Numer. Math. 53, 225–235 (1988)

Numerische Mathematik

Stabilized Mixed Methods for the Stokes Problem

Franco Brezzi[1] and Jim Douglas, Jr.[2]

[1] Dipartimento di Meccanica Strutturale, Università di Pavia, and Istituto di Analisi Numerica del C.N.R. di Pavia, Italy

[2] Department of Mathematics, Purdue University, West Lafayette. IN 47907, USA

Dedicated to Ivo Babuška on the occasion of his sixtieth birthday

Summary. The solution of the Stokes problem is approximated by three stabilized mixed methods, one due to Hughes, Balestra, and Franca and the other two being variants of this procedure. In each case the bilinear form associated with the saddle-point problem of the standard mixed formulation is modified to become coercive over the finite element space. Error estimates are derived for each procedure.

Subject Classifications. AMS(MOS): 65N30; CR: G1.8.

1. Introduction

Consider the Stokes problem

$$-\mu \Delta \tilde{q} + \nabla p = \tilde{f}, \qquad x \in \Omega, \tag{1.1a}$$

$$\operatorname{div} \tilde{q} = 0, \qquad x \in \Omega, \tag{1.1b}$$

$$\tilde{q} = \tilde{0}, \qquad x \in \partial\Omega, \tag{1.1c}$$

where Ω is a bounded domain in R^k, $k=2$ or 3. A mixed formulation of (1.1) is given by the finding of $\{\tilde{q}, p\} \in \tilde{H}_0^1(\Omega) \times (L^2(\Omega)/R)$ such that

$$a(\tilde{q}, \tilde{v}) - (\operatorname{div} \tilde{v}, p) = (\tilde{f}, \tilde{v}), \qquad \tilde{v} \in \tilde{H}_0^1(\Omega), \tag{1.2a}$$

$$(\operatorname{div} \tilde{q}, w) = 0, \qquad w \in L^2(\Omega), \tag{1.2b}$$

where

$$a(\tilde{q}, \tilde{v}) = \mu \sum_{i=1}^{k} (\nabla q_i, \nabla v_i) = \mu \sum_{i,j=1}^{k} (\partial q_i/\partial x_j, \partial v_i/\partial x_j).$$

A standard mixed method for approximating the solution of (1.2) would depend on choosing a pair of spaces $\tilde{V}_h \subset \tilde{H}_0^1(\Omega)$ and $W_h \subset L^2(\Omega)$ such that the "inf-sup" condition

$$\inf_{s \in W_h} \sup_{\tilde{r} \in \tilde{V}_h} \frac{(\operatorname{div} \tilde{r}, s)}{\|\tilde{r}\|_1 \, \|s\|_0} \geqq \alpha > 0,$$

α independent of h, holds. As can be seen in the very recent book of Girault and Raviart [2], there are quite a few such spaces known for this problem; however, most of these combinations employ some basis functions that are not found in many of the engineering code packages that are most commonly used. For this reason it can be convenient to modify the form of (1.2) so that the associated bilinear form is coercive over $\tilde{V}_h \times W_h$; then, almost any pair of spaces can be chosen for $\tilde{V}_h \times W_h$, and the resulting method can be implemented easily and rapidly within the framework of many existing engineering codes.

Hughes et al. [3] proposed to modify the saddlepoint problem as follows. Let $\mathscr{T}_h$ denote the polygonalization of Ω into polygons T of diameter roughly equal to h. Let $W_h \subset H^1(\Omega)$, rather than $L^2(\Omega)$, and test Eq. (1.1a) against ∇w, $w \in W_h$, over each "triangle" $T \in \mathscr{T}_h$, multiply the result by a (small) multiple of $h_T^2 = (\operatorname{diam} T)^2$, and add their sum over triangles to (1.2b). Thus, their method consists of finding $\{\tilde{q}_h, p_h\} \in \tilde{V}_h \times W_h \subset \tilde{H}_0^1(\Omega) \times H^1(\Omega)$ such that

$$a(\tilde{q}_h, \tilde{v}) - (\operatorname{div} \tilde{v}, p_h) = (\tilde{f}, \tilde{v}), \qquad \tilde{v} \in \tilde{V}_h, \tag{1.3a}$$

$$(\operatorname{div} \tilde{q}_h, w) + \alpha \sum_T h_T^2 [(\nabla p_h, \nabla w)_T - (\Delta \tilde{q}_h, \nabla w)_T] = \alpha \sum_T h_T^2 (\tilde{f}, \nabla w)_T, \qquad w \in W_h, \tag{1.3b}$$

where the subscript T indicates that the inner product is to be extended over the set T only; in (1.3) and below, we assume a scaling such that the viscosity μ equals one.

The constant α should be chosen so that the bilinear form

$$A(\{\tilde{q}, p\}, \{\tilde{v}, w\}) = a(\tilde{q}, \tilde{v}) - (\operatorname{div} \tilde{v}, p) + (\operatorname{div} \tilde{q}, w) + \alpha \sum_T h_T^2 [(\nabla p, \nabla w)_T - (\Delta \tilde{q}, \nabla w)_T]$$

is coercive over $\tilde{V}_h \times W_h$ with respect to the norm

$$[\|q\|_1^2 + \sum_T h_T^2 \|\nabla p\|_{0,T}^2]^{1/2}. \tag{1.4}$$

This can be done as follows. Assume a shape regularity for $\mathscr{T}_h$ (i.e., assume that the ratio of the diameter of the circumscribed ball for $T \in \mathscr{T}_h$ to that of the inscribed ball is bounded, independently of $T \in \mathscr{T}_h$ and $h = \max\{\operatorname{diam} T\colon T \in \mathscr{T}_h\}$). Then, whenever $\tilde{V}_h$ and W_h consist of C^0-piecewise polynomial spaces of fixed degrees over $\mathcal{T}_h$ and if the boundary condition (1.1c) holds on $\tilde{V}_h$, there exists a constant Q such that

$$\begin{aligned} A(\{\tilde{q}, p\}, \{\tilde{q}, p\}) &= a(\tilde{q}, \tilde{q}) + \alpha \sum_T h_T^2 [(\nabla p, \nabla p)_T - (\Delta \tilde{q}, \nabla p)_T] \\ &\geqq a(\tilde{q}, \tilde{q}) + \alpha \sum_T h_T^2 [(\nabla p, \nabla p)_T - Q\, h_T^{-1} \|\nabla \tilde{q}\|_{0,T} \|\nabla p\|_{0,T}] \\ &\geqq (1 - \tfrac{1}{2}\alpha Q^2) \|\nabla \tilde{q}\|_0^2 + \tfrac{1}{2}\alpha \sum_T h_T^2 \|\nabla p\|_{0,T}^2. \end{aligned}$$

Hence, we have the desired coercivity, in fact over $\tilde{V}_h \times H^1$, for small α.

Two observations can be made easily. First, the coercivity over $\tilde{V}_h \times W_h$ implies immediately the unique solvability of (1.3), without the imposition of an "inf-sup" or related condition. Second, (1.3) is *not* a penalty procedure, since the solution of the differential problem (1.1) satisfies the equations of (1.3); consequently, no loss in accuracy with respect to the natural norm (1.4) or those given in a duality argument should be expected. These two points, and the accompanying convenience in the choice of approximation spaces $\tilde{V}_h \times W_h$, were the prime motivations for the introduction of (1.3) by Hughes, Balestra, and Franca.

The remainder of the paper consists first of an analysis of the convergence of the solution of (1.3) to that of (1.1) in Sect. 2, the presentation in Sect. 3 of a technique for piecewise-linear $\tilde{V}_h$ and W_h for which the terms involving $(\Delta \tilde{q}_h, \nabla w)_T$ and $(\tilde{f}, \nabla w)_T$ are dropped, the presentation in Sect. 4 of a variant of (1.3) in which certain of the added terms are replaced by ones on boundary triangles. The proofs in Sect. 2 are simpler than the corresponding development in the paper of Hughes et al. [3], and the results are slightly more general, in that they include L^2 estimates for the error. The method of Sect. 3 is a penalty procedure and the results obtained here simplify and generalize earlier results for the same method obtained by Brezzi and Pitkäranta [1]. We point out that this method introduces a penalty error of order $O(h)$ independent of the choice of the discretization space; consequently, though the use of higher order spaces is feasible for the method without stability problems, such usage is not recommended. The method of Sect. 4 is possibly a bit cheaper computationally than (1.3) and maintains the same error estimates.

The concepts introduced by Hughes et al. [3] can be applied to other mixed finite methods, such as for second order elliptic equations and for the equations for linear elasticity. In the case of the second order elliptic equation, nothing very useful seems to be gained; however, for the elasticity equations convenience in the choice of approximation spaces does result, though at perhaps an increased computational cost over the application of mixed spaces designed for the problem. Neither of these applications will be discussed further here.

2. Analysis of the Error in the Method of Hughes, Balestra, and Franca

Consider first the case of a convex polygonal domain, let $\tilde{V}_h$ consist of C^0-piecewise linear vector functions over a triangulation $\mathcal{T}_h$ consistent with $\partial\Omega$

and vanishing on $\partial\Omega$, and let W_h consist of C^0-piecewise linear functions over the same $\mathcal{T}_h$. The coercivity of A over $\tilde{V}_h \times W_h$ implies the existence and uniqueness of a solution $\{\tilde{q}_h, p_h\}$ to (1.3). Let $\{\tilde{r}, s\} \in \tilde{V}_h \times W_h$ be an optimal order correct interpolation of $\{\tilde{q}, p\}$. Now, subtract (1.3) from (1.2), shift $\{\tilde{q}, p\}$ to $\{\tilde{r}, s\}$, and integrate the term $(\operatorname{div} \tilde{v}, \tilde{q}-\tilde{r})$ by parts:

$$a(\tilde{r}-\tilde{q}_h, \tilde{v})-(\operatorname{div} \tilde{v}, s-p_h)=a(\tilde{r}-\tilde{q}, \tilde{v})-(\operatorname{div} \tilde{v}, s-p), \qquad \tilde{v} \in \tilde{V}_h, \tag{2.1 a}$$

$$(\operatorname{div}(\tilde{r}-\tilde{q}_h), w)+\alpha \sum_T h_T^2 [(\nabla(s-p_h), \nabla w)_T-(\Delta(\tilde{r}-\tilde{q}_h), \nabla \tilde{v})_T]$$
$$=(\tilde{q}-\tilde{r}, \nabla w)+\alpha \sum_T h_T^2 [(\nabla(s-p), \nabla w)_T-(\Delta(\tilde{r}-\tilde{q}), \nabla w)_T], \qquad w \in W_h. \tag{2.1 b}$$

Take $\tilde{v}=\tilde{r}-\tilde{q}_h$ and $w=s-p_h$; add the equations and use the coercivity of A over $\tilde{V}_h \times W_h$ and the regularity

$$\|\tilde{q}\|_2+\|p\|_1 \leqq Q\,\|\tilde{f}\|_0 \qquad (p \text{ normalized so that } (p, 1)=0)$$

of the Stokes problem on the convex polygon Ω; it follows that

$$\begin{aligned}\rho\Big[\|\tilde{r}-\tilde{q}_h\|_1^2+\sum_T h_T^2\,\|\nabla(s-p_h)\|_{0,T}^2\Big] &\leqq a(\tilde{r}-\tilde{q}, \tilde{r}-\tilde{q}_h)+(\operatorname{div}(\tilde{r}-\tilde{q}_h), p-s)+(\tilde{q}-\tilde{r}, \nabla(s-p_h)) \\ &\quad+\alpha \sum_T h_T^2 [(\nabla(s-p), \nabla(s-p_h))_T-(\Delta(\tilde{r}-\tilde{q}), \nabla(s-p_h))_T].\end{aligned} \tag{2.2}$$

Let the shape regularity condition on $\mathcal{T}_h$ be expressed by the inequality

$$\|\nabla \chi\|_{0,T} \leqq Q\, h_T^{-1}\, \|\chi\|_{0,T}, \qquad \chi \in W_h; \tag{2.3}$$

the analogue of this inequality is valid on $\tilde{V}_h$ as well. A simple calculation shows that, for α small enough

$$\begin{aligned}&\|\tilde{q}-\tilde{q}_h\|_1+\Big[\sum_T h_T^2\,\|\nabla(p-p_h)\|_{0,T}^2\Big]^{1/2} \\ &\leqq Q\Big\{\|\tilde{q}-\tilde{r}\|_1^2+\|p-s\|_0^2+\sum_T \big[h_T^{-2}\,\|\tilde{q}-\tilde{r}\|_{2,T}^2+h_T^2\,\|\tilde{q}-\tilde{r}\|_{2,T}^2+h_T^2\,\|\nabla(p-s)\|_{0,T}^2\big]\Big\}^{1/2} \\ &\leqq Q\,\|\tilde{f}\|_0\, h.\end{aligned} \tag{2.4}$$

A very similar argument shows that

$$\|\tilde{q}-\tilde{q}_h\|_1+\Big[\sum_T h_T^2\,\|\nabla(p-p_h)\|_{0,T}^2\Big]^{1/2} \leqq Q\{\|\tilde{q}\|_{j+1}+\|p\|_j\}\, h^j, \tag{2.5}$$

when the shape regularity condition holds for $\mathcal{T}_h$ and the space $\tilde{V}_h \times W_h$ is chosen to be of the form

$$\tilde{V}_h=\{\tilde{v} \in C^0(\Omega)\colon\ \tilde{v}|_T \in P_m(T),\ T \in \mathcal{T}_h\}, \qquad m \geqq j, \tag{2.6 a}$$

$$W_h=\{w \in C^0(\Omega)\colon\ w|_T \in P_n(T),\ T \in \mathcal{T}_h\}, \qquad n \geqq j-1. \tag{2.6 b}$$

Note that the error estimate (2.5) is not given in terms of the data function $\tilde{f}$; a polygonal domain does not lead to a shift theorem that bounds $\|\tilde{q}\|_{j+1}$ and $\|p\|_j$ in terms of $\|\tilde{f}\|_{j-1}$. For a smooth boundary the shift theorem is valid; however, it cannot be expected that the boundary condition $\tilde{q}_h=\tilde{0}$ can be applied on $\tilde{V}_h$. It is possible to define a procedure over a space $\tilde{V}_h \subset \tilde{H}^1(\Omega)$, rather than $\tilde{H}_0^1(\Omega)$, by introducing a method based on ideas corresponding to those discussed by Nitsche [4] for the Dirichlet problem; however, we shall not treat this extension here.

Let us turn to error estimates in $L^2(\Omega)$, with Ω being polygonal and convex and $\tilde{V}_h \times W_h$ being given by (2.6). We wish to prove the following theorem.

Theorem 1. *Let $\mathscr{T}_h$ satisfy* (2.3). *Then,*

$$\|\tilde{q}-\tilde{q}_h\|_0 \leqq Q\{\|\tilde{q}\|_{j+1}+\|p\|_j\}\, h^{j+1}.$$

If, in addition to the shape regularity of $\mathscr{T}_h$ required by (2.3), *there exists a positive constant γ such that $h_T \geqq \gamma h$ for $T\in\mathscr{T}_h$, then*

$$\|p-p_h\|_0 \leqq Q\{\|\tilde{q}\|_{j+1}+\|p\|_j\}\, h^j.$$

Proof. Consider $\|q-q_h\|_0$ first. Let

$$\begin{aligned} -\Delta\tilde{\rho}+\nabla\sigma &= \tilde{q}-\tilde{q}_h, && x\in\Omega,\\ \operatorname{div}\tilde{\rho} &= 0, && x\in\Omega,\\ \tilde{\rho} &= \tilde{0}, && x\in\partial\Omega, \end{aligned}$$

so that $\|\tilde{\rho}\|_2+\|\sigma\|_1 \leqq Q\,\|\tilde{q}-\tilde{q}_h\|_0$. Then,

$$\begin{aligned} \|\tilde{q}-\tilde{q}_h\|_0^2 &= (-\Delta\tilde{\rho}+\nabla\sigma, \tilde{q}-\tilde{q}_h) = a(\tilde{q}-\tilde{q}_h, \tilde{\rho}) - (\operatorname{div}(\tilde{q}-\tilde{q}_h), \sigma)\\ &= a(\tilde{q}-\tilde{q}_h, \tilde{\rho}-\tilde{\omega}) - (\operatorname{div}(\tilde{q}-\tilde{q}_h), \sigma-\tau) + a(\tilde{q}-\tilde{q}_h, \tilde{\omega}) - (\operatorname{div}(\tilde{q}-\tilde{q}_h), \tau) \end{aligned}$$

for $\{\tilde{\omega},\tau\}\in\tilde{V}_h\times W_h$. First,

$$\begin{aligned} |a(\tilde{q}-\tilde{q}_h, \tilde{\rho}-\tilde{\omega}) - (\operatorname{div}(\tilde{q}-\tilde{q}_h), \sigma-\tau)| &\leqq Q\,\|\tilde{q}-\tilde{q}_h\|_1\,[\|\tilde{\rho}-\tilde{\omega}\|_1+\|\sigma-\tau\|_0]\\ &\leqq Q\,h\,\|\tilde{q}-\tilde{q}_h\|_1\,\|\tilde{q}-\tilde{q}_h\|_0 \end{aligned}$$

for properly chosen $\tilde{\omega}$ and τ. Next,

$$\begin{aligned} a(\tilde{q}-\tilde{q}_h, \tilde{\omega}) &= (\operatorname{div}\tilde{\omega}, p-p_h) = (\operatorname{div}(\tilde{\omega}-\tilde{\rho}), p-p_h) = -(\tilde{\omega}-\tilde{\rho}, \nabla(p-p_h))\\ &\leqq Q\,h\,\|q-q_h\|_0\,\{\sum_T h_T^2\,\|\nabla(p-p_h)\|_{0,T}^2\}^{1/2}. \end{aligned}$$

Finally,

$$\begin{aligned} -(\operatorname{div}(\tilde{q}-\tilde{q}_h), \tau) &= \alpha\sum_T h_T^2\,[(\nabla(p-p_h), \nabla\sigma+\nabla(\tau-\sigma))_T - (\Delta(\tilde{q}-\tilde{r})+\Delta(\tilde{r}-\tilde{q}_h), \nabla\tau)_T]\\ &\leqq Q\,h\,\|\tilde{q}-\tilde{q}_h\|_0\,\{\|\tilde{r}-\tilde{q}_h\|_1 + (\sum_T h_T^2\,[\|\nabla(p-p_h)\|_{0,T}^2+\|\tilde{q}-\tilde{r}\|_{2,T}^2])^{1/2}. \end{aligned}$$

It follows that $\|\tilde{q}-\tilde{q}_h\|_0 \leqq Q\{\|\tilde{q}\|_{j+1}+\|p\|_j\}\, h^{j+1}$.

To analyze $p-p_h$, add the assumption that $\mathscr{T}_h$ is quasi-regular both in size and shape; then, the inequality (2.5) can be written in the form

$$\|\tilde{q}-\tilde{q}_h\|_1+h\|p-p_h\|_1 \leqq Q\{\|\tilde{q}\|_{j+1}+\|p\|_j\} h^j.$$

Let the dual problem be changed to

$$\begin{aligned} -\Delta\tilde{\rho}+\nabla\sigma&=\tilde{0}, && x\in\omega,\\ \operatorname{div}\tilde{\rho}&=p-p_h, && x\in\Omega,\\ \tilde{\rho}&=\tilde{0}, && x\in\partial\Omega,\end{aligned}$$

so that $\|\tilde{\rho}\|_1+\|\sigma\|_0 \leqq Q\|p-p_h\|_0$. Then, with $\tilde{\omega}\in\tilde{V}_h$ and $\tau\in W_h$,

$$\begin{aligned}\|p-p_h\|_0^2&=(p-p_h, \operatorname{div}\tilde{\omega})+(p-p_h, \operatorname{div}(\tilde{\rho}-\tilde{\omega}))\\ &=a(\tilde{q}-\tilde{q}_h, \tilde{\omega})-(\nabla(p-p_h), \tilde{\rho}-\tilde{\omega})\\ &\leqq Q\{\|\tilde{q}-\tilde{q}_h\|_1+h\|p-p_h\|_1\}\|p-p_h\|_0,\end{aligned}$$

and it follows that $\|p-p_h\|_0 \leqq Q\{\|\tilde{q}\|_{j+1}+\|p\|_j\} h^j$, as was to be shown.

3. A Penalty Stabilization

Consider the modification of the Stokes problem given by

$$-\Delta\tilde{q}^h+\nabla p^h=\tilde{f}, \qquad x\in\Omega, \tag{3.1a}$$

$$\operatorname{div}\tilde{q}^h-h^2\Delta p^h=0, \qquad x\in\Omega, \tag{3.1b}$$

$$\tilde{q}^h=\tilde{0}, \qquad x\in\partial\Omega, \tag{3.1c}$$

$$\partial p^h/\partial n=0, \qquad x\in\partial\Omega, \tag{3.1d}$$

and the associated weak problem given by the finding of $\tilde{q}^h\in\tilde{H}_0^1(\Omega)$ and $p^h\in H^1(\Omega)$ such that

$$a(\tilde{q}^h, \tilde{v})-(\operatorname{div}\tilde{v}, p^h)=(\tilde{f}, \tilde{v}), \qquad \tilde{v}\in\tilde{H}_0^1(\Omega), \tag{3.2a}$$

$$(\operatorname{div}\tilde{q}^h, w)+h^2(\nabla p^h, \nabla w)=0, \qquad w\in H^1(\Omega). \tag{3.2b}$$

The problem (3.2) is a penalized version of the Stokes problem; it is this problem that will be approximated by what amounts to a stabilized mixed method. First, let us analyze the difference between the solution of (3.2) and (1.2). Let

$$\tilde{\rho}=\tilde{q}-\tilde{q}^h, \qquad \pi=p-p^h,$$

so that

$$a(\tilde{\rho}, \tilde{v})-(\operatorname{div}\tilde{v}, \pi)=0, \qquad \tilde{v}\in\tilde{H}_0^1(\Omega), \tag{3.3a}$$

$$(\operatorname{div}\tilde{\rho}, w)+h^2(\nabla\pi, \nabla w)=h^2(\nabla p, \nabla w), \qquad w\in H^1(\Omega). \tag{3.3b}$$

Test (3.3) by selecting $\tilde{v}=\tilde{\rho}$ and $w=\pi$; then, it follows that

$$\|\tilde{\rho}\|_1 + h\|\nabla\pi\|_0 \leqq h\|\nabla p\|_0 \leqq Q\|\tilde{f}\|_0 h. \tag{3.4}$$

Note that this implies that $\|\operatorname{div}\tilde{q}^h\|_0 = O(h)$, as $\operatorname{div}\tilde{q}=0$, and also that $\|\tilde{\rho}\|_2 = O(1)$, by the strong form of (3.3a).

L^2-estimates can be derived for $\tilde{\rho}$ and π by duality. Let $\tilde{\psi}\in\tilde{H}_0^1(\Omega)$ and $\theta\in H^1(\Omega)$ be determined by the equations

$$\begin{aligned} -\Delta\tilde{\psi} + \nabla\theta &= \tilde{\rho}, & x&\in\Omega,\\ \operatorname{div}\tilde{\psi} &= 0, & x&\in\Omega,\\ \tilde{\psi} &= \tilde{0}, & x&\in\partial\Omega, \end{aligned}$$

with $(\pi, 1)=0$. Then, $\|\tilde{\psi}\|_2 + \|\theta\|_1 \leqq Q\|\tilde{\rho}\|_0$ and

$$\begin{aligned} \|\tilde{\rho}\|_0^2 &= a(\tilde{\rho},\tilde{\psi}) + (\nabla\theta,\tilde{\rho}) = (\operatorname{div}\tilde{\psi},\pi) + (\nabla\theta,\tilde{\rho}) = -(\operatorname{div}\tilde{\rho},\theta)\\ &= h^2(\nabla(\pi-p),\nabla\theta) = -h^2(\nabla p^h,\nabla\theta) \leqq Q h^2\|\nabla p^h\|_0\|\nabla\theta\|_0\\ &\leqq Q h^2\|\tilde{\rho}\|_0\|\tilde{f}\|_0, \end{aligned}$$

so that

$$\|\tilde{\rho}\|_0 \leqq Q\|\tilde{f}\|_0 h^2. \tag{3.5}$$

Next, consider the dual problem

$$\begin{aligned} -\Delta\tilde{\psi} + \nabla\theta &= \tilde{0}, & x&\in\Omega,\\ \operatorname{div}\tilde{\psi} &= \pi, & x&\in\Omega,\\ \tilde{\psi} &= \tilde{0}, & x&\in\partial\Omega, \end{aligned}$$

with $(\pi, 1)=0$. Then,

$$\|\pi\|_0^2 = (\operatorname{div}\tilde{\psi},\pi) = a(\tilde{\rho},\tilde{\psi}) \leqq \|\rho\|_1\|\tilde{\psi}\|_1 \leqq Q\|\tilde{f}\|_0 h\|\pi\|_0$$

and

$$\|\pi\|_0 \leqq Q\|\tilde{f}\|_0 h. \tag{3.6}$$

We can now consider the discrete problem. Let $\{\tilde{q}_h, p_h\}\in\tilde{V}_h\times W_h$ be the solution of the equations

$$a(\tilde{q}_h,\tilde{v}) - (\operatorname{div}\tilde{v}, p_h) = (\tilde{f},\tilde{v}), \qquad \tilde{v}\in\tilde{V}_h, \tag{3.7a}$$

$$(\operatorname{div}\tilde{q}_h, w) + h^2(\nabla p_h,\nabla w) = 0, \qquad w\in W_h. \tag{3.7b}$$

Thus,

$$\begin{aligned} a(\tilde{q}^h - \tilde{q}_h,\tilde{v}) - (\operatorname{div}\tilde{v}, p^h - p_h) &= 0, & \tilde{v}&\in\tilde{V}_h,\\ (\operatorname{div}(\tilde{q}^h - \tilde{q}_h), w) + h^2(\nabla(p^h - p_h),\nabla w) &= 0, & w&\in W_h. \end{aligned}$$

Set $\varphi=\{\tilde{v}, w\}$, $\|\varphi\|_\Phi = [a(\tilde{v},\tilde{v}) + h^2\|\nabla w\|_0^2]^{1/2}$, and

$$A(\varphi,\zeta) = A(\{\tilde{v}, w\},\{\tilde{u}, z\}) = a(\tilde{v},\tilde{u}) - (\operatorname{div}\tilde{v}, w) + (\operatorname{div}\tilde{u}, z) + h^2(\nabla w,\nabla z).$$

Then, $A(\varphi, \varphi) = a(\tilde{v}, \tilde{v}) + h^2(\nabla w, \nabla w) = \|\varphi\|_\Phi^2$, so that it follows that

$$\begin{aligned}
&\|\tilde{q}^h - \tilde{q}_h\|_1^2 + h^2 \|\nabla(p^h - p_h)\|_0^2 \\
&\quad \leqq Q A(\{\tilde{q}^h - \tilde{q}_h, p^h - p_h\}, \{\tilde{q}^h - \tilde{q}_h, p^h - p_h\}) \\
&\quad \leqq Q \inf[A(\{\tilde{q}^h - \tilde{v}, p^h - w\}, \{\tilde{q}^h - \tilde{v}, p^h - w\}) : \tilde{v} \in \tilde{V}_h,\ w \in W_h] \\
&\quad \leqq Q \{\|\tilde{q}^h\|_2^2 + \|p^h\|_1^2\} h^2,
\end{aligned} \tag{3.8}$$

so that, by (3.4) and the remark following,

$$\|\tilde{q} - \tilde{q}_h\|_1 + h \|\nabla(p - p_h)\|_0 \leqq Q \|\tilde{f}\|_0 h. \tag{3.9}$$

Again we employ duality to derive L^2-estimates for the error $\{\tilde{r}, s\} = \{\tilde{q} - \tilde{q}_h, p - p_h\}$. First, let

$$\begin{aligned}
-\Delta \tilde{\psi} + \nabla \theta &= \tilde{r}, \qquad x \in \Omega, \\
\operatorname{div} \tilde{\psi} &= 0, \qquad x \in \Omega,
\end{aligned}$$

with $\tilde{\psi} = \tilde{0}$ on $\partial\Omega$ and $(\theta, 1) = 0$. Then, with $\{\tilde{\chi}, \eta\} \in \tilde{V}_h \times W_h$,

$$\begin{aligned}
\|\tilde{r}\|_0^2 &= a(\tilde{r}, \tilde{\psi}) - (\operatorname{div} \tilde{r}, \theta) = a(\tilde{r}, \tilde{\psi} - \tilde{\chi}) + a(\tilde{r}, \tilde{\chi}) - (\operatorname{div} \tilde{r}, \theta - \eta) - (\operatorname{div} \tilde{r}, \eta) \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Now,

$$\mathrm{I} + \mathrm{III} \leqq Q \|\tilde{r}\|_1 \{\|\tilde{\psi}\|_2 + \|\theta\|_1\} h \leqq Q \|\tilde{r}\|_1 \|\tilde{r}\|_0 h.$$

Next,

$$\begin{aligned}
\mathrm{II} &= (\operatorname{div} \tilde{\chi}, s) = (\operatorname{div}(\tilde{\chi} - \tilde{\psi}), s) = -(\tilde{\chi} - \tilde{\psi}, \nabla s) \\
&\leqq Q \|\tilde{\psi}\|_2 \|s\|_1 h^2 \leqq Q \|\tilde{r}\|_0 \|s\|_1 h^2.
\end{aligned}$$

Finally,

$$\mathrm{IV} = h^2(\nabla s, \nabla \eta) \leqq h^2 \|\nabla s\|_0 \|\nabla \eta\|_0 \leqq Q h^2 \|s\|_1 \|\tilde{r}\|_0.$$

Thus, it follows that

$$\|\tilde{r}\|_0 \leqq Q \|\tilde{f}\|_0 h^2. \tag{3.10}$$

Similarly, if the dual problem is changed to be

$$\begin{aligned}
-\Delta \tilde{\psi} + \nabla \theta &= \tilde{0}, \qquad x \in \Omega, \\
\operatorname{div} \tilde{\psi} &= s, \qquad x \in \Omega,
\end{aligned}$$

with the usual boundary conditions and normalization, then

$$\begin{aligned}
\|s\|_0^2 &= (\operatorname{div} \tilde{\psi}, s) = (\operatorname{div}(\tilde{\psi} - \tilde{\chi}), s) + a(\tilde{r}, \tilde{\chi}) \\
&\leqq Q[\|s\|_0 \|s\|_1 h + \|\tilde{r}\|_1 \|s\|_0],
\end{aligned}$$

and

$$\|s\|_0 \leqq Q \|\tilde{f}\|_0 h. \tag{3.11}$$

Thus, the same asymptotic error estimates hold for the penalty stabilized method as for the method of Hughes et al. [3] when C^0-piecewise-linear (or -bilinear or -trilinear) spaces are used for the velocity components and the pressure. Note that no gain in rate of convergence will occur if higher order spaces are used. For the linear elements this method is somewhat less expensive in computational requirements than the other procedure.

4. A Modification of the Hughes, Balestra, and Franca Method

If the divergence is taken of (1.1 a), then one sees that

$$\Delta p = \operatorname{div} \tilde{f}, \qquad x \in \Omega. \tag{4.1 a}$$

Let the outer normal to Ω be denoted by ν; then

$$\partial p/\partial \nu = \tilde{f} \cdot \nu + \nu \cdot \Delta \tilde{q}, \qquad x \in \partial\Omega, \tag{4.1 b}$$

so that, if (4.1) is tested against $w \in H^1(\Omega)$,

$$(\nabla p, \nabla w) - \langle \nu \cdot \Delta \tilde{q}, w \rangle = (\tilde{f}, \nabla w). \tag{4.2}$$

Now, assume that the triangulation $\mathcal{T}_h$ is quasi-regular both in shape and size near the boundary $\partial\Omega$; it can be less regular in the interior of Ω. Then modify the approximation procedure, which is not limited to linear elements, to become the finding of $\{\tilde{q}_h, p_h\} \in \tilde{V}_h \times W_h$ such that

$$a(\tilde{q}_h, \tilde{v}) - (\operatorname{div} \tilde{v}, p_h) = (\tilde{f}, \tilde{v}), \qquad \tilde{v} \in \tilde{V}_h, \tag{4.3 a}$$

$$(\operatorname{div} \tilde{q}_h, w) + \alpha h^2 [(\nabla p_h, \nabla w) - \sum_T \langle \nu \cdot \Delta \tilde{q}_h, w \rangle_{\partial T \cap \partial\Omega}] = \alpha h^2 (\tilde{f}, \nabla w), \quad w \in W_h. \tag{4.3 b}$$

Here, it is important that the normalization $(p_h, 1) = 0$ be understood, as it simplifies the proof of coercivity of the bilinear form

$$A(\{\tilde{q}, p\}, \{\tilde{v}, w\}) = a(\tilde{q}, \tilde{v}) - (\operatorname{div} \tilde{v}, p) + (\operatorname{div} \tilde{q}, w) + \alpha h^2 [(\nabla p, \nabla w) - \sum_T \langle \nu \cdot \Delta \tilde{q}, w \rangle_{\partial T \cap \partial\Omega}]$$

over $\tilde{V}_h \times W_h$. Note that scaling and the assumed quasi-regularity show that over $\tilde{V}_h \times W_h$

$$|\langle \nu \cdot \Delta \tilde{q}, p \rangle_{\partial T \cap \partial\Omega}| \leq Q h^{-1} \|\tilde{q}\|_{1,T} \|p\|_{1,T},$$

so that

$$h^2 \sum_T \langle \nu \cdot \Delta \tilde{q}, p \rangle_{\partial T \cap \partial\Omega} \leq Q \|\tilde{q}\|_1 \cdot h \|p\|_1$$

and

$$A(\{\tilde{q}, p\}, \{\tilde{q}, p\}) \geq \rho [\|\tilde{q}\|_1^2 + h^2 \|p\|_1^2]$$

for sufficiently small α. Consequently, there exists a unique solution of (4.3) when α is so chosen.

We shall present the error analysis for the case of C^0-piecewise-linear approximation spaces; the extension to higher order spaces is quite analogous to that given above for the Hughes, Balestra, and Franca procedure. Let $\{\tilde{\chi}, \eta\} \in \tilde{V}_h \times W_h$ and write the error equations in the form

$$a(\tilde{\chi}-\tilde{q}_h, \tilde{v})-(\operatorname{div} \tilde{v}, \eta-p_h)=a(\tilde{\chi}-\tilde{q}, \tilde{v})-(\operatorname{div} \tilde{v}, \eta-p),$$
$$(\operatorname{div}(\tilde{\chi}-\tilde{q}), w)+\alpha h^2 [(\nabla(\eta-p_h), \nabla w)+\sum_T \langle v \cdot \Delta(\tilde{\chi}-\tilde{q}_h), w\rangle_{\partial T \cap \partial\Omega}]$$
$$=(\tilde{q}-\tilde{\chi}, \nabla w)+\alpha h^2(\nabla(\eta-p), \nabla w)+\alpha h^2 \langle v \cdot \Delta(\tilde{\chi}-\tilde{q}), w\rangle_{\partial\Omega},$$

where we note that $v \cdot \Delta(\tilde{\chi}-\tilde{q}) \in L^2(\partial\Omega)$. Choose the test function $\{\tilde{\chi}-\tilde{q}_h, \eta-p_h\}$. Then, all terms resulting in the expression above were present in (2.2) with the exception one coming from the last term, which we can estimate as follows:

$$|\langle v \cdot \Delta(\tilde{\chi}-\tilde{q}), \eta-p_h\rangle_{\partial\Omega}| \leqq Q \|\tilde{q}\|_2 \|\eta-p_h\|_1.$$

Thus, it again follows that

$$\|\tilde{q}-\tilde{q}_h\|_1+h\|p-p_h\|_1 \leqq Q \|\tilde{f}\|_0 h. \tag{4.4}$$

L^2-estimates can be derived in the usual way. Let us begin with $\tilde{q}-\tilde{q}_h$. Let

$$-\Delta\tilde{\psi}+\nabla\theta=\tilde{q}-\tilde{q}_h, \qquad x\in\Omega,$$
$$\operatorname{div}\tilde{\psi}=0, \qquad x\in\Omega,$$

with $\tilde{\psi}=\tilde{0}$ on $\partial\Omega$ and $(\theta, 1)=0$. Then, with $\{\tilde{\chi}, \eta\} \in \tilde{V}_h \times W_h$,

$$\|\tilde{q}-\tilde{q}_h\|_0^2=a(\tilde{q}-\tilde{q}_h, \tilde{\psi}-\tilde{\chi})+a(\tilde{q}-\tilde{q}_h, \tilde{\chi})-(\operatorname{div}(\tilde{q}-\tilde{q}_h), \theta-\eta)-(\operatorname{div}(\tilde{q}-\tilde{q}_h), \eta)$$
$$=a(\tilde{q}-\tilde{q}_h, \tilde{\psi}-\tilde{\chi})+a(\tilde{q}-\tilde{q}_h, \tilde{\chi})-(\operatorname{div}(\tilde{q}-\tilde{q}_h), \theta-\eta)+\alpha h^2(\nabla(p-p_h), \nabla\eta)$$
$$+\alpha h^2 \langle v \cdot \Delta(\tilde{q}-\tilde{q}_h), \eta\rangle_{\partial\Omega}.$$

The only new term in this relation is the last one, and it can be bounded by (where $\tilde{\xi} \in \tilde{V}_h$)

$$\alpha h^2 \langle v \cdot \Delta(\tilde{q}-\tilde{\xi}), \eta\rangle+\alpha h^2 \langle v \cdot \Delta(\tilde{\xi}-\tilde{q}_h), \eta\rangle$$
$$\leqq Q\{h^2 \|\tilde{q}\|_2 \|\eta\|_1+h\|\tilde{\xi}-\tilde{q}_h\|_1 \|\eta\|_1\}$$
$$\leqq Q\{\|\tilde{q}\|_2+\|p\|_1\} h^2 \|\tilde{q}-\tilde{q}_h\|_0.$$

Thus,

$$\|\tilde{q}-\tilde{q}_h\|_0 \leqq Q \|\tilde{f}\|_0 h^2. \tag{4.5}$$

The argument for the L^2-estimate for $p-p_h$ in Sect. 2 did not use the second error equation; the result for $p-p_h$ remains valid for this method:

$$\|p-p_h\|_0 \leqq Q \|\tilde{f}\|_0 h. \tag{4.6}$$

As pointed out in the introduction, the method of this section should be a bit cheaper than the original one, since there are many fewer boundary elements than there are in total, so that the assembly of the matrix involves fewer terms. There should be little difference in the work required to solve the algebraic equations for the two variants.

References

1. Brezzi, F., Pitkäranta, J.: On the Stabilization of Finite Element Approximations of the Stokes Problem. In: Efficient Solutions of Elliptic Systems, Notes on Numerical Fluid Mechanics, Vol. 10 (W. Hackbusch, ed.), pp. 11–19. Braunschweig, Wiesbaden: Viewig 1984
2. Girault, V., Raviart, P.A.: Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Berlin Heidelberg New York: Springer 1986
3. Hughes, T.J.R., Franca, L.P., Balestra, M.: A New Finite Element Formulation for Computational Fluid Mechanics: V. Circumventing the Babuška-Brezzi condition: A Stable Petrov-Galerkin Formulation of the Stokes Problem Accommodating Equal Order Interpolation. Comput. Methods Appl. Mech. Eng. **59**, 85–99 (1986)
4. Nitsche, J.A.: Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. Abh. Math. Semin. Univ. Hamburg **36**, 9–15 (1970/1971)

Received December 30, 1986/December 2, 1987