < c(x) < M$. Let $\{\phi_j\}_1^{Nk}$ be a real basis for S_Δ and let

$$u = \sum_{r=1}^{Nk} \alpha_r\, \phi_r.$$

The collocation version of (3.8) is

$$\sum_{r=1}^{Nk} \alpha_r \phi_r''(\xi_l) + \lambda c(\xi_l) \sum_{r=1}^{Nk} \alpha_r \phi_r(\xi_l) = 0, \qquad l = 1, 2, \ldots, N k \tag{3.9}$$

where

$$\xi_{(i-1)k+j} \equiv \xi_{i,j} \qquad i = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, k. \tag{3.10}$$

Let

$$B_{jr} = \phi_r''(\xi_j), \quad A_{jr} = \phi_r(\xi_j),$$
$$C_{jr} = \operatorname{diag}(c(\xi_j)), \quad W_{jr} = \operatorname{diag}(w_j)$$

where $w_{(i-1)k+j} \equiv w_{i,j}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{Nk})^T$.

With this notation we rewrite equation (3.9) in matrix form as

$$B\alpha + \lambda C A \alpha = 0. \tag{3.9'}$$

Thus

$$A^T W B \alpha + \lambda A^T W C A \alpha = 0. \tag{3.11}$$

Applying Lemma 2.1 we see that the matrix A is non-singular. Obviously W and C are positive definite and commute, hence $A^T W C A$ is positive definite. We observe that

$$\begin{aligned}(A^T W B \alpha)_\sigma &= \sum_{j=1}^{Nk} A^T_{\sigma j} (W B \alpha)_j \\ &= \sum_{j=1}^{Nk} \phi_\sigma(\xi_j) \sum_{r=1}^{Nk} \alpha_r w_j \phi_r''(\xi_j) \\ &= \sum_{r=1}^{Nk} \alpha_r \sum_{j=1}^{Nk} \phi_\sigma(\xi_j) w_j \phi_r''(\xi_j),\end{aligned}$$

hence

$$(A^T W B)_{\sigma r} = \langle \phi_\sigma, \phi_r'' \rangle. \tag{3.12}$$

Thus by Lemma 3.1

$$(A^T W B)_{\sigma r} = -(\phi_\sigma', \phi_r') - P_k \sum_{i=1}^{N} \phi_\sigma^{(k+1)} \overline{\phi}_r^{(k+1)} (\Delta_i)^{2k+1}. \tag{3.13}$$

Since the $\{\phi_j\}_1^{Nk}$ form a real basis

$$(\phi_\sigma', \phi_r') = (\phi_r', \phi_\sigma'), \qquad \phi_r^{k+1} = \overline{\phi}_r^{k+1}, \quad \text{and} \quad \phi_\sigma^{k+1} = \overline{\phi}_\sigma^{k+1}.$$

Thus (3.12) shows that $A^T W B$ is real and symmetric. Therefore the eigenvalues of the problem (3.11) (and hence of (3.9)) are real, the associated eigenvectors are complete, and may be chosen orthonormal with respect to $\langle \cdot, c \cdot \rangle$.

Let ψ be a discrete eigenfunction associated with the discrete eigenvalue λ, that is

$$\psi''(\xi_j) + \lambda c(\xi_j) \psi(\xi_j) = 0 \qquad j = 1, 2, \ldots, N k. \tag{3.14}$$

Multiplying (3.14) by $w_j \overline{\psi}(\xi_j)$ and the conjugate of (3.14) by $w_j \psi(\xi_j)$, summing on j, and recalling that λ is real, we see that

$$\lambda = \frac{-\langle \overline{\psi}, \psi'' \rangle - \langle \psi, \overline{\psi}'' \rangle}{2 \sum_{j=1}^{Nk} w_j c(\xi_j) |\psi(\xi_j)|^2}. \tag{3.15}$$

Thus by Corollary 3.1

$$\lambda = \frac{\int_0^1 |\psi'|^2 + P_k \sum_{i=1}^{N} |\psi_i^{(k+1)}|^2 (\Delta_i)^{2k+1}}{\sum_{j=1}^{Nk} c(\xi_j)\, w_j |\psi(\xi_j)|^2},$$

and

$$\lambda \geqq \frac{\|\psi'\|^2}{m\,\|\psi\|_\Delta^2}.$$

Moreover by Lemma 3.2

$$\lambda \geqq \frac{\|\psi'\|^2}{mA^2\,\|\psi\|^2}. \tag{3.16}$$

We combine (3.9'), (3.15), and (3.16) to verify that

$$\frac{\alpha^T(-B\,\alpha)}{\alpha^T\, CA\alpha} \geqq \frac{\|\psi'\|^2}{mA^2\,\|\psi\|^2}, \tag{3.17}$$

where

$$\psi = \sum_{i=1}^{Nk} \alpha_i\, \phi_i.$$

We summarize the above as

Lemma 3.3. The eigenvalues of the discrete problem (3.9) are real and positive; the eigenfunctions $\{\psi_j\}_1^{Nk}$ are complete and may be chosen orthonormal with respect to the norm $\langle\cdot, c\cdot\rangle$. If λ_n is the n-th eigenvalue of the discrete problem then

$$\lambda_n \geqq \frac{1}{mA^2}\, n^2 \pi^2 \qquad n = 1, 2, \ldots, Nk. \quad \square \tag{3.18}$$

Remark. (3.17) actually shows $\lambda_n \geqq \frac{1}{mA^2} \lambda_m^R$ where λ_m^R is the Rayleigh-Ritz-Galerkin approximate to the n-th eigenvalue of problem (3.8) when $c(x) \equiv 1$.

Lemma 3.4. Let $V = V(x, s;\, \Delta) \in S_\Delta$ be the solution of

$$V''(\xi_j) - s\, c(\xi_j)\, V(\xi_j) = f(\xi_j) \qquad j = 1, 2, \ldots, N\,k \tag{3.19}$$

where $s \in \mathscr{D} \equiv \{a + b\,i \,|\, |b| \geqq 1 - a\}$. Then

$$\|V\|_\Delta \leqq \frac{K_1}{|s|} \|f\|_\Delta, \tag{3.20}$$

and

$$\|V\|_\infty \leqq \frac{K_1\, K_\Delta}{|s|} \|f\|_\Delta, \tag{3.21}$$

where K_1 depends only on the bounds on $c(x)$, and K_Δ depends on the partition Δ.

Proof. $s \in \mathscr{D}$ implies (by Lemma 3.3) that $-s$ is not an eigenvalue of (3.8), hence (3.19) has a unique solution. Let $\{\psi_r\}_1^{Nk}$ be a basis of eigenvectors orthonormal with respect to $\langle\cdot, c\cdot\rangle$ as in Lemma 3.3. Let

$$V = \sum_{r=1}^{Nk} \alpha_r\, \psi_r. \tag{3.22}$$

Using (3.22) we rewrite (3.19) as

$$\sum_{r=1}^{Nk} [\alpha_r \psi_r''(\xi_j) - s\, c(\xi_j)\, \alpha_r \psi_r(\xi_j)] = \sum_{r=1}^{Nk} f_r\, c(\xi_j)\, \psi_r(\xi_j), \qquad j = 1, 2, \ldots, N\,k, \tag{3.23}$$

where $f_r \equiv \langle f, \psi_r\rangle$.

Remark. Observe that $\sum_{r=1}^{Nk} f_r \psi_r(x)$ is the S_Δ interpolant of $f(x)/c(x)$. Hence the right hand side of (3.23) is $f(\xi_j)$. Thus

$$\sum_{r=1}^{Nk} [\alpha_r c(\xi_j)(\lambda_r - s)\psi_r(\xi_j)] = \sum_{r=1}^{Nk} f_r c(\xi_j)\psi_r(\xi_j) \qquad j=1,2,\ldots,N, \tag{3.24}$$

where λ_r is the (negative) eigenvalue associated with ψ_r.

Let $a+bi=s\in\mathscr{D}$. To obtain a bound on

$$|\alpha_r| = |f_r|/|\lambda_r - a - bi|$$

we consider two cases. When $a<0$, we have $|b|>1-a>|a|$. Hence $2b^2 \geqq a^2+b^2=|s|^2$ or $1/|b| \leqq \sqrt{2}/|s|$. Thus

$$|\alpha_r| \leqq |f_r|/|bi| \leqq 2|f_r|/|s| \quad \text{for} \quad a<0.$$

When $a \geqq 0$, $|\lambda_r - a - bi| > |-a-bi| = |s|$ and

$$|\alpha_r| \leqq 2|f_r|/|s| \quad \text{for} \quad a \geqq 0.$$

Thus

$$\begin{aligned}
\|V\|_\Delta^2 &= \langle V, V\rangle \\
&\leqq \frac{1}{m}\langle V, cV\rangle \\
&= \frac{1}{m}\sum_{j=1}^{Nk} w_j \sum_{i=1}^{Nk} \alpha_i \psi_i(\xi_j) \sum_{l=1}^{Nk} \bar{\alpha}_l\, c(\xi_j)\, \bar{\psi}_l(\xi_j) \\
&= \frac{1}{m}\sum_{i=1}^{Nk} |\alpha_i|^2 \\
&\leqq \frac{1}{m}\sum_{i=1}^{Nk} 4\,|f_i|^2/|s|^2 \\
&= \frac{4}{m|s|^2}\langle f, \frac{1}{c} f\rangle \\
&\leqq \frac{4}{|s|^2}\frac{1}{m^2}\|f\|_\Delta^2
\end{aligned}$$

and (3.20) follows with $K_1 = \dfrac{2}{m}$.

Since S_Δ is a finite dimensional space all norms on S_Δ are equivalent. In particular there exists a constant K_Δ such that $\|V\|_\infty \leqq K_\Delta \|V\|_\Delta$ for all $V\in S_\Delta$. Hence (3.21) follows from (3.20). ▯

Lemma 3.5. If $V\in S_\Delta$ is the solution of (3.19) with $s\in\mathscr{D} \equiv \{a+bi \,|\, |b| \geqq 1-a\}$ then

$$\|V'\| \leqq \frac{K_2}{|s|^{\frac{1}{2}}}\|f\|_\Delta. \tag{3.25}$$

Proof. Multiplying (3.19) by $\bar{V}(\xi_j)\,w_j$ and summing on j shows

$$\langle V, V''\rangle = \langle V, f\rangle + \langle V, s\,c\,V\rangle.$$

Multiplying the conjugate of (3.19) by $V(\xi_j)\,w_j$ and summing on j shows

$$\langle V'', V\rangle = \langle f, V\rangle + \langle s\,c\,V, V\rangle.$$

Adding these equations and applying Lemma 3.2 gives

$$(V', V') + P_k \sum_{i=1}^{N} |V_i^{(k+1)}|^2 (\Delta_i)^{2k+1} = -\mathrm{Re}\langle V, f\rangle - \mathrm{Re}\langle V, s\,c\,V\rangle.$$

Hence

$$\|V'\|^2 \leqq \|V\|_\Delta \|f\|_\Delta + |s|\; M \|V\|_\Delta^2$$

and by Lemma (3.4)

$$\|V'\|^2 \leqq \frac{K_1}{|s|} \|f\|_\Delta^2 + \frac{K_1^2 M}{|s|} \|f\|_\Delta^2.$$

Thus (3.25) follows with $K_2 = \left[\frac{2}{m} + \frac{4M^2}{m}\right]^{\frac{1}{2}}$. □

Lemma 3.6. Let

$$\mathscr{D}_2 = \{a + b\,i \mid |b| \geqq 1 - a\} \cap \{s \mid |s|^{\frac{1}{2}} \geqq 2[B\Lambda + Q\Lambda]\, K_2\}.$$

If $s \in \mathscr{D}_2$ and $V = V(x, s; \Delta) \in S_\Delta$ is a solution of

$$\begin{aligned} &V''(\xi_j) + b(\xi_j)\, V'(\xi_j) + q(\xi_j)\, V(\xi_j) \\ &\quad - s\, c(\xi_j)\, V(\xi_j) = -f(\xi_j, s) - c(\xi_j)\, U_0(\xi_j) \qquad j = 1, \ldots, N k \end{aligned} \tag{3.26}$$

then

$$\|V'(\cdot, s; \Delta)\| \leqq \frac{K_3}{|s|^{\frac{1}{2}}} \|f(\cdot, s) + c(\cdot)\, U_0(\cdot)\|_\Delta, \tag{3.27}$$

and

$$\|V(\cdot, s; \Delta)\|_\infty \leqq \frac{K_3}{2|s|^{\frac{1}{2}}} \|f(\cdot, s) + c(\cdot)\, U_0(\cdot)\|_\Delta. \tag{3.28}$$

Proof. We rewrite (3.26) as

$$\begin{aligned} V''(\xi_j) - s\, c(\xi_j)\, V(\xi_j) &= -b(\xi_j)\, V'(\xi_j) - q(\xi_j)\, V(\xi_j) \\ &\quad - f(\xi_j, s) - c(\xi_j)\, U_0(\xi_j) \qquad j = 1, 2, \ldots, N k. \end{aligned} \tag{3.29}$$

Thus, applying Lemma 3.5,

$$\begin{aligned} \|V'\| &\leqq \frac{K_2}{|s|^{\frac{1}{2}}} \{\|b\, V'\|_\Delta + \|q\, V\|_\Delta + \|f + c\, U_0\|_\Delta\} \\ &\leqq \frac{K_2}{|s|^{\frac{1}{2}}} \{(B\Lambda + Q\Lambda)\, \|V'\| + \|f + c\, U_0\|_\Delta\} \end{aligned}$$

where $s \in \{a + bi \mid |b| \geqq 1 - a\}$.

If we choose $s \in \mathscr{D}_2$ then $[K_2(B\Lambda + Q\Lambda)]/|s|^{\frac{1}{2}} \leqq \frac{1}{2}$ and (3.27) follows with $K_3 = 2K_2$. Since

$$\begin{aligned} |V(x, s; \Delta)| &= \left|\tfrac{1}{2} \int_0^1 \mathrm{sgn}(x - y)\; V'(y, s; \Delta)\, dy\right| \\ &\leqq \tfrac{1}{2} \|V'(\cdot, s; \Delta)\| \end{aligned}$$

(3.28) follows from (3.27). □

Corollary 3.1. The transformed discrete Equations (2.12) have a unique solution for . $s \in \mathscr{D}_2$.

Proof. The Equations (2.12) form a finite dimensional linear system, hence the corollary follows immediately from the a priori estimate (3.28). □

Lemma 3.7. Let

$$\mathscr{D}_3=\mathscr{D}_2 \cap \{s \,\|s\,| \geqq 2\, K_1\, Q\} \cap \{s\|\, s\,|^{\frac{1}{2}} \geqq \Delta\, K_3\}.$$

If $s \in \mathscr{D}_3$ and $V=V(x, s; \Delta) \in S_\Delta$ is the solution of (2.12) or (3.29) then

$$\|V(\cdot, s)\|_\Delta \leqq \frac{K_4}{|s|} \|f(\cdot, s)+c(\cdot)\, U_0(\cdot)\|_\Delta. \tag{3.30}$$

Proof. Apply Lemma 3.4 to (3.29).

Thus

$$\|V\|_\Delta \leqq \frac{K_1}{|s|} \{\|b\, V'\|_\Delta+\|q\, V\|_\Delta+\|f+c\, U_0\|_\Delta\}$$

and by (3.27), since $\|V'\|_\Delta \leqq \Delta\, \|V'\|$,

$$\|V\|_\Delta \leqq \frac{K_1}{|s|} \left\{\frac{\Delta\, K_3}{|s|^{\frac{1}{2}}} \|f+c\, U_0\|_\Delta+Q\|V\|_\Delta+\|f+c\, U_0\|_\Delta\right\}.$$

Since $s \in \mathscr{D}_3$ (3.30) follows with $K_4=4\, K_1$. □

Since we are concerned with problem (1.2) for finite time, $0 \leqq t \leqq T<\infty$, we can modify $f(x, t)$ smoothly so that $f(x, t) \equiv 0$ for $t \geqq 2\, T$. This modification leaves the solution $u(x, t)$ unchanged on $[0, T]$ and shows that the Laplace transform of $f(x, t)$

$$\hat{f}(x, s)=\int_0^\infty e^{-st} f(x, t)\, dt=\int_0^{2T} e^{-st} f(x, t)\, dt$$

is, for each fixed x, an entire function of s.

Lemma 3.8. Let problem (1.2) have homogeneous initial data, i.e. $u_0(x) \equiv 0$ in (1.2b). Let $U(x, t, \Delta)$ be the solution of the associated collocation equations (2.7) with $U_0(x) \equiv 0$. If $U_t(x, t, \Delta)$ is absolutely continuous on finite intervals of $[0, \infty]$ and $|\, U_{tt}(x, t; \Delta)| \leqq K\, e^{bt}$ a.e. on $[0, \infty)$ then

$$U(x, t; \Delta)=\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\, \hat{U}(x, s; \Delta)\, ds \tag{3.31}$$

where $\hat{U}$ is the solution of transformed collocation equations and σ is sufficiently large.

Proof. Formally, (3.31) follows from the inversion theorem for the Laplace transform. It is sufficient to show that the integral in (3.31) converges. After integrating

$$\hat{U}(x, s; \Delta)=\int_0^\infty e^{-st}\, U(x, t; \Delta)\, dt$$

twice by parts we obtain

$$\hat{U}(x, s; \Delta)=-\frac{1}{s^2}\, U_t(x, t; \Delta)\,|_{t=0}+\frac{1}{s^2} \int_0^\infty e^{-st}\, U_{tt}(x, t; \Delta)\, dt.$$

Thus

$$\|\hat{U}(x, s; \Delta)\|_\infty \leqq \frac{K}{|s|^2}$$

for $\{s\,|\, \mathrm{Re}(s) \geqq b_0>b\}$ and (3.31) follows for $\sigma \in \mathscr{D}_3 \cap \{\sigma\,|\,\sigma \geqq b_0>b\}$. □

Obviously we can obtain analogs of Lemmas 3.4–3.7 for $\hat{u}(x, s)$, the Laplace transform of the solution $u(x, t)$ of (1.2). The proofs are essentially the same. Also for homogeneous initial data, we have Lemma 3.8 for $u(x, t)$.

Lemma 3.9. Let $f(x, t) \equiv 0$ in problem (1.2). For fixed θ, $\frac{\pi}{2} < \theta \leqq \frac{3\pi}{4}$, and $\alpha \geqq 1$ (specified below) let

$$\Gamma = \{s \mid s = \alpha + r\, e^{\pm i\theta} \text{ if } \operatorname{Re} s < \sigma;\ r \in [0, \infty)\}$$
$$\cup \{s \mid \operatorname{Re} s = \sigma,\ -A \leqq \operatorname{Im} s \leqq A,\ A = (\sigma - \alpha) \tan \theta\}$$

(see Fig. 1). If $U(x, t; \Delta)$ is the solution of the associated collocation equations (2.7)

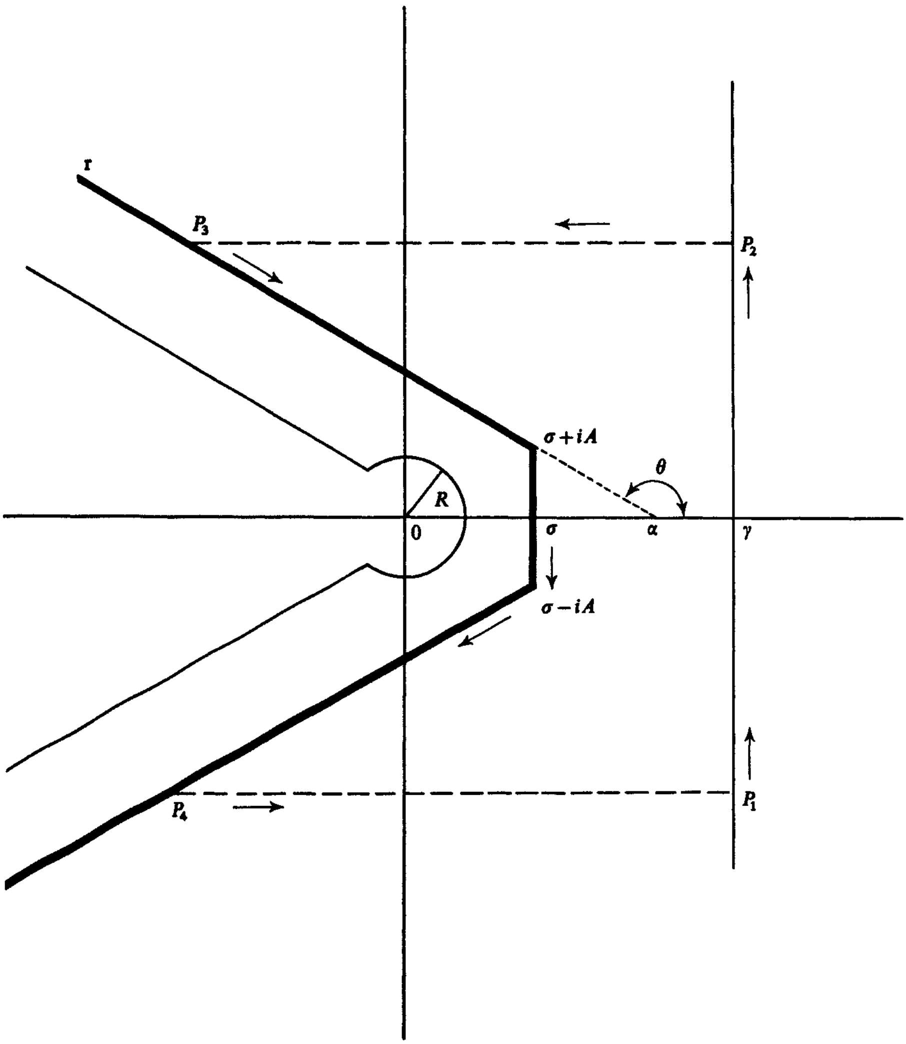

Fig. 1.

then

$$U(x,t;\Delta)=\frac{1}{2\pi i}\int_{\Gamma} e^{st}\hat{U}(x,s;\Delta)\,ds=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\hat{U}(x,s:\Delta)\,ds \quad (3.32)$$

where $\hat{U}$ is the solution of the transformed collocation equations and $\gamma\geqq\sigma$.

Proof. For each fixed x, $\hat{U}(x,s;\Delta)$ is analytic for $s\in\mathcal{D}_3$. If we choose α sufficiently large so that $\Gamma\subset\mathcal{D}_3$ the estimate (from (3.29) and (3.21))

$$\|\hat{U}(\cdot,s;\Delta)\|_\infty\leqq\frac{K_1 K_\Delta}{|s|}\|U_0(\cdot,s)\|_\Delta$$

shows that the integral over Γ converges. It remains to show that the two integrals in (3.32) are equal. By Cauchy's theorem the integral over $P_1P_2P_3P_4P_1$ is zero (see Fig. 1). Thus it suffices to show that the integrals over the lines P_2P_3, and P_4P_1 go to zero as $P_2\nearrow\infty$ and $P_1\searrow-\infty$. We consider P_2P_3, the integral over P_4P_1 is treated similarly. Let Im $P_2=p$. Hence

$$\int_{P_2P_3} e^{st}\hat{U}(x,s;\Delta)\,ds=\int_{\gamma}^{0}+\int_{0}^{\alpha-p\tan\left(\theta-\frac{\pi}{2}\right)} e^{st}\hat{U}(x,s;\Delta)\,ds.$$

However, with $s=x+ip$

$$\left|\int_{\gamma}^{0} e^{st}\hat{U}(x,s;\Delta)\,ds\right|\leqq\int_{0}^{\gamma} e^{xt}\frac{\overline{K}}{|s|}\,dx\leqq\frac{\overline{K}}{pt}(e^{\gamma t}-1)$$

and

$$\left|\int_{0}^{\alpha-p\tan\left(\theta-\frac{\pi}{2}\right)} e^{st}\hat{U}(x,s;\Delta)\,ds\right|\leqq\int_{\alpha-p\tan\left(\theta-\frac{\pi}{2}\right)}^{0} e^{xt}\frac{\overline{K}}{|s|}\,dx\leqq\frac{K}{pt}.$$

Thus as $p\nearrow\infty$ the integral over the line P_2P_3 goes to zero. □

Under the hypothesis of Lemma 3.9 we obtain a similar result for $u(x,t)$. Thus when $f(x,t)\equiv 0$ we have

$$U(x,t;\Delta)-u(x,t)=\frac{1}{2\pi i}\int_{\Gamma} e^{st}(\hat{U}(x,s;\Delta)-\hat{u}(x,s))\,ds. \quad (3.33)$$

Deforming the contour in Lemma 3.9 depends on knowing an a priori bound of the form $\frac{|s|}{K}$ on the Laplace transform of $U(x,t;\Delta)$ (or $u(x,t)$). The proof given applies when a bound of the form $\frac{K}{|s|^p}$ (some $p>0$) is known. In particular if $f(x,t)=F(x)\frac{t^k}{k!}$ (k a positive integer), and we have a bound on $F(x)$ (e.g. $\|F(x)\|_\infty\leqq C$), then

$$|\hat{f}(x,s)|\leqq\frac{C}{|s|^{k+1}}.$$

This remark gives the more general

Lemma 3.10. Suppose $f(x,t)$ is a polynomial in t with bounded coefficients that are functions of x. Then the results of Lemma 3.9 hold. In particular, Equation (3.33) is valid. □

4. Global Estimates

In this section we obtain global error estimates for the continuous time collocation method for problem (1.2). In order to obtain optimal order error estimates for $t \in [0, T]$ (see Theorem 4.1) we let $U_0(x; \Delta)$, the initial data for the collocation equations, be the "elliptic interpolant" of $u_0(x)$. That is

$$(U_0'' + b\, U_0' + q\, U_0)(\xi_j) = (u_0'' + b\, u_0' + q\, u_0)(\xi_j) \qquad j = 1, 2, \ldots, N\, k.$$

In general, the differential operator L defined by

$$Lu = \frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + q(x)\, u$$

is not invertible. However, we may shift the problem (1.2) by letting

$$v(x, t) = e^{\lambda t} u(x, t).$$

Equation (1.2) becomes

$$\begin{aligned} c\, v_t &= (L - \lambda)\, v + e^{-\lambda t} f \\ v(x, 0) &= u_0(x) \qquad\qquad 0 \leqq x \leqq 1 \\ v(0, t) &= v(1, t) = 0 \qquad 0 \leqq t \leqq T. \end{aligned} \tag{1.2s}$$

For suitable λ, $L - \lambda$ is invertible and the elliptic interpolant, $V_0(x; \Delta)$, of $v(x, 0)$ exists for $|\Delta|$ sufficiently small. A short computation shows that the shifted collocation equations of problem (1.2) are the same as the collocation equations for the shifted analytic problem, i.e., let

$$V(x, t; \Delta) = e^{\lambda t} U(x, t; \Delta), \qquad 0 < t \leqq T,$$

where $V(x, t; \Delta)$ is the solution of the collocation equations associated with problem (1.2s). With $U_0(x; \Delta) \equiv V_0(x; \Delta)$ the collocation equations (2.7) have a solution. Thus, all results obtained for $V(x, t; \Delta)$, $(V(x, t; \Delta) - v(x, t))$ apply mutatis mutandis to $U(x, t; \Delta)$, $(U(x, t; \Delta) - u(x, t))$ with an appropriate factor $e^{T\lambda}$.

Thus we assume the elliptic interpolant of u_0 exists (for if not we apply the above shifting procedure) and now prove

Theorem 4.1. Let $u(x, t)$ be the solution of (1.2). Assume that the coefficients $b(x)$, $q(x)$, $c(x)$ have bounded derivatives through order $k+2$ and

$$u, u_t, u_{tt} \in L^\infty [0, T; W^{k+4, \infty}].$$

Let U_0, the initial data for the collocation equations (2.7), be the elliptic interpolant of $u_0(x)$; i.e.

$$(U_0'' + b\, U_0' + q\, U_0)(\xi_j) = (u_0'' + b\, u_0' + q\, u_0)(\xi_j), \qquad j = 1, 2, \ldots, N\, k.$$

Then the collocation equations (2.7) have a unique solution $U(x, t; \Delta)$ and

$$\begin{aligned} \|U - u\|_{L^\infty[0, T;\, L^\infty[0,1]]} \leqq C\, |\Delta|^{k+2} \cdot \{ &\|u\|_{L^\infty[0,T;\, W^{k+4, \infty}]} + \|u_t\|_{L^\infty[0, T;\, W^{k+4, \infty}]} \\ &\|u_{tt}\|_{L^\infty[0, T;\, W^{k+4, \infty}]}\} \end{aligned} \tag{4.1}$$

where C is a constant that depends on the coefficients of the differential equation but is independent of the partition Δ.

Proof. A solution to the collocation equations (2.7) or (2.9) exists locally in time. This fact, coupled with the uniform bound proved below, gives global existence and uniqueness of a solution to the collocation equations.

With the differential operator L defined above we rewrite (1.2) as

$$\begin{aligned} c\,u_t - L\,u &= f(x,t) \\ u(x,0) &= u_0(x). \end{aligned} \tag{4.2}$$

Let $\phi(x,t) = u(x,t) - u_0(x)$, then (4.2) becomes

$$\begin{aligned} c\,\phi_t - L\,\phi &= L\,u_0 + f(x,t) \equiv -g(x,t) \\ \phi(x,0) &\equiv 0. \end{aligned} \tag{4.3}$$

We denote by $\Phi(x,t;\Delta)$ the solution of the collocation equations

$$c(\xi_{ij})\,\Phi_t(\xi_{ij},t;\Delta) - L\,\Phi(\xi_{ij},t;\Delta) = L\,u_0(\xi_{ij}) + f(\xi_{ij},t)\quad \Phi(x,0;\Delta)\equiv 0 \tag{4.4}$$

associated with (4.3). That is, if we choose U_0, the initial data of the collocation equations (2.7), so that

$$L\,U_0(\xi_i) = L\,u_0(\xi_i) \qquad i = 1, 2, \ldots, k\,N \tag{4.5}$$

(i.e. U_0 is the elliptic interpolant of u_0) then (2.7), and (4.2)–(4.4) show

$$U - u = \Phi - \phi + U_0 - u_0. \tag{4.6}$$

We use the Laplace transform with the problem (4.3) to estimate the term $\Phi - \phi$. Let $s = a_0 + b\,i$ and $L_0 = L - a_0\,c$. The Laplace transform of (4.3) is

$$L\,\hat{\phi} - s\,c\,\hat{\phi} = \hat{g}(x,s)$$

or

$$L_0\,\hat{\phi} - i\,b\,c\,\hat{\phi} = \hat{g}(x,s).$$

Let $W(x,s;\Delta)$ denote the solution of the collocation equations

$$L_0\,W(\xi_i) = L_0\,\hat{\phi}(\xi_i) \qquad i = 1, 2, \ldots, k\,N. \tag{4.7}$$

At the collocation points $\{\xi_i\}_{i=1}^{kN}$ we have

$$L(\hat{\Phi} - W) - s\,c(\hat{\Phi} - W) = \hat{g}(\xi_i, s) - L\,W + s\,c\,W = i\,b\,c(W - \hat{\phi}).$$

If a_0 is sufficiently large then Lemma 3.6 and the estimate

$$\|\hat{\Phi} - W\|_\infty \leqq \|(\hat{\Phi} - W)'\|$$

imply

$$\|\hat{\Phi} - W\|_\infty \leqq \frac{K\,|b|}{|s|^{\frac{1}{2}}}\,\|W - \hat{\phi}\|_\infty.$$

Thus

$$\|\hat{\Phi} - \hat{\phi}\|_\infty \leqq \|\hat{\Phi} - W\|_\infty + \|W - \hat{\phi}\|_\infty \leqq \left(1 + \frac{K\,|b|}{|s|^{\frac{1}{2}}}\right)\|W - \hat{\phi}\|_\infty. \tag{4.8}$$

Since W satisfies (4.7) the results of de Boor and Swartz [2] show that

$$\|W - \hat{\phi}\|_\infty \leqq C\,\|\hat{\phi}(\cdot,s)\|_{W^{k+4,\infty}}\,|\Delta|^{k+2}$$

and hence

$$\|\hat{\Phi}-\hat{\phi}\|_\infty \leqq \left(1+\frac{K|b|}{|s|^{\frac{1}{2}}}\right)C\|\hat{\phi}(\cdot,s)\|_{W^{k+4,\infty}}|\Delta|^{k+2}. \tag{4.9}$$

Recall

$$\frac{\partial^p}{\partial x^p}\hat{\phi}(x,s)=\int_0^\infty e^{-st}\frac{\partial^p}{\partial x^p}\phi(x,t)\,dt \qquad p=1,\ldots,k+4. \tag{4.10}$$

We integrate (4.10) by parts twice to obtain

$$\begin{aligned}\frac{\partial^p}{\partial x^p}\hat{\phi}(x,s)&=\frac{1}{s^2}\frac{\partial}{\partial t}\left(\frac{\partial^p}{\partial x^p}\phi(s,t)\Big|_{t=0}+\frac{1}{s^2}\int_0^\infty e^{-st}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^p}{\partial x^p}\phi(x,t)\right)\right)dt\\&=\frac{1}{s^2}\frac{\partial}{\partial t}\left(\frac{\partial^p}{\partial x^p}u(x,t)\Big|_{t=0}+\frac{1}{s^2}\int_0^\infty e^{-st}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^p}{\partial x^p}u(x,t)\right)\right)dt,\\&\qquad p=1,2,\ldots,k+4.\end{aligned} \tag{4.11}$$

Since we are concerned with finite T we may assume that $f(x,t)$ decays smoothly and rapidly to 0 on $[T,T+1]$ in such a way that

$$\left\|\left(\frac{\partial}{\partial t}\right)^i u\right\|_{L^\infty[T,T+1;W^{k+4,\infty}]}\leqq K\left\|\left(\frac{\partial}{\partial t}\right)^i u\right\|_{L^\infty[0,T;W^{k+4,\infty}]} \qquad i=0,1,2$$

(see, e.g., A. Friedman [8]). For $t>T+1$, $u(x,t)$ and all its derivatives decay exponentially. This observation, coupled with the assumption

$$u,\,u_t,\,u_{tt}\in L^\infty[0,T;W^{k+4,\infty}[0,1]]$$

and Equation (4.11), shows that

$$\begin{aligned}&\|\hat{\phi}(\cdot,s)\|_{W^{k+4,\infty}}\\&\quad\leqq\frac{K_1}{|s|^2}\{\|u\|_{L^\infty[0,T;W^{k+4,\infty}]}+\|u_t\|_{L^\infty[0,T;W^{k+4,\infty}]}+\|u_{tt}\|_{L^\infty[0,T;W^{k+4,\infty}]}\}.\end{aligned} \tag{4.12}$$

From (4.9), since $|b|=0(|s|)$,

$$\begin{aligned}\|\hat{\Phi}-\hat{\phi}\|_\infty&\leqq\frac{K_2}{|s|^{\frac{1}{2}}}|\Delta|^{k+2}\\&\quad\cdot\{\|u\|_{L^\infty[0,T;W^{k+4,\infty}]}+\|u_t\|_{L^\infty[0,T;W^{k+4,\infty}]}+\|u_{tt}\|_{L^\infty[0,T;W^{k+4,\infty}]}\}.\end{aligned} \tag{4.13}$$

However,

$$\|\Phi-\phi\|_\infty=\left\|\frac{1}{2\pi i}\int_{a_0-i\infty}^{a_0+i\infty}e^{st}(\hat{\Phi}-\hat{\phi})\,ds\right\|$$

and by (4.13)

$$\begin{aligned}\|\Phi-\phi\|_\infty&\leqq K_3|\Delta|^{k+2}\\&\quad\cdot\{\|u\|_{L^\infty[0,T;W^{k+4,\infty}]}+\|u_t\|_{L^\infty[0,T;W^{k+4,\infty}]}+\|u_{tt}\|_{L^\infty[0,T;W^{k+4,\infty}]}.\end{aligned} \tag{4.14}$$

For the elliptic interpolant, U_0, the results of de Boor and Swartz [2] show

$$\|U_0-u_0\|\leqq C_1\|u_0\|_{W^{k+4,\infty}}|\Delta|^{k+2}. \tag{4.15}$$

Combining (4.6), (4.14), and (4.15) we obtain the result (4.1). □

Theorem 4.2. Let $u(x, t)$ be the solution of (1.2). Assume that the coefficients $b(x)$, $q(x)$, $c(x)$ have bounded derivatives through order $k+2$ and let

$$u, u_t, u_{tt} \in L^\infty[0, T; W^{k+4,\infty}[0, 1]].$$

Let U_0, the initial data for the collocation equations (2.7), be the interpolant of u at the collocation points, i. e.

$$U_0(\xi_j) = u_0(\xi_j) \qquad i = 1, 2, \ldots, N k.$$

Then the collocation equations (2.7) have a unique solution $U(x, t, \Delta)$ and for $t \in [\tau, T]$, $\tau > 0$,

$$\|U - u\|_{L^\infty[\tau, T; L^\infty[0,1]]} \leqq C|\Delta|^{k+2} \cdot \{\|u\|_{L^\infty[0, T; W^{k+4,\infty}]} + \|u_t\|_{L^\infty[0, T; W^{k+4,\infty}]} + \|u_{tt}\|_{L^\infty[0, T; W^{k+4,\infty}]}\} \tag{4.16}$$

where C is a constant that depends on τ and the coefficients of the differential equation, but is independent of the partition Δ.

Proof. We break problem (1.2) into two problems:

$$\begin{aligned} c(x)\, u_{1t} - L\, u_1 &= f(x, t) \\ u_1(x, 0) &= 0 \end{aligned} \tag{I}$$

$$\begin{aligned} c(x)\, u_{2t} - L\, u_2 &= 0 \\ u_2(x, 0) &= u_0(x). \end{aligned} \tag{II}$$

Thus $u = u_1 + u_2$. Since the elliptic interpolant of $u_1(x, 0)\,(=0)$ is simply 0 the estimate (4.16) for problem (I) follows from Theorem 4.1. In fact, for problem (I), the estimate (4.16) holds for all t, $0 \leqq t \leqq T$.

It remains to consider problem II and the effect of using the values of $u_0(x)$ at the collocation points as initial data.

Let $W(x, s; \Delta)$ be the solution of the collocation equations

$$W''(\xi_j) + b(\xi_j)\, W'(\xi_j) = \hat{u}_2''(\xi_j, s) + b(\xi_j)\, \hat{u}_2'(\xi_j, s) \qquad j = 1, \ldots, N k,$$

where $\hat{u}_2(x, s)$ is the Laplace transform of $u_2(x, t)$. The results of de Boor and Swartz [2] show

$$\|W(\cdot, s; \Delta) - \hat{u}_2(\cdot, s)\|_\infty \leqq C_1 \|\hat{u}_2(\cdot, s)\|_{W^{k+4,\infty}} |\Delta|^{k+2} \tag{4.17}$$

where C_1 is independent of Δ.

With U_2 denoting the solution of the collocation equations associated with problem (II) we have, at the collocation points,

$$\begin{aligned} &[(\hat{U}_2 - W)'' + b(\hat{U}_2 - W)' + q(\hat{U}_2 - W) - s\, c(\hat{U}_2 - W)](\xi) \\ &= [q(\hat{u}_2 - W) - s\, c(\hat{u}_2 - W)](\xi). \end{aligned}$$

Hence by Lemma 3.6, for $s \in \mathscr{D}_2$

$$\|\hat{U}_2 - W\|_\infty \leqq \frac{K_3}{|s|^{\frac{1}{2}}} [Q + |s| M] \|W - \hat{u}_2\|_\infty. \tag{4.18}$$

Let Γ be the curve described in Lemma 3.9. Then Lemma 3.9 and (3.32) show

$$\|(u_2-U_2)(\cdot,t)\|_{L^\infty[0,1]} \leqq \frac{1}{2\pi}\int_\Gamma |e^{st}|\,\|\hat{U}_2-\hat{u}_2\|_{L^\infty}\,ds. \tag{4.19}$$

From (4.17) and (4.19)

$$\|\hat{U}_2-\hat{u}_2\|_\infty \leqq \|\hat{U}_2-W\|_\infty+\|W-\hat{u}_2\|_\infty$$
$$\leqq\left\{\frac{K_3}{|S|^{\frac{1}{2}}}[Q+|s|M]+1\right\}C_1\|\hat{u}_2(\cdot,s)\|_{W^{k+4,\infty}}|\Delta|^{k+2}$$

Thus

$$\|(u_2-U_2)(\cdot,t)\|_\infty \leqq C_2\int_\Gamma |e^{st}|\,|s|^{\frac{1}{2}}\,\|\hat{u}_2\|_{W^{k+4,\infty}}|\Delta|^{k+2}\,ds. \tag{4.20}$$

However, $\|\hat{u}_2(\cdot,s)\|_{W^{k+4,\infty}}$ grows at worst as a polynomial in $|s|^{\frac{1}{2}}$ of degree $k+2$ whose coefficients depend on the coefficients of the differential equation and u_0.

Thus when $t\geqq\tau>0$ the integral on the right side of (4.20) converges and we obtain (4.16) for problem (II). Combining the results for problems (I) and (II) completes the proof. □

5. Estimates at the Knots

Lemma 5.1. Let $g(t)$ have $p-1$ continuous derivatives on $(0,\infty)$, let $g^{(p-1)}$ be absolutely continuous on finite intervals of $(0,\infty)$ and $|g^{(p)}(t)|\leqq K\,e^{bt}$ a.e. on $(0,\infty)$. Further suppose that

$$\lim_{t\to0+} g^j(t)=0 \qquad j=0,1,\ldots,p-1.$$

Then for $\mathrm{Re}(s)\geqq\varrho>\max[0,b]$, there exists a constant $M=M(\varrho)$ so that

$$|\hat{g}(s)|\leqq\frac{M}{|s|^p}. \tag{5.1}$$

Proof. This lemma follows immediately from the well known (see e.g. Widder [12]) formula

$$\hat{g}^p(s)=s^p\,\hat{g}(s)-\sum_{j=1}^{p}s^{p-j}g^{(j-1)}(0+). \qquad \square \tag{5.2}$$

Theorem 5.1. Let $u(x,t)$ be the solution of (1.2). Assume that the coefficients $b(x)$, $c(x)$, $q(x)\in C^{2k+2}[0,1]$, and $u_0(x)$, $f(\cdot,t)\in C^{2k+2}[0,1]$. Let

$$R(j)=[j/2]+1=\begin{cases}\dfrac{j}{2}+1 & j \text{ even.}\\[2mm] \dfrac{j+1}{2}+1 & j \text{ odd}\end{cases}. \tag{5.3}$$

Assume $u(x,\cdot)$ and its first $2k+2$ derivatives with respect to x are in $C^{R(k)}[0,T]$ with

$$\left|\left(\frac{\partial}{\partial t}\right)^l\left(\frac{\partial}{\partial x}\right)^i u(x,t)\right|\leqq Me^{\beta t} \qquad \begin{matrix} i=0,1,\ldots,2k+2\\ l=0,1,\ldots,R(k)\end{matrix}$$

for some $M>0$, $\beta>0$. Let U_0, the initial data of the collocation equations (2.7) be the "super-elliptic" interpolant for the problem (1.2). That is, let the differential operator L be defined by

$$L\,u \equiv \frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + q(x)\,u$$

and assume L is invertible.

We choose p so that

$$p = [k/2]. \tag{5.4}$$

Let $\{\phi_j(x)\}_0^p$ be given by

$$\begin{aligned} \phi_0(x) &\equiv u_0(x) \\ c(x)\,\phi_j(x) &\equiv L\,\phi_{j-1}(x) + \frac{\partial^{j-1}}{\partial t^{j-1}} f(x,t)\Big|_{t=0} \qquad j=1,2,\ldots,p. \end{aligned} \tag{5.5}$$

Let $\{\Phi_j(x;\Delta)\}_0^p$ be given by the system of collocation problems:

$$\begin{aligned} &L\,\Phi_p|(\xi_j) = L\,\phi_p|(\xi_j) \qquad j=1,2,\ldots,N\,k. \\ &L\,\Phi_m(\xi) = c(\xi_j)\Phi_{m+1}(\xi_j) - \frac{\partial^m}{\partial t^m} f(\xi_j,t)\Big|_{t=0} \qquad \begin{array}{l} m=p-1,p-2,\ldots,0; \\ j=1,2,\ldots,N\,k. \end{array} \end{aligned} \tag{5.6}$$

Set

$$U_0(x;\Delta) = \Phi_0(x;\Delta). \tag{5.7}$$

Then the collocation equations have a unique solution $U(x,t,\Delta)$, and at the knots $\{x_j\}_{j=1}^{N-1}$.

$$|(U-u)(x_j,t)| \leqq C(T)\,|\Delta|^{2k} \tag{5.8}$$

where $C(T)$ is a constant that is independent of the partition Δ and depends only on the coefficients of the differential equation and on the quantities

$$\left\| \frac{\partial^i}{\partial t^i} u(x,t) \right\|_{W^{k+j+2,\infty}} \qquad j=0,1,\ldots,k; \qquad i=0,1,\ldots,R(k-j). \tag{5.9}$$

Proof. As in the proof of Theorem 4.1 we rewrite (1.2) as

$$\begin{aligned} c\,u_t - L\,u &= f(x,t) \\ u(x,0) &= u_0(x). \end{aligned} \tag{5.10}$$

Let

$$\phi(x) \equiv u(x,t) - \sum_{j=0}^{p} \phi_j(x)\,\frac{t^j}{j!} \tag{5.11}$$

where $\{\phi_j(x)\}_0^p$ is defined by (5.5).

Thus

$$\begin{aligned} c\,\phi_t - L\,\phi &= f(x,t) - \sum_{j=0}^{p=1} [c\,\phi_{j+1}(x) - L\,\phi_j]\frac{t^j}{j!} + \frac{t^p}{p!} L\,\phi|(x) \\ &= f(x,t) - \sum_{j=0}^{p=1} \frac{t^j}{p!}\left(\frac{\partial}{\partial t}\right)^j f(x,t)\Big|_{t=0} + \frac{t^p}{p!} L\,\phi_p = \frac{t^p}{p!} F(x,t) \end{aligned} \tag{5.12}$$

where $F(x,t)$ is determined by the Taylor series expansion for $f(x,t)$ and $L\,\phi_p$. Moreover,

$$\phi(x,0) = 0.$$

Let $\Phi(x, t; \Delta)$ satisfy the collocation equations

$$\begin{aligned} &c(\xi_j)\, \Phi_t(\xi_j, t; \Delta) - L\, \Phi(\xi_j, t; \Delta) = \frac{t^p}{p!} F(\xi_j, t), \quad j = 1, 2, \ldots, N\,k; \\ &\Phi(x, 0; \Delta) = 0. \end{aligned} \tag{5.13}$$

The collocations equations (2.7) and (5.11)-(5.13) show

$$U(x, t, \Delta) - u(x, t) = \Phi(x, t, \Delta) - \phi(x, t) + \sum_{j=0}^{p} (\Phi_j(x, \Delta) - \phi_j(x)) \frac{t^j}{j!}. \tag{5.14}$$

We use the Laplace transform of problem (5.12) and its associated collocation equations (5.13) to estimate the term $\Phi - \phi$ at the knots $\{x_j\}_{j=1}^{N-1}$. Applying the results of de Boor and Swartz [2] directly to $\hat{\Phi} - \hat{\phi}$ at the knot x_j $(j = 1, \ldots, N-1)$ shows that

$$|(\hat{\Phi} - \hat{\phi})\ (x_j, s)| \leqq |\Delta|^{2k} \sum_{j=0}^{k} |G_{k-j}(s)|\, \|\hat{\phi}(\cdot, s)\|_{W^{k+j+1, \infty}} \tag{5.15}$$

where $G_j(s)$ depends on the Green's function associated with the operator $L - s\,c$. A direct computation shows that $|G_j(s)|$ grows as a polynomial of order j (degree $< j$) in the variable $|s|^{\frac{1}{2}}$. That is

$$|G_j(s)| = 0(|s|^{(j-1)/2}) \qquad j = 0, 1, \ldots, k.$$

Choose j, $0 \leqq j \leqq k$, and consider the term

$$|G_{k-j}(s)|\, \|\phi(\cdot, s)\|_{W^{k+j+2, \infty}}.$$

From (5.12) (or (5.11) and (5.5)) and our assumptions on $u(x, t)$ and $f(x, t)$ we find that

$$\left(\frac{\partial}{\partial t}\right)^l \left(\frac{\partial}{\partial x}\right)^i \phi(x, t)\bigg|_{t=0} = 0, \qquad \begin{aligned} &i = 0, 1, \ldots, 2\,k+2; \\ &l = 0, 1, \ldots, R(k). \end{aligned}$$

Thus $\phi(x, t)$ and its first $k+j+2$ derivatives with respect to x satisfy the hypothesis of Lemma 5.1 with $p = R(k)$. Hence, for Re(s) sufficiently large,

$$\|\hat{\phi}(\cdot, s)\|_{W^{k+j+2, \infty}} \leqq \frac{M}{|s|^{R(k-j)}}, \qquad j = 0, 1, \ldots, k,$$

and

$$|G_{k-j}(s)|\, \|\hat{\phi}(\cdot, s)\|_{W^{k+j+2, \infty}} \leqq \frac{\text{const}}{|s|^{r'}},$$

where

$$r' = \begin{cases} \frac{3}{2} & j \text{ even} \\ 2 & j \text{ odd} \end{cases}.$$

Since we have chosen p as in (5.4) we have

$$|(\hat{\Phi} - \hat{\phi})\ (x_j, s)| \leqq \frac{C}{|s|^{\frac{3}{2}}} |\Delta|^{2k} \cdot j = 1, \ldots, N-1.$$

Hence

$$|(\Phi - \phi)\,(x_j, t)| \leqq \frac{1}{2\pi} \int_{a-i\infty}^{a+i\infty} e^{at} \frac{C}{|s|^{\frac{3}{2}}} |\Delta|^{2k} ds \leqq c_1 e^{at} |\Delta|^{2k}. \tag{5.16}$$

From the results of Cerutti [4] the system (5.6) satisfies

$$|\Phi_i(x_j;\Delta)-\phi_i(x_j)|\leqq c_2|\Delta|^{2k} \quad j=1,\ldots,N-1, \quad i=1,\ldots,p. \tag{5.17}$$

Hence, combining (5.16), (5.17) and (5.12) we have

$$|(U-u)(x_j,t)|\leqq c_1|\Delta|^{2k}+\sum_{j=0}^{p} c_2|\Delta|^{2k}\frac{t^j}{j!}\leqq c_3(T)|\Delta|^{2k}$$

which proves (5.8). □

Theorem 5.2. Let $u(x,t)$ be the solution of (1.2) and assume the hypothesis of Theorem 5.1. Let U_0, the initial data for the collocation equation (2.7) be the interpolant of u at the collocation points, i.e.

$$U_0(\xi_j)=u_0(\xi_j) \quad j=1,2,\ldots,N\,k.$$

Then the collocation equations have a unique solution $U(x,t;\Delta)$ and at a knot x_i, $i=1,2,\ldots,N-1$,

$$\begin{aligned}&|(U-u)(x_i,t)|\leqq C|\Delta|^{2k}\cdot\\&\{\|u\|_{L^\infty[0,T;W^{k+4,\infty}]}+\|u_t\|_{L^\infty[0,T;W^{k+4,\infty}]}\,\|u_{tt}\|_{L^\infty[0,T;W^{k+4,\infty}]}\}\end{aligned} \tag{5.18}$$

for $0<\tau\leqq t\leqq T$, where C depends on τ and T but is independent of the partition Δ.

Proof. Using linearity we rewrite problem (1.2) as two problems:

$$\begin{aligned}c(x)\,u_{1t}-L\,u_1&=f_1(x,t)\\u_1(x,0)&=0\end{aligned} \tag{I}$$

and

$$\begin{aligned}c(x)\,u_{2t}-L\,u_2&=P(x,t)\\u_2(x,0)&=u_0(x)\end{aligned} \tag{II}$$

where $P(x,t)$ is a polynomial in t with coefficients depending on x; specifically

$$P(x,t)\equiv\sum_{j=0}^{p-1}\left(\frac{\partial}{\partial t}\right)^j f(x,t)\bigg|_{t=0}\frac{t^j}{j!},$$

and p is given by (5.4). With $f_1(x,t)=f(x,t)-P(x,t)$ we have

$$u(x,t)=u_1(x,t)+u_2(x,t).$$

Furthermore

$$\left(\frac{\partial}{\partial t}\right)^j f_1(x,t)\bigg|_{t=0}=0 \quad j=0,1,\ldots,p-1.$$

It follows immediately from (5.5) and (5.6) that the super elliptic interpolant of problem (I) is $U_1(x,0;\Delta)\equiv 0$. Thus, as a consequence of Theorem 5.1, the estimate (5.18) holds (in fact for all t, $0\leqq t\leqq T$) for problem (I).

It remains to consider problem II and the effect of using the values of $u_0(x)$ at the collocation points as initial data. As in proof of Theorem 4.2 we apply the results of de Boor and Swartz [2] to the operator $L_s\,\hat{u}_2\equiv\hat{u}''+b\,\hat{u}_2+q\,\hat{u}_2-s\,c\,\hat{u}_2$

to obtain an estimate for $|(\hat{U}_2-\hat{u}_2)(x_j, s)|$. In particular we have

$$|(\hat{U}_2-\hat{u}_2)(x_j, s)| \leqq |\Delta|^{2k} \sum_{j=0}^{k} |G_{k-j}(s)| \, \|\hat{u}_2\|_{W^{k+j+2},\infty} \tag{5.19}$$

where $|G_j(s)|$ is a polynomial of order j in the variable $|s|^{\frac{1}{2}}$. Moreover $\|\hat{u}_2\|_{W^{k+j+2},\infty}$ is bounded by a polynomial of degree $k+j$ in $|s|^{\frac{1}{2}}$. Since $P(x, t)$ is a polynomial in t with bounded functions of x as coefficients we apply Lemma 3.10 to obtain

$$(U_2-u_2)(x_j, t) = \int_\Gamma e^{st}(\hat{U}-\hat{u})(x_j, s)\, ds.$$

Thus, using (5.19),

$$|(U_2-u_2)(x_j, t)| \leqq \left| \int_\Gamma e^{st} |G(s)| \, \|\hat{u}_2\|_{W^{2k+2},\infty} |\Delta|^{2k} ds \right|.$$

Since this integral converges for $t \geqq \tau > 0$ we combine problems (I) and (II) to obtain (5.17). □

6. A-Stability

We consider in more detail the special case when the eigenvalues of the operator L, $Lv \equiv v'' + b(x)\, v' + q(x)\, v$, are all strictly negative (i.e. $Lv = \lambda c v$, $v(0) = v(1) = 0$, $v \not\equiv 0$ implies $\lambda \leqq -\lambda_0 < 0$). This condition will be satisfied, for example, when $q(x) \leqq 0$. From previous estimates (e.g. Lemma 3.7) we know that the eigenvalues, λ, of the discrete problem

$$\begin{aligned} &L V(\xi_j; \Delta) = \lambda c(\xi_j) V(\xi_j; \Delta), \quad j = 1, 2, \ldots, Nk \\ &V \in S_\Delta \end{aligned} \tag{6.1}$$

lie in $C/\mathscr{D}_3$ (see Fig. 1). Recall that this fact is independent of the partition Δ. It follows that for s on the path $\Gamma \subset \mathscr{D}_3$ or to the right of Γ (see Fig. 1, p. 238 or Fig. 2, p. 249) that the problem

$$L V(\xi_j) - s\, c(\xi_j) V(\xi_j) = g(\xi_j) \quad j = 1, 2, \ldots, Nk \tag{6.2}$$

has a unique solution $V(x; \Delta) \in S_\Delta$.

Such estimates also hold for $v(x)$, the solution of the continuous problem,

$$Lv = \lambda c v, \quad v(0) = v(1) = 0. \tag{6.3}$$

Hence

$$Lv - s c v = g \tag{6.4}$$

has a unique solution for s on the path Γ or to the right of Γ. Further if $s \in \{s \mid \operatorname{Re} s > -\lambda_0\}$ then (6.4) also has a unique solution. In particular, for s in the closed trapezoid $A\,B\,C\,D$ (see Fig. 2) (6.4) has a unique solution. The results of de Boor and Swartz [2] assure us that for each $s_0 \in A\,B\,C\,D$ there is a $\delta(s_0) > 0$ so that for all partitions with $|\Delta| \leqq \delta(s_0)$, (6.2) has a unique solution. That is $L_\Delta - s_0 c$, the discrete operator associated with problem (6.2), is invertible for $|\Delta| \leqq \delta(s_0)$. Since $(L_\Delta - s\, c)^{-1}$ depends continuously on s and the invertible operators form an open set (in the uniform operator topology) $(L_\Delta - s\, c)^{-1}$ exists in some open

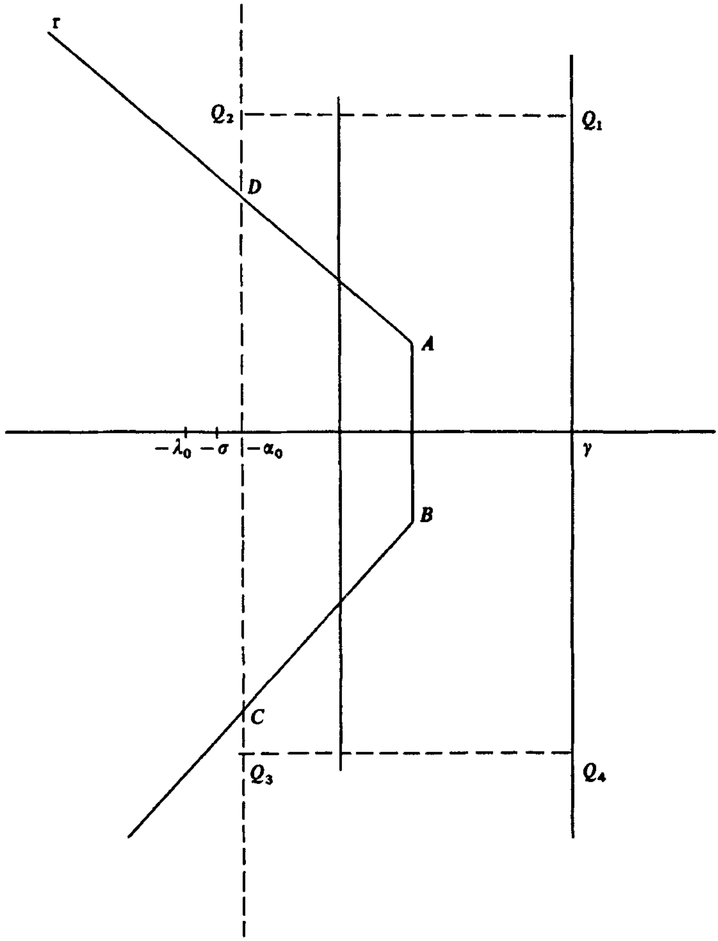

Fig. 2.

neighborhood of s_0 for partitions Δ with $|\Delta| \leq \frac{\delta(s_0)}{2}$. In this manner we construct an open cover of the compact set $A\ B\ C\ D$. Thus there exists a $\delta_0 > 0$ so that (6.2) has a unique solution for all s with $\mathrm{Re}(s) > -\alpha_0$ and for all partitions with $|\Delta| < \delta_0$.

Using linearity we break problem (1.2) into three parts

$$\begin{aligned} c\,u_{1t} &= L\,u_1 + f_1(x,t) \\ u_1(x,0) &= 0 \end{aligned} \qquad \text{(I)}$$

$$\begin{aligned} c\,u_{2t} &= L\,u_2 + f_0(x) \\ u_2(x,0) &= 0 \end{aligned} \qquad \text{(II)}$$

$$\begin{aligned} c\,u_{3t} &= L\,u_3 \\ u_3(x,0) &= u_0(x). \end{aligned} \qquad \text{(III)}$$

In addition to the assumption on the location of the eigenvalues of L we assume that

$$|f_1(x,t)| \leqq M e^{-\sigma t}, \qquad 0<\alpha_0<\sigma<\lambda_0 \tag{6.5}$$

uniformly in x, where M is a positive constant. Thus, $\hat{f}_1(x,s)$, the Laplace transform of $f_1(x,t)$, is analytic in the half plane $\{s \mid \operatorname{Re}(s) > -\sigma\}$ for each fixed x.

The Laplace transform of problem I is

$$s\,c\,\hat{u}_1 = L\,\hat{u}_1 + \hat{f}_1(x,s), \tag{6.6}$$

and similarly the discrete problem associated with problem I has Laplace transform

$$s\,c\,\hat{U}_1(\xi_j,s) = L\,\hat{U}_1(\xi_j,s) + \hat{f}_1(\xi_j,s), \qquad j=1,2,\ldots,N\,k. \tag{6.7}$$

Under the hypothesis of Theorem 4.1

$$\|U_1(\cdot,t;\Delta)-u_1(\cdot,t)\|_\infty = \left\| \frac{1}{2\pi i}\int_{\gamma+i\infty}^{\gamma-i\infty} e^{st}\,(\hat{U}_1(\cdot,s;\Delta)-\hat{u}_1(x,s))\,ds \right\|_\infty . \tag{6.8}$$

In this case, with zero initial data and $\hat{f}_1(x,s)$ analytic in the half-plane $\{s \mid \operatorname{Re} s > -\sigma\}$, $\hat{u}_1$ and $\hat{U}_1$ are also analytic in $\{s \mid \operatorname{Re} s > -\alpha_0\}$. Since the integrals along Q_1Q_2 and Q_3Q_4 (see Fig. 2) go to zero as $Q_1 \nearrow \gamma+i\infty$ and $Q_4 \searrow \gamma-i\infty$, we have, by Cauchy's theorem,

$$\int_{-\alpha_0-i\infty}^{-\alpha_0+i\infty} e^{st}\,(\hat{U}_1-\hat{u}_1)\,d\,s = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,(\hat{U}_1-\hat{u}_1)\,ds.$$

However, as in the proof of Theorem 4.1,

$$\left\| \frac{1}{2\pi i}\int_{-\alpha_0-i\infty}^{-\alpha+i\infty} e^{st}(\hat{U}_1-\hat{u}_2)\,ds \right\|_\infty \leqq \frac{1}{2\pi}\,e^{-\alpha_0 t}\int_{-\alpha_0-i\infty}^{-\alpha_0+i\infty} \frac{K}{|s|^{\frac{3}{2}}}\,ds\,|\Delta|^{k+2}$$

and hence

$$\|U_1(\cdot,t;\Delta)-u_1(\cdot,t)\|_\infty \leqq e^{-\alpha_0 t}\,|\Delta|^{k+2}\,K(u,u_t,u_{tt}). \tag{6.9}$$

Moreover, since $\|u_1(\cdot,t)\|_\infty \leqq t\,M\,e^{-\sigma t} \leqq C\,e^{-\alpha_0 t}$, we see that

$$\|U_1(\cdot,t;\Delta)\|_\infty \leqq C_1\,e^{-\alpha_0 t}. \tag{6.9b}$$

The Laplace transforms of problem II and its discrete analog are

$$s\,c\,\hat{u}_2 = L\,\hat{u}_2 + \frac{f_0(x)}{s} \tag{6.10}$$

and

$$s\,c(\xi_j)\,\hat{U}_2(\xi_j) = L\,\hat{U}_2(\xi_j) + \frac{f_0(\xi_j)}{s} \qquad j=1,2,\ldots,N\,k. \tag{6.11}$$

In this case (6.8) also holds but $\hat{f}_0$ has a simple pole at the origin. In order to replace the path $(\gamma-i\infty,\gamma+i\infty)$ by the path $(-\alpha_0-i\infty,\,-\alpha_0+i\infty)$ it is necessary to compute the residues of $\hat{u}_2$ and $\hat{U}_2$ in the trapezoid $A\;B\;C\;D$. We observe that $\hat{U}_2(x,s;\hat{\Delta})$ and $\hat{u}_2(x,s)$ are analytic except possibly at $s=0$. Hence we need only consider the residue at $s=0$.

Lemma 6.1. For $s \in A\ B\ C\ D$ (see Fig. 2) the solution of

$$L\,v - s\,c\,v = -\frac{f_0(x)}{s}, \qquad v(0) = v(1) = 0 \tag{6.12}$$

satisfies

$$\|v(\cdot, s)\|_\infty \leqq \|v'(\cdot, s)\| \leqq \frac{C}{|s|}\|f_0(s)\|. \tag{6.13}$$

Proof. Since s is not in the spectrum of L there exists a constant $K(s)$, which depends continuously on s so that

$$\|v\| \leqq K(s)\frac{\|f_0\|}{|s|}. \tag{6.14}$$

Multiplying (6.12) by v and integrating by parts gives

$$-(v', v') + (b\,v', v) + (q\,v, v) - (s\,c\,v, v) = -\left(\frac{f_0}{s}, v\right)$$

and hence

$$\begin{aligned}\|v'\|^2 &\leqq B\|v'\|\|v\| + Q\|v\|^2 + |s|\,M\|v\|^2 + \frac{1}{|s|}\|f_0\|\|v\| \\ &\leqq \frac{1}{2}\|v'\|^2 + \frac{B^2}{2}\|v\|^2 + (Q + |s|\,M)\|v\|^2 + \frac{1}{|s|}[\|f_0\|^2 + \|v\|^2].\end{aligned} \tag{6.15}$$

Since s is in the compact set $A\ B\ C\ D$, $K(s)$ in (6.14) is uniformly bounded. Using (6.14) in (6.15) gives the result (6.13). □

Corollary 6.1. Let $\phi(x)$ be the solution of the two-point boundary value problem

$$\begin{aligned}L\,\phi &= -f_0(x) \\ \phi(0) &= \phi(1) = 0.\end{aligned} \tag{6.16}$$

Then

$$\lim_{s\to 0} s\,\hat{u}_2(x, s) = \phi(x) \tag{6.17}$$

and

$$\operatorname{Res}(\hat{u}_2, 0) = \phi(x). \tag{6.18}$$

Proof. Since $\hat{u}_2(x, s)$ is analytic in $A\ B\ C\ D\backslash\{0\}$ and has (at worst) a simple pole at $s = 0$ we compute the residue at $s = 0$ by the formula

$$\operatorname{Re} s\,(\hat{u}_2, 0) = \lim_{s\to 0} s\,\hat{u}_2(x, s).$$

Multiplying (6.12) by s we have

$$s^2 c\,\hat{u}_2 = L\,s\,\hat{u}_2 - L\,\phi,$$

and (6.17), (6.18) follow from (6.13) and the continuity of L^{-1}. □

Lemma 6.2. For $s \in A\ B\ C\ D$ (see Fig. 2) the solution of

$$\begin{cases} L\,V(\xi_j; \Delta) - s\,c(\xi_j)\,V(\xi_j; \Delta) = -\dfrac{f_0(\xi_j)}{s}, & j = 1, 2, \ldots, N\,k \\ V \in S_\Delta \end{cases} \tag{6.19}$$

satisfies

$$\|V(\cdot, s; \Delta)\|_\infty \leqq \frac{K_\Delta}{|s|}\|f_0(\cdot)\|_\infty. \tag{6.20}$$

Proof. For s in the compact trapezoid $A\ B\ C\ D$ the norm of the inverse of the discrete operator associated with $L - s\,c$ is bounded (for $|\Delta| < \delta_0$). Since this norm depends continuously on s it is uniformly bounded i.e.

$$\|V(\cdot, s; \Delta)\|_\infty \leqq \frac{K_\Delta}{|s|} \|f_0(\cdot)\|_\infty, \quad s \in A\ B\ C\ D. \quad \square$$

Following the argument of the corollary to Lemma 6.1, we obtain

Corollary 6.2. Let $\Phi(x; \Delta)$ be the solution of the collocation equations,

$$\begin{aligned} L\,\Phi(\xi_j; \Delta) &= -f_0(\xi_j) \qquad j = 1, 2, \ldots, N\,k, \\ \Phi(x, \Delta) &\in S_\Delta. \end{aligned} \tag{6.21}$$

Then

$$\lim_{s \to 0} s\,\hat{U}_2(x, s, \Delta) = \Phi(x; \Delta) \tag{6.22}$$

and

$$\operatorname{Res}(\hat{U}_2, 0) = \Phi(x; \Delta). \quad \square \tag{6.23}$$

Thus

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} (\hat{U}_2 - \hat{u}_2)\, ds &= \frac{1}{2\pi i} \int_{-\alpha_0 - i\infty}^{-\alpha_0 + i\infty} e^{st} (\hat{U}_2 - \hat{u}_2)\, ds + \operatorname{Res}(e^{st}(\hat{U}_2 - \hat{u}_2), 0) \\ &= \frac{1}{2\pi i} \int_{-\alpha_0 - i\infty}^{-\alpha_0 + i\infty} e^{st} (\hat{U}_2 - \hat{u}_2)\, ds + (\Phi - \phi). \end{aligned}$$

Under the hypothesis of Theorem 4.1 we conclude

$$\|U_2(\cdot, t; \Delta) - u_2(\cdot, t)\|_\infty \leqq e^{-\alpha_0 t} |\Delta|^{k+2} K(u, u_t, u_{tt}) + \|\Phi - \phi\|_\infty. \tag{6.24}$$

Thus, using the results of de Boor and Swartz [2] we obtain

$$\|U_2(\cdot, t; \Delta) - u_2(\cdot, t)\|_\infty \leqq e^{-\alpha_0 t} |\Delta|^{k+2} K(u, u_t, u_{tt}) + K'(f_0) |\Delta|^{k+2}. \tag{6.24b}$$

Furthermore, since $\|u_2(\cdot, t) - \phi(x)\|_\infty \leqq t\,M e^{-\alpha t} \leqq C e^{-\alpha_0 t}$ we see that

$$\|U_2(\cdot, t; \Delta) - \Phi(x; \Delta)\|_\infty \leqq C_2 e^{-\alpha_0 t}. \tag{6.24c}$$

To treat problem III let

$$w = u_3 - u_0$$

and

$$W = U_3 - U_0,$$

where $U_3(x, t; \Delta)$ is the solution of the associated discrete problem, and U_0 is the elliptic interpolant of u_0. Thus we are led to

$$\begin{aligned} c\,w_t &= L\,w + L\,u_0 \\ w(x, 0) &= 0 \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} c(\xi_j)\, W_t(\xi_j) &= L\,W(\xi_j) + L\,U_0(\xi_j) \qquad j = 1, 2, \ldots, N\,k \\ W(x, 0; \Delta) &= 0. \end{aligned} \tag{6.26}$$

With U_0, the elliptic interpolant of u_0 (6.26) is the discrete version of (6.25) and the discussion of problem II applies. In particular,

$$W-w=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}(\hat{W}-\hat{w})\,ds=\frac{1}{2\pi i}\int_{-\alpha_0-i\infty}^{-\alpha_0+i\infty} e^{st}(\hat{W}-\hat{w})\,ds-U_0+u_0. \quad (6.27)$$

Thus,

$$\|U_3(\cdot,t;\Delta)-u_3(\cdot,t)\|_\infty=\|W-w+U_0-u_0\|_\infty \quad (6.28)$$

$$=\left\|\frac{1}{2\pi i}\int_{-\alpha_0-i\infty}^{-\alpha_0+i\infty} e^{st}(\hat{W}-\hat{w})\,ds\right\|_\infty \leqq e^{-\alpha_0 t}|\Delta|^{k+2}K(u,u_t,u_{tt}).$$

Moreover, since $\|u_3(\cdot,t)\|_\infty \leqq t\,Me^{-\sigma t}\leqq Ce^{-\alpha_0 t}$ we see that

$$\|U_3(\cdot,t;\Delta)\|_\infty \leqq C_3\,e^{-\alpha_0 t}. \quad (6.28\text{b})$$

Combining (6.9), (6.24) and (6.28) gives

Theorem 6.1. Let U_0 be the elliptic interpolant of u_0 and in addition to the hypothesis of Theorem 4.1 assume:

1° the eigenvalues λ of L satisfy $\lambda\leqq-\lambda_0<0$;

2° $f(x,t)=f_0(x)+f_1(x,t)$ where $|f_1(x,t)|\leqq M\,e^{-\sigma t}$ $\quad 0<\alpha_0<\sigma$.

Then the collocation equations (2.7) have a unique solution $U(x,t;\Delta)$. Moreover, let $\phi(x)$ be the solution of the two-point boundary value problem (6.16) and let $\Phi(x;\Delta)$ be the corresponding solution of the collocation equations (6.21). Then

$$\|U(\cdot,t;\Delta)-\Phi(\cdot;\Delta)\|_\infty \leqq (C_1+C_2+C_3)\,e^{-\alpha_0 t} \quad (6.29)$$

and

$$\|U(\cdot,t;\Delta)-u(\cdot,t)\|_\infty \leqq K|\Delta|^{k+2}e^{-\alpha_0 t}+K'(f_0)|\Delta|^{k+2}. \quad (6.30)$$

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