Invent. math. 74, 169–198 (1983)

Inventiones mathematicae

An intrinsic characterization of Kähler manifolds

Reese Harvey* and H. Blaine Lawson, Jr.*

Department of Mathematics, Rice University, Post Office Box 1892, Houston, TX 77251, USA
Department of Mathematics, S.U.N.Y., Stony Brook, NY 11794, USA

Table of contents

§ 1. Introduction 169
§ 2. $d_{1,1}$ has closed range 172
§ 3. Positive currents of bidimension 1,1 174
§ 4. Kähler manifolds 175
§ 5. Some general problems 176
§ 6. Non-singular families of curves 177
§ 7. Elliptic surfaces 180
§ 8. The Kähler rank of a surface 186
§ 9. Non-elliptic Hopf surfaces 188
A) Hopf surfaces of class 1 188
B) Hopf surfaces of class 0 191
§ 10. Surfaces of Bombieri, Hirzebruch, and Inoue 196

§ 1. Introduction

This paper is an investigation of the Kähler condition for a compact complex manifold. The first main result is a characterization of those compact complex manifolds which admit Kähler metrics. The main idea of this characterization is the following. Recall that the Kähler form ω of any hermitian metric on a complex manifold X has the property that it restricts to be the volume form on every complex curve. Hence, if the metric is Kählerian, i.e., if $d\omega=0$, then no compact complex curve can be a boundary in X. More generally, it is true that each positive current T of bidimension $1,1$ has mass (or "weighted volume") equal to $T(\omega)$, and hence cannot be the $(1,1)$-component of a boundary unless $T=0$. (More specifically, if $T=\pi_{1,1}dS$, then $\mathbf{M}(T)=T(\omega)=(\pi_{1,1}dS)(\omega)=(dS)(\omega)=S(d\omega)=0$, and so $T=0$.) The first main theorem of this paper asserts that this intrinsic condition actually characterizes Kähler manifolds. That is, *if a compact manifold carries no positive* $(1,1)$*-components of boundaries, then*

* Research supported by NSF Grants MPS 75-05270 and MCS 8301365

the manifold supports a Kähler metric. It should be noted that the question: "Which complex manifolds admit a Kähler metric?" is equivalent to asking: "Which complex manifolds can be calibrated?" This is a natural question to arise in the context of Calibrated Geometries (Harvey-Lawson [7]).

The key technical result required for the proof of this characterization theorem is established in Sect. 2. Here it is proved that the space of (1, 1)-components of boundaries is a closed subspace of the space of all (1, 1)-currents on a compact manifold.

A brief description of positive currents of bidimension 1, 1 is given in Sect. 3. The Characterization Theorem is proved in Sect. 4.

The remainder of the paper is concerned with improvements on, and applications of, the Characterization Theorem. In Sect. 5, we enunciate a set of problems aimed at relaxing the condition we have given for a manifold to be Kähler (and thereby strengthening the consequences of being non-Kähler). Examples of solutions to these problems are given in subsequent sections of the paper.

In Sect. 6 we examine the extent to which the condition of being Kähler persists under twisted products. For example, we show the following. *Let X be compact and suppose $f\colon X \to Y$ is a holomorphic submersion with 1-dimensional fibres onto a Kähler manifold Y. Then there exists a Kähler metric on X if and only if the fibre of f is not a (1, 1)-component of a boundary.* Examples where the fiber bounds are provided by the Calabi-Eckmann manifolds $S^1 \times S^{2n+1} \xrightarrow{f} \mathbb{P}^n(\mathbb{C})$. The above condition on the fibre of f is equivalent to the requirement that for any volume form ω on Y, the pullback $f^*\omega$ is not the (1,1)-component of an exact form on X.

When X has dimension two, the result above can be substantially strenthened. In Sect. 7 the following is proved.

Theorem. *Let X be an elliptic surface. Then the following are equivalent.*
(1) *X admits a Kähler metric.*
(2) *The first Betti number of X is even.*
(3) *The general fibre of X does not bound.*

By the "general fibre of X" we mean any non-singular fibre of the elliptic fibration $f\colon X \to C$. Condition (3) means that the class of the fibre in $H_2(X; \mathbb{R})$ is not zero. This condition is equivalent to the condition that the map $f^*\colon H^2(C; \mathbb{R}) \to H^2(X; \mathbb{R})$ is not zero. The equivalence of (1) and (2) in the Theorem above is a result of Miyaoka [10].

It would be interesting to find a direct proof that a compact complex surface with even first Betti number carries no (non-trivial) positive currents T which are (1, 1)-components of boundaries. For two different classes of currents we give such a direct proof. First, for currents T which are smooth such a direct proof is given in Proposition 37 and the preceeding paragraph. Second, for currents T which are d-closed Theorem 26 provides a direct proof. The result just mentioned (Theorem 26) raises the following question.

Question. If T is a positive (1, 1) component of a boundary on a compact complex surface, then is T necessarily d-closed?

An affirmative answer to this question combined with Theorem 26 would have as a consequence the theorem ([15] and [13]):

A surface with even first Betti number admits a Kähler metric,

without taking special note of K3 surfaces and relying on Kodaira's classification.

It is quite important to note that our Characterization Theorem has a positive consequence for non-Kähler manifolds. It asserts that any non-Kähler manifold carries an analytic object analogous to the one carried by a Kähler manifold (the Kähler form). Namely, *any non-Kähler manifold carries a positive bidimension* $(1, 1)$*-current which is the* $(1, 1)$*-component of a boundary.*

These objects appear to be intimately related to the geometry of a non-Kähler manifold. In fact, on any complex manifold X, there are three cones of particular interest. Let $P(X)$ denote the convex cone of positive $(1, 1)$-currents on X. Then set

$$P_{\mathrm{bdy}_{1,1}}(X) \equiv \{T \in P(X)\colon T \text{ is a } (1,1)\text{-component of a boundary}\},$$

$$P_{\mathrm{bdy}}(X) \equiv \{T \in P(X)\colon T \text{ is a boundary}\},$$

$$P_{\mathrm{closed}}(X) \equiv \{T \in P(X)\colon dT = 0\}.$$

The behavior of these cones under meromorphic transformations is an interesting phenomenon to study.

One can begin to see the relationship of these cones to the geometry of a surface X from the following fact (Theorem 43). Let $\mathscr{P}^{\infty}_{\mathrm{bdy}_{1,1}}(X)$ denote the set of smooth currents (i.e., C^{∞} forms) in $P_{\mathrm{bdy}_{1,1}}(X)$. Then the *intrinsically defined open set* $B(X) = \{x \in X\colon \phi_x \neq 0$ for some $\phi \in \mathscr{P}^{\infty}_{\mathrm{bdy}_{1,1}}(X)\}$ carries a complex analytic foliation $\mathscr{F}(X)$ intrinsically defined by $\mathscr{P}^{\infty}_{\mathrm{bdy}_{1,1}}(X)$.

Motivated by examples, we introduce the concept of the *Kähler rank* of a surface X. If X is Kähler then the *Kähler rank* is 2. If the complement of the foliation set $B(X)$ is a curve in X then the *Kähler rank* is 1. Otherwise the *Kähler rank* is zero. In particular, the *Kähler rank* of an elliptic non-Kähler surface is one.

In §§ 9 and 10 we investigate the Kähler rank and the cones defined above on each of the known non-Kähler surfaces. The results are quite interesting. For example, when X is an elliptic Hopf surface the three cones coincide and are exactly the set of positive currents for the elliptic fibration. For Hopf surfaces which are not elliptic there are two classes to consider denoted class 1 and class 0. For class 1 the elliptic fibration is replaced with the distinguished complex analytic foliation $\mathscr{F}(X)$ given above, and the cones are again just the foliation currents for $\mathscr{F}(X)$. In this case $B(X) = X$ so that all of X is foliated. For a Hopf surface of class 0 we have succeeded in showing that the only d-closed positive current on X is a distinguished torus (and its positive multiples). Although we have shown that $P_{\mathrm{bdy}_{1,1}}(X)$ contains no smooth currents its general structure remains unknown for class 0. In particular, Hopf surfaces of class 0 are the only known surfaces of Kähler rank zero.

The non-elliptic Hopf surfaces provide examples of the following phenomenon. Those of class 1 are "smoothly non-Kähler" but not "geometrically non-

Kähler" (in the sense that there exist a smooth positive form that bounds but no positive combination of complex curves that bounds). On the other hand, those of class 0 are "geometrically non-Kähler" but not "smoothly non-Kähler" (in the sense that there exists a complex curve that bounds but no smooth positive form that bounds).

Note that in describing the cone $P_{\text{closed}}(X)=P_{\text{bdy}}(X)$ on a Hopf surface X, we are, in particular, describing all complex curves and all closed positive C^∞ $(1,1)$-forms on X.

Similar remarks apply to our analyses of these cones on Inoue surfaces and Inoue-Hirzebruch surfaces. We refer the reader to § 10.

We would like to remark that an important inspiration for this work was the work of Dennis Sullivan [14].

For complex dimension 3 or more an interesting, strictly larger, class of complex manifolds (co-Kähler) has been characterized by M.L. Michelsohn [9].

§ 2. $d_{1,1}$ has closed range

Let $\mathscr{H}$ denote the sheaf of germs of pluriharmonic functions (i.e. locally real parts of holomorphic functions) on a complex manifold X. Define $d^c \equiv i(\partial-\bar\partial)$ the conjugate differential and recall that the exterior derivative $d=\partial+\bar\partial$. Thus $\bar\partial=\frac{1}{2}(d+i\,d^c)$ and $\partial=\frac{1}{2}(d-i\,d^c)$. The operators d and d^c are real and hence $dd^c u=0$ for any pluriharmonic function u. Conversely, if $dd^c u=0$ on a simply connected manifold X, then u is the real part of a holomorphic function $f=u+iv$ on X. First note that a 0,1-form α vanishes if and only if the imaginary part $\frac{1}{2i}(\alpha-\bar\alpha)$ vanishes. Thus $\bar\partial(u+iv)=0$ if and only if

$$0=\operatorname{Im}\bar\partial(u+iv)=\operatorname{Im}\tfrac{1}{2}(d+i\,d^c)(u+iv)=\tfrac{1}{2}(dv+d^c u).$$

Consequently, a solution v to the equation

$$dv=-d^c u$$

provides the holomorphic function $f\equiv u+iv$.

More is true.

(1) **Proposition.**

$$0\to\mathscr{H}\to\mathscr{E}_{\mathbb{R}}\xrightarrow{dd^c}\mathscr{E}^{1,1}_{\mathbb{R}}\xrightarrow{d}[\mathscr{E}^{1,2}\oplus\mathscr{E}^{2,1}]_{\mathbb{R}}\xrightarrow{d}[\mathscr{E}^{1,3}\oplus\mathscr{E}^{2,2}\oplus\mathscr{E}^{3,1}]_{\mathbb{R}}\xrightarrow{d}\dots$$

is a fine resolution of the sheaf of germs of pluriharmonic functions.

Here $\mathscr{E}^{p,q}$ denotes the sheaf of germs of smooth forms of bidegree p,q and the subscript $\mathbb{R}$ denotes germs of real valued forms.

The proof is standard and omitted. Exactness at $\mathscr{E}_{\mathbb{R}}$ was noted above.

Remark. It is of some interest that the adjoint sequence is not locally exact, even for dimension 2.

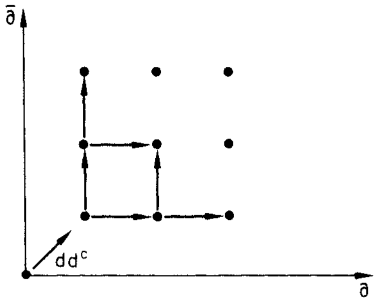

Fig. 1

(2) **Corollary.**

a) $H^1(X, \mathscr{H}) = \{\psi \in \mathscr{E}^{1,1}(X)_{\mathbb{R}}\colon d\psi = 0\}/dd^c \mathscr{E}(X)_{\mathbb{R}}$.

b) $H^2(X, \mathscr{H}) = \{\psi \in [\mathscr{E}^{1,2}(X) \oplus \mathscr{E}^{2,1}(X)]_{\mathbb{R}}\colon d\psi = 0\}/d\mathscr{E}^{1,1}(X)_{\mathbb{R}}$.

These cohomology groups are finite dimensional vector spaces if X is compact.

(3) **Proposition.** *If X is a compact complex manifold, then $H^p(X, \mathscr{H})$ is finite dimensional for $p = 0, 1, \ldots$.*

Proof. The exact sequence of sheaves: $0 \to \mathbb{R} \to \mathcal{O} \xrightarrow{\mathrm{Re}} \mathscr{H} \to 0$ gives rise to a long exact sequence of cohomology groups. The result is thus a direct consequence of the well established finite dimensionality of $H^*(X; \mathbb{R})$ and $H^*(X; \mathcal{O})$.

In particular, combining Corollary 2b) with Proposition 3 this proves that the image of

$$d\colon \mathscr{E}^{1,1}(X)_{\mathbb{R}} \to Z(X) \equiv \{\psi \in [\mathscr{E}^{1,2}(X) \oplus \mathscr{E}^{2,1}(X)]_{\mathbb{R}}\colon d\psi = 0\}$$

has finite codimension. Therefore, by a standard consequence of the open mapping theorem the image of d in $Z(X)$ is closed. This proves the next result.

(4) **Lemma.**

$$d\colon \mathscr{E}^{1,1}(X)_{\mathbb{R}} \to [\mathscr{E}^{1,2}(X) \oplus \mathscr{E}^{2,1}(X)]_{\mathbb{R}} \text{ has closed range.}$$

This Lemma can be dualized as follows. Let $\mathscr{E}'_{p,q}(X)$ denote the dual space of the Frechet space $\mathscr{E}^{p,q}(X)$. An element $T \in \mathscr{E}'_{p,q}(X)$ has compact support and dimension $p+q$; it is called a *current of bidimension p, q*. As before the subscript $\mathbb{R}$ indicates that only real currents are included. Now we compute the adjoint of the operator d in Lemma 4. The domain $\mathscr{E}^{1,1}(X)_{\mathbb{R}}$ has dual space $\mathscr{E}'_{1,1}(X)_{\mathbb{R}}$, and the target $[\mathscr{E}^{1,2}(X) \oplus \mathscr{E}^{2,1}(X)]_{\mathbb{R}}$ has dual space $[\mathscr{E}'_{1,2}(X) \oplus \mathscr{E}'_{2,1}(X)]_{\mathbb{R}}$. Let π denote the natural projection

$$\pi\colon \mathscr{E}'_2(X)_{\mathbb{R}} \to \mathscr{E}'_{1,1}(X)_{\mathbb{R}}$$

sending a current T of dimension 2 to the component $T_{1,1}$ of bidimension 1, 1 (in general $T=T_{2,0}+T_{1,1}+T_{0,2}$). If $\psi\in\mathscr{E}^{1,1}(X)_{\mathbb{R}}$ and $S\in[\mathscr{E}'_{1,2}(X)\oplus\mathscr{E}'_{2,1}(X)]_{\mathbb{R}}$ then

$$(S, d\psi)=(dS, \psi)=((\pi\circ d)(S), \psi).$$

Thus $\pi\circ d$ is the adjoint of the operator d in Lemma 4.

(5) **Definition.** The differential operator

$$d_{1,1}\colon [\mathscr{E}'_{1,2}(X)\oplus\mathscr{E}'_{2,1}(X)]_{\mathbb{R}}\to\mathscr{E}'_{1,1}(X)_{\mathbb{R}}$$

is defined by $d_{1,1}\equiv\pi\circ d$ restricted to $[\mathscr{E}'_{1,2}(X)\oplus\mathscr{E}'_{2,1}(X)]_{\mathbb{R}}$.

(6) **Lemma.** *The operator*

$$d_{1,1}\colon [\mathscr{E}'_{1,2}(X)\oplus\mathscr{E}'_{2,1}(X)]_{\mathbb{R}}\to\mathscr{E}'_{1,1}(X)_{\mathbb{R}}$$

has closed range.

Proof. As noted above $d_{1,1}$ is the adjoint operator to the operator

$$d\colon \mathscr{E}^{1,1}(X)_{\mathbb{R}}\to[\mathscr{E}^{1,2}(X)\oplus\mathscr{E}^{2,1}(X)]_{\mathbb{R}}$$

from a Frêchet space to a Frêchet space. Hence the closed range theorem is applicable (see Schaefer [12], Sect. 7 of Chap. IV). This theorem says that the adjoint of a map with closed range has closed range, so that Lemma 6 is a consequence of Lemma 4.

§ 3. Positive currents of bidimension 1, 1

Suppose X is a complex manifold. The space of compactly supported currents $\mathscr{E}'_{1,1}(X)$ of bidimension 1, 1 consists of those currents T which can be locally expressed as

$$T=\sum_{j,k} T^{jk}\frac{i}{2}\,\partial/\partial z^j\wedge\partial/\partial\bar{z}^k,$$

where each T^{jk} is a distribution. Such a current T is said to be *positive* if $\sum T^{jk}w_j\bar{w}_k$ is a non-negative measure for each $w\in\mathbb{C}^n$. This definition is independent of the choice of coordinates. One can show (e.g. Harvey [4] Lemma 1.20) that if T is positive then each T^{jk} is a Radon measure. Let $\mathscr{M}^{\mathrm{cpt}}_{1,1}(X)$ denote the subspace of $\mathscr{E}'_{1,1}(X)$ consisting of those currents T whose coefficients T^{jk} are Radon measures. Let $P_{1,1}(X)$ denote the cone of positive currents in $\mathscr{M}^{\mathrm{cpt}}_{1,1}(X)$. The structure of currents with measure coefficients is very pretty. First, consider an (auxiliary) hermitian metric on X. Then $G_{\mathbb{C}}(1, T_xX)$, the grassmannian of complex 1-dimensional subspaces of the tangent space to X, can be considered as the compact submanifold of $\Lambda^2 T_xX$ consisting of unit simple 2 vectors of bidegree 1, 1. Similarly the grassmannian $G(2, T_xX)$ of oriented real two-dimensional subspaces of T_xX is contained in $\Lambda^2 T_xX$.

Now if T has measure coefficients there exists a non-negative Radon measure $\|T\|$ called the *total variation measure of* T and a 2-vector field $\vec{T}$, which is

$\|T\|$-measurable, such that $T = \|T\| \vec{T}$. (That is, $T(\phi) = \int \phi(\vec{T_x})\, d\|T\|(x)$ for any exterior 2-form ϕ on X.) Furthermore, for $\|T\|$ a.e. x the 2-vector $\vec{T}(x)$ belongs to the boundary of the convex hull of $G(2, T_x X)$. The *mass* of T, denoted $M(T)$, is defined to be the total variation measure $\|T\|(X)$. The reader is referred to Federer [2] for more details. One can show without difficulty that

(7) *A 2-dimensional current T is positive if and only if $\vec{T}(x)$ belongs to the convex hull of $G_{\mathbb{C}}(1, T_x X)$ for $\|T\|$ – a.e. x.*

That is, T is positive iff $\vec{T}(x)$ is a positive 1, 1 vector $\|T\|$ - almost everywhere.

An important class of positive currents is the set of finite sums $T = \sum n_j [V_j]$ where each V_j is a compact 1-dimensional complex subvariety of X and each n_j is a positive integer. Such currents are called (positive) *holomorphic chains.*

§ 4. Kähler manifolds

Suppose X is a Kähler manifold with Kähler form ω.

(8) **Theorem** (Wirtinger's Inequality). *The inequality $\omega(\xi) \leqq 1$ holds for all ξ in the convex hull of the real grassmannian $G(2, T_x X)$, and equality is attained if and only if ξ lies in the convex hull of the complex grassmannian $G_{\mathbb{C}}(1, T_x X)$.* For the proof see Federer [2] or Harvey [4]. In particular, because of (7), if $T \in P_{1,1}(X)$ is positive of bidimension 1, 1, then

$$T(\omega) = \int_X \omega(\vec{T})\, \|T\| = \int_X \|T\| = M(T).$$

A Kähler form ω is characterized by the following three properties:

(9) $$\omega \in \mathscr{E}^{1,1}(X)_{\mathbb{R}},$$

(10) $$d\omega = 0,$$

(11) $$T(\omega) = M(T) \quad \text{for each } T \in P_{1,1}(X).$$

These properties can be dualized as follows. Consider

$$d_{1,1}\colon\ [\mathscr{E}'_{1,2}(X) \oplus \mathscr{E}'_{2,1}(X)]_{\mathbb{R}} \to \mathscr{E}'_{1,1}(X)_{\mathbb{R}}$$

as in Sect. 2. Let $B_{1,1}(X)$ denote the range of $d_{1,1}$. That is $T \in B_{1,1}(X)$ if and only if T is the bidimension 1,1 component of a boundary dS with $S \in \mathscr{E}'_3(X)$. Now condition (10) that $d\omega = 0$ can be reformulated as

(10)′ $$\omega \text{ vanishes on } B_{1,1}(X),$$

since ω vanishes on $d\mathscr{E}'_3(X)$ if and only if ω vanishes on $B_{1,1}(X)$. Note that $d\mathscr{E}'_3(X)_{\mathbb{R}} \cap \mathscr{E}'_{1,1}(X)_{\mathbb{R}}$ is a proper subspace of $B_{1,1}(X)$. The property (11) contains information independent of the hermitian metric on X

(11)′ $$T(\omega) > 0 \quad \text{for each } T \in P_{1,1}(X) \text{ with } T \neq 0.$$

(12) **Proposition.** *Suppose X admits a Kähler metric. Then:*

(13) $$P_{1,1}(X) \cap B_{1,1}(X) = \{0\}.$$

That is, there are no positive currents with compact support which are bidimension 1, 1 *components of boundaries.*

Proof. Let ω denote a Kähler form on X. Conditions (10)′ and (11)′ say that ω is zero on $B_{1,1}(X)$ and strictly positive on $P_{1,1}(X) - \{0\}$.

If the manifold is compact then (13) characterizes Kähler manifolds.

(14) **Theorem.** *Suppose X is a compact complex manifold. If there are no (non-trivial) positive currents which are bidimension* 1, 1 *components of boundaries, then there exists a Kahler metric for X.*

Proof. The hypothesis is that $P_{1,1}(X) \cap B_{1,1}(X) = \{0\}$. Choose any hermitian metric h on X and let $\psi \equiv -\operatorname{Im} h$. Then $\psi \in \mathscr{E}^{1,1}(X)_{\mathbb{R}}$ and

(15) $$K \equiv \{T \in P_{1,1}(X) \colon \ T(\psi) = 1\}$$

is a compact base for the cone $P_{1,1}(X)$. More precisely, K is weakly compact in the space of bidimension 1, 1 currents with measure coefficients, and hence K is weakly compact in $\mathscr{E}'_{1,1}(X)_{\mathbb{R}}$. Lemma 6 says that the image of $d_{1,1}$, namely $B_{1,1}(X)$, is a weakly closed subspace of $\mathscr{E}'_{1,1}(X)_{\mathbb{R}}$. Thus the Hahn-Banach separation Theorem is applicable (see Schaefer [12], p. 65). This Theorem implies that there exists a form $\omega \in \mathscr{E}^{1,1}(X)_{\mathbb{R}}$ which is zero on the subspace $B_{1,1}(X) \subset \mathscr{E}'_{1,1}(X)_{\mathbb{R}}$ and strictly positive on K. Hence (10)′ and (11)′ are valid. Now $\omega(B_{1,1}) = 0$ implies $d\omega = 0$. Furthermore, choose $\vec{T} \in G_{\mathbb{C}}(1,n) \subset \Lambda^{1,1} T_x X$ and set $T \equiv \delta_x \vec{T}$. Then $T \in K$ and hence $\omega(\delta_x \vec{T}) > 0$. Consequently, the 1, 1 form ω has rank n at each point x, and is positive. This completes the proof that ω is a Kähler form.

Remark. Suppose X is a compact complex manifold. Sullivan [14] noted that there are no (non-trivial) positive 2-currents which bound if and only if there exists a closed smooth 2-form whose (1, 1)-component is positive definite.

§ 5. Some general problems

It is very useful, whenever possible, to relax the condition that we have given for a manifold to be Kähler. This, of course, strengthens the consequences of being non-Kähler. There are several natural ways to relax our condition. They lead to the formulation of some interesting problems in complex geometry. We state these problems here, and in subsequent sections we shall solve them for certain interesting classes of manifolds.

Recall (cf. [4]) that a *positive holomorphic* 1-*chain* on a complex manifold X is a finite sum $\sum n_j [C_j]$ where each n_j is a positive integer and each C_j is a complex curve in X. The positive holomorphic 1-chains are examples of d-closed, positive (1, 1)-currents.

We shall say that a complex manifold X has *Property NK* if there exists a non-trivial positive current which is the bidimension-$(1,1)$ component of a boundary. Of course, our main theorem asserts that for compact manifolds, Property NK is equivalent to being non-Kähler.

Problem 1. Describe general classes of complex manifolds for which Property NK implies the existence of a holomorphic 1-chain

(a) which bounds, or, at least,

(b) which is the $(1,1)$-component of a boundary.

Manifolds in such a class will be Kähler if and only if they carry no holomorphic 1-chains which bound (in case (a)) or are the $(1,1)$-components of boundaries (in case (b)).

It is also natural to reduce our criterion to the class of smooth differential forms.

Problem 2. Describe general classes of complex manifolds for which Property NK implies the existence of a (non-trivial) smooth positive bidimension-$(1,1)$ current

(a) which bounds or, at least,

(b) which is the $(1,1)$-component of a boundary.

To test whether a given manifold in such a class is Kähler it suffices to check the pointwise non-negative, smooth $(n-1,n-1)$-forms, to see if one is a boundary.

In general, the set of positive $(1,1)$-currents which are boundaries (or nearly boundaries) constitute an important analytic-geometric object on a complex manifold. The essential relationship with the geometry of the manifold will be clear when we discuss non-Kähler surfaces in detail below. We pose the following general problem.

Problem 3. On a given complex manifold, describe all of the positive bidimension-$(1,1)$ currents which

(a) bound

(b) are the $(1,1)$-components of boundaries

(c) are d-closed.

Part (c) is, of course, a generalization of the problem of describing all the complex curves in a given complex manifold.

Later we will use the notations

$$P_{\mathrm{bdy}}(X),\ P_{\mathrm{bdy}_{1,1}}(X) \text{ and } P_{\mathrm{closed}}(X) \tag{16}$$

for the set of currents in (a), (b), and (c), respectively.

§ 6. Non-singular families of curves

In this section we shall treat a class of manifolds where Problems 1, 2 and 3 above can be handled. This is the class of manifolds which fibre over a manifolds of one-lower dimension. The first main result allows one to build Kähler manifolds inductively by a sequence of holomorphic submersions.

We assume that X is a compact connected complex manifold.

(17) **Theorem.** *Suppose $f\colon X \to Y$ is a holomorphic submersion with 1-dimensional fibres onto a Kähler manifold Y. Then there exists a Kähler metric on X if and only if the fibre of f is not a* $(1,1)$*-component of a boundary.*

Note A. Any two fibres of f are homologous. Hence, if one fibre is a $(1,1)$-component of a boundary, then they all are. That is, if $F_p \equiv f^{-1}(p) = \pi_{1,1} dS$, then for any $q \in Y$, $F_q = F_p + dS_0 = \pi_{1,1} dS + dS_0 = \pi_{1,1} d(S+S_0)$.

Note B. This condition on the fibres of the submersion is a necessary assumption. The Hopf surface $X = (\mathbb{C}^2 - \{0\})/2\mathbb{Z} \simeq S^1 \times S^3$ admits a holomorphic submersion onto $\mathbb{P}^1(\mathbb{C})$ with Kähler fibres (tori), however, X is certainly not a Kähler surface. Of course, in this case the fibres are actually boundaries.

Note C. The argument of Note A above actually proves that in Theorem 17 the condition on the fibre can be replaced by the corresponding condition on any $(1,1)$-cycle homologous to the fibre. Let $\omega \in \mathscr{E}^{n-1,n-1}(Y)$ be a smooth volume form on Y of total integral 1. Then the pull-back $f^*\omega$, considered as a $(1,1)$-current on X, is homologous to the fibre (which is essentially the pull-back of the δ-function). Consequently, Theorem 17 can be restated as follows.

$(17)^\infty$ **Theorem.** *Suppose $f\colon X \to Y$ is as in Theorem* 17. *Then there exists a Kähler metric on X if and only if the pull-back of a volume form on Y is not the* $(1,1)$*-component of a boundary.*

Note D. The question remains whether X (as in Thm. 17) is Kähler if and only if the fibre does not bound. When $\dim_{\mathbb{C}}(X) = 2$, this is true, as we shall see in § 7.

Proof. Suppose X is not Kähler. Then by Theorem 14 there exists a positive current T of bidimension $1,1$ on X which is the $1,1$ component of a boundary, say S. That is $T = (dS)_{1,1}$.

The push forward f_* maps currents of bidimension p,q to currents of bidimension p,q, and maps positive currents to positive currents. Therefore $f_*(T)$ is a positive current of bidimension $1,1$ which is the $1,1$ component of the boundary $d(f_*(S))$. Since Y is assumed to be Kähler, Theorem 14 implies that $f_*(T)$ must vanish.

The next lemma is needed to complete the proof of the Theorem.

(18) **Lemma.** *Suppose $f\colon X \to Y$ is a holomorphic submersion with one-dimensional fibres, and suppose T is a positive current of bidimension* $1,1$ *on X. Then the push-forward $f_*(T)$ of T to Y is zero if and only if $T = \|T\|\,\vec{F}$ where $\vec{F}$ is the field of unit 2-vectors tangent to the fibres. If, in addition, T satisfies the equation $\partial\bar{\partial}T = 0$, then*

$$T = f^*(\mu)$$

for some non-negative density μ on Y.

Proof. Suppose $T = \|T\|\,\vec{F}$. Then for any 2-form ϕ on Y, we have $(f^*\phi)(\vec{F}) \equiv 0$. Hence, $(f_* T)(\phi) = T(f^*\phi) = 0$ and we conclude that $f_* T = 0$.

Conversely, suppose $f_* T = 0$ and write $T = \|T\|\,\vec{T}$ where $\vec{T}$ is the associated field of positive $(1,1)$-vectors. Then for any positive $(1,1)$-form ω on Y we have that

$$(f_* T)(\omega) = \int_X (f^* \omega)(\vec{T})\, d\|T\| = 0.$$

From positivity we know that $(f^*\omega)(\vec{T}) \geqq 0$. Consequently, we conclude that $(f^*\omega)(\vec{T}) = 0$ $\|T\|$-almost everywhere. Since this holds for all positive (1, 1)-forms on Y, it is easy to see that $\vec{T} = \vec{F}$, $\|T\|$-a.e., as claimed. This argument is as follows. Fix a point $x \in X$ and choose an isomorphism $T_x X \xrightarrow{\approx} \mathbb{C}^n$ so that the differential f_* becomes the linear projection $f_*(z_1, \ldots, z_n) = (z_2, \ldots, z_n)$. Then $\vec{T}_x$ can be written as $\vec{T}_x = \sum \vec{T}^{jk} \frac{1}{2i}\left(\frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \bar{z}_k}\right)$ where $\vec{T}^{jk}$ is a hermitian positive semi-definite matrix. Choose any covector $a = a_2 dz_2 + \ldots + a_n dz_n$ and set $\omega = \frac{1}{2i} a \wedge \bar{a}$. (Note that ω is of the form $f^*\omega_0$ for $\omega_0 \in T_{fx} Y$.) Then $\omega(\vec{T}_x) = \sum a_j \vec{T}^{jk} \bar{a}_k = 0$ for all such a. We conclude, since $\vec{T}^{jk}$ is semi-definite, that $\vec{T}^{jk} = 0$ for $j \geqq 2$ or $k \geqq 2$, that is, $\vec{T}_x = \frac{i}{2}\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1}\right) = \vec{F}_x$ as claimed.

Recall now that any positive (1, 1)-current can be thought of as an exterior $(n-1, n-1)$-form with measure coefficients. Under this correspondence, a current of type $T = \|T\|\, \vec{F}$ is written as

$$T = \|T\|\, f^*(\Omega)$$

where Ω is a smooth volume form on Y. The equation $\partial\bar{\partial} T = 0$ in this case becomes the equation

$$(\partial\bar{\partial}\|T\|) \wedge f^*(\Omega) = 0.$$

This implies that, in the fiber directions, the measure $\|T\|$ is harmonic. Since the fibres are compact, we conclude that $\|T\|$ is independent of fibre coordinates, i.e., $\|T\|$ is the pull-back of a measure μ_0 on Y. Thus $T = f^*(\mu_0 \Omega) = f^*(\mu)$, and the lemma is proved.

We now complete the proof of Theorem 17. As before, if X is not Kähler, then there exists a positive (1, 1)-current T with $T = d_{1,1} S$ for some 3-dimensional current S. Since f is holomorphic, we have that $f_* T = d_{1,1}(f_* S)$. Thus, since Y is Kähler, we conclude that $f_* T = 0$. Lemma 18 then implies that $T = f^*(\mu)$ for some non-negative density μ on Y. Let $c = \int_Y \mu = \|T\|$, and recall that any two densities with the same total mass are homologous on Y. Hence, for any point $y \in Y$, the Dirac density δ_y at y has the property that

$$c\,\delta_y - \mu = dR$$

for some current R on Y. Pulling back by f we have that

$$c[f^{-1}(y)] - T = df^*(R).$$

Therefore, the fibre $[f^{-1}(y)]$ is the (1, 1)-component of a boundary. This completes the proof.

The next corollary to Lemma 18 is an example of a solution to Problem 3b) above.

(19) **Theorem.** *Suppose f: $X \to Y$ is a holomorphic submersion, with 1-dimensional fibers, of a non-Kähler manifold X onto a Kähler manifold Y. Then the cone of all positive currents which are* 1, 1 *components of boundaries is equal to* $\{T\colon T = f^*(\mu)$ *for some non-negative density* μ *on* $Y\}$.

Note that on the non-Kähler manifold X, this cone recaptures the submersion Y intrinsically.

§ 7. Elliptic surfaces

In this section we shall give a complete answer to Problems 1, 2 and 3 for elliptic surfaces. Recall that an *elliptic surface* is a compact complex surface X which admits a holomorphic map f: $X \to Y$ onto a curve Y such that for almost all $y \in Y$, the fibre $f^{-1}(y)$ is a non-singular elliptic curve. Note that f need not be a submersion (it can have singular fibres), so the theorems of the last section do not immediately apply. Nevertheless, dimension 2 is sufficiently special that quite strong results can be proved. In the process we will establish some useful general propositions on complex surfaces.

Our first main result is the following.

(20) **Theorem.** *For a compact elliptic surface f: $X \to Y$ the following are equivalent.*

(1) *X admits a Kähler metric.*
(2) *The first Betti number of X is even.*
(3) *The general fibre of f does not bound in $H_2(X, \mathbb{R})$.*

The equivalence of (1) and (2) is a Theorem of Miyaoka [10]. The argument here is quite different from that of [10] and establishes the further equivalence: (1)$\Leftrightarrow$(3).

Note that, as seen in §6, condition (3) is equivalent to

(3)′ *The pull-back $f^*\omega$ of a volume form ω on Y is not exact on X.*

Proof of Theorem 20. The proof will be given in several stages. We begin by establishing some results for arbitrary (not necessarily elliptic) surfaces.

Suppose X is a compact complex surface. Let $q \equiv h^{0,1} \equiv \dim_{\mathbb{C}} H^1(X; \mathcal{O})$ denote the *irregularity* of X, and let $b_1 \equiv \dim_{\mathbb{R}} H^1(X; \mathbb{R})$ denote the *first Betti number* of X. Kodaira [8] has shown that:

(21) $$\Delta \equiv 2q - b_1 = \text{either } 0 \text{ or } 1,$$

and consequently

(22) $$\Delta = 0 \Leftrightarrow b_1 \text{ is even}.$$

Of course, if X is Kähler, then b_1 is even (and so $\Delta = 0$). Our first step towards proving the converse is the following.

(23) *Proposition. Suppose X has even first Betti number. Then there are no (non-trivial) positive* 1, 1 *currents that bound.*

Proof. As noted above, the exact sequence

$$0 \to \mathbb{R} \to \mathcal{O} \xrightarrow{\mathrm{Re}} \mathscr{H} \to 0$$

of sheaves induces the long exact sequence:

$$0 \to H^1(\mathbb{R}) \to H^1(\mathcal{O}) \to H^1(\mathscr{H}) \to H^2(\mathbb{R}) \to H^2(\mathcal{O}) \dots.$$

Note that $0 \to H^1(\mathbb{R}) \to H^1(\mathcal{O})$ is surjective if and only if $2q = b_1$. Therefore, by (21) and (22),

(24) $$H^1(\mathscr{H}) \to H^2(\mathbb{R}) \textit{ is injective} \Leftrightarrow b_1 \textit{ is even.}$$

Recall that $H^1(\mathscr{H}) \cong \{T \in \mathscr{E}'_{1,1}(X)_{\mathbb{R}}\colon\ dT = 0\}/i\partial\bar\partial\mathscr{E}'(X)_{\mathbb{R}}$. Now suppose that $T \in \mathscr{E}'_{1,1}(X)_{\mathbb{R}}$ and that $T = dS$. Then T determines a class in $H^1(\mathscr{H})$ and the image of that class in $H^2(\mathbb{R})$ is zero. Hence, the hypothesis that b_1 is even implies that the class determined by T in $H^1(\mathscr{H})$ must vanish. That is, $T = i\partial\bar\partial\phi$ for some $\phi \in \mathscr{E}'(X)$. If T is positive then ϕ is plurisubharmonic and hence constant by the maximum principal. Thus, $T = 0$ and the proof is complete.

The Proposition can be strengthened by the following observation.

(25) **Proposition.** *Suppose that T is a real $(1, 1)$-current on a complex surface X, and that $T = (dS)_{1,1}$ for some real current S. If $dT = 0$, then, in fact, $T = dS$.*

Proof. $T = \bar\partial S^{1,0} + \partial S^{0,1}$ and since $dT = 0$, $\partial S^{1,0}$ is a holomorphic 2-form. Hence

$$0 = \int d(S^{1,0} \wedge \overline{\partial S^{1,0}}) = \int \partial S^{1,0} \wedge \overline{\partial S^{1,0}}$$

which proves that $\partial S^{1,0} = 0$.

Combined with Proposition 23 we have proved the following.

(26) **Theorem.** *Suppose X has even first Betti number. If T is a positive current on X which is the $1, 1$ component of a boundary and if $dT = 0$, then $T = 0$.*

See Theorem 38 for a dual interpretation.

On an elliptic surface the positive currents which are $(1, 1)$-components of boundaries can be completely described.

(27) **Proposition.** *Suppose $X \xrightarrow{f} Y$ is an elliptic fibration. Then each positive current T which is the $(1, 1)$-component of a boundary is of the form*

(28) $$T = f^*(\mu)$$

for some non-negative measure (density) μ on Y.

Note. The current $f^*(\mu)$ in Eq. (28) is defined by first restricting to the regular points, where f is submersive. Here $f^*(\mu)$ is well-defined and gives a positive current of finite mass. We then take the natural extension of this current to all of X.

Since any current of type (28) is d-closed, Proposition 27 enables us to apply Theorem 26. This proves that an elliptic surface with even first Betti number carries no positive (1, 1)-components of boundaries. Applying our main result (Theorem 14) then proves Miyaoka Theorem:

(29) **Corollary.** *An elliptic surface is Kähler if and only if b_1 is even.*

A current of the form (28) is cohomologous to $c[F]$ where F is any non-singular fibre and $c=\int_Y \mu$. Thus we also conclude:

(30) **Corollary.** *An elliptic surface is Kähler if and only if the generic fibre does not bound (in real homology).*

These two corollaries constitute Theorem 20 above. It remains only to establish the proposition.

Proof of Proposition (27). The proof of Lemma 18 shows that for some non-negative density μ on Y, the current $T-f^*(\mu)$ is positive and supported in the singular fibres of f. This current is also $\partial\bar{\partial}$-closed. Consequently, by a local calculation (the next lemma), $T-f^*(\mu)$ is of the form

$$T-f^*(\mu)=\sum_j c_j[C_j] \tag{31}$$

where each c_j is a positive constant, and each C_j is an irreducible component of a singular fibre.

(32) **Lemma.** *Suppose T is a positive bidimension* (1, 1)*-current on a complex manifold, and assume that $\partial\bar{\partial}T=0$. If T is supported in a complex curve C, then T can be written as a sum*

$$T=\sum h_j[C_j]$$

where each C_j is an irreducible component of C and each h_j is a non-negative harmonic function on C_j (i.e., the pull-back of hj to the normalization of Cj is harmonic).

Proof. At each manifold point of $\operatorname{supp} T$ we may choose coordinates $z\equiv(z^1,\dots,z^n)\equiv(z^1,z')$ with $\operatorname{supp} T=\{z'=0\}$. Then

$$T=\delta_0(z')\sum_{j,k} h^{jk}(z^1)\frac{i}{2}\frac{\partial}{\partial z^j}\wedge\frac{\partial}{\partial \bar{z}^k}$$

with each $h^{jk}(z^1)$ a measure. Now

$$\begin{aligned}0=\partial\bar{\partial}T&=\frac{i}{2}\delta_0(z')\frac{\partial^2 h^{11}}{\partial z^1\partial\bar{z}^1}\\&+\operatorname{Re}\sum_{j\geq 2} i\frac{\partial\delta_0(z')}{\partial z^j}\frac{\partial h^{j1}(z^1)}{\partial\bar{z}^1}+\sum_{j,k\geq 2}\frac{i}{2}\frac{\partial^2\delta_0(z')}{\partial z^j\ \partial\bar{z}^k}h^{jk}(z^1).\end{aligned}$$

Therefore, $h^{11}(z^1)$ is a non-negative harmonic function, $h^{j1}(z^1)$ is holomorphic, and each $h^{jk}=0$ for $j,k\geq 2$. Moreover T positive implies $h^{j1}=0$ for $j\geq 2$. Thus $T=\delta_0(z)\,h^{11}(z^1)\,\frac{i}{2}\frac{\partial}{\partial z^1}\wedge\frac{\partial}{\partial\bar{z}^1}$ with $h^{11}(z^1)\geq 0$ and harmonic. Since $h^{11}(z)$ is harmonic and ≥ 0 on C_j minus its singular points, it can be extended to a non-negative super-harmonic function $\tilde{h}_j$ on the normalization $\tilde{C}_j$ of C_j. Set $T' = \sum [\pi_j]_* (\tilde{h}_j [C_j])$ where $\pi_j\colon \tilde{C}_j \to C_j$ is the natural projection (i.e., $T' = \sum h_j [C_j]$). Fix a singular point $a \in C$ and let $\tilde{a}_j \in \tilde{C}_j$ be the point (if any) with $\pi_j(\tilde{a}_j) = a$. On $\tilde{C}_j$, $i\partial\bar{\partial}\tilde{h}_j = -k_j[\tilde{a}_j]$ for some $k_j \geqq 0$ because $\tilde{h}_j$ is superharmonic. Since $\pi_*[\tilde{a}_j] = [a]$, $i\partial\bar{\partial} T' = -\sum k_j [a]$ near a. Thus, near a, both $T - T'$ and $i\partial\bar{\partial}(T - T') = \sum k_j [a]$ have support in a. It follows easily that $\sum k_j = 0$, and so each $k_j = 0$ and $T = T'$. This completes the proof of the lemma.

Equation (31) now follows immediately from Lemma 32 since each $\tilde{C}_j$ is compact. In particular, the current T is d-closed. T is also the (1, 1)-component of a boundary, and thus by Proposition 25, T must be d-exact. This gives the following equation in $\mathbb{R}$-homology:

$$0 = [T] = [f^* \mu] + \sum_j c_j [C_j]. \tag{33}$$

The homology class $[f^* \mu]$ is exactly the class $c[F]$ where F denotes a non-singular fibre and where $c = \int_Y \mu \geqq 0$.

To simplify the discussion we consider for the moment, the case where there is only one singular fibre, say $f^{-1}(y_0)$. Then there are positive constants $\gamma_1, \gamma_2, \ldots$ and γ so that

$$f^*(\delta_{y_0}) = \sum_j \gamma_j C_j \sim \gamma F \tag{34}$$

(where "$\sim$" means "homologous over $\mathbb{R}$").

We now consider the intersection pairing "$\cdot$" on the set of curves $\{C_j\}$. The matrix of intersection numbers consists of integers $[C_j] \cdot [C_k]$ which are $\geqq 0$ for $j \neq k$. Furthermore, since the non-singular fibres are connected, so is the singular fibre. This implies that the matrix $[C_j] \cdot [C_k]$ is *connected* in the sense that each pair of distinct indices is joined by a sequence of indices for which each successive pairing is strictly positive.

The fibre class $[F] \cong \sum \gamma_j [C_j]$ has the property that $[F] \cdot [C_j] = 0$ for all j. (This is evident geometrically.) It now follows from a lemma in linear algebra, which is basic for surface theory, [1, Proposition 1.1 on p. 350], that any linear combination $x = \sum c_j [C_j]$ with all $c_j \geqq 0$ and $x \cdot x = 0$, must be a scalar multiple of $[F]$. From Eq. (33) we see that $0 = [T] = c[F] + \sum c_j [C_j]$, and so $(\sum c_j [C_j])^2 = 0$. Consequently there is a $t \geqq 0$ so that $c_j = t\gamma_j$ for all j. Using (31) and (34) we conclude that

$$\begin{aligned} T &= f^*(\mu) + t \sum \gamma_j C_j \\ &= f^*(\mu) + t f^*(\delta_{y_0}) \\ &= f^*(\mu') \end{aligned}$$

where $\mu' = \mu + t\delta_{y_0}$ is again a non-negative density.

In the case where there are several singular fibres, the intersection matrix decomposes into a direct sum of connected matrices, each of which corresponds to a single fibre. The argument then proceeds exactly as before. This completes the proof of Proposition 27 and Theorem 20.

The discussion above enables us to give the following example of a solution to problem 3.

(35) **Theorem.** *Suppose $f: X \to Y$ is a non-Kähler elliptic surface. Then the following convex cones coincide.*

$P_{\mathrm{bdy}_{1,1}} \equiv$ *All positive* $(1,1)$*-components of boundaries on* X.

$P_{\mathrm{bdy}} \equiv$ *All positive* $(1,1)$*-boundaries on* X.

$P_f \equiv \{f^*(\mu) \in \mathscr{E}'_{1,1}(X)$: μ *is a non-negative density on* $Y\}$.

Proof. It is evident that $P_{\mathrm{bdy}} \subseteq P_{\mathrm{bdy}_{1,1}}$ and Proposition 27 asserts that $P_{\mathrm{bdy}_{1,1}} = P_f$. However, $P_f \subseteq P_{\mathrm{bdy}}$ by Theorem 20(3′).

(36) **Theorem.** *Let $f: X \to Y$ be a non-Kähler elliptic surface. Then the cone of positive d-closed currents on X consists of all elements of the form*

$$T = f^*(\mu) + \sum t_j C_j$$

where $f^*(\mu)$ belongs to P_f, where $t_j \geqq 0$ and where $C_1, C_2, \ldots$ denote the irreducible components of the singular fibres of f.

Proof. Let T be a d-closed positive $(1,1)$-current on X, and let Ω be a volume form on Y. Then $M(f_* T) = (f_* T)(\Omega) = T(f^*\Omega) = 0$ since, by Theorem 20(3′), the form $f^*\Omega$ is exact on X. The theorem now follows from the arguments of Lemmas 18 and 32.

Some remarks on non-elliptic surfaces. We would like to be able to give a direct proof that for a compact surface X

$$\text{Property } NK \Leftrightarrow b_1(X) \text{ is odd.}$$

Of course, by our main theorem this is equivalent to the conjecture of Kodaira [11, p. 85]

$$b_1(X) \text{ is even} \Leftrightarrow X \text{ is Kähler.}$$

Perhaps it is worth remarking that there is an elementary argument that

$$\text{Property } (NK)^\infty \Rightarrow b_1(X) \text{ is odd.}$$

This is to say that if $b_1(X)$ is even, then there are no *smooth* positive $(1,1)$-currents which are $(1,1)$-components of boundaries. This follows from Proposition 23 and the next result.

(37) **Proposition.** *Suppose X is an arbitrary compact complex surface. If T is smooth, positive and the* $1,1$ *component of a boundary then T is a boundary, and in addition T is simple* (i.e. *of rank* $\leqq 1$) *at each point.*

Proof. Suppose $T = (dS)^{1,1}$ for some current S. Then $-\partial T = \bar\partial[\partial S^{1,0}]$ so that $\partial S^{1,0}$ is smooth. Similarly, we have that $\bar\partial S^{0,1}$ is smooth, and so dS is smooth. Now by deRham we may assume S is smooth without changing dS. We then see that:

$$\begin{aligned} 0 &= \int d(S \wedge dS) = \int dS \wedge dS \\ &= \int T \wedge T + 2\int (dS)^{2,0} \wedge \overline{(dS)^{2,0}}. \end{aligned}$$

Therefore $T \wedge T \equiv 0$ and $(dS)^{2,0} \equiv 0$.

Utilizing Kodaira's beautiful classification of surfaces, if X is non-algebraic and non-elliptic with first Betti number even, then X is either a torus (which is obviously Kähler) or X is a K3 surface. Thus Theorem 20 settles the Question above except for the case where X is a K3 surface with no meromorphic functions. Recently, as an important application of Yau's solution to the Calabi conjecture, Todorov [15], Siu [13] gave an argument that each K3 surface is Kähler.

Perhaps it is worth remarking that the dual interpretation of Theorem 26 above says: If a compact surface X has $b_1(X)$ even, then there exists a real 2-form ω on X such that:

(i) $d\omega=0$,

(ii) $\omega^{1,1}$ is positive definite,

(iii) $\omega^{2,0}=\partial\alpha^{1,0}$ for some $(1,0)$-form $\alpha^{1,0}$.

More generally, the existence of such a "weakened" Kähler form can be characterized as follows.

(38) **Theorem.** *Suppose X is a compact complex manifold. The manifold X admits a real 2-form $\omega\in\mathscr{E}^2(X)$ with*

(i) $d\omega=0$,

(ii) $\omega^{1,1}$ *positive definite,*

(iii) $\omega^{2,0}=\partial\alpha$ *for some* $1,0$ *form* α,

if and only if X does not support a (non-trivial) positive, d-closed current which is the bidimension 1, 1 *component of a boundary.*

Only the proof of the difficult half of the theorem is sketched. As before, $P_{1,1}$ denotes the positive currents of bidimension $1,1$ and $B_{1,1}$ denotes the bidimension $1,1$ components of boundaries. Let $Z_{1,1}$ denote the d-closed currents of bidimension $1,1$. Let $A^\perp$ denote the annihilator of A. Then $B_{1,1}^\perp=\{\omega\in\mathscr{E}_{\mathbb{R}}^{1,1}(X)\colon d\omega=0\}$ and will be denoted $Z^{1,1}$. Moreover, $Z_{1,1}^\perp=\{\omega\in\mathscr{E}_{\mathbb{R}}^{1,1}(X)\colon \omega=(d\alpha)^{1,1}$ for some $\alpha\in\mathscr{E}_{\mathbb{R}}^{1}(X)\}$ will be denoted $B^{1,1}$. The hypothesis is that $P_{1,1}\cap B_{1,1}\cap Z_{1,1}=\{0\}$. The Hahn-Banach Theorem implies that there exist $\tilde{\omega}\in\mathscr{E}_{\mathbb{R}}^{1,1}(X)$ with $\tilde{\omega}\in(B_{1,1}\cap Z_{1,1})^\perp$ and $\tilde{\omega}$ positive definite. Now $(B_{1,1}\cap Z_{1,1})^\perp=B_{1,1}^\perp+Z_{1,1}^\perp=Z^{1,1}+B^{1,1}$ (assume for the moment that this sum is a closed subspace). Thus, for some α, $\tilde{\omega}-(d\alpha)^{1,1}$ is closed. Equivalently,

$$\omega\equiv\tilde{\omega}-\partial\alpha^{1,0}-\bar{\partial}\alpha^{0,1}$$

is closed. Since $\omega^{1,1}=\tilde{\omega}$ and $\omega^{2,0}=-\partial\alpha^{1,0}$, properties (ii) and (iii) follow.

It remains to verify that $Z^{1,1}+B^{1,1}$ is a closed subspace of $\mathscr{E}_{\mathbb{R}}^{1,1}(X)$. First note that

$$0\to\mathscr{H}\xrightarrow{(-d)\oplus id}\operatorname{Re}\Omega^1\oplus\mathscr{E}^0\xrightarrow{id+d}\mathscr{E}^1\xrightarrow{d^{1,1}}\mathscr{E}^{1,1}\xrightarrow{dd^c}\mathscr{E}^2$$

is an exact sequence of sheaves. This can be used to show that the image of $d^{1,1}$ (i.e., $B^{1,1}$) has finite codimension in the closed space $\ker(dd^c)$. Since $Z^{1,1}$ is a closed subspace of the kernel of dd^c this implies that $B^{1,1}+Z^{1,1}$ is closed.

§ 8. The Kähler rank of a surface

Our characterization of Kähler manifolds says that on a non-Kähler manifold there exists a (non-trivial), 1, 1 component of a boundary. In the last section we characterized the cone of all such currents in the elliptic case. In particular, this cone contains a geometric object $[F]$, the current corresponding to integration over the generic fiber; and an analytic object Ω, the pull back of a volume form on the base. In this section we consider compact complex surfaces which admit a form such as Ω.

First, we establish some notation which will be useful in the remaining sections. Let P denote the cone of all positive currents of bidimension 1, 1 on the surface. Recall that

$P_{\mathrm{bdy}} \equiv \{T \in P\colon\ T \text{ is a boundary}\}$,
$P_{\mathrm{bdy}_{1,1}} \equiv \{T \in P\colon\ T \text{ is a } (1,1)\text{-component of a boundary}\}$,
$P_{\mathrm{closed}} \equiv \{T \in P\colon\ T \text{ is } d\text{-closed}\}$.

Proposition 25 asserts that on a compact surface

$$P_{\mathrm{closed}} \cap P_{\mathrm{bdy}_{1,1}} = P_{\mathrm{bdy}}.$$

Now we turn our attention to the subcones of the above cones consisting of those currents that are smooth. The subcone is denoted by using the superscript ∞. Proposition 37, which is basic to this section, says that:

$$P^{\infty}_{\mathrm{bdy}_{1,1}} = P^{\infty}_{\mathrm{bdy}}$$

and that any form $\varphi \in P^{\infty}_{\mathrm{bdy}}$ is simple at every point of X. Suppose $x \in X$ is a point such that $\varphi_x \neq 0$. Note that for any other $\psi \in P^{\infty}_{\mathrm{bdy}}$ the sum $\varphi + \psi$ is also in $P^{\infty}_{\mathrm{bdy}}$, and so the 1, 1 vectors φ_x, ψ_x and $\varphi_x + \psi_x$ are all simple. This implies that $\psi_x = \lambda \varphi_x$ for some $\lambda > 0$. In particular, the complex line

$$\vec{\mathscr{F}}_x = \ker(\varphi_x)$$

is defined independently of the choice of form $\varphi \in P^{\infty}_{\mathrm{bdy}}$. Since $d\varphi = 0$, we see that this line field $\vec{\mathscr{F}}$ *is integrable.*

This leads us to consider the following intrinsically defined set:

$$\mathscr{B}(X) \equiv \{x \in X\colon\ \exists\, \varphi \in P^{\infty}_{\mathrm{bdy}}(X) \text{ with } \varphi_x \neq 0\}. \tag{39}$$

The argument given above proves the following.

(40) **Theorem.** *The open subset $\mathscr{B}(X) \subseteq X$ carries an intrinsically defined complex analytic foliation $\mathscr{F}$. This foliation has the (defining) property that $\varphi|_{\mathscr{F}} \equiv 0$ for any $\varphi \in P^{\infty}_{\mathrm{bdy}}(X)$.*

While a surface may contain no complex curves, it still may have such an analytic foliation.

Motivated by the above discussion and the structure of $P^{\infty}_{\mathrm{bdy}_{1,1}} = P^{\infty}_{\mathrm{bdy}}$ for a non-Kähler elliptic surface, we introduce the concept of Kähler rank.

(41) **Definition.** Suppose X is a compact complex surface. If the complement of the open subset $\mathscr{B}(X)$ in X is contained in a complex curve, then X is said

to have *Kähler rank one*. If X is Kähler we say that the *Kähler rank* of X is two. Otherwise, the *Kähler rank* is zero.

Note that for a non-Kähler elliptic surface the Kähler rank is one. In fact by Theorem 35, the complement of $\mathscr{B}(X)$ in X is contained in the union of the singular fibers of the elliptic surface X.

Many interesting questions remain. Is the Kähler rank of a surface a bimeromorphic invariant? If so, it is vaguely analogous to the transcendence degree of the meromorphic function field over $\mathbb{C}$ (which is 0, 1, or 2) and orthogonal to the other basic numerical invariant κ, the canonical dimension (which is $-1, 0, 1$ or 2).

We now briefly recall some relevant facts concerning nonelliptic surfaces. Suppose X is a compact surface with no meromorphic functions (i.e., $M(X) = \mathbb{C}$). Then for any holomorphic line bundle L on X, $\dim H^0(X, L) \leqq 1$, for otherwise the ratio of two independent sections of L would yield a non-constant meromorphic function. In particular, the *geometric genus* $p_g = \dim H^2(\mathcal{O}) = \dim H^0(\Omega^2)$ is either 0 or 1. If $p_g = 1$, then Kodaira has proved that X is either a complex torus (which of course is Kähler) or a K3-surface. We shall assume that $p_g = 0$.

In this case, it is a fact that (cf. [1]): $b_1 = q = 1$, $\chi(\mathcal{O}) = b_2^+ = 0$, and $b_2 = b_2^- = \chi(\mathbb{R})$ ($\equiv$ the Euler characteristic). Furthermore,

$$\begin{gathered} h^{1,0} \equiv \dim H^0(\Omega^1) = 0 \qquad h^{1,1} \equiv \dim H^1(\Omega^1) = b_2. \\ h^1 \equiv \dim_{\mathbb{R}} H^1(\mathscr{H}) = b_2 + 1. \end{gathered} \tag{42}$$

In particular, the sequence

$$\begin{array}{ccccccccc} 0 \to & H^1(\mathbb{R}) & \to & H^1(\mathcal{O}) & \to H^1(\mathscr{H}) \to H^2(\mathbb{R}) \to 0 \\ & \| & & \| & \\ & \mathbb{R} & & \mathbb{C} & \end{array} \tag{43}$$

is exact. This has the following consequence which will be useful to us later.

(44) **Lemma.** *Suppose $M(X) = \mathbb{C}$ and $p_g = 0$. Then given any two non-trivial currents, $T_0, T_1 \in P_{\mathrm{bdy}}(X)$, there exist constants $\alpha_0 > 0$ and $\alpha_1 > 0$ so that*

$$\alpha_0 T_0 - \alpha_1 T_1 = i\partial\bar{\partial}\phi$$

for some function ϕ on X.

Proof. Recall that $H^1(\mathscr{H}) = \{T \in \mathscr{E}'^{1,1}(X) \colon dT = 0\} / i\partial\bar{\partial}\mathscr{E}'^0(X)$ from Corollary 2a. The image of a class T in $H^2(\mathbb{R})$ is zero if and only if T is a boundary. Now given a non-trivial $T \in P_{\mathrm{bdy}}(X)$, the class of T in $H^2(\mathbb{R})$ is zero, but its class in $H^1(\mathscr{H})$ is not. (For if $T = i\partial\bar{\partial}\psi$, then ψ is a plurisubharmonic function on X, and therefore constant.)

By the sequence (43) we see that the kernel of the map $H^1(\mathscr{H}) \to H^2(\mathbb{R})$ is 1-dimensional over $\mathbb{R}$. Consequently, given two non-trivial currents $T_0, T_1 \in P_{\mathrm{bdy}}(X)$, there are real constants α_0, α_1 so that $[\alpha_0 T_0 - \alpha_1 T_1] = 0$ in $H^1(\mathscr{H})$, i.e., $\alpha_0 T_0 - \alpha_1 T_1 = i\partial\bar{\partial}\phi$ for some ϕ. Both α_0 and α_1 are of the same sign since otherwise either ϕ or $-\phi$ would be plurisubharmonic. This completes the proof.

Recall now that for a non-elliptic non-algebraic surface X with $p_g=0$ we have $b_1(X)=1$. It is a theorem of Kodaira that if, in addition, $b_2(X)=0$ and X contains a curve, then X is a non-elliptic *Hopf surface*. (We shall study these surfaces in detail in Sect. 9.) If $b_2(X)=0$ and X contains no curves, then Inoue and Bogomolov have proved that X is an *Inoue surface*. (We shall study these in Sect. 10.)

The remaining case to consider is the one where X is a non-elliptic non-algebraic surface with $p_g=0$ and $b_2(X)>0$. Little is known in this case. There are, however, some examples: the so-called Inoue-Hirzebruch surfaces (see Sect. 11), and some recent examples given by Kato.

§ 9. Non-elliptic Hopf surfaces

In this section we shall concentrate on the Hopf surfaces of non-elliptic type. Every such surface has a finite cover by a *primary* Hopf surface. This is a surface of the form

$$X=(\mathbb{C}^2-\{0\})/\mathbb{Z}$$

where $\mathbb{Z}$ is generated by a certain biholomorphism Φ of $\mathbb{C}^2-\{0\}$. The allowable biholomorphisms Φ fall into two classes.

Class 1. Here we have

$$\Phi(z_1,z_2)=(\alpha_1 z_1,\alpha_2 z_2)$$

where α_1 and α_2 are complex constants with $0<|\alpha_1|\leqq|\alpha_2|<1$. Hopf surfaces in this class are elliptic if and only if $\alpha_1^p=\alpha_2^q$ for some $p,q\in\mathbb{Z}$.

Class 0. Here Φ is of the form

$$\Phi(z_1,z_2)=(\alpha^m z_1+\lambda z_2^m,\alpha z_2)$$

for non-zero constants α and λ with $|\alpha|<1$ and for some positive integer m. Hopf surfaces of this class are never elliptic.

A. Hopf surfaces of Class 1

We begin with (non-elliptic) surfaces of class 1. Let $r\geqq 1$ be defined by

$$|\alpha_1|=|\alpha_2|^r, \tag{45}$$

and consider the plurisubharmonic function φ in $\mathbb{C}^2$ given by

$$\varphi=\operatorname{Log}(|z_1|^2+|z_2|^{2r}). \tag{46}$$

Since $\varphi(\Phi(z))=\varphi(z)+\operatorname{Log}|\alpha_1|^2$, the 1-form

$$\tfrac{1}{2}d^c\varphi\in\mathscr{E}^1(X) \tag{47}$$

is well defined on the quotient X. Furthermore, its exterior derivative

(48) $$\Omega \equiv \tfrac{1}{2} dd^c \varphi = i\partial\bar{\partial}\varphi \in \mathscr{E}^{1,1}(X)$$

is a positive smooth 1, 1 current on X which bounds. Theorem 40 says that there is an intrinsic complex foliation $\mathscr{F}$ defined on $B(X)$ which is determined by the simple 2-form Ω on the set where Ω is not zero. More explicitly,

(49) $$\Omega = \frac{|z_2|^{2(r-1)}}{(|z_1|^2 + |z_2|^{2r})^2} i(z_2 dz_1 - r z_1 dz_2) \wedge (\bar{z}_2 d\bar{z}_1 - r\bar{z}_1 d\bar{z}_2)$$

vanishes exactly on the z_1-axis. Replacing the φ defined above by

(50) $$\tilde{\varphi} \equiv \mathrm{Log}(|z_1|^2 + |z_2|^{2r})(|z_1|^{2/r} + |z_2|^2)$$

and defining

$$\tilde{\Omega} \equiv i\partial\bar{\partial}\tilde{\varphi}$$

we obtain a positive smooth 1, 1 current on X which bounds and is never zero. Thus each (non-elliptic) Hopf surface X of class 1 is of Kähler rank 1 with $B(X) = X$.

The holomorphic vector field

(51) $$V(z) \equiv r z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}$$

is well defined and never zero on X. Moreover, V is in the kernel of Ω. Thus the complex foliation $\mathscr{F}$ is in fact a holomorphic foliation. The induced foliation on $\mathbb{C}^2 - \{0\}$ can be described by the flow lines

$$\{\Psi_t(z)\colon t \in \mathbb{C}\}$$

of the holomorphic flow Ψ_t determined by V:

$$\Psi_t(z) \equiv (e^{rt} z_1, e^t z_2), \qquad t \in \mathbb{C}.$$

Alternatively, each flow line L_c is the graph of the multivalued function

$$z_1 = c z_2^r, \qquad \text{for some } c \in \mathbb{C},$$

over the z_2-axis which we denote L_∞. Thus each leaf L of $\mathscr{F}$ on X when pulled back to $\mathbb{C}^2 - \{0\}$ is of the form

(52) $$\bigcup \{L_c\colon \Phi^N(L_c) = L_C \text{ for some } N \in Z\}$$

where C is fixed.

The function

(53) $$\pi(z) = \frac{|z_1|^2}{|z_1|^2 + |z_2|^{2r}}$$

is Φ-invariant and gives a well defined map $\pi\colon X \to [0, 1]$.

(54) **Lemma.** *The fibers of π are exactly the closures of the leaves of the foliation $\mathscr{F}$ on X.*

Proof. The fibers of π are clearly unions of leaves of $\mathscr{F}$. Let L denote a leaf pulled back to $\mathbb{C}^2-\{0\}$. It will suffice to show that its closure $\bar{L}$ is of the form

$$|z_1|=\rho|z_2|^r \quad \text{for some } \rho\in[0,\infty]$$

where $\rho=\infty$ defines the set $z_2=0$.

First note that:

(55) $$L_c=L_C \text{ if and only if } C=e^{2\pi kir}c \text{ for some } k\in\mathbb{Z},$$

so that L contains

$$z_1=e^{2\pi kir}cz_1^r, \quad k\in\mathbb{Z}$$

for some $c\in\mathbb{C}$.

Hence, if r is irrational, then $\bar{L}$ is of the form $|z_1|=\rho|z_2|^r$ with $\rho=|c|$. Second, let θ be defined by

$$\alpha_1=e^{i\theta}\alpha_2^r, \tag{56}$$

and note that:

(57) $$\Phi(L_c)=L_C \quad \text{where } C=e^{i\theta}c.$$

If X is not elliptic then either r is irrational or $\theta/2\pi$ is irrational. If $\theta/2\pi$ is irrational again we have that $\bar{L}$ is of the form

$$|z_1|=\rho|z_2|^r \quad \text{with } \rho\equiv|c|,$$

completing the proof.

(58) **Theorem.** *Let X be a non-elliptic Hopf surface of class* 1. *Let*

$$\Omega\equiv i\partial\bar{\partial}\,\mathrm{Log}(|z_1|^2+|z_2|^{2r})(|z_1|^{2/r}+|z_2|^2)$$

denote the positive 1, 1 *form defined on X above. All of the following cones agree.*

1. $P_{\mathrm{bdy}_{1,1}}$
2. P_{bdy}
3. P_{closed}
4. The d-closed positive foliation currents for $\mathscr{F}$ on X
5. $\{\pi^*(f)\,\Omega$: *f is a non-negative generalized function on* $[0,1]\}$

Proof. Since $H^2(X,\mathbb{R})=0$, $P_{\mathrm{bdy}}=P_{\mathrm{closed}}$. Moreover, if $T\in P_{\mathrm{closed}}$, then $T(\Omega)=0$ since Ω bounds. Thus $P_{\mathrm{bdy}}=P_{\mathrm{closed}}=\{T$: T is a positive d-closed foliation current for $\mathscr{F}$ on $X\}$. If $T\in P_{\mathrm{bdy}_{1,1}}$, then we also have that $T(\Omega)=0$ (since $d\Omega=0$), and again conclude that T is a positive foliation current for $\mathscr{F}$ on X, but which is only dd^c-closed. Each positive foliation current T for $\mathscr{F}$ on X must be of the form

$$T=g\Omega$$

where g is a non-negative generalized function on X (i.e. a non-negative current of dimension four on X).

It remains to show that: if $dd^c g \wedge \Omega = 0$ then

$$g = \pi^*(f) \quad \text{for some} \quad f \geqq 0 \quad \text{on} \quad [0,1].$$

First, assume that g is continuous on X. Then g is harmonic on the leaves of $\mathscr{F}$ and bounded. Since the leaves of $\mathscr{F}$ are easily seen to be of the form $\mathbb{C}$ or $\mathbb{C}^*$ we conclude by Liouville's theorem that g is constant on the leaves and hence constant on the leaf closures. Thus $g = \pi^*(f)$ by Lemma 54.

More generally, introduce coordinates ρ, θ_1, θ_2, θ_3 defined by

$$z_2 = e^{\theta_3 + i\theta_2}, \qquad z_1 = \rho e^{r\theta_3 + i\theta_1}. \tag{59}$$

Then

$$\Phi(\rho, \theta_1, \theta_2, \theta_3) \equiv (\rho, \theta_1 + a_1, \theta_2 + a_2, \theta_3 + \operatorname{Log}|\alpha_2|)$$

where $a_j \equiv \operatorname{Arg}(\alpha_j)$ $j = 1, 2$.

Moreover, $\Psi_t(\rho, \theta_1, \theta_2, \theta_3) \equiv (\rho, \theta_1 + r \operatorname{Im} t, \ \theta_2 + \operatorname{Im} t, \theta_3 + \operatorname{Re} t)$ defines the leaves:

$$\rho = \text{constant},$$
$$\theta_1 - r\theta_2 = \text{constant}.$$

It suffices to show that for each smooth function $\varphi(\rho) \geqq 0$ with $\int \varphi(\rho)\, d\rho = 1$

$$G(\theta_1, \theta_2, \theta_3) \equiv \int g(\rho, \theta_1, \theta_2, \theta_3)\, \varphi(\rho)\, d\rho$$

is a constant.

First we note that $dd^c g \wedge \Omega = 0$ becomes

$$(A^2 + B^2)(g) = 0.$$

which implies that $(A^2 + B^2)(G) = 0$, where $A \equiv r\dfrac{\partial}{\partial\theta_1} + \dfrac{\partial}{\partial\theta_2}$ and $B \equiv \dfrac{\partial}{\partial\theta_3}$. Choose $\psi \geqq 0$, $\int \psi(\sigma)\, d\sigma = 1$, $\psi \in C_0^\infty(\mathbb{R})$. Then smoothing G by the approximate identity based on ψ we define

$$G_\varepsilon(\sigma) = \int_{\mathbb{R}} \psi(\sigma)\, G(\theta + \varepsilon\sigma)\, d\sigma.$$

Then G_ε is smooth on $S^1 \times S^1 \times S^1$, harmonic on the dense leaves which are either $\mathbb{C}$ or $\mathbb{C}^*$, and hence each G_ε is constant. Thus

$$G \equiv \lim_{\varepsilon \to 0} G_\varepsilon \qquad \text{is constant.}$$

B. Hopf surfaces of class 0

We now consider the non-elliptic Hopf surfaces of class 0. First note that the parameter $\lambda \in \mathbb{C}^*$ used in the above description of those surfaces, can be chosen arbitrarily. That is, let $\Phi_j\colon \mathbb{C}^2 \to \mathbb{C}^2$, $j = 1, 2$, be the biholomorphism given by

$$\Phi_j(z_1, z_2) = (\alpha^m z_1 + \lambda_j z_2^m, \alpha z_2)$$

where $0<|\alpha|<1$ and where $\lambda_1, \lambda_2 \in \mathbb{C}^*$ are arbitrary. Set

$$X_j \equiv (\mathbb{C}^2 - \{0\})/\langle \Phi_j \rangle.$$

(60) **Proposition.** *The surfaces X_1 and X_2 are holomorphically equivalent.*

Proof. Define $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ by $F(z_1, z_2) \equiv ((\lambda_2/\lambda_1) z_1, z_2)$. Since $F \circ \Phi_1 = \Phi_2 \circ F$, F induces a biholomorphism $\tilde{F}\colon X_1 \to X_2$.

We now fix a Hopf surface $X = (\mathbb{C}^2\{0\})/\langle \Phi \rangle$ where $\Phi(z_1, z_2) = (\alpha^m z_1 + \lambda z_2, \alpha z_2)$. Here α satisfies $0<|\alpha|<1$ and $\lambda \in \mathbb{C}^*$ may be chosen to our convenience. The first step in our analysis will be to construct a holomorphic vector field on X.

Choose a value a for $\log \alpha$. Then $\alpha^t = e^{t \log \alpha} = e^{ta}$ is defined for all $t \in \mathbb{C}$. Note that

$$\text{(61)} \qquad \operatorname{Re}(a) < 0$$

since $|\alpha|<1$. We now consider the complex flow on $\mathbb{C}^2$ given by

$$\text{(62)} \qquad \begin{aligned} \Phi_t(z_1, z_2) &= (\alpha^{tm} z_1 + t\lambda \alpha^{(t-1)m} z_2^m, \alpha^t z_2) \\ &= (e^{mat} z_1 + t\lambda \alpha^{-m} e^{mat} z_2^m, e^{at} z_2) \end{aligned}$$

for all $t \in \mathbb{C}$. Note that $\Phi_1 = \Phi$. Since Φ_t gives a group action of $\mathbb{C}$ on $\mathbb{C}^2$, Φ_t commutes with $\Phi_1 = \Phi$ for all t. Hence this flow descends to a complex flow on X.

The complex vector field corresponding to this flow is

$$\mathscr{V}(z) = (maz_1 + \lambda \alpha^{-m} z_2^m) \frac{\partial}{\partial z_1} + az_2 \frac{\partial}{\partial z_2}.$$

Set $V = \operatorname{Re} \mathscr{V}$ and note that

$$\begin{aligned} \langle V(z), z \rangle &= \operatorname{Re}\{ma|z_1|^2 + \lambda \alpha^{-m} z_2^m \bar{z}_1 + a|z_2|^2\} \\ &= (\operatorname{Re} a)(|z_1|^2 + |z_2|^2) + |\lambda|\,|\alpha|^{-m} |z_2|^m |z_1|. \end{aligned}$$

Suppose $|z_2| \leqq 1$, and choose λ so that

$$|\lambda|\,|\alpha|^{-m} \leqq -\operatorname{Re}(a).$$

Then

$$\begin{aligned} \langle V(z), z \rangle &\leqq \operatorname{Re}(a)(|z_1|^2 - |z_1|\,|z_2| + |z_2|^2) \\ &\leqq \tfrac{1}{2} \operatorname{Re}(a)(|z_1|^2 + |z_2|^2) \end{aligned}$$

which implies the following.

(63) **Lemma.** *V is transverse to all spheres of radius $\leqq 1$ about the origin in $\mathbb{C}^2$.*

(Since $\Phi = \Phi_1$, we have that Φ_t, $t \in \mathbb{R}$, gives an S^1-action on the manifold X. Lemma 63 then proves that X is diffeomorphic to $S^1 \times S^3$.)

We now return to the complex flow associated to $\mathscr{V}=V+iJV$. Writing $\Phi_t(z_1, z_2)=(z_1(t), z_2(t))$, we note from Eq. (62) that

$$\frac{z_1(t)}{z_2(t)^m}=\frac{z_1}{z_2^m}+\frac{\lambda}{\alpha^m}t. \tag{64}$$

Consider now the subset M^2 of the unit sphere $S^3=\{|z_1|^2+|z_2|^2=1\}$ given by

$$M^2=\left\{z\in S^3\colon z_2=0 \text{ or } \operatorname{Im}\left(\frac{\bar{\lambda}z_1}{\bar{\alpha}^m z_2^m}\right)=0\right\}. \tag{65}$$

This surface is an immersed torus. It can be explicitly parametrized by

$$F(\varphi, \psi)\equiv\left(\frac{\lambda}{|\lambda|}\cos\varphi\, e^{i\psi}, \frac{\alpha}{|\alpha|}\sin\varphi\, e^{im\psi}\right)$$

for $\varphi, \psi\in[0, 2\pi)$. The image of F has an m-fold self-intersection along the curve $\gamma=S^3\cap\{z_1\text{-axis}\}$.

(66) **Lemma.** *Each orbit of the holomorphic flow Φ_t, $t\in\mathbb{C}$, meets the submanifold $M^2-\gamma$ exactly once (and transversely) with the exception of the z_1-axis which meets M^2 in the singular curve γ.*

Proof. By (64) we see that (for $z_2=0$) the equation $\rho\equiv\operatorname{Im}(\bar{\lambda}z_1(t)/\bar{\alpha}^m z_2^m(t))=0$ uniquely determines $\operatorname{Im}(t)$. Now the real flow Φ_t is transverse to the unit sphere, and so there is a unique $\operatorname{Re}(t)$ such that $\Phi_t(z)\in M^2$.

To see that $\Phi_t(z)$ meets M transversely when $z_2\neq 0$, we consider the defining equations $r=|z_1|^2+|z_2|^2=1$ and $\rho=0$ for M^2. It is easy to calculate that on M^2, $(d\rho)(V)=0$ and $(d\rho)(JV)=|\lambda|^2/|\alpha|^{2m}$. Furthermore, by (63) we have $(dr)(V)$ is never zero on $r=1$. Hence, $(dr\wedge d\rho)(V\wedge JV)$ is never zero on M^2. This completes the proof.

The orbits of the flow Φ_t, $t\in\mathbb{C}$, give a complex analytic foliation $\mathscr{F}$ of the Hopf manifold $X=(\mathbb{C}^2-\{0\})/\Phi$. As we shall see, this foliation has only one compact leaf, the (image of the) z_1-axis. The surface $M-\gamma$ is a cross-section of the remainder of the foliation.

We now consider the positive foliation currents for $\vec{\mathscr{F}}$. These are the currents (of finite mass) of the form

$$T=\mu\vec{\mathscr{F}}$$

where $\vec{\mathscr{F}}$ is the field of unit oriented tangent 2-vectors to $\mathscr{F}$ and where μ is a non-negative Radon measure on X. Suppose we consider a local "flow-box", i.e., a local complex coordinate system (w_1, w_2) on X in which the leaves of $\mathscr{F}$ are given by the equation $w_2=$constant. Then the foliation current T can be expressed (dually) as a differential form

$$T=\mu i dw_2\wedge d\bar{w}_2$$

(with some Radon measure $\mu\geqq 0$ as coefficient). Note that in this representation

(67) $$dT=0 \Leftrightarrow \frac{\partial \mu}{\partial w_1}=\frac{\partial \mu}{\partial \bar{w}_1}=0 \Leftrightarrow \mu=\mu(w_2).$$

A major step in our analysis of X is the following.

(68) **Proposition.** *Let T be a positive d-closed foliation current for the complex analytic foliation $\mathscr{F}$ of X. Then*

$$T=r[z_1-\text{axis}]$$

for some $r>0$.

Proof. Let χ denote the characteristic function of an open subset of M^2 whose closure is disjoint from $\gamma=M^2 \cap \{z_1-\text{axis}\}$. Extend χ to all of X by requiring χ to be constant on the leaves of $\mathscr{F}$. (This extended function is the characteristic function of an open $\mathscr{F}$-saturated subset of X.)

We now observe that χT is also a positive d-closed foliation current. To see this, consider a small "flow box", i.e., a local complex coordinate system (w_1, w_2) on X as above. In these coordinates $\chi=\chi(w_2)$ is the characteristic function of an open subset in the w_2-plane. Hence, by (67) we conclude that $d(\chi T)=0$.

Note that the cycle M is transversal to $\mathscr{T}$ on the support of the current χT. (See Lemma 66 and the definition of χ.) This means that we can pair χT with M. From the positivity of χT we see that this pairing $(\chi T, M)$ is zero if and only if $\chi T=0$. However, since $H_2(X)=0$, we can write $M=dS$ for some 3-chain S. Hence, $(\chi T, dS)\equiv(d(\chi T), S)=0$, and we conclude that $\operatorname{supp}(T)\subset[z_1\text{-axis}]$. Since T is a flat 2-current with support in the 2-torus $\mathbf{T}\equiv[z_1\text{-axis}]\subset X$, it is a standard result of Federer [2], that $T=r\mathbf{T}$ for some constant r. Since T was positive, we must have $r>0$ (assuming, of course, that $\mathbf{T}$ is oriented canonically as a complex curve.) This completes the proof of Proposition 68.

This brings us to our main result.

(69) **Theorem.** *Let $X=(\mathbb{C}^2\{0\})/\langle\Phi\rangle$ be a Hopf suface of class 0. Then*

$$P_{\text{bdy}}(X)=P_{\text{closed}}(X)=\{r\mathbf{T}:\ r\geqq 0\}$$

where $\mathbf{T}\cong\{z_1-\text{axis}\}/\langle\alpha^m\rangle$. Moreover, $P_{\text{bdy}_{1,1}}(X)$ contains no smooth currents; so that each Hopf surface of class 0 has Kähler rank zero.

Remark. The structure of $P_{\text{bdy}_{1,1}}(X)$ is not completely understood.

Hence, not only is the elliptic curve $\mathbf{T}$ the only complex curve on X, it is the only d-closed positive $(1,1)$-current on X.

Proof. The last part of the theorem follows from the first part by Proposition 37. To prove the first part let T be a positive d-closed $(1,1)$-current on X. We shall show that T is a positive foliation current for the orbit foliation $\mathscr{F}$. The theorem will then follow from Proposition 68. (Note that since $H_2(X;\mathbb{R})=0$, we have $P_{\text{bdy}}(X)=P_{\text{closed}}(X)$.)

We first observe that by averaging T over the S^1-action on X, we can assume that T is Φ_t-invariant for $t\in\mathbb{R}$. (That is we can replace T by

$$\int_0^1 (\Phi_t)_*(T)\,dt$$

and recall that since $\Phi_1 = \Phi$, the biholomorphisms Φ_t for $t \in \mathbb{R}/\mathbb{Z}$, give an S^1-action in X.) This averaging procedure preserves the property of being (or not being) a foliation current for the orbit foliation.

We now observe that by Lemma 44, there is a positive constant r such that

$$T - r\mathbf{T} + i\partial\bar{\partial} F_0 \tag{70}$$

for some generalized function F_0 on X. We lift T and F_0 to the universal covering space $\mathbb{C}^2 - \{0\}$. Here Eq. (70) becomes

$$T = i\partial\bar{\partial} F \tag{71}$$

where

$$F \equiv r \log|z_2| + F_0.$$

Since T is positive, Eq. (71) means that F is a plurisubharmonic function in $\mathbb{C}^2 - \{0\}$. Therefore, by a standard result (see [4]), F extends across 0 as a plurisubharmonic function. Thus, F is lower semicontinuous on $\mathbb{C}^2$ (with values in $\mathbb{R} \cup \{-\infty\}$), and in particular, we know that

(72) *F is bounded from above in a neighborhood of* 0 *in* $\mathbb{C}^2$.

Note that by integrating over the S^1-action, we can assume that the function F_0 on X is Φ_t-invariant. This implies that

$$F(\Phi_t(z)) = F(z) + \mathrm{Re}(a)\, t$$

for all $t \in \mathbb{R}$. (Recall that $a = \log \alpha$ and $\mathrm{Re}(a) < 0$.) Furthermore, since $\Phi_{t+is} = \Phi_t \circ \Phi_{is}$, we have that

$$F(\Phi_{t+is}(z)) = F(\Phi_{is}(z)) + \mathrm{Re}(a)\, t \tag{73}$$

for all $t, s \in \mathbb{R}$. The expression in (73) is a subharmonic function in the (t, s)-plane which may be $\equiv -\infty$. We consider $z \in \mathbb{C}^2 - \{z_1\text{-axis}\}$ for which the function

$$f(s) \equiv F(\Phi_{is}(z))$$

is not identically $-\infty$ and such that

$$f'(0) = (J\,V)\,\phi(z)$$

is defined. (Note that $\Phi_*(\partial/\partial t) = V$ and $\Phi_*(\partial/\partial s) = J\,V$.) Since the function $f(s) + \mathrm{Re}(a)t$ is subharmonic and linear in t, it follows that $f(s)$ *is convex*, that is, $f''(s)$ is a non-negative measure. This implies that $f(s) \geqq f(0) + f'(0)s$ for all s, and therefore,

$$F(\Phi_{t+is}(z)) \geqq F(z) + bs + \mathrm{Re}(a)\, t \tag{74}$$

for all s and t, where

$$b = f'(0) = (J\,V)\,\phi(z). \tag{75}$$

We shall now show that if T is not a foliation current, then (64) contradicts the boundedness of F (near 0) established in (72). Examination of (62) shows that for a sequence of complex numbers $\tau_k = t_k + i s_k$, $k = 1, 2, 3, \ldots$, the points $\zeta_k \equiv \Phi_{\tau_k}(z) \to 0$ in $\mathbb{C}^2$ provided that $\mathrm{Re}(a\tau_k) \to -\infty$. By (74) we have that $F(\zeta_k) \geqq b s_k + \mathrm{Re}(a) t_k$. Hence, we look for a sequence of solutions to the system of real equations

$$\begin{aligned} \mathrm{Re}(a)\, t_k - \mathrm{Im}(a)\, s_k &= -k \\ \mathrm{Re}(a)\, t_k + b\, s_k &= k \end{aligned} \tag{76}$$

for $k = 1, 2, 3, \ldots$. These solutions exist unless $b = -\mathrm{Im}(a)$. The existence of these solutions implies F is not bounded from above near 0 in $\mathbb{C}^2$. Hence, we must have that $b = (JV)\phi(z) = -\mathrm{Im}(a)$ at all points z where this derivative is defined. In particular, the convex function $f(s)$ is linear. Thus the function $F(\Phi_{t+is})$ is linear in t and s. In particular, we conclude that $T = dd^c F$ vanishes on the orbits of Φ_τ, and so T is a foliation current as claimed. This completes the proof.

§ 10. Surfaces of Bombieri, Hirzebruch, and Inoue

The surfaces S_M, $S^{+}_{N,p,q,r;t}$, and $S^{(-)}_{N,p,q,r}$ of Bombieri and Inoue (see e.g. [1]) are all of the form

$$X \equiv \mathbb{H} \times \mathbb{C}/G \tag{77}$$

where the group G acts properly and discontinuously on $\mathbb{H} \times \mathbb{C}$ with no fixed points. (Here $\mathbb{H}$ denotes the upper half plane.) Moreover, the 2nd Betti number b_2 of X vanishes.

In this section we shall investigate these surfaces from the point of view of this paper (see [1] and its references for the standard definitions and facts).

The only other information about such surfaces which will be needed is the following.

(78) The action of G on $\mathbb{H} \times \mathbb{C}$ preserves the factors of the product.

The group H given by the induced action of G on $\mathbb{H}$ is a subgroup of $Bih(\mathbb{H}) = SL(2, \mathbb{R})$ and contains:

$$\begin{aligned} &h_0(z) = \alpha z && \alpha > 1 \text{ irrational} \\ &h_1(z) = z + \alpha && a \in \mathbb{R},\ a \neq 0 \\ &h_2(z) = z + \alpha a. && \end{aligned} \tag{79}$$

Choose coordinates $z = x + iy \in \mathbb{H}$ and $w \in \mathbb{C}$. Then the fact that $H \subset SL(2, \mathbb{R})$ implies that the forms

$$\tau \equiv \frac{dx}{y} \quad \text{and} \quad \omega \equiv d\tau = \frac{dx \wedge dy}{y^2} \tag{80}$$

are defined on the quotient manifold X.

(81) **Proposition.** *The Kähler rank of each compact complex surface defined by* (77) *is one. The foliation set $B(X)$ is all of X and the canonical foliation is holomorphic with leaves biholomorphic to $\mathbb{C}$ or $\mathbb{C}^*$.*

Proof. Since ω is never vanishing and positive with $\omega = d\tau$ on X, the Kähler rank is one and $B(X) = X$, with the leaves parameterized by $\{z\} \times \mathbb{C}$ under the quotient map.

(82) **Theorem.** *Suppose X is one of the compact complex surfaces given by* (77). *The cones*

$$P_{\mathrm{bdy}_{1,1}}(X) = P_{\mathrm{closed}}(X) = P_{\mathrm{bdy}}(X)$$

all agree and equal the cone $P(X)$ of positive closed foliation currents for the canonical foliation of X. This cone $P(X)$ can be described explicitly by

$$P(X) = \left\{ T\colon\ T \equiv \phi(y) \frac{dx \wedge dy}{y^2} \ \text{and}\ \phi \in P \right\}$$

where P is the set of non-negative generalized functions on $\mathbb{R}^+$ invariant under the transformation $y \to \alpha y$.

Remark. Note that the standard fact that X contains no curves is a special case of this Theorem.

The Theorem follows immediately from two independent lemmas.

(83) **Lemma.** *Suppose T is a positive current of bidimension* 1, 1 *on X which is either d-closed or the bidimensional* 1, 1 *component of a boundary on X. Then T is a positive d-closed foliation current for the canonical foliation on X.*

(84) **Lemma.** *Each positive d-closed foliation current T, for the canonical foliation on X, is of the form*

$$T \equiv \phi(y) \frac{dx \wedge dy}{y^2}$$

where $\phi(y) \geqq 0$ is a generalized function on $\mathbb{R}^+$ invariant under the transformation $y \to \alpha y$.

Proof of Lemma (83). If T is d-closed then $T(\omega) = T(d\tau) = 0$. If $T = (dS)_{1,1}$ then $T(\omega) = (dS)(\omega) = 0$ also. Since $T(\omega) = 0$, and both T and ω are positive, we conclude that the $dw \wedge d\bar{w}$-component of T must vanish identically. Hence, T can be expressed in the form

$$T = a_{11}\, i\, dz \wedge d\bar{z} + a_{12}\, i\, dz \wedge d\bar{w} + a_{21}\, i\, dw \wedge d\bar{z}.$$

Positivity now implies that $a_{11} \geqq 0$ and $a_{12} = a_{21} = 0$. Consequently, T is of the form $T = f\omega$, where $f \equiv 2a_{11} y^2$ is a non-negative generalized function on $\mathbb{H} \times \mathbb{C}$ which is G-invariant (since T and ω are).

Note that since $dd^c T = 0$, the function f is harmonic in the variable w. Thus for each non-negative function $\psi \in C_0^\infty(\mathbb{H})$, the integral $\int_{\mathbb{H}} f \psi\, \omega$ defines a non-negative harmonic function on $\mathbb{C}$, which by Harnack's Inequality must be constant. Consequently, f is a generalized function of z alone, so that $T=f\omega$ is d-closed. This completes the proof of Lemma 83.

Proof of Lemma (84). Since $T=f\omega$ is independent of z it is determined by a non-negative measure $\mu(=f\omega)$ on $\mathbb{H}$. This measure μ on $\mathbb{H}$ is invariant under the subgroup H of $SL(2,\mathbb{R})$ determined by the action of G on $\mathbb{H}\times\mathbb{C}$.

For each $\psi(y)\geqq 0$, where $\psi\in C_0^\infty(\mathbb{R}^+)$, the push-forward of $\psi\mu$ to the x-axis defines a non-negative measure $\nu=\pi_*(\psi\mu)$ on $\mathbb{R}$ which is invariant under translation by a and αa (α irrational) because of (79). Hence $\nu=c\,dx$, for some $c\in[0,\infty)$ by Weyl's Lemma. Consequently

$$\mu=f\frac{dx\wedge dy}{y^2}$$

with $f(y)\geqq 0$ depending only on y. This completes the proof.

Similar results can be obtained for the Inoue-Hirzebruch surfaces (see [1] and its references). Such a surface X has the property that there are two connected sets of curves in X, say C^+ and C^-. Moreover, $Y\equiv X-(C^+\cup C^-)$ can be expressed as a quotient $Y=\mathbb{H}\times\mathbb{C}/G$. Again the forms $\tau\equiv\frac{dx_1}{y_1}$ and $\omega\equiv d\tau$ are invariant under G. Moreover, they extend to X with $d\tau=\omega$ and ω positive. In particular, each Inoue-Hirzebruch surface has Kähler rank one.

References

1. Bombieri, E., Husemoller, D.: Classification and embeddings of surfaces. Proc. of Symp. in Pure Math. AMS **29**, 329–420 (1975)
2. Federer, H.: Geometric measure theory. Berlin-Heidelberg-New York: Springer 1967
3. Gunning, R.C., Rossi, H.: Analytic functions of several complex variables. Englewood Cliffs, N.J.: Prentice-Hall 1965
4. Harvey, R.: Holomorphic chains and their boundaries. Proc. Symp. Pure Math. **30**, Part I, AMS, Providence, R.I. (1977), pp. 309–382
5. Harvey, R., Knapp, A.W.: Positive p, p forms, Wirtinger's inequality, and currents. Proc. Tulane Conf. on Value Distributions, Part A, pp. 49–62. New York: Marcel Dekker 1973
6. Harvey, R., Lawson, Jr., H.B.: Calibrated foliations. Amer. J. Math. **104**, 607–643 (1982)
7. Harvey, R., Lawson, Jr., H.B.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)
8. Kodaira, K.: On the structure of compact complex analytic surfaces, I. Amer. J. Math. **86**, 751–798 (1964)
9. Michelsohn, M.L.: On the existence of special metrics in complex geometry. Acta Math. **143**, 261–295 (1983)
10. Miyaoka, Y.: Kähler metrics on elliptic surfaces. Proc. Japan Acad. **50**, 533–536 (1974)
11. Morrow, J., Kodaira, K.: Complex manifolds. New York: Holt-Rinehart and Winston 1971
12. Schaefer, H.H.: Topological vector spaces. Berlin-Heidelberg-New York: Springer 1971
13. Siu, Y.-T.: Every $K3$ surface is Kähler. Invent. Math. (to appear)
14. Sullivan, D.: Cycles for the dynamical study of foliated manifolds and complex manifolds. Invent. Math. **36**, 225–255 (1976)
15. Todorov, A.N.: Applications of the Kähler-Einstein-Calabi-Yau metric to moduli of $K3$ surfaces. Invent. Math. **61**, 251–265 (1980)

Oblatum 15-XI-1982 & 11-IV-1983