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Inventiones mathematicae

All nuclear C^*-algebras are amenable

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1. Introduction

A Banach algebra A is called amenable if any bounded derivation δ of A into a dual Banach A-bimodule X^* is of the form $\delta(a)=ax-xa$ for some $x\in X^*$ ([13, p. 60]). A few years ago Connes proved that amenable C^*-algebras are necessarily nuclear ([6]). In the present paper we will prove the converse implication; All nuclear C^*-algebras are amenable. A partial result was obtained recently by Bunce and Paschke [2]. They proved, that any bounded derivation δ of a nuclear C^*-algebra A into a dual Banach A-bimodule X^* is of the form $\delta(a)=ax-xa$ for some $x\in X^*$, provided that X^{**} is weakly sequentially complete. Their proof as well as ours relies heavily on the fact that the second dual of a nuclear C^*-algebra A is an approximately finite dimensional von Neumann algebra. (cf. [3, 6, 9]). Our main new tool is the generalization of Grothendieck's inequality to bilinear forms on C^*-algebras proved by Pisier [16] for C^*-algebras, having the bounded approximation property, and by the author [11] for general C^*-algebras. This inequality is in Sect. 2 used to prove, that for any approximately finite dimensional von Neumann algebra M, there exists a mean m on the semigroup of isometries $I(M)$ in M (considered as a discrete semigroup), such that

$$\int_{I(M)} V(au^*,u)\,dm(u)=\int_{I(M)} V(u^*,ua)\,dm(u),$$

for all separately σ-weakly continuous bilinear forms V on M and all $a\in M$. It is essential to consider $I(M)$ and not only the unitary group $U(M)$. The mean can be concentrated on $U(M)$ if and only if M is finite (cf. Prop. 2.4(2)). Applying the above result to $M=A^{**}$, the second dual of a nuclear C^*-algebra, one gets quite easily, that A has a virtual diagonal in the sense of [14], which implies that A is amenable (cf. Theorem 3.1). In [18], Rosenberg proved that the C^*-algebras O_n, $n=2,3,\ldots,\infty$ constructed by Cuntz ([7]) are amenable but not strongly amenable in the sense of [13, p. 70]. He also proved that the tensor product $O_n\otimes K$ of O_n with the compact operators K on an infinite dimensional

Hilbert space is strongly amenable. In Theorem 3.3 we show that $A \otimes K$ is strongly amenable for any amenable C^*-algebra A. The problem whether amenable C^*-algebras are stable isomorphic to strongly amenable C^*-algebras was brought to our attention by L.G. Brown.

In the last section (Sect. 4) we use the generalized Grothendieck inequality to prove that any derivation δ from a C^*-algebra A into its dual is of the form $\delta(x) = x\varphi - \varphi x$ for a $\varphi \in A^*$. This is the affirmative answer to a problem raised in [2].

2. Bilinear forms on injective von Neumann algebras

We shall make repeatedly use of the following generalization of Grothendieck's inequality to C^*-algebras (cf. [11, Theorem 1.1]): Let V be a bounded bilinear form on a C^*-algebra A. Then there exist four states $\varphi_1, \varphi_2, \psi_1, \psi_2$ on A such that

$$|V(x,y)| \leqq \|V\| (\varphi_1(x^* x) + \varphi_2(x x^*))^{\frac{1}{2}} (\psi_1(y^* y) + \psi_2(y y^*))^{\frac{1}{2}}$$

for all $x, y \in A$. Moreover, if A is a von Neumann algebra, and V is separately σ-weakly continuous, then $\varphi_1, \varphi_2, \psi_1, \psi_2$ can be chosen normal (cf. [11, Prop. 2.3]). In [16, cor. 2.2] Pisier proved a similar inequality for C^*-algebras having the bounded approximation property. Using that nuclear C^*-algebras have the metric approximation property (cf. [4]), Pisier's result is in fact sufficient to prove Theorem 2.1 below for $M = A^{**}$, where A is a nuclear C^*-algebra, and hence also sufficient to prove our main result (Theorem 3.1). However, in Sect. 4 we will need the inequality for general C^*-algebras.

For any unital C^*-algebra, we let $U(A)$ (resp. $I(A)$) be the group of unitary operators in A (resp. the semigroup of isometries in A). Following [10], a mean on a semigroup G is a state m on the algebra $l^\infty(G)$ of all bounded functions on G (no topology will be taken into account). For $f \in l^\infty(G)$, we will often write $m(f)$ in the form

$$\int_G f(g)\, dm(g).$$

We let $B(A, A)$ denote the set of bounded bilinear forms on a C^*-algebra A, and we let $B_\sigma(M, M)$ denote the set of separately σ-weakly continuous bilinear forms on a von Neumann algebra M.

Theorem 2.1. *Let M be an injective von Neumann algebra. There exists a mean m on the semigroup $I(M)$ of isometries on M, such that*

$$\int_{I(M)} V(au^*, u)\, dm(u) = \int_{I(M)} V(u^*, ua)\, dm(u)$$

for all $V \in B_\sigma(M, M)$ and all $a \in M$.

Lemma 2.2. *Let M be a von Neumann algebra, and let p be the largest finite projection in the center of M. Let m be a mean on $I(M)$, such that for all positive, normal functionals φ on M,*

$$\int_{I(M)} \varphi(uu^*)\, dm(u) = \varphi(p).$$

Then for any $V \in B_\sigma(M,M)$, the maps

$$a \to \int_{I(M)} V(a u^*, u)\, dm(u)$$

and

$$a \to \int_{I(M)} V(u^*, u a)\, dm(u)$$

are σ-weakly continuous functionals on M.

Proof. Let $V \in B_\sigma(M,M)$. By the generalized Grothendieck inequality [11, Prop. 2.3], there exist four normal states $\varphi_1, \varphi_2, \psi_1, \psi_2$ on M such that

$$|V(x,y)| \leqq \|V\| (\varphi_1(x^* x) + \varphi_2(x x^*))^{\frac{1}{2}} (\psi_1(y^* y) + \psi_2(y y^*))^{\frac{1}{2}}$$

for all $x, y \in M$.
For $i=1,2$, put

$$\varphi_i'(x) = \int_{I(M)} \varphi_i(u x u^*)\, dm(u), \qquad x \in M$$

$$\psi_i'(x) = \int_{I(M)} \psi_i(u x u^*)\, dm(u), \qquad x \in M.$$

Since for any positive, normal functional φ on a finite von Neumann algebra N, the convex hull of the set $\{u^* \varphi u \mid u \in U(N)\}$ is relatively $\sigma(N_*, N)$-compact (cf. [19, Chap. V, proof of Theorem 2.4]) it follows that the restrictions of φ_i' and ψ_i' to pM are normal functionals. But by the assumption on m,

$$\begin{aligned}\varphi_i'(1-p) &= \int_{I(M)} \varphi_i(u(1-p)u^*)\, dm(u)\\ &= \int_{I(M)} ((1-p)\varphi_i)(u u^*)\, dm(u)\\ &= ((1-p)\varphi_i)(p)\\ &= 0\end{aligned}$$

and in the same way $\psi_i'(1-p)=0$. This shows that φ_i' and ψ_i' vanish on the properly infinite part of M. Hence φ_i' and ψ_i' are normal. Put now

$$\omega_1(a) = \int_{I(M)} V(a u^*, u)\, dm(u), \qquad a \in M,$$

$$\omega_2(a) = \int_{I(M)} V(u^*, u a)\, dm(u), \qquad a \in M.$$

Then

$$\begin{aligned}|\omega_1(a)| &= |V(a u^*, u)|\\ &\leqq \|V\| (\varphi_1(u a^* a u^*) + \varphi_2(a a^*))^{\frac{1}{2}} (\psi_1(u^* u) + \psi_2(u u^*))^{\frac{1}{2}}\\ &\leqq \sqrt{2}\, \|V\| (\varphi_1(u a^* a u^*) + \varphi_2(a a^*))^{\frac{1}{2}}.\end{aligned}$$

Hence using the Hölder inequality, we get

$$\begin{aligned}|\omega_1(a)| &\leqq \sqrt{2}\, \|V\| \int_{I(M)} (\varphi_1(u a^* a u^*) + \varphi_2(a a^*))^{\frac{1}{2}}\, dm(u)\\ &= \sqrt{2}\, \|V\| (\varphi_1'(a^* a) + \varphi_2(a a^*))^{\frac{1}{2}}, \qquad a \in M.\end{aligned}$$

In the same way one gets

$$|\omega_2(a)| \leqq \sqrt{2}\,\|V\|\,(\psi_1(a^* a) + \psi_2'(a a^*))^{\frac{1}{2}}, \qquad a \in M.$$

This proves that ω_1 and ω_2 are σ-strong* continuous, which implies that $\omega_1, \omega_2 \in M_*$.

Lemma 2.3. *Let M_α be an increasing set of von Neumann algebras, such that each $I(M_\alpha)$ has a mean m_α satisfying the condition in Theorem 2.1. Let M be the σ-weak closure of $\bigcup_\alpha M_\alpha$. Then $I(M)$ also has a mean satisfying the conditions in Theorem 2.1.*

Proof. Let $V \in B_\sigma(M, M)$. Since the restriction of V to $M_\alpha \times M_\alpha$ belongs to $B_\sigma(M_\alpha, M_\alpha)$, we have

$$\int_{I(M_\alpha)} V(a u^*, u)\, dm_\alpha(u) = \int_{I(M_\alpha)} V(u^*, u a)\, dm_\alpha(u)$$

for all $a \in M_\alpha$. Each m_α can be extended to a mean $\bar{m}_\alpha$ on $I(M)$ by putting

$$\bar{m}_\alpha(f) = m_\alpha(f_\alpha), \qquad f \in l^\infty(I(M)),$$

where f_α is the restriction of f to $I(M_\alpha)$.

For $V \in B_\sigma(M, M)$, $a \in M_\alpha$ and $\beta \geqq \alpha$ we get

$$\int_{I(M)} V(a u^*, u)\, d\bar{m}_\beta = \int_{I(M)} V(u^*, u a)\, d\bar{m}_\beta.$$

Let m be a weak* clustering point for the net $(\bar{m}_\alpha)_\alpha$, then

$$\int_{I(M)} V(a u^*, u)\, dm = \int_{I(M)} V(u^*, u a)\, dm \qquad (*)$$

for all $V \in B_\sigma(M, M)$ and all $a \in \bigcup_\alpha M_\alpha$.

If M is a finite von Neumann algebra, the mean m satisfies automatically the condition in Lemma 2.2, and hence the equality $(*)$ is valid for all $a \in M$.

Assume now that M is not finite, and let p be the largest finite projection in the center of M. Since $(1-p)M$ is properly infinite, it contains an infinite type I factor. Hence we can choose a sequence of isometries $(w_n)_{n \in \mathbb{N}}$ in $(1-p)M$, such that $w_n^* \to 0$ σ-strongly. Put $v_n = p + w_n$. Then $v_n \in I(M)$, and $v_n v_n^*$ converges σ-weakly to p. Put

$$m_n(f) = \int_{I(M)} f(v_n u)\, dm(u), \qquad f \in l^\infty(I(M)).$$

and let m' be a weak* clustering point for this sequence. Let $V \in B_\sigma(M, M)$, and put

$$V_n(x, y) = V(x v_n^*, v_n y), \qquad x, y \in M.$$

For $a \in \bigcup_\alpha M_\alpha$, we have

$$\int_{I(M)} V_n(a u^*, u)\, dm(u) = \int_{I(M)} V_n(u^*, u a)\, dm(u)$$

which is equivalent to,

$$\int_{I(M)} V(a u^*, u)\, dm_n(u) = \int_{I(M)} V(u^*, u a)\, dm_n(u).$$

Hence

$$\int_{I(M)} V(a u^*, u)\, dm'(u) = \int_{I(M)} V(u^*, u a)\, dm'(u). \qquad (**)$$

Let φ be a normal state on M. Since $u u^* \geqq p$ for any $u \in I(M)$, we have

$$\int_{I(M)} \varphi(u u^*)\, dm'(u) \geqq \varphi(p).$$

On the other hand

$$\begin{aligned}\int_{I(M)} \varphi(u u^*)\, dm'(u) &\leqq \limsup_n \int_{I(M)} \varphi(v_n u u^* v_n^*)\, dm(u)\\ &\leqq \limsup_n \varphi(v_n v_n^*)\\ &= \varphi(p).\end{aligned}$$

This shows that m' satisfies the condition in Lemma 2.2, and we can conclude that $(**)$ is valid for all $a \in M$.

Proof of Theorem 2.1. Assume first that M is finite dimensional, and let m_0 be the normalized Haar measure on the continuous functions on $U(M)$. Let m be a Hahn-Banach extension of m_0 to $l^\infty(U(M))$ satisfying $\|m\| = 1$. Since $m(1) = m_0(1) = 1$, m is a mean on $U(M)$ (considered as a discrete group). Using that m is right invariant on the continuous functions, we get

$$\int_{U(M)} V(v u^*, u)\, dm(u) = \int_{U(M)} V(u^*, u v)\, dm(u)$$

for $V \in B_\sigma(M, M) = B(M, M)$ and $v \in U(M)$. Since $M = \operatorname{span} U(M)$ and $I(M) = U(M)$ it follows that Theorem 2.1 holds for all finite dimensional von Neumann algebras. Using [5] and Lemma 2.3 one gets that it holds for any injective von Neumann algebra with separable predual. The result can be extended to countable generated injective von Neumann algebras, because such an algebra is a direct sum of injective von Neumann algebras with separable predual. Finally by [9, Theorem 4] we get that Theorem 2.1 is valid for any injective von Neumann algebra.

Remark. In [12] de la Harpe proved that a von Neumann algebra M with separable predual is injective if and only if the unitary group $U(M)$ is amenable in the sense that the space $C_r^b(U(M))$ of right uniformly continuous functions on $U(M)$ admits a left invariant mean (or equivalently, the space $C_l^b(U(M))$ of left uniformly continuous functions on $U(M)$ admits a right invariant mean). Here $U(M)$ is considered as a topological group in the strong operator topology. The separability condition on M_* is not essential (use [9, Theorem 4]). Unfortunately de la Harpe's result cannot be used to give a more direct proof of Theorem 2.1. The reason is that when V is a separately σ-weakly continuous bilinear form on a von Neumann algebra M, the function

$$f(u) = V(u^*, u), \qquad u \in U(M)$$

is not in general left uniformly continuous: Take f. inst. $M=B(H)$, where H is a Hilbert space with basis $(e_k)_{k=0,1,2,\dots}$, and let $(u_n)_{n\in\mathbb{N}}$ be the unitary operators defined in [12, p. 226]. Then $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the left uniform structure on $U(M)$, and $u_n^* e_0 = e_n$ for all $n\in\mathbb{N}$. Let $a\in B(H)$ be given by

$$a\,e_k = (-1)^k e_k, \qquad k=0,1,2,\dots$$

and put

$$V(x,y) = (y\,a\,x\,e_0, e_0), \qquad x,y\in B(H).$$

Since the sequence $V(u_n^*, u_n) = (-1)^n$ is not convergent, the function $f(u) = V(u^*, u)$ cannot be left uniformly continuous on $U(M)$.

For the sake of completeness, we will show that the conditions (1) and (2) in Theorem 2.1 actually characterize the class of injective von Neumann algebras.

Proposition 2.4. *Let M be a von Neumann algebra.*

(1) There exists a mean m on $I(M)$, such that

(i) $$\int_{I(M)} V(a u^*, u)\, dm(u) = \int_{I(M)} V(u^*, u\,a)\, dm(u)$$

for all $V\in B_\sigma(M,M)$ and all $a\in M$ if and only if M is injective.

(2) There exists a mean m on $U(M)$, such that

(ii) $$\int_{U(M)} V(a u^*, u)\, dm(u) = \int_{U(M)} V(u^*, u\,a)\, dm(u)$$

for all $V\in B_\sigma(M,M)$ and all $a\in M$ if and only if M is injective and finite.

Proof. 1) The "if part" is already proved. Assume now that M is a von Neumann algebra on a Hilbert space H, and m is a mean on $I(M)$ which satisfies (i). We can define a linear map E of $B(H)$ into itself by

$$\varphi(E(x)) = \int_{I(M)} \varphi(u^* x\, u)\, dm(u), \qquad \varphi\in B(H)_*.$$

Clearly $\|E\|\leqq 1$. Using (i) on the bilinear form

$$V_{x,\varphi}(a,b) = \varphi(a\,x\,b), \qquad a,b\in M,$$

we get for $x\in B(H)$, $a\in M$ and $\varphi\in B(H)_*$:

$$\begin{aligned}\varphi(a\,E(x)) &= \int_{I(M)} \varphi(a u^* x u)\, dm(u)\\ &= \int_{I(M)} \varphi(u^* x u\, a)\, dm(u) = \varphi(E(x)\,a).\end{aligned}$$

Hence $E(x)\in M'$ for all $x\in B(H)$. Using that $u^* u = 1$ for $u\in I(M)$, one easily gets that $E(x') = x'$ for all $x'\in M'$. Thus E is a projection of norm 1 of $B(H)$ onto M'. This implies that M is injective (cf. [8, Theorem 5.1 and Theorem 5.3]).

2) The "if part" follows from Theorem 2.1, because $U(M) = I(M)$, when M is finite. Assume next that M is a von Neumann algebra, and m is a mean on $U(M)$ satisfying (ii). The proof of the "only if part" in (1) can also be applied in this case to show that M is injective. If M is not finite, there exists a non zero

central projection p in M, such that pM is properly infinite. Let φ be a normal state on pM. Put

$$\varphi'(x) = \int_{U(M)} \varphi(u x u^*)\, dm(u), \qquad x \in pM.$$

Then φ' is a state on pM. Using (ii) on the bilinear form $V_{x,\varphi}(a,b) = \varphi(b x a)$, we get for all $a \in M$

$$\varphi'(a x) = \int_{U(M)} \varphi(u a x u^*)\, dm(u) = \int_{U(M)} \varphi(u x a u^*)\, dm(u) = \varphi'(x a).$$

Hence φ' is a tracial state on pM. This contradicts that pM is properly infinite. Thus M is finite.

3. The Main Results

Let A be a Banach algebra, and let $A \bigotimes_\pi A$ denote the completion of the algebraic tensor product $A \odot A$ in the projective tensor norm (cf. [19, p. 189]). The Banach space $A \bigotimes_\pi A$ is a Banach A-bimodule, when left and right multiplication are defined by

$$a(b \otimes c) = ab \otimes c,$$
$$(b \otimes c)a = b \otimes ca.$$

Let p: $A \bigotimes_\pi A \to A$ be the bounded linear map for which

$$p(a \otimes b) = ab, \qquad a, b \in A.$$

Following Johnson [14] a virtual diagonal for A is an element $\omega \in (A \bigotimes_\pi A)^{**}$ for which

1) $a\omega = \omega a \qquad \forall a \in A$
2) $p^{**}(\omega) a = a \quad \forall a \in A.$

Here $(A \bigotimes_\pi A)^{**}$ and A^{**} are considered as dual Banach A-bimodules in the usual way. Note that when A is a C^*-algebra, Condition 2) is equivalent to $p^{**}(\omega) = 1$ (unit in A^{**}). By [14, Theorem 1.3] a Banach algebra is amenable if and only if it admits a virtual diagonal.

When E is a subset of a dual Banach space, we let $w^* \operatorname{co}(E)$ denote the weak* closed convex hull of E.

Theorem 3.1. *Any nuclear C^*-algebra A has a virtual diagonal ω in the weak* closed convex hull of $\{a^* \otimes a \mid a \in A, \|a\| \leq 1\}$. In particular all nuclear C^*-algebras are amenable.*

Proof. Let A be a nuclear C^*-algebra. Then A^{**} is an injective von Neumann algebra. We can identify $B(A,A)$ with $(A \bigotimes_\pi A)^*$, by putting $\langle V, a \otimes b \rangle = V(a,b)$ for $V \in B(A,A)$ and $a, b \in A$.

It follows from [15, Lemma 2.1] that any bilinear form on A has a unique extension to a separately σ-weakly continuous bilinear form $\tilde{V}$ on A^{**}. Let m

be a mean on $I(A^{**})$, which satisfies the conditions in Theorem 2.1. For $V\in B(A,A)$ put

$$\omega(V)=\int_{I(A^{**})}\tilde{V}(u^*,u)\,dm(u).$$

Clearly $\omega\in B(A,A)^*=(A\bigotimes_\pi A)^{**}$. We will show that ω is a virtual diagonal for A. For $a\in A$, we let L_a and R_a denote left and right multiplication with a on $A\bigotimes_\pi A$, i.e.

$$L_a(b\otimes c)=a\,b\otimes c,$$
$$R_a(b\otimes c)=b\otimes c\,a.$$

By definition ω is a virtual diagonal for A if and only if

(1) $$L_a^{**}\,\omega=R_a^{**}\,\omega,\qquad a\in A$$

(2) $$p^{**}(\omega)=1\quad(\text{unit in } A^{**}),$$

where $p(a\otimes b)=a\,b$. For $V\in B(A,A)\cong(A\bigotimes_\pi A)^*$, we have

$$(L_a^*\,V)(x,y)=V(a\,x,y),$$
$$(R_a^*\,V)(x,y)=V(x,y\,a).$$

Hence for all $a\in A$:

$$\begin{aligned}(L_a^{**}\,\omega)(V)&=\omega(L_a^*\,V)\\&=\int_{I(A^{**})}\tilde{V}(a\,u^*,u)\,dm(u)\\&=\int_{I(A^{**})}\tilde{V}(u^*,u\,a)\,dm(u)\\&=\omega(R_a^*\,V)=(R_a^{**}\,\omega)(V).\end{aligned}$$

This proves (1).

For $\varphi\in A^*$, put $V_\varphi(x,y)=\varphi(x\,y)$, $x,y\in A$. Since

$$\langle p^*\,\varphi,x\otimes y\rangle=\varphi(x\,y)=V_\varphi(x,y),$$

we have $p^*\,\varphi=V_\varphi$. Let $\tilde{\varphi}$ be the unique extension of φ to a normal functional on A^{**}.

Clearly

$$\tilde{V}_\varphi(x,y)=\tilde{\varphi}(x\,y),\qquad x,y\in A^{**}.$$

Hence for all $\varphi\in A^*$

$$\begin{aligned}\langle p^{**}\,\omega,\varphi\rangle&=\langle\omega,V_\varphi\rangle\\&=\int_{I(A^{**})}\tilde{\varphi}(u^*\,u)\,dm(u)=\tilde{\varphi}(1).\end{aligned}$$

This shows that $p^{**}\omega$ is the unit in A^{**}. Hence ω is a virtual diagonal. If $\omega\notin w^*\operatorname{co}\{a^*\otimes a\,|\,a\in A,\ \|a\|\leq 1\}$, there exist a $W\in B(A,A)$ and $a\,c\in\mathbb{R}$, such that

$$\operatorname{Re}W(a^*,a)\leq c<\operatorname{Re}\omega(W)$$

for all a in the unitball of A. Let $u \in I(A^{**})$. By Kaplansky's density theorem there exists a net (a_α) in the unitball of A, such that $a_\alpha \to u$ σ-strong*. By [11, Cor. 2.4], the extension $\tilde{W}$ of W to A^{**} is jointly σ-strong* continuous. Hence

$$\operatorname{Re} \tilde{W}(u^*, u) \leqq v < \operatorname{Re} \omega(W)$$

for all $u \in I(A^{**})$. This contradicts that

$$\omega(W) = \int_{I(A^{**})} \tilde{W}(u^*, u)\, dm(u).$$

Hence $\omega \in w^* \operatorname{co}\{a^* \otimes a \mid a \in A,\ \|a\| \leqq 1\}$.

Corollary 3.2. *Let A be a nuclear C^*-algebra and let δ be a derivation of A into a dual Banach A-bimodule X^*. Then there exists an x in the weak* closed convex hull of $\{a^* \delta(a) \mid a \in A,\ \|a\| \leqq 1\}$ such that $\delta(a) = a x - x a$ for all $a \in A$.*

Proof. There is a unique bounded linear map Φ: $A \underset{\pi}{\otimes} A \to X^*$, such that

$$\Phi(a \otimes b) = a\, \delta(b), \qquad a, b \in A.$$

Let Φ_* be the restriction of Φ^* to $X \subseteqq X^{**}$ and put $\tilde{\Phi} = (\Phi_*)^*$. Then $\tilde{\Phi}$ is a $\sigma((A \otimes A)^{**}, (A \otimes A)^*) - \sigma(X^*, X)$ continuous extension of Φ to $(A \otimes A)^{**}$. It follows from the proof of [14, Theorem 1.3] that for any virtual diagonal $\omega \in (A \otimes A)^{**}$:

$$\delta(a) = a\, \tilde{\Phi}(\omega) - \tilde{\Phi}(\omega)\, a, \qquad a \in A.$$

By Theorem 3.1 ω can be chosen in $w^* \operatorname{co}\{a^* \otimes a \mid a \in A,\ \|a\| \leqq 1\}$, which implies that $\tilde{\Phi}(\omega) \in w^* \operatorname{co}\{a^* \delta(a) \mid a \in A,\ \|a\| \leqq 1\}$.

Recall that a unital C^*-algebra is strongly amenable, if and only if for any derivation δ of A into a unital dual Banach A-bimodule X^*, there exists $x \in w^* \operatorname{co}\{u^* \delta(u) \mid u \in U(A)\}$ such that $\delta(a) = a x - x a$ for all $a \in A$. Moreover, a non unital C^*-algebra A is strongly amenable if and only if $\tilde{A} = A + \mathbb{C}1$ is strongly amenable (cf. [13, p. 70–72]).

Theorem 3.3. *Let A be a nuclear C^*-algebra and let $K(H)$ be the compact operators on an infinite dimensional Hilbert space H, then $A \otimes K(H)$ is strongly amenable.*

For the proof we need a few lemmas:

Lemma 3.4. *Let A be a unital C^*-algebra. If A has a virtual diagonal $\omega \in w^* \operatorname{co}\{u^* \otimes u \mid u \in U(A)\}$, then A is strongly amenable.*

Proof. The statement follows from the proof of Corollary 3.2.

Remark. The converse of Lemma 3.4 is also true. This can be proved by the same arguments as in the proof of the "only if" part of [14, Theorem 1.3].

Lemma 3.5. *Let A be a C^*-algebra. If there exists a mean m on $U(A^{**})$, such that for all $V \in B_\sigma(A^{**}, A^{**})$ and all $a \in A$*

$$\int_{U(A^{**})} V(a u^*, u)\, dm(u) = \int_{U(A^{**})} V(u^*, u a)\, dm(u)$$

then A is strongly amenable.

Proof. Assume first that A is unital. For $V \in B(A,A)$ we let $\tilde{V}$ denote its unique extension to a separately σ-weakly continuous bilinear form on A^{**}. We can define $\omega \in B(A,A)^* = (A \underset{\pi}{\otimes} A)^{**}$ by

$$\omega(V) = \int_{U(A^{**})} \tilde{V}(u^*, u)\, dm(u).$$

Then ω is a virtual diagonal for A (cf. proof of Theorem 3.1). If $\omega \notin w^* \operatorname{co}\{u^* \otimes u \mid u \in U(A)\}$ there exist $W \in B(A,A)$ and $c \in \mathbb{R}$, such that

$$\operatorname{Re} \omega(W) < c \leqq \operatorname{Re} W(u^*, u), \qquad \forall u \in U(A).$$

By [19, Theorem 4.11] $U(A)$ is σ-strong*-dense in $U(A^{**})$. Moreover, the extension $\tilde{W}$ of W to $A^{**} \times A^{**}$ is jointly σ-strong* continuous ([11, Cor. 2.4]). Hence

$$\operatorname{Re} \omega(W) < c \leqq \operatorname{Re} \tilde{W}(u^*, u) \qquad \forall u \in U(A^{**}).$$

This contradicts the definition of $\omega(W)$. Hence $\omega \in w^* \operatorname{co}\{u^* \otimes u \mid u \in U(A)\}$, which implies that A is strongly amenable by Lemma 3.4.

Assume next that A is not unital, and put $B = A + \mathbb{C}1$. There exists a central projection p in B^{**} such that $A^{**} = pB^{**}$ and $(1-p)B^{**}$ is one dimensional. We can identify $U(A^{**})$ with the partial isometries in B^{**}, which have support and range projections equal to p. Let m be a mean on $U(A^{**})$ that satisfies the condition in the lemma, and define $\omega \in (B \underset{\pi}{\otimes} B)^{**} = B(B,B)^*$ by

$$\omega(V) = \int_{U(A^{**})} \tilde{V}(u^*, u)\, dm(u) + \tilde{V}(1-p, 1-p)$$

where $\tilde{V}$ is the usual extension of $V \in B(B,B)$ to $B^{**} \times B^{**}$. Since $a(1-p) = (1-p)a = 0$ for all $a \in A$, it is easily verified that

$$L_a^{**} \omega = R_a^{**} \omega, \qquad \forall a \in A$$

(cf. proof of Theorem 3.1). Since L_1^{**} and R_1^{**} are the identity on $(B \underset{\pi}{\otimes} B)^{**}$ it follows that

$$L_b^{**} \omega = R_b^{**} \omega, \qquad \forall b \in A + \mathbb{C}1 = B.$$

If $\omega \notin w^* \operatorname{co}\{u^* \otimes u \mid u \in U(B)\}$, we can as in the first part of the proof find $W \in B(B,B)$ and $c \in \mathbb{R}$, such that

$$\operatorname{Re} \omega(W) < c \leqq \operatorname{Re} \tilde{W}(v^*, v), \qquad \forall v \in U(B^{**}).$$

If $u \in U(A^{**})$, then $u_1 = u + (1-p)$ and $u_2 = u - (1-p)$ are unitary operators in B^{**}. Moreover,

$$\tilde{W}(u^*, u) + \tilde{W}(1-p, 1-p) = \tfrac{1}{2}(\tilde{W}(u_1, u_1) + \tilde{W}(u_2, u_2)).$$

Hence $\operatorname{Re}(\tilde{W}(u^*, u) + \tilde{W}(1-p, 1-p)) \geqq c$. This implies that $\operatorname{Re} \omega(W) \geqq c$, which is a contradiction. Hence $\omega \in w^* \operatorname{co}\{u^* \otimes u \mid u \in U(B)\}$. Since $p(u^* \otimes u) = 1$ for all $u \in U(B)$, we have $p^{**}(\omega) = 1$. Therefore ω is a virtual diagonal. Hence by Lemma 3.5, B is strongly amenable, which implies that A is strongly amenable.

Lemma 3.6. *Let M be an injective von Neumann algebra, and let p be a projection in M, such that $p \sim 1-p$. Then there exists a mean m on $U(M)$, such that for all $V \in B_\sigma(M,M)$ and all $a \in pMp$:*

$$\int_{U(M)} V(au^*, u)\,dm(u) = \int_{U(M)} V(u^*, ua)\,dm(u).$$

Proof. Let F_2 be the algebra of complex 2×2-matrices, and let $(e_{ij})_{i,j=1,2}$ be the matrix units in F_2. The assumption $p \sim 1-p$ implies that M can be written in the form

$$M = N \otimes F_2$$

in such a way that $p = 1 \otimes e_{11}$. Thus

$$pMp = \{x \otimes e_{11} \mid x \in N\}.$$

Since N is also injective, we get by Theorem 2.1 that there exists a mean m on $I(N)$, such that for all $V \in B_\sigma(N,N)$ and all $a \in N$:

$$\int_{I(N)} V(au^*, u)\,dm(u) = \int_{I(N)} V(u^*, ua)\,dm(u).$$

To each $u \in I(N)$, we associate the two unitary operators u_1, u_2 in $M = N \otimes F_2$ given by

$$u_1 = \begin{pmatrix} u & 1-uu^* \\ 0 & u^* \end{pmatrix}, \qquad u_2 = \begin{pmatrix} u & -(1-uu^*) \\ 0 & -u^* \end{pmatrix}.$$

We can now define a mean $\bar{m}$ on $U(M)$ by

$$\bar{m}(f) = \tfrac{1}{2} \int_{I(N)} (f(u_1) + f(u_2))\,dm(u), \qquad f \in l^\infty(U(M)).$$

Let $W \in B_\sigma(M,M)$ and $b \in pMp$. The operator b has the form $b = a \otimes e_{11}$ for an $a \in N$. Let $V \in B_\sigma(N,N)$ be defined by

$$V(x,y) = W(x \otimes e_{11}, y \otimes e_{11}), \qquad x, y \in N.$$

By 2×2-matrix computation, one gets that for all $u \in I(N)$:

$$b u_1^* = b u_2^* = a u^* \otimes e_{11}$$
$$u_1 b = u_2 b = u a \otimes e_{11}$$

and

$$\tfrac{1}{2}(u_1 + u_2) = u \otimes e_{11}.$$

Hence

$$\tfrac{1}{2}(W(bu_1^*, u_1) + W(bu_2^*, u_2)) = W(au^* \otimes e_{11}, u \otimes e_{11}) = V(au^*, u)$$

and

$$\tfrac{1}{2}(W(u_1^*, u_1 b) + W(u_2^*, u_2 b)) = V(u^*, ua).$$

Therefore

$$\int_{U(M)} W(b\,v^*, v)\,d\bar{m}(v) = \int_{I(N)} V(a\,u^*, u)\,dm(u)$$
$$= \int_{I(N)} V(u^*, u\,a)\,dm(u)$$
$$= \int_{U(M)} W(v^*, v\,b)\,d\bar{m}(v).$$

Hence the mean $\bar{m}$ satisfies the conditions in the lemma.

Proof of Theorem 3.3. Let A be a nuclear C^*-algebra and let $K = K(H)$ be the compact operators on an infinite dimensional Hilbert space, and put $B = A \otimes K$. We will show that B satisfies the condition in Lemma 3.5. We have

$$B^{**} = A^{**} \hat{\otimes} B(H)$$

(von Neumann algebra tensor product), and B^{**} is an injective von Neumann algebra. Let $(E, \leqq)$ denote the set of finite dimensional projections in $B(H)$ with the usual ordering. Since $\dim H = +\infty$, we can for each $e \in E$ find a projection $f \geqq e$, such that $f \sim 1 - f$ in $B(H)$. Using Lemma 3.6 on $M = B^{**}$ and $p = 1 \otimes f$ we get that there exists a mean m_e on $U(B^{**})$, such that for all $V \in B_\sigma(B^{**}, B^{**})$ and all $a \in A^{**} \otimes e\,B(H)\,e$:

$$\int_{U(B^{**})} V(a\,u^*, u)\,dm_e(u) = \int_{U(B^{**})} V(u^*, u\,a)\,dm_e(u).$$

Let m be a weak* clustering point for the net $(m_e)_{e \in E}$. Then

$$\int_{U(B^{**})} V(a\,u^*, u)\,dm(u) = \int_{U(B^{**})} V(u^*, u\,a)\,dm(u) \qquad (*)$$

for $V \in B_\sigma(B^{**}, B^{**})$ and $a \in \bigcup_{e \in E} A^{**} \otimes e\,B(H)\,e$.

Since $\bigcup_{e \in E} A \otimes e\,B(H)\,e$ is uniformly dense in $B = A \otimes K$ the formula $(*)$ holds for all $a \in B$. Hence by Lemma 3.5, B is strongly amenable.

4. Derivations of a C^*-Algebra Into Its Dual

In [2, Sect. 3], Bunce and Paschke proved that all derivations from a semifinite von Neumann algebra M into its predual M_* are of the form $\delta(x) = x\,\varphi - x\,\varphi$ for a $\varphi \in M_*$. Using the generalized Grothendieck inequality, we can show that this is true for all von Neumann algebras. As a corollary one gets that any derivation from a C^*-algebra into its dual is also given by a commutator. As in the previous sections $I(M)$ denotes the semigroup of isometries in M.

Theorem 4.1. *Let M be a von Neumann algebra, and let δ be a derivation from M to M_*. Then there exists a $\varphi \in M_*$ in the norm closure of $\{u^*\,\delta(u) \mid u \in I(M)\}$, such that $\delta(x) = x\,\varphi - \varphi\,x$ that $\delta(x) = x\,\varphi - \varphi\,x$ for all $x \in M$.*

Proof. The weak and norm closures of a convex set in M_* coincide. Hence, if M is finite, the Theorem follows [2, proof of Lemma 3.1]. Assume next that M

is properly infinite. From the first part of the proof of [2, Theorem 3.2] we get that δ is $\sigma(M, M_*)-\sigma(M_*, M)$-continuous. Hence the bilinear form on M given by

$$V(x,y)=\langle\delta(x),y\rangle, \qquad x,y\in M$$

is separately σ-weakly continuous. By [11, Prop. 2.3], there exist four normal states $\varphi_1, \varphi_2, \psi_1, \psi_2$ on M, such that

$$|\langle\delta(x),y\rangle|\leqq\|\delta\|\,(\varphi_1(x^*x)+\varphi_2(xx^*))^{\frac{1}{2}}(\psi_1(y^*y)+\psi_2(yy^*))^{\frac{1}{2}}$$

for $x, y\in M$. Since M is properly infinite, we can choose a sequence of isometries $v_n\in I(M)$, such that v_n^* converges σ-strongly to 0. Put $\chi_n=v_n^*\,\delta(v_n)$ and let $\chi\in M^*$ be a $\sigma(M^*, M)$-clustering point for χ_n. For $a\in M$ we have

$$\begin{aligned}|\chi_n(a)|&=|\langle\delta(v_n), a\,v_n^*\rangle|\\&\leqq\sqrt{2}\,\|\delta\|\,(\psi_1(v_n a^* a v_n)+\psi_2(aa^*))^{\frac{1}{2}}.\end{aligned}$$

Since

$$\limsup_{n\to\infty}\psi_1(v_n a^* a v_n)\leqq\|a\|^2\limsup_{n\to\infty}\psi_1(v_n v_n^*)=0$$

we get $|\chi(a)|\leqq\sqrt{2}\,\|\delta\|\,\psi_2(aa^*)^{\frac{1}{2}}$. This shows that χ is σ-strong* continuous, i.e. $\chi\in M_*$. Let $u\in U(M)$. Then

$$\begin{aligned}u^*\chi_n u+u^*\delta(u)&=u^* v_n^*\,\delta(v_n)\,u+u^* v_n^* v_n\,\delta(u)\\&=u^* v_n^*\,\delta(v_n u).\end{aligned}$$

Hence, for $a\in M$

$$\begin{aligned}|\langle u^*\chi_n u+u^*\delta(u), a\rangle|&=|\langle\delta(v_n u), a u^* v_n^*\rangle|\\&\leqq\sqrt{2}\,\|\delta\|\,(\psi_1(v_n u a^* a u^* v_n^*)+\psi_2(aa^*))^{\frac{1}{2}}\end{aligned}$$

and therefore

$$|\langle u^*\chi u+u^*\delta(u), a\rangle|\leqq\sqrt{2}\,\|\delta\|\,\psi_2(aa^*)^{\frac{1}{2}}.$$

Using [1, Theorem II.3] we conclude that the convex hull of

$$\{u^*\chi u+u^*\delta(u)\,|\,u\in U(M)\}$$

is weakly relatively compact in M_*. Let K be the norm closure (=weak closure) of

$$\mathrm{co}\{u^*\chi u+u^*\delta(u)\,|\,u\in U(M)\}$$

then K is weakly compact. For $u\in U(M)$, put

$$\alpha_u(\varphi)=u\,\varphi\,u^*+u\,\delta(u^*), \qquad \varphi\in M_*.$$

Then $(\alpha_u)_{u\in U(M)}$ is a group of affine transformations on M_* leaving K invariant. Moreover,

$$\|\alpha_u(\varphi)-\alpha_u(\psi)\|=\|\varphi-\psi\|, \qquad \varphi,\psi\in M_*.$$

Hence by Ryll-Nardzewski's fixed point theorem [10, Appendix 2], there exists $\chi_0\in K$ such that $\alpha_u(\chi_0)=\chi_0$ for all $u\in U(M)$. One gets easily that $\delta(u)=u\chi_0-\chi_0 u$

for all $u \in U(M)$, and hence $\delta(x) = x\chi_0 - \chi_0 x$ for all $x \in M$. Since $u^* \chi u + u^* \delta(u)$ is a weak clustering point for the sequence $u^* v_n^* \delta(v_n u)$ for all $u \in U(M)$, it follows that χ_0 is in the weak closure (= norm closure) of $\operatorname{span}\{v^* \delta(v) | v \in I(M)\}$. This proves the theorem in the properly infinite case.

In the general case $M = pM \oplus (1-p)M$, where pM is finite and $(1-p)M$ is properly infinite. We may assume that $p \neq 0$ and $1-p \neq 0$.

Since

$$(2p-1)\,\delta(p) = \delta(p^2 - p) = 0$$

we have $\delta(p) = 0$. Hence for $x \in M$

$$\delta(px) = p\,\delta(x) \quad \text{and} \quad \delta((1-p)x) = (1-p)\,\delta(x).$$

Thus $\delta(pM) \subseteqq pM_*$ and $\delta((1-p)M) \subseteqq (1-p)M_*$. By the first part of the proof, there exist

$$\varphi_1 \in \overline{\mathrm{co}}\{v^* \delta(v) | v^* v = p\}$$

and

$$\varphi_2 \in \overline{\mathrm{co}}\{w^* \delta(w) | w^* w = 1-p\}$$

(norm closures), such that

$$\begin{aligned} \delta(y) &= y\varphi_1 - \varphi_1 y, && y \in pM \\ \delta(z) &= z\varphi_2 - \varphi_2 z, && z \in (1-p)M. \end{aligned}$$

Put $\varphi = \varphi_1 + \varphi_2$. Then one gets easily

$$\delta(x) = x\varphi - \varphi x, \qquad x \in M.$$

When $v^* v = p$ and $w^* w = 1-p$, then $v+w$, and $v-w$ are isometries in M. Moreover,

$$v^* \delta(v) + w^* \delta(w) = \tfrac{1}{2}((v+w)^* \delta(v+w) + (v-w)^* \delta(v-w)).$$

Hence $\varphi \in \overline{\mathrm{co}}\{u^* \delta(u) | u \in I(M)\}$. This completes the proof.

Corollary 4.2. *Let δ be a derivation of a C^*-algebra A into its dual A^*. Then there exists $\varphi \in A^*$, such that $\|\varphi\| \leqq \|\delta\|$, and*

$$\delta(x) = x\varphi - \varphi x, \qquad x \in A.$$

Proof. By [17], δ is norm continuous. Since δ is weakly compact (cf. [1, Theorem II 8]) it can be extended to a $\sigma(A^{**}, A^*) - \sigma(A^*, A^{**})$ continuous linear map $\tilde{\delta}$: $A^{**} \to A^*$. Using that the multiplication in A^{**} is separately σ-weakly continuous, one gets by a two step argument, that $\tilde{\delta}$ is also a derivation. Hence by Theorem 4.1 there exists $\varphi \in A^*$, such that $\|\varphi\| \leqq \|\delta\|$, and $\tilde{\delta}(x) = x\varphi - \varphi x$, $x \in A^{**}$.

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