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Analysis of Some Low-Order Finite Element Schemes for Mindlin-Reissner and Kirchhoff Plates

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Dedicated to Professor Ivo Babuška on the occasion of his 60th birthday

Summary. We set up a framework for analyzing mixed finite element methods for the plate problem using a mesh dependent energy norm which applies both to the Kirchhoff and to the Mindlin-Reissner formulation of the problem. The analysis techniques are applied to some low order finite element schemes where three degrees of freedom are associated to each vertex of a triangulation of the domain. The schemes proceed from the Mindlin-Reissner formulation with modified shear energy.

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1. Introduction

In the Mindlin-Reissner formulation, the total energy of the plate is expressed by the functional

$$F(u, \theta)=a(\theta, \theta)+(1/t)^2 \int_\Omega |\theta-\nabla u|^2\,dx-2\int_\Omega f u\,dx, \tag{1.1}$$

where t is the thickness of the plate, u and $\theta=(\theta_1, \theta_2)$ stand for the transverse deflection and rotations, respectively, f is proportional to the external load (with the constant of proportionality depending on t), and

$$a(\theta, \varphi)=\int_\Omega \left\{\lambda \operatorname{div}\theta \operatorname{div}\varphi+\mu \sum_{i,j=1}^{2}\left(\frac{\partial \theta_i}{\partial x_j}+\frac{\partial \theta_j}{\partial x_i}\right)\left(\frac{\partial \varphi_i}{\partial x_j}+\frac{\partial \varphi_j}{\partial x_i}\right)\right\} dx,$$

where λ and μ are positive constants. Physically, the first and the second term in functional (1.1) correspond to the bending energy and the shear energy, respectively.

It is well-known that straightforward finite element methods of minimizing the energy (1.1) often give poor results if t is small. The origin of the failure is in the shear energy term, which forces the Kirchhoff constraint $\theta=\nabla u$ in the limit $t\to 0$. In low-order finite element spaces such a constraint may cause "locking". For example if both u and θ are interpolated by continuous piecewise linear functions on a triangulation of Ω, then in the limit $t \to 0$ the transverse deflection in the finite element model is "locked" to be globally a polynomial of degree one.

In this paper we consider some low-order finite element methods where locking is avoided and convergence preserved even in the limit $t \to 0$. The main principle in the methods to be considered is the following: Given a triangulation of Ω, interpolate the transverse deflection u by using reduced cubic Hermitean (or Zienkiewicz) triangles with continuity of u and ∇u imposed at the vertices of the triangulation, and let the rotations be continuous and piecewise linear on the same triangulation. Then impose the Kirchhoff constraint $\theta = \nabla u$ at all vertices of the triangulation (thus leaving three degrees of freedom per vertex) and minimize, over the resulting finite element space, a functional which agrees with (1.1) except for the shear energy term, which is modified in various ways so as to prevent locking. A scheme of this type was proposed first by Fried et al. [14]. Somewhat similar schemes may also be derived from the Kirchhoff formulation of the plate problem by relaxing the Kirchhoff constraint $\theta = \nabla u$ so that it is imposed only at discrete points. In such schemes the shear energy term in (1.1) is neglected altogether, cf. [21, 6, 16, 17].

The aim of this paper is to present a convergence analysis of schemes based on the above philosophy where the shear energy is not neglected, i.e., the Mindlin-Reissner formulation is used, but with modified shear energy. For previous error analyses of finite element methods proceeding from the Mindlin-Reissner formulation of the plate problem, or from the corresponding mixed formulation, the reader is referred to [5, 9, 12, 13, 15, 17]; cf. also [1, 16] for closely related work. Our analysis is parallel with the previous ones in that we make use of a mixed variational formulation of the plate problem where shear stresses occur explicitly. The main new feature in our analysis is the use of a mesh dependent (energy) norm. The norm is chosen so as to make passing from the Mindlin-Reissner model to the Kirchhoff model (corresponding to the limit $t \to 0$) "soft" in the sense that no abrupt change of norm occurs at $t = 0$. It turns out that once the norm is properly chosen, the analysis fits easily into the framework of Babuška [2, 3] and Brezzi [8].

The analysis is carried out in two steps as follows. We denote by (u^t, θ^t) the exact solution of the plate problem for given $t > 0$ and let $(u^0, \theta^0) = \lim_{t \to 0} (u^t, \theta^t)$. We then compare the finite element solution first to the Kirchhoff solution (u^0, θ^0), and then estimate separately the difference between (u^0, θ^0) and (u^t, θ^t). This approach is natural here since discrete Kirchhoff constraints are imposed in the finite element scheme, and thus it is not clear whether the "exact" solution is (u^t, θ^t) or (u^0, θ^0). However, our approach is actually not based on the assumed Kirchhoff constraints but rather on the assumption that the thickness of the plate is not substantially larger than the smallest mesh spacing in the finite element model. It turns out that for such "numerically thin" plates the natural (mesh dependent) energy norm for finite element analysis is independent of t and thus in particular remains the same in the limit $t \to 0$.

The plan of the paper is as follows. In Sect. 2 we introduce the basic notation and establish some estimates relating the Mindlin-Reissner and Kirchhoff solutions of the plate problem. In Sect. 3 we introduce the mesh dependent norms and set up a basic framework for the analysis of mixed finite element methods for Mindlin-Reissner equations. Section 4 is denoted to the analysis of four variants of the finite element scheme described above.

Throughout the paper, we assume for simplicity that the domain Ω is a convex polygon and that the plate is clamped, i.e., $u=\theta=0$ on $\partial\Omega$. The treatment of more general domains and boundary conditions is possible, though not straightforward, see [19].

2. Preliminaries and Some Basic Estimates

For $T\subset\mathbf{R}^2$ and $s\geqq 0$ we let $H^s(T)$ and $H_0^s(T)$ denote the usual Sobolev spaces of index s. The seminorm and the norm of $[H^s(T)]^n$, $n=1,2$, are denoted by $|\cdot|_{s,T}$ and $\|\cdot\|_{s,T}$, respectively. The subscript indicating domain is omitted if $T=\Omega$. By $H^{-s}(T)$ we mean the dual of $H_0^s(T)$ and by $(\cdot,\cdot)$ the inner product of either $L_2(\Omega)$ or $[L_2(\Omega)]^2$.

Let $V=H_0^1(\Omega)$ and $W=[H_0^1(\Omega)]^2$. Then the plate problem to be considered is formulated for $t>0$ as: Find $(u^t,\theta^t)\in V\times W$ which minimizes the energy (1.1) in $V\times W$. We note that, because of the well-known Korn inequality

$$a(\varphi,\varphi)\geqq c\,\|\varphi\|_1^2,\qquad \varphi\in W,\ c=\text{const.}>0, \tag{2.1}$$

(u^t,θ^t) is uniquely determined for any $t>0$ so far as $f\in H^{-1}(\Omega)$. Moreover, one has the following equivalent mixed variational formulation of the problem: Find the triple $(u^t,\theta^t,\gamma^t)\in V\times W\times Q$, where $Q=[L_2(\Omega)]^2$, such that

$$\begin{aligned} a(\theta^t,\varphi)+(\gamma^t,\varphi)&=0, && \varphi\in W\\ -(\gamma^t,\nabla v)&=(f,v), && v\in V\\ (\theta^t-\nabla u^t,\zeta)&=t^2(\gamma^t,\zeta), && \zeta\in Q. \end{aligned} \tag{2.2}$$

Here $\gamma^t=(1/t)^2(\theta^t-\nabla u^t)$ has the physical meaning of a shear stress.

We consider in parallel with (2.2) the corresponding Kirchhoff formulation of the plate problem: Find the triple $(u^0,\theta^0,\gamma^0)\in H_0^2(\Omega)\times W\times[H^{-1}(\Omega)]^2$ such that

$$\begin{aligned} a(\theta^0,\varphi)+(\gamma^0,\varphi)&=0, && \varphi\in W,\\ -(\gamma^0,\nabla v)&=(f,v), && v\in H_0^2(\Omega),\\ (\theta^0-\nabla u^0,\zeta)&=0, && \zeta\in[H^{-1}(\Omega)]^2, \end{aligned} \tag{2.3}$$

or equivalently

$$\begin{aligned} a(\nabla u^0,\nabla v)&=(f,v), && v\in H_0^2(\Omega),\\ \theta^0&=\nabla u^0,\\ (\gamma^0,\varphi)&=-a(\theta^0,\varphi), && \varphi\in W. \end{aligned}$$

This system is obviously uniquely solvable if $f \in H^{-2}(\Omega)$ and one has the basic energy estimate

$$\|u^0\|_2 + \|\theta^0\|_1 + \|\gamma^0\|_{-1} \leqq C\|f\|_{-2}, \quad f \in H^{-2}(\Omega). \tag{2.4}$$

We note that in general $(u^0, \theta^0, \gamma^0)$ is not a solution to (2.2) for $t=0$. However, if $f \in H^{-1}(\Omega)$ and if one knows additionally that $\gamma^0 \in Q$, then $H_0^2(\Omega)$ can be replaced by V in (2.3) by continuity, and so in that case $(u^0, \theta^0, \gamma^0)$ satisfies (2.2) for $t=0$. The assumption that $\gamma^0 \in Q$ will be needed in the analysis below. For $f \in H^{-1}(\Omega)$, this is assured by our assumption that Ω is a convex polygon, for one has then the regularity estimate (cf. [7])

$$\|u^0\|_3 + \|\theta^0\|_2 + \|\gamma^0\|_0 \leqq C\|f\|_{-1}, \quad f \in H^{-1}(\Omega). \tag{2.5}$$

Let us write the above two plate problems more compactly as follows: Given $t \geqq 0$, set $V^t = V$, $Q^t = Q$ if $t > 0$ or $V^t = H_0^2(\Omega)$, $Q^t = [H^{-1}(\Omega)]^2$ if $t=0$, and then find the triple $(u^t, \theta^t, \gamma^t) \in V^t \times W \times Q^t$ such that

$$\mathscr{B}_t(u^t, \theta^t, \gamma^t; v, \varphi, \zeta) = (f, v), \quad (v, \varphi, \zeta) \in V^t \times W \times Q^t, \tag{2.6}$$

where $\mathscr{B}_t$ is a bilinear form defined on $[V^t \times W \times Q^t]^2$ by

$$\begin{aligned} \mathscr{B}_t(u, \theta, \gamma; v, \varphi, \zeta) = a(\theta, \varphi) - (\theta - \nabla u, \zeta) + (\varphi - \nabla v, \gamma) + t^2(\gamma, \zeta), \\ (u, \theta, \gamma), \quad (v, \varphi, \zeta) \in V^t \times W \times Q^t. \end{aligned} \tag{2.7}$$

Then if the space $V^t \times W \times Q^t$ is supplied with the norm

$$|||v, \varphi, \zeta|||_t = \begin{cases} (\|\varphi\|_1^2 + (1/t)^2\|\varphi - \nabla v\|_0^2 + t^2\|\zeta\|_0^2)^{1/2}, & \text{if } t > 0, \\ (\|v\|_2^2 + \|\varphi\|_1^2 + \|\zeta\|_{-1}^2)^{1/2}, & \text{if } t = 0, \end{cases} \tag{2.8}$$

it is easy to see that for all (u, θ, γ), $(v, \varphi, \zeta) \in V^t \times W \times Q^t$,

$$|\mathscr{B}_t(u, \theta, \gamma; v, \varphi, \zeta)| \leqq C|||u, \theta, \gamma|||_t\, |||v, \varphi, \zeta|||_t,$$

and for all $(u, \theta, \gamma) \in V^t \times W \times Q^t$,

$$\sup_{\substack{(v, \varphi, \zeta) \in V^t \times W \times Q^t \\ |||v, \varphi, \zeta|||_t = 1}} \mathscr{B}_t(u, \theta, \gamma; v, \varphi, \zeta) \geqq c|||u, \theta, \gamma|||_t,$$

where C and c are positive constants independent of t. Thus the variational problem (2.6) is well posed in the usual sense [3]. In finite element analysis, however, difficulties arise in this formulation, first of all because the norm $|||\cdot|||_0$ is ruled out in finite element subspaces where only C^0-continuity is imposed, and secondly because the use of $|||\cdot|||_t$ for small positive values of t leads to locking, i.e. no convergence rates are obtained for simple schemes. For these reasons we choose to work below with a mesh dependent norm which may be considered a compromise between the two norms in (2.8). For other approaches, cf. [9, 13, 17].

We complete this section by stating and proving some estimates relating the solution of (2.6) for $t>0$ to that corresponding to $t=0$ for given $f\in H^{-1}(\Omega)$. For related estimates, cf. [12].

Theorem 2.1. *There is a constant C independent of t and f such that if $f\in H^{-1}(\Omega)$, the following estimates hold*

$$\|\theta^t-\theta^0\|_1+t\|\gamma^t-\gamma^0\|_0+\|\gamma^t-\gamma^0\|_{-1}\leqq Ct\|f\|_{-1},$$
$$\|u^t-u^0\|_1+\|\theta^t-\theta^0\|_0\leqq Ct^2\|f\|_{-1},$$

Proof. For $f\in H^{-1}(\Omega)$, (2.2) holds also at $t=0$ (by our assumption that Ω is convex). Therefore if $t>0$,

$$\mathscr{B}_t(u^t-u^0,\theta^t-\theta^0,\gamma^t-\gamma^0;v,\varphi,\zeta)=-t^2(\gamma^0,\zeta),\qquad (v,\varphi,\zeta)\in V\times W\times Q.\quad (2.9)$$

Setting here $v=u^t-u^0$, $\varphi=\theta^t-\theta^0$ and $\zeta=\gamma^t-\gamma^0$, we have

$$\begin{aligned}&a(\theta^t-\theta^0,\theta^t-\theta^0)+t^2\|\gamma^t-\gamma^0\|_0^2\\&\quad=-t^2(\gamma^0,\gamma^t-\gamma^0)\leqq(t^2/2)\|\gamma^0\|_0^2+(t^2/2)\|\gamma^t-\gamma^0\|_0^2\end{aligned}$$

and so by (2.1) and (2.5),

$$\|\theta^t-\theta^0\|_1+t\|\gamma^t-\gamma^0\|_0\leqq Ct\|f\|_{-1}.$$

On the other hand since by (2.9)

$$a(\theta^t-\theta^0,\varphi)+(\gamma^t-\gamma^0,\varphi)=0,\qquad \varphi\in W,$$

and since $|a(\theta,\varphi)|\leqq C\|\theta\|_1\|\varphi\|_1$ for $\theta,\varphi\in W$, it follows that $\|\gamma^t-\gamma^0\|_{-1}\leqq C\|\theta^t-\theta^0\|_1$. Hence, the first part of the assertion follows.

To prove the second part of the assertion we use a duality argument. For $g\in H^{-1}(\Omega)$ and $d\in Q$ given, let $(\rho,\psi,\eta)\in H_0^2(\Omega)\times W\times[H^{-1}(\Omega)]^2$ be such that

$$\begin{aligned}&\mathscr{B}_0(v,\varphi,\zeta;\rho,\psi,\eta)\\&\quad=(v,g)+(\varphi,d),\qquad (v,\varphi,\zeta)\in H_0^2(\Omega)\times W\times[H^{-1}(\Omega)]^2.\end{aligned}\quad (2.10)$$

Then $\psi=\nabla\rho$ and

$$a(\nabla\rho,\nabla v)=(g,v)+(d,\nabla v),\qquad v\in H_0^2(\Omega),$$

so by the regularity estimate (2.5)

$$\|\rho\|_3+\|\psi\|_2\leqq C(\|g\|_{-1}+\|d\|_0).$$

Moreover

$$(\eta,\varphi)=-a(\psi,\varphi)+(d,\varphi),\qquad \varphi\in W,$$

so

$$\begin{aligned}\|\eta\|_0&=\sup_{\substack{\varphi\in W\\ \|\varphi\|_0=1}}[-a(\psi,\varphi)+(d,\varphi)]\\&\leqq C\|\psi\|_2+\|d\|_0\\&\leqq C_1(\|g\|_{-1}+\|d\|_0).\end{aligned}\quad (2.11)$$

Choose now $v=u^t-u^0$, $\varphi=\theta^t-\theta^0$, $\zeta=\gamma^t-\gamma^0$ in (2.10) and apply (2.9) to obtain

$$(u^t-u^0, g)+(\theta^t-\theta^0, d)=-t^2(\gamma^0, \eta).$$

By (2.11) and since $\|\gamma^t\|_0 \leqq \|\gamma^0\|_0 + C\|f\|_{-1} \leqq C_1\|f\|_{-1}$ by (2.5) and by the first part of the theorem proved above, this implies that

$$(u^t-u^0, g)+(\theta^t-\theta^0, d) \leqq C t^2 \|f\|_{-1}(\|g\|_{-1}+\|d\|_0),$$

where C is independent of t, f, g and d. Since this inequality holds for arbitrary $g \in H^{-1}(\Omega)$ and $d \in Q$, the second part of the assertion follows. □

Remark. If $\|f\|_{-1}$ is replace by $\|u^0\|_3$ in the first estimate of Theorem 2.1, then the assumption that $u^0 \in H^3(\Omega)$ is sufficient for the validity of the estimate even on a non-convex polygonal domain. This is clear since we needed in fact only the assumption that the solution to (2.3) satisfies (2.2) at $t=0$. It is also clear that the second estimate of the theorem cannot be extended in this way. □

3. FEM Approximation: The Basic Stability Conditions

In this section we consider general finite element approximation of system (2.2). Let $\{\mathscr{C}^h\}_{0<h<h_0}$ be a family of subdivisions of Ω into triangles T such that $h=\max_{T\in C^h} h_T$, $h_T=\text{diam}(T)$. We assume the family $\{\mathscr{C}^h\}$ to be locally regular in the sense that if α is any angle of any $T\in\mathscr{C}^h$, then $\alpha>\alpha_0$, where α_0 is an absolute, positive constant. We make also the usual assumption that if $T_1, T_2\in\mathscr{C}^h$ and if $\partial T_1\cap\partial T_2$ is non-empty, then $\partial T_1\cap\partial T_2$ is either a common side or a common vertex of T_1 and T_2. Finally, we assume that the parameter t, indicating the thickness of the plate, is related to h in such a way that

$$t \leqq C h_T, \qquad T\in\mathscr{C}^h, \tag{3.1}$$

where C is a constant. This means that the plates to be considered are "numerically thin", see Remark 3.2 below. Below we let C, c, C_i and c_i stand for positive constants independent of t and h.

Assume now that to each $\mathscr{C}^h$ there are associated some finite element spaces $V_h\subset V$, $W_h\subset W$, $Q_h\subset Q$, and $S_h\subset V_h\times W_h$ in such a way that for all $T\in\mathscr{C}^h$, $0<h<h_0$, and for some given k independent of h, $V_{h|T}\subset\mathscr{P}_k$, $W_{h|T}\subset\mathscr{P}_k^2$, and $Q_{h|T}\subset\mathscr{P}_k^2$, where $\mathscr{P}_k$ stands for the space of polynomials of degree $\leqq k$. In analogy with (2.6) we may then define the finite element approximation of the solution to (2.2) as the triple $(u_h, \theta_h, \gamma_h)\in S_h\times Q_h$ which satisfies

$$\mathscr{B}_t(u_h, \theta_h, \gamma_h; v, \varphi, \zeta)=(f, v), \qquad (v, \varphi, \zeta)\in S_h\times Q_h, \tag{3.2}$$

where $\mathscr{B}_t$ is defined by (2.7).

Equation (3.2) defines a mixed finite element scheme for solving the plate problem. Let us analyze this scheme following the lines of Babuška [2, 3] and

Brezzi [8]. To this end, we supply first the space $V \times W \times Q$ with a norm. We propose here a mesh dependent norm defined as

$$|||v, \varphi, \zeta|||_h = \{\|\varphi\|_1^2 + \square\varphi - \nabla v\square_{1,h}^2 + \square\zeta\square_{-1,h}^2\}^{1/2},$$

where $\square\cdot\square_{s,h}$, $s \in \mathbf{R}$, stands for a weighted L_2-norm defined by

$$\square\zeta\square_{s,h} = \{\sum_{T \in \mathscr{C}^h} h_T^{-2s} \|\zeta\|_{0,T}^2\}^{1/2}.$$

In order to carry out successful finite element error analysis in this norm we need in general the assumption that the following two basic conditions hold for some positive constants C and c independent of t and h [2, 3]:

$$|\mathscr{B}_t(u, \theta, \gamma; v, \varphi, \zeta)| \\ \leqq C\,|||u, \theta, \gamma|||_h\,|||v, \varphi, \zeta|||_h, \quad (u, \theta, \gamma), (v, \varphi, \zeta) \in V \times W \times Q, \tag{3.3}$$

$$\sup_{\substack{(v, \varphi, \zeta) \in S_h \times Q_h \\ |||v, \varphi, \zeta|||_h = 1}} \mathscr{B}_t(u, \theta, \gamma; v, \varphi, \zeta) \geqq c\,|||u, \theta, \gamma|||_h, \quad \forall (u, \theta, \gamma) \in S_h \times Q_h. \tag{3.4}$$

We note that if $\gamma, \zeta \in Q$, then $(\gamma, \zeta) \leqq \square\gamma\square_{1,h}\square\zeta\square_{-1,h}$ and $t^2(\gamma, \zeta) \leqq C\square\gamma\square_{-1,h}\square\zeta\square_{-1,h}$ by assumption (3.1). Therefore condition (3.3) holds, and we are thus left with condition (3.4), which is the basic stability condition for the finite element scheme (3.2). In general, the stability condition (3.4) can be verified only after specifiying the finite element spaces in more detail. However, even without doing this we can split the stability condition into simpler conditions in the spirit of Brezzi [8]. The following splitting result turns out to be useful.

Theorem 3.1. *Suppose that for some positive constants c and C, the following two conditions hold:*

a) *For each $(u, \theta) \in S_h$, there is $\zeta \in Q_h$ such that*

$$\square\zeta\square_{-1,h}^2 \leqq C(\|\theta\|_1^2 + \square\theta - \nabla u\square_{1,h}^2) \quad \textit{and} \quad (\theta - \nabla u, \zeta) \geqq c\square\theta - \nabla u\square_{1,h}^2 - C\|\theta\|_1^2.$$

b) *For each $\gamma \in Q_h$, there is $(v, \varphi) \in S_h$ such that*

$$\|\varphi\|_1^2 + \square\varphi - \nabla v\square_{1,h}^2 \leqq C\square\gamma\square_{-1,h}^2, \quad \textit{and} \quad (\varphi - \nabla v, \gamma) \geqq c\square\gamma\square_{-1,h}^2.$$

Then (3.4) *holds.*

Proof. Let $(u, \theta, \gamma) \in S_h \times Q_h$ be given. Then by condition a), there is $\rho \in Q_h$ such that

$$\square\rho\square_{-1,h}^2 \leqq C(\|\theta\|_1^2 + \square\theta - \nabla u\square_{1,h}^2), \quad (\theta - \nabla u, \rho) > c\square\theta - \nabla u\square_{1,h}^2 - C\|\theta\|_1^2.$$

Similarly, by conduction b), there is $(w, \xi) \in S_h$ such that

$$\|\xi\|_1^2 + \square\xi - \nabla w\square_{1,h}^2 \leqq C\square\gamma\square_{-1,h}^2, \quad (\xi - \nabla w, \gamma) > c\square\gamma\square_{-1,h}^2.$$

Set now $(v, \varphi, \zeta)=(u, \theta, \gamma)+\delta(v, \xi, -\rho)$ where $\delta>0$ is so far unspecified. Then

$$\begin{aligned}\mathscr{B}_t(u, \theta, \gamma; v, \varphi, \zeta) \geqq\; & a(\theta, \theta)+t^2\,\|\gamma\|_0^2\\ & +c\,\delta\,\Box\theta-\nabla u\Box_{1,h}^2-C\,\delta\,\|\theta\|_1^2\\ & +c\,\delta\,\Box\gamma\Box_{-1,h}^2+\delta\, a(\theta, \xi)\\ & -\delta\, t^2(\gamma, \rho).\end{aligned}$$

Upon using here the inequalities

$$|\delta\, a(\theta, \xi)| \leqq C_1\,\delta\,\|\theta\|_1\,\|\xi\|_1 \leqq (c\,\delta/2)\,\Box\gamma\Box_{-1,h}^2+C_2\,\delta\,\|\theta\|_1^2$$

and

$$\begin{aligned}|\delta\, t^2(\gamma, \rho)| & \leqq C_3\,\delta\,\Box\rho\Box_{-1,h}\, t\,\|\gamma\|_0\\ & \leqq (t^2/2)\,\|\gamma\|_0^2+C_4\,\delta^2\,\Box\theta-\nabla u\Box_{1,h}^2+C_4\,\delta^2\,\|\theta\|_1^2,\end{aligned}$$

it follows that

$$\begin{aligned}\mathscr{B}_t(u, \theta, \gamma; v, \varphi, \zeta) \geqq\; & (c_1-C\delta-C_2\delta-C_4\delta^2)\,\|\theta\|_1^2\\ & +\delta(c-C_4\delta)\,\Box\theta-\nabla u\Box_{1,h}^2\\ & +(c\,\delta/2)\,\Box\gamma\Box_{-1,h}^2+(t^2/2)\,\|\gamma\|_0^2\end{aligned}$$

and thus

$$\mathscr{B}_t(u, \theta, \gamma; v, \varphi, \zeta) \geqq c_2\,|||u, \theta, \gamma|||_h^2,$$

if δ is small enough. Together with the fact that $|||v, \varphi, \zeta|||_h \leqq (1+C\delta)\,|||u, \theta, \gamma|||_h$, this completes the proof. □

Remark 3.1. Condition a) of Theorem 3.1 holds in general if Q_h is sufficiently large compared with V_h. Assume for example that $Q_h=\{\zeta\in Q\colon \zeta_{|T}\in Q^T,\ T\in\mathscr{C}^h\}$, where for all $T\in\mathscr{C}^h$, $Q^T\supset\mathscr{P}_0^2\oplus\{\nabla v_{|T}\colon v\in V_h\}$. Then if $(u, \theta)\in S_h$ is given, once can choose $\zeta\in Q_h$ so that $\zeta_{|T}=h_T^{-2}(\pi_h\theta-\nabla u)_{|T}$, $T\in\mathscr{C}^h$, where π_h stands for the L_2-projection into Q_h. Then $\Box\zeta\Box_{-1,h}\leqq\Box\theta-\nabla u\Box_{1,h}+\Box\theta-\pi_h\theta\Box_{1,h}$ and $(\theta-\nabla u, \zeta)=\Box\pi_h\theta-\nabla u\Box_{1,h}^2\geqq(1/2)\,\Box\theta-\nabla u\Box_{1,h}^2-\Box\theta-\pi_h\theta\Box_{1,h}^2$. On the other hand since $Q^T\supset\mathscr{P}_0^2\ \forall\, T\in\mathscr{C}^h$, standard approximation theory implies that $\|\theta-\pi_h\theta\|_{0,T}\leqq C h_T|\theta|_{1,T}\ \forall\, T\in\mathscr{C}^h$. Thus $\Box\theta-\pi_h\theta\Box_{1,h}\leqq C|\theta|_1$, and accordingly, condition a) holds.

Regarding condition b), this holds in general if S_h is large enough compared with Q_h. This is often not the case a priori. However, in practive it is easy to force the condition by enriching the space W_h with suitable "bubble" functions, see Sect. 4 below and cf. also [17] where similar techniques are proposed for a higher-order scheme. There is also a way of avoiding stability condition b) by modifying the bilinear form $\mathscr{B}_t$, see Sect. 4 below. □

We close this section by characterizing the norm $|||\cdot|||_h$ in the finite element subspaces. We need here some additional mesh dependent (semi) norms analogous to those introduced in [4]. Let $\zeta \in Q$ be such that $\zeta_{|T} \in [H^1(T)]^2$ for all $T \in \mathscr{C}^h$ and let $v \in V$ be such that $v_{|T} \in H^2(T)$ for all $T \in \mathscr{C}^h$. Then set

$$|\zeta|_{1,h}^2 = \sum_{T \in \mathscr{C}_h} \{|\zeta|_{1,T}^2 + h_T^{-1} \int_{\partial T \setminus \partial \Omega} |[\zeta]|^2 \, ds\},$$

where $[\zeta]$ denotes the jump of ζ across ∂T, and let

$$|v|_{2,h} = |\nabla v|_{1,h}, \qquad \|v\|_{2,h} = \{\|v\|_1^2 + |v|_{2,h}^2\}^{1/2}.$$

Then we have

Lemma 3.1. *There is a positive constant c such that for all* $(v, \varphi, \zeta) \in S_h \times Q_h$, $|||v, \varphi, \zeta|||_h \geqq c \|v\|_{2,h}$.

Proof. Since $|||v, \varphi, \zeta|||_h^2 \geqq \|\varphi\|_1^2 + h_0^{-2} \|\varphi - \nabla v\|_0^2$, it follows that $|||v, \varphi, \zeta|||_h \geqq c\|v\|_1$. On the other hand, by simple local inverse estimates $|\varphi - \nabla v|_{1,h} \leqq C ⦀\varphi - \nabla v⦀_{1,h}$ if $(v, \varphi) \in S_h$, so

$$|v|_{2,h} \leqq |\varphi - \nabla v|_{1,h} + |\varphi|_{1,h} \leqq C ⦀\varphi - \nabla v⦀_{1,h} + |\varphi|_1, \qquad (v, \varphi) \in S_h.$$

Upon combining these estimates, the assertion follows. □

Remark 3.2. Assume that the stability condition (3.4) holds and that $(u_h, \theta_h, \gamma_h) \in S_h \times Q_h$ is the solution to (3.2). Then it follows from Lemma 3.1 that

$$\|u_h\|_{2,h} + \|\theta_h\|_1 + ⦀\gamma_h⦀_{-1,h} \leqq C \|f\|_{-2,h},$$

where

$$\|f\|_{-2,h} = \sup_{(v,\varphi) \in S_h} \frac{(f, v)}{\|v\|_{2,h}}.$$

This may be viewed as the discrete analogue of the basic energy estimate (2.4) corresponding to the Kirchhoff plate model. This confirms that so far as condition (3.1) holds, the plate should be considered "numerically thin". On the other hand if (3.1) is violated, estimate (3.3) and Theorem 3.1 still remain valid provided that the underlying norm $⦀\cdot⦀_{s,h}$ is redefined by

$$⦀\zeta⦀_{s,h}^2 = \sum_{T \in \mathscr{C}_h} [\max(h_T, t)]^{-2s} \|\zeta\|_{0,T}^2.$$

In particular in the "numerically thick" regime where $h_T \leqq t \ \forall T \in \mathscr{C}_h$, the norm $|||\cdot|||_h$ then reduces to the natural energy norm $|||\cdot|||_t$ of the continuous problem, as defined by (2.8). □

4. Some Low-Order Schemes

We apply now the techniques of the previous section to analyze some relatively simple methods for solving the plate problem. The methods to be considered are all based on a single finite element which we first introduce.

If $T \in \mathscr{C}^h$, let

$$\mathscr{V}^T = \left\{ p \in \mathscr{P}_3 : 6p(a^0) - 2\sum_{i=1}^{3} p(a^i) + \sum_{i=1}^{3} \nabla p(a^i) \cdot (a^i - a^0) = 0 \right\},$$

where a^0 is the midpoint and $\{a^1, a^2, a^3\}$ are the vertices of T. Then $\mathscr{V}^T \supset \mathscr{P}_2$ and for the degrees of freedom of $\mathscr{V}^T$ one can take the set $\{v(a^i), \nabla v(a_i); i = 1, 2, 3\}$. The finite element so obtained is usually referred to as the Zienkiewicz triangle [11]. We construct now another element by setting

$$\mathscr{Y}^T = \{(p, q) \in \mathscr{V}^T \times \mathscr{P}_1^2 : q(a^i) = \nabla p(a^i), i = 1, 2, 3\}$$

and choosing for the degrees of freedom of $\mathscr{Y}^T$ the set $\sigma_T = \{p(a^i), q(a^i); i = 1, 2, 3\}$. The resulting element $\{T, \mathscr{Y}^T, \sigma_T\}$ is similar to the Discrete Kirchhoff triangle discussed in [6]. Below we use this element as the basis of approximation schemes.

To define our approximation schemes we need the following subspaces:

$$Y_h = \{(v, \varphi) \in V \times W : (v, \varphi)_{|T} \in \mathscr{Y}^T \ \forall T \in \mathscr{C}^h\},$$
$$Q_h = \{\zeta \in Q : \zeta_{|T} \in \mathscr{P}_0^2 \ \forall T \in \mathscr{C}^h\},$$
$$B_h = \{\varphi \in W : \varphi_{|T} \subset \mathscr{P}_3^2 \text{ and } \varphi_{|\partial T} = 0, \ \forall T \in \mathscr{C}^h\}.$$

We note that if $(\varphi, v) \in Y_h$ and if x is a vertex of triangulation $\mathscr{C}^h$ then ∇v is continuous at x and the "discrete Kirchhoff condition" $\varphi = \nabla v$ holds at x. In fact, we could define Y_h alternatively as

$$Y_h = \{(v, \varphi) \in V_h \times W_h : \varphi(x) = \nabla v(x), \ x \in \Xi_h\},$$

where Ξ_h is the set of vertices in triangulation $\mathscr{C}^h$, and the subspaces $V_h \subset V$ and $W_h \subset W$ are defined as

$$V_h = \{v \in V : v_{|T} \in \mathscr{V}^T \ \forall T \in \mathscr{C}^h \text{ and } \nabla v \text{ is continuous at } x \in \Xi_h\},$$
$$W_h = \{\varphi \in W : \varphi_{|T} \in \mathscr{P}_1^2 \ \forall T \in \mathscr{C}^h\}.$$

Assuming that the space Y_h is constructed in terms of the finite element $\{T, \mathscr{Y}^T, \sigma_T\}$, the natural degrees of freedom of Y_h are $\{v(x), \varphi(x) : x \in \Xi_h \setminus \partial\Omega\}$.

The "bubble" space $\mathscr{B}_h$ is needed only in the formulation of the first finite element scheme below. As will be seen, the "bubbles" have the effect of forcing the stability condition b) in Theorem 3.1.

In the above notation, our first finite element scheme is the following:

Scheme 1. Set $S_h = Y_h \oplus \{0\} \times B_h$. Then find $(u_h, \theta_h) \in S_h$ which minimizes in S_h the functional

$$F(v, \varphi) = a(\varphi, \varphi) + t^{-2}((\pi_h(\varphi - \nabla v), \pi_h(\varphi - \nabla v)) - 2(f, v),$$

where $\pi_h : Q \to Q_h$ is the L_2-projection into Q_h.

We note that if $(u_h, \theta_h)=(u_h, \psi_h)+(0, \xi_h)$ where $(u_h, \psi_h)\in Y_h$ and $\xi_h\in B_h$, then Scheme 1 corresponds to solving the system

$$a(\psi_h, \varphi)+t^{-2}(\pi_h(\psi_h+\xi_h-\nabla u_h), \pi_h(\varphi-\nabla v))=(f, v), \qquad (v, \varphi)\in Y_h, \tag{4.1}$$

$$a(\xi_h, \eta)+t^{-2}(\pi_h(\psi_h+\xi_h-\nabla u_h), \pi_h \eta)=0, \qquad \eta\in B_h. \tag{4.2}$$

Here we have taken into account the fact that for any $\eta\in B_h$ and $\zeta\in Q_h$,

$$\int_T \frac{\partial \eta}{\partial x_i}\cdot\zeta\, dx=0, \qquad i=1, 2,\ T\in\mathscr{C}^h,$$

so in particular $a(\varphi, \eta)=0$ whenever $\eta\in B_h$ and $\varphi\in W$ is such that $\varphi_{|T}\in\mathscr{P}_1^2$, $T\in\mathscr{C}^h$.

Let us solve (4.2) for $\pi_h \xi_{h|T}$ on a given $T\in\mathscr{C}^h$. To this end, let $\varphi_T^1\in W$ and $\varphi_T^2\in W$ be defined on Ω so that $\varphi_T^i(x)=0$, $i=1, 2$, if $x\in\Omega\setminus T$ and $\varphi_T^1(x)=(p(x), 0)$ and $\varphi_2^T(x)=(0, p(x))$ if $x\in T$, where $p\in\mathscr{P}_3$ is such that $p_{|\partial T}=0$ and $\int_T p\, dx=\text{area}(T)$. Further let A be a symmetric 2×2 matrix defined by

$$A=(a(\varphi_T^i, \varphi_T^j))_{i, j=1, 2}. \tag{4.3}$$

Then if $(\pi_h \xi_h)(x)\equiv(\xi_1, \xi_2)$ and $\pi_h(\psi_h-\nabla u_h)(x)\equiv(z_1, z_2)$ on T, it follows from (4.2) that

$$\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}=[-\mathrm{I}+t^2(\text{area}(T)\, I+t^2 A)^{-1} A]\begin{bmatrix}z_1\\ z_2\end{bmatrix}, \tag{4.4}$$

where I denotes the 2×2 identity matrix.

Upon collecting the local Eqs. (4.4) we obtained a global relationship of the form

$$\pi_h \xi_h=-\pi_h(\psi_h-\nabla u_h)+t^2 \Lambda_h \pi_h(\psi_h-\nabla u_h), \tag{4.5}$$

where $\Lambda_h\colon Q_h\to Q_h$ is a linear operator. Inserting then this expression into (4.1) we obtain a scheme for computing (u_h, ψ_h) directly. Now, knowing (u_h, ψ_h) we may either compute ξ_h from (4.5), or simply ignore ξ_h and let (u_h, ψ_h) take the role of the finite element solution. Noting that Λ_h is a symmetric operator in Q (since matrix A in (4.4) is symmetric), the latter simplified scheme is the following:

Scheme 2. Take $S_h=Y_h$. Then find $(u_h, \theta_h)\in S_h$ which minimizes in S_h the functional

$$F(v, \varphi)=a(\varphi, \varphi)+(\Lambda_h \pi_h(\varphi-\nabla v), \pi_h(\varphi-\nabla v))-2(f, v),$$

where $\Lambda_h\colon Q_h\to Q_h$ is defined as above.

Let us now generalize Scheme 2. We note first that by (4.3) through (4.5) and by the assumption (3.1), operator Λ_h in (4.5) satisfies

$$C\|\zeta\|_{1,h}^2\geqq(\Lambda_h\zeta, \zeta)\geqq c\|\zeta\|_{1,h}^2, \qquad \zeta\in Q_h. \tag{4.6}$$

It turns out that Scheme 2 always works if $\Lambda_h: Q_h \to Q_h$ is a symmetric operator satisfying (4.6). The simplest choice is then such that

$$\Lambda_h \zeta_{|T} = c_T h_T^{-2} \zeta_{|T}, \qquad T \in \mathscr{C}^h,$$

where $c \leqq c_T \leqq C$ for some fixed constants c and C. The resulting scheme then takes the following form:

Scheme 3. Find $(u_h, \theta_h) \in S_h := Y_h$ which minimizes in S_h the functional

$$F(v, \varphi) = a(\varphi, \varphi) + \sum_{T \in \mathscr{C}_h} c_T h_T^{-2} \int_T |\pi_h(\varphi - \nabla v)|^2 \, dx - 2(f, v),$$

where $0 < c \leqq c_T \leqq C$ for some fixed constants c and C.

We obtain still another scheme by simply dropping the projection π_h from the shear energy term of the functional in Scheme 3. This final "trick" in fact brings the scheme close to the original proposal by Fried et al. [14].

Scheme 4. Find $(u_h, \theta_h) \in S_h := Y_h$ which minimizes in S_h the functional

$$F(v, \varphi) = a(\varphi, \varphi) + \sum_{T \in \mathscr{C}^h} c_T h_T^{-2} \int_T |\varphi - \nabla v|^2 \, dx - 2(f, v),$$

where $0 < c \leqq c_T \leqq C$, for some fixed constants c and C.

We carry out below the error analysis of the above four schemes. Since we have imposed discrete Kirchhoff conditions in the finite element subspaces, it is natural to compare the finite element solution (u_h, θ_h) first to the exact Kirchhoff solution (u^0, θ^0). An estimate relating (u_h, θ_h) and (u^t, θ^t) is then obtained applying Theorem 2.1. The main result of this section is the following,

Theorem 4.1. *Let (u_h, θ_h) be defined by any of the above four schemes. Then we have the error bounds*

$$\|u^0 - u_h\|_{2,h} + \|\theta^0 - \theta_h\|_1 \leqq C h \|f\|_{-1},$$

and

$$\|u^0 - u_h\|_1 + \|\theta^0 - \theta_h\|_0 \leqq C h^2 \|f\|_{-1}.$$

By combining these estimates with those of Theorem 2.1 we have:

Corollary 4.1. *Under the assumptions of Theorem 4.1 one has the error estimates*

$$\|\theta^t - \theta_h\|_1 \leqq C(h + t) \|f\|_{-1},$$

and

$$\|u^t - u_h\|_1 + \|\theta^t - \theta_h\|_0 \leqq C(h^2 + t^2) \|f\|_{-1}.$$

The rest of this section is devoted to the proof of Theorem 4.1. We start from the closely related Schemes 1 and 2. The first task here is to formulate Scheme 1 as a mixed method of the type (3.2). This is in fact easy: If $(u_h, \theta_h) = (u_h, \psi_h) + (0, \xi_h)$, where $(u_h, \psi_h) \in Y_h$ and $\xi_h \in B_h$ satisfy (4.1) and (4.2), and if $\gamma_h = t^{-2} \pi_h(\theta_h - \nabla u_h)$, then the triple $(u_h, \theta_h, \gamma_h)$ satisfies (3.2). The main task is now to verify that the stability condition (3.4) holds. We apply here Theorem 3.1.

Let us show that condition a) of Theorem 3.1 holds. To this and, let us split the local finite element subspace $\mathscr{Y}^T$, for a given $T\in\mathscr{C}^h$ with vertices $\{a^1, a^2, a^3\}$, as $\mathscr{Y}^T=\mathscr{Y}_1^T\oplus\mathscr{Y}_2^T$, where

$$\mathscr{Y}_1^T=\{(p,q)\in\mathscr{Y}^T: q=0 \text{ or } q\in\mathscr{P}_0^2 \text{ and } p(a^i)=0,\ i=1,2,3\},$$
$$\mathscr{Y}_2^T=\{(p,q)\in\mathscr{Y}^T: p(a^i)=0,\ i=1,2,3,\ \int_T q\,dx=0\}.$$

Further set

$$W_i^T=\{w=q-\nabla p: (p,q)\in\mathscr{Y}_i^T\},\qquad i=1,2.$$

Then it is easy to see that W_1^T is two-dimensional and that the mapping $w\to\|\pi_T w\|_{0,T}$, where π_T denotes the local averaging operator:

$$(\pi_T w)(x)=(1/\mathrm{area}(T))\int_T w\,dx,\qquad x\in T,\ w\in[L_2(T)]^2, \tag{4.7}$$

defines a norm in W_1^T. Thus in particular,

$$\|\pi_T w\|_{0,T}\geqq c\,\|w\|_{0,T},\qquad w\in W_1^T. \tag{4.8}$$

On the other hand since the mapping $(p,q)\to|q|_{1,T}$ obviously defines a norm in $\mathscr{Y}_2^T$, which is equivalent to the norm

$$(p,q)\to\left\{\sum_{i=1}^3|q(a^i)|^2\right\}^{1/2}=\left\{\sum_{i=1}^3|\nabla p(a^i)|^2\right\}^{1/2},$$

it follows in particular that

$$\begin{aligned}h_T^{-1}\|q-\nabla p\|_{0,T}&\leqq h_T^{-1}\|q\|_{0,T}+h_T^{-1}\|\nabla p\|_{0,T}\\&\leqq C|q|_{1,T},\qquad (p,q)\in\mathscr{Y}_2^T.\end{aligned} \tag{4.9}$$

Let now $(p_T, q_T)\in\mathscr{Y}^T$ be given and let $(p_T,q_T)=(p_T^{(1)},q_T^{(1)})+(p_T^{(2)},q_T^{(2)})$, where $(p_T^{(i)},q_T^{(i)})\in\mathscr{Y}_i^T$, $i=1,2$, and let $\zeta_T=h_T^{-2}\pi_T(q_T^{(1)}-\nabla p_T^{(1)})$, where π_T is defined by (4.7). Then by (4.9) and since $|q_T^{(2)}|_{1,T}=|q_T|_{1,T}$,

$$\begin{aligned}h_T^2\|\zeta_T\|_{0,T}^2&\leqq 2h_T^{-2}\{\|q_T-\nabla p_T\|_{0,T}^2+\|q_T^{(2)}-\nabla p_T^{(2)}\|_{0,T}^2\}\\&\leqq C\{h_T^{-2}\|q_T-\nabla p_T\|_{0,T}^2+|q_T|_{1,T}^2\},\end{aligned}$$

and by (4.8) and (4.9)

$$\begin{aligned}&\int_T(q_T-\nabla p_T)\cdot\zeta_T\,dx\\&\quad\geqq h_T^{-2}\|\pi_T(q_T^{(1)}-\nabla p_T^{(1)})\|_{0,T}^2-h_T^{-2}\|q_T^{(1)}-\nabla p_T^{(1)}\|_{0,T}\|q_T^{(2)}-\nabla p_T^{(2)}\|_{0,T}\\&\quad\geqq c\,h_T^{-2}\|q_T^{(1)}-\nabla p_T^{(1)}\|_{0,T}^2-C|q_T|_{1,T}^2\\&\quad\geqq c_1\,h_T^{-2}\|q_T-\nabla p_T\|_{0,T}^2-C_1|q_T|_{1,T}^2.\end{aligned}$$

Therefore assuming that $(p_T, q_T)=(u, \psi)_{|T}$, $T\in\mathscr{C}^h$, for a given $(u, \psi)\in Y_h$, and defining $\zeta\in Q_h$ so that $\zeta_{|T}=\zeta_T$, $T\in\mathscr{C}^h$, we conclude summing the above inequalities over T that

$$\Box\zeta\Box^2_{-1,h}\leqq c(\Box\psi-\nabla u\Box^2_{1,h}+|\psi|^2_1)$$

and

$$(\psi-\nabla u, \zeta)\geqq c\Box\psi-\nabla u\Box^2_{1,h}-C|\psi|^2_1.$$

Now if finally $\theta=\psi+\xi$ where $\xi\in B_h$, these inequalities imply further that

$$\begin{aligned}\|\zeta\|^2_{-1,h}&\leqq C(\Box\theta-\nabla u\Box^2_{1,h}+\Box\xi\Box^2_{1,h}+|\psi|^2_1)\\&\leqq C_1(\Box\theta-\nabla u\Box^2_{1,h}+|\theta|^2_1),\end{aligned}$$

and

$$\begin{aligned}(\theta-\nabla u, \zeta)&\geqq c\Box\psi-\nabla u\Box^2_{1,h}-C|\psi|^2_1-\Box\xi\Box_{1,h}\Box\zeta\Box_{-1,h}\\&\geqq c_1(\Box\psi-\nabla u\Box^2_{1,h}+\Box\xi\Box^2_{1,h})-C_1(|\psi|^2_1+\Box\xi\Box^2_{1,h})\\&\geqq c_2\Box\theta-\nabla u\Box^2_{1,h}-C_2|\theta|^2_1.\end{aligned}$$

Hence, condition a) of Theorem 3.1 holds if $S_h=Y_h\oplus\{0\}\times B_h$.

Condition b) is more easily verified: Given $\zeta\in Q_h$, let $\xi\in B_h$ be defined so that $\pi_h\xi_{|T}=h_T^2\zeta_{|T}$, $T\in\mathscr{C}^h$. Then the choice $(v, \varphi)=(0, \xi)$ yields the desired inequalities in condition b).

Having thus verified that the stability condition (3.4) holds, we can now prove the error estimates of Theorem 4.1 for Scheme 1. First, comparing (2.2) with $t=0$ and (3.2) we have

$$\mathscr{B}_t(u_h-u^0, \theta_h-\theta^0, \gamma_h-\gamma^0; v, \varphi, \zeta)=-t^2(\gamma^0, \zeta), \qquad (v, \varphi\zeta)\in S_h\times Q_h,$$

and therefore for any $(\tilde{u}, \tilde{\theta}, \tilde{\gamma})\in S_h\times Q_h$,

$$\begin{aligned}&\mathscr{B}_t(u_h-\tilde{u}, \theta_h-\tilde{\theta}, \gamma_h-\tilde{\gamma}; v, \varphi, \zeta)\\&\quad=\mathscr{B}_t(u^0-\tilde{u}, \theta^0-\tilde{\theta}, \gamma^0-\tilde{\gamma}; v, \varphi, \zeta)-t^2(\gamma^0, \zeta), \qquad (v, \varphi, \zeta)\in S_h\times Q_h.\end{aligned}\tag{4.10}$$

Now since (3.4) holds, we may choose here (v, φ, ζ) so that $|||v, \varphi, \zeta|||_h=1$ and

$$\mathscr{B}_t(u_h-\tilde{u}, \theta_h-\tilde{\theta}, \gamma_h-\tilde{\gamma}, v, \varphi, \zeta)\geqq c|||u_h-\tilde{u}, \theta_h-\tilde{\theta}, \gamma_h-\tilde{\gamma}|||_h.$$

To estimate the right side in (4.10), we take $\tilde{\gamma}=0$ (or $\tilde{\gamma}=\pi_h\gamma^0$) and let $\tilde{u}\in V_h$ and $\tilde{\theta}\in W_h$ be the natural interpolants of u^0 and θ^0, respectively. Since $\theta^0=\nabla u^0$, this choice guarantees that $(\tilde{u}, \tilde{\theta})\in Y_h$. Applying then the standard interpolation error estimates (cf. [11, 4]):

$$\|\theta^0-\tilde{\theta}\|_0+h\|\theta^0-\tilde{\theta}\|_1\leqq Ch^2|\theta^0|_2,$$

$$\|u^0-\tilde{u}\|_1+h\|u^0-\tilde{u}\|_{2,h}\leqq Ch^2|u^0|_3,$$

and recalling (3.3) and (2.5), we have

$$|\mathscr{B}_t(u^0-\tilde{u}, \theta^0-\tilde{\theta}, \gamma^0-\tilde{\gamma}; v, \varphi, \zeta)| \leqq C|||u^0-\tilde{u}, \theta^0-\tilde{\theta}, \gamma^0-\tilde{\gamma}|||_h |||v, \varphi, \zeta|||_h$$
$$= C|||u^0-\tilde{u}, \theta^0-\tilde{\theta}, \gamma^0-\tilde{\gamma}|||_h$$
$$\leqq C_1 h \|f\|_{-1}.$$

Finally by (3.1) and (2.5)

$$t^2|(\gamma^0, \zeta)| \leqq Ct\|\gamma^0\|_0 ⦀\zeta⦀_{-1,h} \leqq C_1 h\|f\|_{-1}.$$

Combining now these inequalities, we see that

$$|||u_h-\tilde{u}, \theta_h-\tilde{\theta}, \gamma_h-\tilde{\gamma}|||_h \leqq Ch\|f\|_{-1},$$

and thus by Lemma 3.1,

$$\|u_h-\tilde{u}\|_{2,h}+\|\theta_h-\tilde{\theta}\|_1 \leqq Ch\|f\|_{-1}.$$

Using finally the triangle inequality and recalling the above interpolation error estimates, we obtain the first part of Theorem 3.1.

The second part of Theorem 3.1 follows, from the same type of duality argument as that used in the proof Theorem 2.1. We omit the rather standard details here, and thus consider Theorem 4.1 to be proved so far as (u_h, θ_h) is determined by Scheme 1.

To see that Theorem 4.1 covers also Scheme 2, let $(u_h, \theta_h) \in S_h$ and $(\tilde{u}, \tilde{\theta}) \in Y_h$ be defined as above and let $(u_h, \theta_h) = (u_h, \psi_h) + (0, \xi_h)$, where $(u_h, \psi_h) \in Y_h$ and $\xi_h \in B_h$. Then

$$|\theta_h-\tilde{\theta}|_1^2 = |\psi_h-\tilde{\theta}|_1^2 + |\xi_h|_1^2 \geqq |\psi_h-\tilde{\theta}|_1^2,$$

and

$$\|\theta_h-\tilde{\theta}\|_0^2 \geqq c(\|\psi_h-\tilde{\theta}\|_0^2 + \|\xi_h\|_0^2) \geqq c\|\psi_h-\tilde{\theta}\|_0^2,$$

so the estimates obtained above remain valid if θ_h is replaced by ψ_h, and thus, Theorem 4.1 holds also for Scheme 2.

Consider now Scheme 3. We may interpret also this scheme as a mixed method of the type (3.2) by defining $\gamma_h \in Q_h$ as follows:

$$\gamma_{h|T} = c_T h_T^{-2} \pi_h(\theta_h - \nabla u_h)_{|T}, \qquad T \in \mathscr{C}^h.$$

The triple $(u_h, \theta_h, \gamma_h)$ then satisfies

$$\mathscr{B}_h(u_h, \theta_h, \gamma_h; v, \varphi, \zeta) = (f, v), \qquad (v, \varphi, \zeta) \in S_h \times Q_h, \tag{4.11}$$

where

$$\mathscr{B}_h(u, \theta, \gamma; v, \varphi, \zeta) = a(\theta, \varphi) - (\theta - \nabla u, \zeta) + (\varphi - \nabla v, \gamma) + \sum_{T \in \mathscr{C}^h} (1/c_T) h_T^2 \int_T \gamma\zeta\, dx. \tag{4.12}$$

We now have to verify that the basic conditions (3.3) and (3.4) hold when $\mathscr{B}_t$ is replaced by the bilinear form (4.12). First, it is obvious from (4.12) that so far as $c_i^T \geqq c > 0$,

$$|\mathscr{B}_h(u, \theta, \gamma; v, \varphi, \zeta)| \\ \leqq C |\!|\!| u, \theta, \gamma |\!|\!|_h |\!|\!| v, \varphi, \zeta |\!|\!|_h, \quad (u, \theta, \gamma), (v, \varphi, \zeta) \in V \times W \times Q, \tag{4.13}$$

so the first condition holds. To prove the second condition, we argue as follows: Given $(u, \theta, \gamma) \in S_h \times Q_h$, pick first $\rho \in Q_h$ so that $\square\rho\square_{-1,h}^2 \leqq C(\|\theta\|_1^2 + \square\theta - \nabla u\square_{1,h}^2)$ and $(\theta - \nabla u, \rho) \geqq c\square\theta - \nabla u\square_{1,h}^2 - C\|\theta\|_1^2$. We know the existence of such ρ from the analysis of Scheme 1 above. Then taking $(v, \varphi, \zeta) = (u, \theta, \gamma) - \delta(0, 0, \rho)$, where $\delta > 0$ is so far unspecified, we have $|\!|\!| v, \varphi, \zeta |\!|\!|_h \leqq (1 + C\delta) |\!|\!| u, \theta, \gamma |\!|\!|_h$ and

$$\begin{aligned} &\mathscr{B}_h(u, \theta, \gamma; v, \varphi, \zeta) \\ &\quad \geqq (c - C\delta) \|\theta\|_1^2 + c\delta\square\theta - \nabla u\square_{1,h}^2 + c\square\gamma\square_{-1,h}^2 - C\delta\square\gamma\square_{-1,h}\square\rho\square_{-1,h} \\ &\quad \geqq (c - C\delta - C_1\delta^2) \|\theta\|_1^2 + \delta(c - C_1\delta)\square\theta - \nabla u\square_{1,h}^2 + (c/2)\square\gamma\square_{-1,h}^2 \\ &\quad \geqq c_1 |\!|\!| u, \theta, \gamma |\!|\!|_h^2, \end{aligned}$$

if δ is small enough. Hence the desired condition holds: there is a positive constant c such that

$$\sup_{(v, \varphi, \zeta) \in S_h \times Q_h} \frac{\mathscr{B}_h(u, \theta, \gamma; v\varphi, \zeta)}{|\!|\!| v, \varphi, \zeta |\!|\!|_h} \geqq c |\!|\!| u, \theta, \gamma |\!|\!|_h \quad \text{for all } (u, \theta, \gamma) \in S_h \times Q_h. \tag{4.14}$$

Having established the basic conditions (4.13) and (4.14), the error analysis of Scheme 3 is simply a copy of that of Scheme 1. We omit repeating these details and consider instead Scheme 4. This scheme can also be written in the form of (4.11) by simply choosing Q_h to be sufficiently large so that if $\gamma_{h|T} = h_T^{-2}(\theta_h - \nabla u_h)_{|T}$, $T \in \mathscr{C}^h$, then $\gamma_h \in Q_h$. The triple $(u_h, \theta_h, \gamma_h)$ then satisfies (4.11). One can choose for example

$$Q_h = \{\zeta \in Q : \zeta_{|T} \in \mathscr{P}_2^2, \ T \in \mathscr{C}^h\}.$$

With these preparations, the error analysis of the scheme is identical with that of Scheme 3. The proof of Theorem 4.1 is now complete.

Remark 4.1. We point out that Scheme 1 and Scheme 3 represent two different philosophies of stabilizing a finite element scheme: In Scheme 1 one forces the stability condition b) of Theorem 3.1 by supplying the finite element with "bubbles" (which have no role in the sense of approximation theory), while in Scheme 3 the stability of γ_h is achieved by modifying the bilinear form. For further applications of these techniques of stabilization, cf. [10, 20] and the references therein. □

Remark 4.2. As is clear from the analysis of Schemes 1 through 3, the discrete Kirchhoff conditions are required here for the stability condition a) of Theorem 3.1 to hold. This is not the case in Scheme 4, however, since for sufficiently large Q_h the condition is bound to hold for any choice of S_h (see Remark 3.1

above). Thus, Scheme 4 works also if one takes for example $S_h = V_h \times W_h$. In fact, in case of Scheme 4 the error estimates of Theorem 4.1 hold for any choice of S_h such that the following approximability condition holds: Given $u \in H^3(\Omega) \cap H_0^2(\Omega)$, there exists $(\tilde{u}, \tilde{\theta}) \in S_h$ such that

$$\|u - \tilde{u}\|_{2,h} + \|\nabla u - \tilde{\theta}\|_1 \leqq C h \|u\|_3,$$

and

$$\|u - \tilde{u}\|_1 + \|\nabla u - \tilde{\theta}\|_0 \leqq C h^2 \|u\|_3.$$

Note that these conditions can hold only if the interpolation polynomials for the transverse deflection are quadratic (at least). This requirement is in fact the basic reason for ruling out the "simplest" schemes where both u and θ are interpolated by piecewise linear functions. □

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