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Chebyshevian Multistep Methods for Ordinary Differential Equations*

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Summary. In this paper some theory of linear multistep methods for $y^{(r)}(x) = f(x, y)$ is extended to include smooth, stepsize-dependent coefficients. Treated in particular is the case where exact integration of a given set of functions is desired.

1. Introduction

For the initial-value problem

$$y^{(r)}(x) = f(x, y), \qquad y^{(j)}(a) = 0, \qquad j = 0, 1, \ldots, r-1 \tag{1.1}$$

consider the multistep method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^r \sum_{i=0}^{k} \beta_i f(x_{n+i}, y_{n+i}) \tag{1.2}$$

or, equivalently, the operator

$$L[y](x) = \sum_{i=0}^{k} \alpha_i y(x+ih) - h^r \sum_{i=0}^{k} \beta_i y^{(r)}(x+ih). \tag{1.3}$$

By definition, (1.2) or (1.3) is of order p if the operator L integrates the set of functions $\{1, x, \ldots, x^{p+r-1}\}$ exactly, i.e. $L[x^i] = 0$, $i = 0, 1, \ldots, p+r-1$.

For many initial-value problems (1.1), where the solution is of periodic or exponential character, the choice of the set $\{1, x, \ldots, x^{p+r-1}\}$ might not be the best. An example is the harmonic oscillation $y''(x) = -\omega^2 y(x)$ with $y(0) = 1$, $y'(0) = i\omega$. Obviously the solution is $y(x) = e^{i\omega x}$. The solution obtained by using Cowell's method yields an orbit which spirals inwards instead of being circular, as is required for the exact solution (Stiefel and Bettis [6]). Better results are obtained in this case if we demand that the operator given by (1.3) integrates the functions $\sin \omega x$, $\cos \omega x$ exactly, i.e. $L[\sin \omega x] = L[\cos \omega x] \equiv 0$ identically in x for all h. Note that one loses the advantage of stepsize-independent coefficients. However, the coefficients can normally be given in a form which allows

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fast recomputation when a change in stepsize is necessary (see Sec. 6 for an example).

The purpose of this paper is to extend some of the theory for linear multistep methods to include stepsize-dependent coefficients. In particular we treat the case where we demand exact integration of a given set of linearly independent functions. Some theory in the trigonometric case has been developed by Gautschi [3], and integration formulas have been proposed by various authors. See e.g. Salzer [5], Gautschi [3], Stiefel and Bettis [6], and Bettis [1].

2. The Order of a Chebyshevian Multistep Method

Let $k \geqq r \geqq 1$, $h_0 > 0$, $H = (0, h_0]$, and $\bar{H} = [0, h_0]$. For $h \in \bar{H}$, define the linear operator $L: C^r[a, b] \to C[a, b]$ by

$$L[y](x) = \sum_{i=0}^{k} \alpha_i(h) y(x+ih) - h^r \sum_{i=0}^{k} \beta_i(h) y^{(r)}(x+ih) \tag{2.1}$$

where

$$\alpha_i, \beta_i: \bar{H} \to \mathbb{R}, \quad i = 0, 1, \ldots, k, \quad \alpha_k: \bar{H} \to \mathbb{R} \backslash \{0\}, \tag{2.2}$$

and for some $m \geqq r$

$$\alpha_i, \beta_i \in C^m(\bar{H}), \quad i = 0, 1, \ldots, k. \tag{2.3}$$

Using (2.3) we write

$$\alpha_i(h) = \sum_{j=0}^{m-1} \alpha_{i,j} h^j + O(h^m), \quad \beta_i(h) = \sum_{j=0}^{m-1} \beta_{i,j} h^j + O(h^m) \tag{2.4}$$

where $\alpha_{i,j}$ and $\beta_{i,j}$ are constants independent of h. If we expand $y(x+ih)$ and $y^{(r)}(x+ih)$ in Taylor's series around x, substitute $\alpha_i(h)$ and $\beta_i(h)$ given by (2.4) in the expression (2.1), and do some rearranging we find for all $y \in C^m[a, b]$

$$L[y] = \sum_{n=0}^{m-1} \left\{ \sum_{s=0}^{n} C_{s,n-s} y^{(s)} \right\} h^n + O(h^m) \tag{2.5}$$

where for $n = 0, 1, 2, \ldots, m-1$

$$C_{s,n} = \frac{1}{s!} \sum_{i=0}^{k} i^s \alpha_{i,n} \qquad s = 0, 1, \ldots, r-1, \tag{2.6a}$$

$$C_{r,n} = \frac{1}{r!} \sum_{i=0}^{k} \{ i^r \alpha_{i,n} - r! \, \beta_{i,n} \}, \tag{2.6b}$$

$$C_{s,n} = \frac{1}{s!} \sum_{i=0}^{k} \{ i^s \alpha_{i,n} - s^{(r)} i^{s-r} \beta_{i,n} \}, \quad s = r+1, r+2, \ldots. \tag{2.6c}$$ [1]

Definition 2.1. An operator L of the form (2.1) is said to be of order $p < m - r$ if in (2.5) $C_{s,n} = 0$ for $0 \leqq s + n \leqq p + r - 1$ and $C_{s,n} \neq 0$ for some $s, n \geqq 0$ such that $s + n = p + r$.

From (2.5) we immediately have the following result.

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1 $s^{(r)} = s(s-1) \ldots (s-r+1)$.

Theorem 2.1. An operator L of the form (2.1) is of order $q<m-r$ if and only if there exists a differential operator T_{q+r} such that for all $y\in C^{q+r}[a,b]$

$$T_{q+r}[y](x)=a_{q+r}y^{(q+r)}(x)+\cdots+a_0 y(x) \tag{2.7}$$

and for all h sufficiently small

$$L[y](x)=T_{q+r}[y](x)h^{q+r}+O(h^{q+r+1}). \tag{2.8}$$

Here $a_0,\ldots,a_{q+r}$ are constants which depend on L, but not on h, x or y.

3. Null Space and Order

Let $\mathcal{N}$ denote the nullspace of L

$$\mathcal{N}=\{y\in C^m[a,b]: L[y]=0 \text{ all } h\in H\}. \tag{3.1}$$

Observe that by definition $\mathcal{N}$ does not depend on h.

We have the following theorem.

Theorem 3.1. Suppose the operator L given by (2.1)–(2.3) is of order $q<m-r$. Then $\mathcal{N}\subset$ Ker T_{q+r}. Here $\mathcal{N}$ is given by (3.1) and T_{q+r} is the differential operator in Theorem 2.1.

Proof. Suppose $g\in\mathcal{N}$. Then $L[g]=0$ for all $h\in H$. Therefore by (2.8) $T_{q+r}[g]=0$. ▮

Corollary 3.1. If L is of order $q<m-r$, and dim $\mathcal{N}=p+r$ then $p\leqq q$.

Remark 3.1. The null space of the operator T_{q+r} given by (2.7) consists of sums of products of polynomials and exponential functions. Hence by Theorem 3.1 they are the only type of functions we can hope to integrate exactly.

Remark 3.2. Suppose the order of L is p, and the dimension of its nullspace $\mathcal{N}$ is $p+r$. Then by Theorem 3.1 $\mathcal{N}=$ Ker T_{p+r}, and moreover $a_{p+r}\neq 0$ in (2.7). Hence the functions integrated exactly are precisely the solutions of the differential equation $T_{p+r}[y]=0$. Consider (2.7) with $p=q$ and define the polynomial f by

$$f(x)=\sum_{j=0}^{p+r} a_j z^j=a_{p+r}\prod_{j=0}^{q}(z-\omega_j)^{r_j}\prod_{j=q+1}^{s}(z-\omega_j)^{r_j}(z-\bar{\omega}_j)^{r_j}. \tag{3.2}$$

Here $a_0,\ldots,a_{p+r}$ are the coefficients in (2.7). Also $\omega_0,\ldots,\omega_q$ are the real roots, and $\omega_{q+1},\ldots,\omega_s$ the complex roots of f. In this case L exactly integrates the functions given by the formulas

$$\begin{aligned} &e^{\omega_j x}, x e^{\omega_j x},\ldots, x^{r_j-1}e^{\omega_j x}, \quad j=0,1,\ldots,s\\ &e^{\bar{\omega}_j x}, x e^{\bar{\omega}_j x},\ldots, x^{r_j-1}e^{\bar{\omega}_j x}, \quad j=q+1,\ldots,s \end{aligned} \tag{3.3}$$

where $\sum_{j=0}^{q} r_j+2\sum_{j=q+1}^{s} r_j=p+r$. Conversely if L exactly integrates the functions given by (3.3), then from (2.6) we have

$$a_{p+r}=\frac{1}{(p+r)!}\left\{\sum_{i=0}^{k} i^{p+r}\alpha_{i,0}-(p+r)^{(r)}\sum_{i=0}^{k} i^p\beta_{i,0}\right\} \tag{3.4}$$

and $a_0, \ldots, a_{p+r-1}$ can be found by comparing coefficients in (3.2). Hence we have an expression for the leading term in the truncation error.

The following example shows that it is possible to have a null space of dimension $p+r$, yet the exponent of h in the leading term of the truncation error is greater than $p+r$.

Example 3.1. Let $r=k=1$, $m=3$, and consider the following variant of the Euler method:

$$L[y](x)=y(x+h)-(1-h^2)y(x)-h(1-h)y'(x).$$

For h sufficiently small and for all $y\in C^3[a,b]$ we obtain

$$L[y]=(\tfrac{1}{2}y''+y'+y)h^2+O(h^3).$$

So the method is of order 1. However, the null space of L as defined in (3.1) is empty. To verify this suppose, on the contrary, that $\mathcal{N}\neq\emptyset$. Then by Remark 3.1 $\mathcal{N}$ contains a function g_p of the form $g_p(x)=x^p e^{wx}$ for some p and w. For $p=0$ we find

$$\begin{aligned} L[g_0](x) &= e^{w(x+h)}-(1-h^2)e^{wx}-wh(1-h)e^{wx}\\ &= e^{wx}[(\tfrac{1}{2}w^2+w+1)h^2+\tfrac{1}{6}w^3h^3+O(h^4)] \end{aligned}$$

where we have expanded e^{wh} in powers of h. Clearly this expression is not zero identically in h for any w. Similarly, for any $p\geqq 1$, we have that $L[g_p](x)$ is not zero identically in x and h.

Remark 3.3. The form (2.8) of the truncation error tells us that the global error analysis of Henrici [4] will be valid also for Chebyshevian multistep methods with minor modifications. In particular, if $\dim\mathcal{N}=p+r$, then the global error will be of order $O(h^p)$ under the usual stability and continuity assumptions. The error constant (Henrici [4, p. 223]) is given by $a_{p+r}/\sum\beta_{i,0}$.

Corollary 3.2 (Consistency condition). If L is of order $p\geqq 0$ then

$$\varrho^{(n)}(1)=O(h^{p+r-n}) \qquad n=0,1,\ldots,r-1. \tag{3.5}$$

Moreover, if $p\geqq 1$, then

$$\varrho^{(r)}(1)=r!\,\sigma(1)+O(h^p). \tag{3.6}$$

Here the polynomials ϱ and σ are given by

$$\varrho(\xi)=\sum_{i=0}^{k}\alpha_i(h)\xi^i, \qquad \sigma(\xi)=\sum_{i=0}^{k}\beta_i(h)\xi^i. \tag{3.7}$$

Proof. We have from (2.4)

$$\varrho^{(n)}(1)=\sum_{j=0}^{p+r-n-1}\left\{\sum_{i=0}^{k} i^{(n)}\alpha_{i,j}\right\}h^j+O(h^{p+r-n}). \tag{3.8}$$

But $C_{s,n}=0$, $0\leqq n+s\leqq p+r-1$. So from (2.6)

$$\sum_{i=0}^{k} i^n\alpha_{i,j}=0, \qquad n\leqq r-1,\ j\leqq p+r-n-1. \tag{3.9}$$

It follows that the double sum in (3.8) vanishes and (3.5) is proved.

Similarly, using (2.4), (2.5), (2.6), and (3.9)

$$\begin{aligned}\varrho^{(r)}(1)-r!\,\sigma(1)&=\sum_{j=0}^{p-1}\left\{\sum_{i=0}^{k} i^{(r)}\alpha_{i,j}\right\}h^j-r!\sum_{j=0}^{p-1}\sum_{i=0}^{k}\beta_{i,j}h^j+O(h^p)\\&=\sum_{j=0}^{p-1}\left\{\sum_{i=0}^{k}(i^r\alpha_{i,j}-r!\,\beta_{i,j})\right\}h^j+O(h^p)\\&=r!\sum_{j=0}^{p-1}C_{r,j}+O(h^p)=O(h^p).\end{aligned}\tag{3.10}$$

Remark 3.5. The discussion of Sections 2 and 3 can easily be extended to operators where the coefficients depend on x. For $h\in\bar{H}$, define the linear operator $L\colon C^r[a,b]\to C[a,b]$ by

$$L[y](x)=\sum_{i=0}^{k}\alpha_i(x+ih)\,y(x+ih)-h^r\sum_{i=0}^{k}\beta_i(x+ih)\,y^{(r)}(x+ih)\tag{3.11}$$

where

$$\alpha_i,\beta_i\colon [a,b+kh_0]\to\mathbb{R},\quad i=0,1,\ldots,k\qquad \alpha_k\colon [a,b+kh_0]\to\mathbb{R}\backslash\{0\},$$

and for some $m\geqq r$

$$\alpha_i,\beta_i\in C^m[a,b\,k\,h_0].$$

Then, expanding α_i and β_i in Taylor's series around x, the analog of (2–4) reads

$$\alpha_i(x+ih)=\sum_{j=0}^{m-1}\alpha_{i,j}(x)h^j+O(h^m),\qquad \beta_i(x+ih)=\sum_{j=0}^{m-1}\beta_{i,j}(x)h^j+O(h^m)$$

where $\alpha_{i,j}(x)$, $\beta_{i,j}(x)$ does not depend on h. Proceeding as in Section 2 we find

$$L[y](x)=\sum_{n=0}^{m-1}\left\{\sum_{s=0}^{n}C_{s,n-s}(x)\,y^{(s)}(x)\right\}h^n+O(h^m).$$

Here $C_{s,n-s}(x)$ is given by (2.6) with $\alpha_{i,j}$ and $\beta_{i,j}$ replaced by $\alpha_{i,j}(x)$ and $\beta_{i,j}(x)$, respectively. Hence all functions integrated exactly by L given by (3.11) are solutios of some linear, homogeneous, ordinary differential equation

$$a_n(x)\,y^{(n)}(x)+\cdots+a_0(x)\,y(x)=0$$

where $a_i\in C^{m-n+i}[a,b]$ $i=0,1,\ldots,n$.

4. An Existence Theorem

Lemma 4.1 (see Henrici [4], Lemma 5.3 and 6.3). Suppose h is fixed, $h\in H$, and $\omega\in C$. Let $n\geqq r$ if $\omega=0$, and $n\geqq 1$ otherwise. Then

$$L[x^m e^{\omega x}]=0,\ m=0,1,\ldots,n-1\quad\text{and}\quad L[x^n e^{\omega x}]\neq 0$$

if and only if the function ϕ given by $\phi(\xi)=\frac{\varrho(\xi)}{\log\xi}-\sigma(\xi)$ has a zero of exact multiplicity s at $\xi=a=e^{\omega h}$, where $s=n$ if $\omega\neq 0$ and $s=n-r$ if $\omega=0$.

Proof. By Leibniz' rule

$$[x^m e^{\omega x}]^{(r)} = \sum_{j=0}^{\min(m,r)} \binom{r}{j} m^{(j)} x^{m-j} \omega^{r-j} e^{\omega x}.$$

So

$$L[x^m e^{\omega x}]$$
$$= \sum_{i=0}^{k} \left\{\alpha_i(h)(x+ih)^m e^{\omega(x+ih)} - h^r \beta_i(h) \sum_{j=0}^{\min(m,r)} \binom{r}{j} m^{(j)} \omega^{r-j} (x+ih)^{m-j} e^{\omega(x+ih)}\right\}$$
$$= e^{\omega x} \sum_{i=0}^{k} \left\{\alpha_i(h)(x+ih)^m (e^{\omega h})^i - \beta_i(h) \sum_{j=0}^{\min(m,r)} \binom{r}{j} m^{(j)} (\omega h)^{r-j} h^j (x+ih)^{m-j} (e^{\omega h})^i\right\}.$$

Hence

$$L[e^{\omega x}] = e^{\omega x} f_0(a), \tag{4.1 a}$$

$$L[x^m e^{\omega x}] = e^{\omega x} f_m(a, h), \qquad m = 1, 2, \ldots \tag{4.1 b}$$

where

$$f_0(\xi) = \varrho(\xi) - (\log \xi)^r \sigma(\xi) = (\log \xi)^r \phi(\xi) \tag{4.2 a}$$

and for $m = 1, 2, 3 \ldots$

$$f_m(\xi, h) = \sum_{i=0}^{k} \left\{\alpha_i(h)(x+ih)^m \xi^i - \beta_i(h) \sum_{j=0}^{\min(m,r)} \binom{r}{j} m^{(j)} (\log \xi)^{r-j} h^j (x+ih)^{m-j} \xi^i\right\}. \tag{4.2 b}$$

But (see Appendix)

$$f_{m+1}(\xi, h) = x f_m(\xi, h) + h\xi \frac{\partial f_m(\xi, h)}{\partial \xi} \tag{4.3}$$

so we can write

$$L[x^m e^{\omega x}] = e^{\omega x} \left[h^m a^m f_0^{(m)}(a) + \sum_{i=0}^{m-1} a_{i,m} f_0^{(i)}(a)\right] \tag{4.4}$$

for some $a_{0,m}, \ldots, a_{m-1,m}$. From (4.1) $L[x^m e^{\omega x}] = 0$, $m = 0, 1, \ldots, n-1$ and $L[x^n e^{\omega x}] \neq 0$, if and only if f_m has a zero at $\xi = a$ for $m = 0, 1, \ldots, n-1$ and $f_n(a, h) \neq 0$. By (4.4), since $a \neq 0$, this is equivalent to saying that f_0 has a zero exact multiplicity n at $\xi = a$. Finally, it follows from (4.2a) that this is equivalent to the statement that ϕ has a zero of exact multiplicity n at $\xi = a \neq 1$, and a zero of exact multiplicity $n - r$ at a if $a = 1$. ∎

We next prove a theorem which in effect says that for any polynomial ϱ satisfying (4.5) with real smooth coefficients, we can find a polynomial σ with real smooth coefficients such that the corresponding operator L exactly integrates the polynomials $1, x, \ldots, x^{r-1}$ as well as any set of $k+1$ functions of the form (3.3) satisfying (4.6).

Theorem 4.1. Let $0 \leqq k' \leqq k$ and suppose ϱ is given by (2.7) with coefficients satisfying (2.2) and (2.3). Furthermore, assume

$$\varrho^{(n)}(1) = 0, \qquad n = 0, 1, \ldots, r-1, \text{ all } h \in H. \tag{4.5}$$

Let a set of functions of the form (3.3) be given where

$$\begin{aligned} &\text{(i) } p = k' + 1 \\ &\text{(ii) } \omega_0 = 0, \quad r_0 \geqq r \\ &\text{(iii) } |\mathrm{Im}(\omega_j h_0)| < \pi, \quad j = 1, 2, \ldots, s. \end{aligned} \tag{4.6}$$

Then there is a unique polynomial σ of degree $\leqq k'$ satisfying (2.2) and (2.3) such that L integrates the functions (3.3) exactly.

Proof. By Lemma 4.1, L integrates the functions given by (3.3) exactly if and only if the function ϕ has zeros of multiplicity r_j at $\xi = e^{\omega_j h}, e^{\overline{\omega}_j h}, j=1, 2, \ldots, s$ and a zero of multiplicity $r_0 - r$ at $\xi = 1$. Hence we want to find a polynomial σ of degree $\leqq k'$ such that

$$\sigma^{(n)}(1) = \psi^{(n)}(1), \qquad n = 0, 1, \ldots, r_0 - r - 1, \tag{4.7a}$$

$$\sigma^{(n)}(e^{\omega_j h}) = \psi^{(n)}(e^{\omega_j h}), \qquad n = 0, 1, \ldots, r_j - 1, \; j = 0, 1, \ldots, s \tag{4.7b}$$

$$\sigma^{(n)}(e^{\overline{\omega}_j h}) = \psi^{(n)}(e^{\overline{\omega}_j h}), \qquad n = 0, 1, \ldots, r_j - 1, \; j = q+1, \ldots, s, \tag{4.7c}$$

where

$$r_0 - r + \sum_{j=1}^{q} r_j + 2 \sum_{j=q+1}^{s} r_j = k' + 1$$

and

$$\psi(\xi) = \varrho(\xi) / \log \xi.$$

This Hermite like interpolation problem has a unique solution σ of degree $\leqq k'$. By Lemma 4.1, the corresponding L will integrate the functions given by (3.3) exactly.

To see that the coefficients $\beta_j(h)$ of σ are real, we define the polynomial $\bar{\sigma}$ by $\bar{\sigma}(\xi) = \overline{\sigma(\bar{\xi})}$. Then

$$\sigma^{(n)}(e^{\omega_j h}) = \psi^{(n)}(e^{\omega_j h}) = \overline{\psi^{(n)}(e^{\overline{\omega}_j h})} = \overline{\sigma^{(n)}(e^{\overline{\omega}_j h})}$$

where n and j take on the values given by (4.7). Hence by the uniqueness of the interpolation problem considered, $\bar{\sigma} = \sigma$ and the $\beta_j(h)$s are real. It remains to show that $\beta_j \in C^m(\bar{H})$. But $\varrho^{(j)}(1) = 0$, $j = 0, 1, \ldots, r-1$ implies ψ is analytic at $\xi = 1$. (We let $\log 1 = 0$, and cut the complex plane along the negative real axis.) By (4.6), ψ is analytic in an open, simply connected set containing $e^{\omega_j h}, e^{\overline{\omega}_j h}$, $j = 0, 1, \ldots, s$ all $h \in H$. Since $\alpha_i \in C^m(\bar{H})$ it follows that $X_{j,n} \in C^m(\bar{H})$ where $X_{j,n}(h) = \psi^{(n)}(e^{\omega_j h})$, and j, n takes on the values in (4.7). But, by Cramer's rule, the unique solution $\beta_j(h)$, $j = 0, 1, \ldots, k$, to the interpolation problem (4.7) can be written as a rational expression in the $X_{j,n}$ and $e^{\omega_j h}$ with denominator bounded away from zero. It follows chat $\beta_j \in C^m(\bar{H})$. ▌

Remark 4.2. In general, it is not possible to construct an L which is exact for more than $p = k+1$ linearly independent functions of the form (3.3) with all $\omega_j \neq 0$, without imposing conditions on ϱ. Since the part of L consisting of $h^r \sum_{i=0}^{k} \beta_i(h) y^{(r)}(x + ih)$ annihilates all polynomials of degree $\leqq r - 1$ for any $\beta_j(h)$, the conditions to be satisfied by ϱ when $\omega_0 = 0$ and $p = k+1$ takes on the simple form (4.5). When all $\omega_j \neq 0$ it follows from Lemma 4.1 that a similar condition on ϱ alone does not exist.

Remark 4.3. The coefficients in the interpolation polynomial of the type considered in the proof of Theorem 4.1 can be given explicitly. Thus the method of proof provides us with an algorithm for finding σ. However, since the formulas are rather complicated and involve calculation with complex numbers, it is not recommended. More powerful algorithms should be (and to some extent have been) developed. See e.g. Bettis [1].

5. The Maximum Order of a Stable Method

Definition 5.1. The operator L given by (2.1)–(2.3) is called stable if for all $h\in H$ the polynomial ϱ given by (3.7) has all its roots on the unit disc and all roots of modulus one are of multiplicity at most r.

For $r=1$ let L be the operator with constant coefficients given by (1.3). For this operator $C_{s,n}$ in (2.6) vanish for all $n\geqq 1$. Hence the definition of order as given in Definition 2.1 and the definition of order as given by Henrici [4] p. 221 coincide.

We need the following lemma.

Lemma 5.1. Let the operator L given by (1.3) be such that no roots of the polynomial $\varrho(\xi)=\sum_{i=0}^{k}\alpha_i\xi^i$ exceed one in modulus. Moreover assume $\varrho(1)=0$. Then the order of L is at most $k+2$.

Remark 5.1. The following is proved in [4] p. 229–232. Suppose no roots of ϱ exceed one in modulus and ϱ has a simple root at $\xi=1$ (see footnote in [4] p. 229). Then the order p of L is at most $k+2$. Thus Lemma 5.1 claims that even if we allow multiple roots at $\xi=1$ the order cannot exceed $k+2$. See also Dahlquist [2].

Proof of Lemma 5.1. We need only make a few changes in the development given by Henrici [4] p. 230–232. Suppose ϱ has a root of multiplicity s at $\xi=1$. Then (5–114) in [4] should read $a_s>0$. Using (5–121) and arguing with b_{k+1} and b_{k+2} as done in [4] the lemma follows. ∎

Theorem 5.1. Suppose L given by (2.1)–(2.4) with $r\leqq 2$ is of order p, and suppose L satisfies the condition of stability. Then $p\leqq k+2$.

Proof. We give a proof for $r=1$. (The proof for $r=2$ is quite similar.) Let the operator $\tilde{L}$ be defined by

$$\tilde{L}[y](x)=\sum_{i=0}^{k}\alpha_i(0)\,y(x+ih)-h\sum_{i=0}^{k}\beta_i(0)\,y'(x+ih).$$

Here $\alpha_i(h)$ and $\beta_i(h)$ are the coefficients in L. Since L is of order p, it follows from (2.6) that

$$C_{n,0}=\left\{\begin{array}{ll}\sum_{i=0}^{k}\alpha_i(0) & n=0\\ \sum_{i=0}^{k}i\alpha_i(0)-\sum_{i=0}^{k}\beta_i(0) & n=1\\ \frac{1}{n!}\left\{\sum_{i=0}^{k}i^n\alpha_i(0)-n\cdot\sum_{i=0}^{k}i^{n-1}\beta_i(0)\right\} & n>1\end{array}\right\}=0,\qquad n=0,1,\ldots,p.$$

Hence the operator $\tilde{L}$ is of order p. If $p > k+2$, it follows from Lemma 5.1 that the polynomial $\varrho_0(\xi) = \sum_{i=0}^{k} \alpha_i(0)\xi^i$ has a root of modulus greater than one. The coefficients $\alpha_i(h)$ are continuous functions of h. Then for h sufficiently small, $\varrho(\xi) = \sum_{i=0}^{k} \alpha_i(h)\xi^i$ will have a root of modulus greater than one. ▌

6. Example Two-Step Method of Order 4

Consider the two-step method

$$y(x+2h) - (1+a(h))y(x+h) + a(h)y(x) = h\{\beta_2(h)y'(x+2h) + \beta_1(h)y'(x+h) + \beta_0(h)y'(x)\} \tag{6.1}$$

where we have assumed that $\varrho(1) = 0$. We desire that (6.1) exactly integrates $e^{\pm\omega_i h}$, where ω_i $i = 1, 2$ are real or imaginary. Let $u_i = e^{\omega_i h}$, $\mu = \omega_1 h$, $\nu = \omega_2 h$. From Lemma 4.1 we have the four equations

$$\varrho(u_i^{\pm 1}) = \log(u_i^{\pm 1})\sigma(u_i^{\pm 1}), \qquad i = 1, 2.$$

Solving this system we find

$$\begin{aligned} \beta_0(h) &= \beta_2(h) = \frac{\dfrac{\sinh\nu}{\nu} - \dfrac{\sinh\mu}{\mu}}{\cosh\nu - \cosh\mu} \\ \beta_1(h) &= 2\,\frac{\dfrac{\sinh\mu}{\mu}\cdot\cosh\nu - \dfrac{\sinh\nu}{\nu}\cosh\mu}{\cosh\nu - \cosh\mu} \\ a(h) &= -1. \end{aligned} \tag{6.2}$$

For small h, numerically more convenient expressions are found using the series expansions

$$\begin{aligned} \beta_0 &= \beta_2 = \frac{1}{3} - \frac{h^2}{90}(\omega_1^2 + \omega_2^2) + \frac{h^4}{2520}(\omega_1^4 + \omega_1^2\omega_2^2 + \omega_2^4) + O(h^6) \\ \beta_1 &= \frac{4}{3} + \frac{h^2}{45}(\omega_1^2 + \omega_2^2) - \frac{h^4}{1260}(\omega_1^4 - 13\omega_1^2\omega_2^2 + \omega_2^4) + O(h^6). \end{aligned} \tag{6.3}$$

The truncation error is given by

$$L[y](x) = -\frac{h^5}{90}\{y^{v}(x) - (\omega_1^2 + \omega_2^2)y'''(x) + \omega_1^2\omega_2^2 y'(x)\} + O(h^6). \tag{6.4}$$

Table 1 shows the expressions that are integrated exactly for various choices of ω_1 and ω_2.

Table 1

ω_1, ω_2 real	$1, e^{\pm\omega_1 x}, e^{\pm\omega_2 x}$
ω_1, ω_2 imaginary	$1, \sin\omega_1 x, \cos\omega_1 x, \sin\omega_2 x, \cos\omega_2 x$
$\omega_1 = \omega_2 = \omega$	$1, e^{\pm\omega x}, xe^{\pm\omega x}$
$\omega_1 = \omega, \omega_2 = 0$	$1, x, x^2, e^{\pm\omega x}$
$\omega_1 = \omega_2 = 0$	$1, x, x^2, x^3, x^4$ (Simpson's method)

7. Appendix

We derive the recurrence relation (4.3). From (4.2) we have

$$f_m(\xi, h) = \sum_{i=0}^{k} \left\{ \alpha_i(h)(x+ih)^m \xi^i - \sum_{j=0}^{\min(m,r)} \binom{r}{j} m^{(j)} (\log \xi)^{r-j} h^j \beta_i(h)(x+ih)^{m-j} \xi^i \right\}.$$

Hence

$$\begin{aligned} h\xi \frac{\partial f_m(\xi,h)}{\partial \xi} = \sum_{i=0}^{k} \Big\{ & i h \alpha_i(h)(x+ih)^m \xi^i \\ & - \sum_{j=0}^{\min(m,r)} \binom{r}{j} m^{(j)} h^j (\log\xi)^{r-j} i h \beta_i(h)(x+ih)^{m-j}\xi^i \\ & - \sum_{j=0}^{\min(m,r)} \binom{r}{j} m^{(j)} h^{j+1} (\log\xi)^{r-j-1}(r-j)\beta_i(h)(x+ih)^{m-j}\xi^i \Big\}. \end{aligned}$$

So

$$\begin{aligned} x f_m(\xi,h) + h\xi\frac{\partial f_m(\xi,h)}{\partial\xi} = \sum_{i=0}^{k} \Big\{ & \alpha_i(h)(x+ih)^{m+1}\xi^i \\ & - \sum_{j=0}^{\min(m,r)} \binom{r}{j} m^{(j)} h^j (\log\xi)^{r-j}\beta_i(h)(x+ih)^{m+1-j}\xi^i \\ & - \sum_{j=1}^{\min(m,r)+1} \binom{r}{j-1}(r-j+1) m^{(j-1)} h^j (\log\xi)^{r-j}\beta_i(h)(x+ih)^{m+1-j}\xi^i \Big\}. \end{aligned} \tag{7.1}$$

Note that the expression $\min(m, r)+1$ in the last sum can be replaced by $\min(m+1, r)$ since $r-j+1$ vanish whenever they are different. Now collecting terms in the last two sums we find

(i) $1 \leqq j \leqq \min(m, r)$.

We have a common factor $(\log\xi)^{r-j}\beta_i(h)(x+ih)^{m+1-j}\xi^i$ which is multiplied by

$$\binom{r}{j} m^{(j)} + \binom{r}{j-1}(r-j+1) m^{(j-1)} = \binom{r}{j} m^{(j-1)}[m-j+1+j] = \binom{r}{j}(m+1)^{(j)}.$$

(ii) $j=0$. We have $m^{(j)} = (m+1)^{(j)} = 1$.

(iii) $j = \min(m+1, r)$. If $m+1 \leqq r$ then $j = m+1$ and we have

$$\binom{r}{j-1}(r-j+1) m^{(j-1)} = \binom{r}{j} j m^{(j-1)} = \binom{r}{j}(m+1)^{(j-1)}.$$

If $r < m+1$, then since $r-j+1=0$ we have the same upper limit in the two sums.

Thus the whole expression given by (7.1) reduces to $f_{m+1}(\xi, h)$.

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