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Gelfand numbers of operators with values in a Hilbert space

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Summary. In the paper we prove two inequalities involving Gelfand numbers of operators with values in a Hilbert space. The first inequality is a Rademacher version of the main result in [Pa-To-1] which relates the Gelfand numbers of an operator from a Banach space X into l_2^n with a certain Rademacher average for the dual operator. The second inequality states that the Gelfand numbers of an operator u from l_1^N into a Hilbert space satisfy the inequality

$$k^{1/2}\, c_k(u) \leqq C\,\|u\|\,(\log(1+N/k))^{1/2}$$

where C is a universal constant. Several applications of these inequalities in the geometry of Banach spaces are given.

0. Introduction

In recent times, much attention has been paid to the study of (bounded linear) operators in Hilbert spaces and Banach spaces by means of geometric quantities, such as n-widths and entropy numbers. In the eighties research activity in this area grew considerably. A great deal of classical problems were solved, interesting new developments started, and deep connections between Banach space geometry and other areas of mathematics were discovered. In this article we prove two new inequalities in the local theory of Banach spaces having striking applications in the geometry of Banach spaces and the theory of Rademacher processes. Moreover, they offer new current research directions. These inequalities involve Gelfand numbers and Rademacher averages for operators with values in a Hilbert space (Theorem 1.2) and Gelfand numbers for operators from l_1^n into a Hilbert space (Theorem 2.2).

Moreover, the results are employed to the study of large subspaces of l_∞^N and l_p^N (Sect. 3) and to the problem of large euclidean subspaces (Sect. 4).

Let us recall some notions. The dual Banach space and the closed unit ball of a Banach space X are denoted by X^* and B_X, respectively. For an operator u from a Banach space X into a Banach space Y the n-th dyadic entropy number of u is defined by

$$e_n(u) =: \inf\{\varepsilon > 0 : \exists\, y_1, \ldots, y_{2^{n-1}} \in Y : u(B_X) \subset \bigcup_1^{2^{n-1}} (y_i + \varepsilon B_y)\}.$$

Moreover, the n-th approximation number of u is defined by

$$a_n(u) =: \inf\{\|u - v\| : \operatorname{rank} v < n\},$$

the n-th Gelfand number by

$$c_n(u) = \inf\{\|u|_Z\| : Z \subset X, \operatorname{codim} Z < n\}$$

and the n-th Kolmogorov number by

$$d_n(u) = \inf_{F \subset Y, \dim F < n} \sup_{x \in B_X} \inf_{y \in F} \|u(x) - y\|.$$

The n-th Kolmogorov number $d_n(u)$ of an operator u may be described as the infimum of all $\varepsilon > 0$ such that there is a subspace $F \subset Y$ with $\dim F < n$ and

$$u(B_X) \subset F + \varepsilon B_Y.$$

Roughly speaking the Kolmogorov numbers $d_n(u)$ deal with that part of the set $u(B_X)$ which lies outside a certain finite dimensional subspace $F \subset Y$.

There is a well-known duality relation (see [Pie]) $d_n(u^*) = c_n(u)$.

We denote by l_p^n the space $\mathbb{R}^n$ equipped with the norm

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad 1 \leqq p \leqq \infty$$

and by B_p^n its unit ball.

There are several constants which enter into the estimates below. These constants are denoted by letters like $a, a_1, b_1, c, c_1, c_2, \ldots$ We did not distinguish carefully between the different constants neither did we try to get good estimates for them. The same letter will be used to denote different universal constants in different parts of the paper.

1. Gelfand numbers and Rademacher averages

This section is devoted to basic estimates for Gelfand numbers and entropy numbers of operators with values in a Hilbert space with Rademacher average.

For this purpose we need the so-called l-norm of an operator from l_2^n into a Banach space X which is defined by

$$l(u) = \left(\int_{\mathbb{R}^n} \|u(x)\|^2 \, d\gamma_n(x)\right)^{1/2}$$

where γ_n denotes the canonical (normalized) Gaussian measure on the euclidean space $\mathbb{R}^n$. Moreover, for any operator u from l_2 into X we define $l(u)$ by

$$l(u) = \operatorname{Sup}\{l(uv) \colon \|v \colon l_2^n \to l_2\| \leqq 1, n = 1, 2, \ldots\}.$$

V. Milman discovered that the l-norm is an appropriate parameter for estimating Gelfand numbers (c.f. [M.1]). The sharp inequality which relates the Gelfand numbers with the l-norm is the main result from [Pa-To.1] which we now recall.

Theorem 1.1 [Pa-To.1]. *Let X be a Banach space and let $u \colon X \to l_2$ be a compact operator. Then*

$$\operatorname*{Sup}_{k \geqq 1} k^{1/2} c_k(u) \leqq a\, l(u^*) \tag{1.1}$$

where a is a universal constant.

Remark. This result solves the following geometrical problem: given an n-dimensional Banach space X and an euclidean norm $\|\cdot\|_2$ on X and $0 < \lambda < 1$, find a subspace E of X with $\dim E \geqq \lambda n$ such that

$$\|x\|_2 \leqq M_* f(1-\lambda)\|x\| \qquad \text{for } x \in E.$$

Here M_* denotes the Levy mean of the dual norm of X,

$$M_* = \Big(\int_{S^{n-1}} \|x\|_*^2 \, d\sigma(x)\Big)^{1/2}.$$

where σ is the normalized rotation invariant measure on the sphere $S^{n-1} = \{x \in \mathbb{R}^n \colon \|x\|_2 = 1\}$. This problem was considered by V. Milman who proved in [M.1] that $f(1-\lambda) \leqq c/(1-\lambda)$ where c is a universal constant. In this geometrical language, Theorem 1.1 states that even

$$f(1-\lambda) \leqq c/(1-\lambda)^{1/2}.$$

This kind of inequality possesses remarkable applications (cf. [M.2], [M.3], [B-M], [M-P.1], [M-P.2], [Pa-To.4], [P.2]...) For more information we refer to a forthcoming book of G. Pisier [P.4].

Now for our main theorem in this section we use a modified notion of the l-norm by taking instead of Gaussian variables, Rademacher variables. Let $f_1, \ldots, f_m$ be an orthonormal basis of l_2^m and let $v \colon l_2^m \to Y$ be an operator. Denote

$$r(v) = \Big(\operatorname*{Average}_{\varepsilon_i = \pm 1} \Big\|\sum_{i=1}^m \varepsilon_i\, v(f_i)\Big\|^2\Big)^{1/2}.$$

It is well-known that $r(v) \leqq c\, l(v)$ where c is a universal constant and actually the two norms $r(v)$ and $l(v)$ are equivalent when Y is a cotype q Banach space (see [Ma-P]). In some cases, a direct application of (1.1) will not give the sharp "logarithmic factor". The following inequality is the announced Rademacher version of Theorem 1.1.

Theorem 1.2. *Let $f_1, \ldots f_m$ be an orthonormal basis of l_2^m. Let X be a Banach space and let $u: X \to l_2^m$ be a rank n operator. Then*

$$k^{1/2} c_k(u) \leqq b \left(\underset{\varepsilon_i = \pm 1}{\text{Average}} \left\| \sum_{i=1}^{m} \varepsilon_i u^*(f_i) \right\|^2 \right)^{1/2} (\log(1+n/k))^{1/2} \tag{1.2}$$

for $1 \leqq k \leqq n \leqq m$, $m = 1, 2, \ldots$, where b is a universal constant.

Remark. (1) If we choose for $(f_i)_{i=1,\ldots,n}$ the canonical basis of $\mathbb{R}^n$, then for the identity operator i from l_1^n into l_2^n, we have $r(i^*) = 1$. Thus from theorem 1.2 we at once conclude the inequality,

$$c_k(i: l_1^n \to l_2^n) \leqq b \left(\frac{\log(1+n/k)}{k} \right)^{1/2} \tag{1.3}$$

for $k = 1, 2, \ldots, n$, $n = 1, 2, \ldots$, where b is a universal constant. Observe that as $l(i^*) \geqq c(\log n)^{1/2}$ for some constant $c > 0$, the estimate (1.3) cannot be obtained directly from Theorem 1.1. It was shown in [G-G] that this estimate (1.3) is optimal. A different proof of the Garnaev and Gluskin result will be given at the end of Sect. 2.

(2) Other related estimates in terms of the dual norm of the operator norm $l(\cdot)$ may be found in [Pa-To.3].

Proof of Theorem 1.2. The proof consists of two steps.

Step 1. Let $f_1, \ldots, f_m$ be as in the statement of Theorem 1.2 and put

$$r(v) = \left(\underset{\varepsilon_i = \pm 1}{\text{Average}} \left\| \sum_{i=1}^{m} \varepsilon_i v(f_i) \right\|^2 \right)^{1/2}$$

for every operator $v: l_2^m \to Y$.

In view of applying (1.1) we compare the Gaussian and Rademacher averages by the following inequality:

$$l(v) \leqq a_1 (\log(1 + n^{1/2} \|v\| r^{-1}(v)))^{1/2} r(v), \tag{1.4}$$

where a_1 is an absolute constant and rank $v = n$. To prove this inequality let $g_1, \ldots, g_m$ be i.i.d. standard Gaussian variables and let $t > 0$ be a number that will be chosen later. We define for each $i = 1, \ldots, m$ the truncated variables g_i' by $g_i' = g_i$ on the set where $|g_i| \leqq t$ and $g_i' = 0$ elsewhere. Now since the variables (g_i') are symmetric and bounded by t, a well known convexity argument gives

$$\left(\mathbb{E} \left\| \sum_{i=1}^{m} g_i' v(f_i) \right\|^2 \right)^{1/2} \leqq t(r(v)).$$

Now, let $g_i''=g_i-g_i'$ and let P be the orthogonal projection in l_2^m onto the orthogonal subspace to Ker v. So we have $v=vP$ and

$$\mathbb{E}\left\|\sum_{i=1}^{m} g_i''\, v(f_i)\right\|^2 \leqq \|v\|^2\, \mathbb{E}\left\|\sum_{i=1}^{m} g_i''\, P(f_i)\right\|^2.$$

From the symmetry of the variables (g_i'') and the parallelogram equality we deduce that

$$\mathbb{E}\left\|\sum_{i=1}^{m} g_i''\, P(f_i)\right\|^2 = \mathbb{E}\sum_{i=1}^{m} |g_i''|^2\, \|P(f_i)\|^2$$
$$= \|g_1''\|_{L^2}^2 \sum_{i=1}^{m} \|P(f_i)\|^2.$$

The latter sum, $\left(\sum_{i=1}^{m} \|P(f_i)\|^2\right)^{1/2}$ is the Hilbert-Schmidt norm of P and is therefore equal to $(\operatorname{rank} P)^{1/2}=n^{1/2}$.

Hence we get

$$\left(\mathbb{E}\left\|\sum_{i=1}^{m} g_i''\, v(f_i)\right\|^2\right)^{1/2} \leqq n^{1/2}\, \|v\|\, \|g_1''\|_{L^2}.$$

An easy computation shows that $\|g_1''\|_{L^2}\leqq(8/\pi)^{1/4}\, t^{1/2}\, e^{-t^2/4}$ for $t\geqq 1$. Therefore from the triangular inequality we get

$$l(v)\leqq(t+n^{1/2}\,\|v\|\, r^{-1}(v)(8/\pi)^{1/4}\, t^{1/2}\, e^{-t^2/4})\, r(v).$$

We now choose $t=2(\log(1+n^{1/2}\,\|v\|\, r^{-1}(v)))^{1/2}$ to minimize the last upper bound of $l(v)$ and conclude the proof of inequality (1.4).

Step 2. In order to control the logarithmic factor involved in (1.4), we introduce a parameter $\rho>0$ and the following renorming: let $u\colon X\to l_2^m$, we set $\|x\|_\rho=\max(\|x\|,\, \rho^{-1}\,\|u(x)\|)$ for every $x\in X$ and write X_ρ for the space X equipped with the norm $\|\cdot\|_\rho$. Moreover let $i\colon X_\rho\to X$ be the identity. Observe that $\|i\|\leqq 1$ and $\|ui\|\leqq\rho$. As shown in [Pa-To.2], this renorming will not affect estimates on $c_k(u)$ since we have the following characterization

$$c_k(u)=\inf\{\rho>0;\, \rho>c_k(ui\colon X_\rho\to l_2)\}. \qquad ((1.5)$$

Therefore using theorem 1.1 and (1.5) we get that if

$$a\, k^{-1/2}\, l((ui)^*)<\rho \quad \text{then} \quad c_k(u)<\rho.$$

For this conclusion, owing to (1.4) it is sufficient to have

$$a a_1 (\log(1+n^{1/2}\,\|ui\|\, r^{-1}((ui)^*)))^{1/2}\, k^{-1/2}\, r((ui)^*)<\rho. \qquad (1.6)$$

To outline the proof we denote $f(\rho)=\rho n^{1/2}\, r^{-1}((ui)^*)$. Since $\|ui\|\leqq\rho$, we can see that (1.6) holds whenever

$$(\log(1+f(\rho)))^{1/2} f(\rho)^{-1}<(a a_1)^{-1}(k/n)^{1/2}. \tag{1.7}$$

Clearly f is a continuous function and since $\|i\|\leqq 1$,

$$f(\rho)\geqq\rho n^{1/2}\, r^{-1}(u^*). \tag{1.8}$$

Let $E=\operatorname{Ker} u$ and $F=\operatorname{range}(u)$ and consider the canonical commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;u\;} & l_2^m \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle j} \\ X/E & \xrightarrow[\;w\;]{} & F. \end{array}$$

Moreover we use a similar diagram for ui instead of u where E and w are substituted by E_ρ and w_ρ respectively. Let P be the orthogonal projection from l_2^m onto F. Clearly

$$\begin{aligned} r((ui)^* = r(w^* P) &\geqq \|w^{*-1}\|^{-1} r(P) \\ &\geqq \|w_\rho^{-1}\|^{-1}(\operatorname{rank} P))^{1/2}. \end{aligned}$$

Now $\operatorname{rank}(P)=\operatorname{rank}(u)$, and $\|w_\rho^{-1}\|=\max(1/\rho,\ \|w^{-1}\|)$. Consequently, for ρ small enough,

$$f(\rho)\leqq(n/\operatorname{rank}(u))^{1/2}=1. \tag{1.9}$$

Observe that if $n<k$ then $c_k(u)=0$ and we have nothing to prove. So we may assume $n\geqq k$. Combining (1.8) and (1.9), for any $\lambda\geqq 1$, we may find ρ such that

$$f(\rho)=\lambda(n/k)^{1/2}(\log(1+n/k))^{1/2}.$$

Set $g(\lambda)=\lambda(n/k)^{1/2}(\log(1+n/k))^{1/2}$. An elementary computation shows that

$$(\log(1+g(\lambda)))^{1/2}\, g(\lambda)^{-1}<(a a_1)^{-1}(k/n)^{1/2}$$

for some universal constant $\lambda=b\geqq 1$.

Hence we can find ρ such that

$$f(\rho)=b(n/k)^{1/2}(\log(1+n/k))^{1/2}$$

and (1.7) is satisfied. Consequently for such a number ρ, we have $c_k(u)<\rho$. Coming back to the definition of f and observing that $r((ui)^*)\leqq r(u^*)$ we obtain

$$\rho\leqq b\, k^{-1/2}(\log(1+n/k))^{1/2}\, r(u^*)$$

which proves (1.2). $\square$

Now from the previous inequalities we provide corresponding inequalities for entropy numbers. First of all we recall a result from [C1]. Let $\alpha>0$ and let $u: X \to Y$ be an operator, then

$$\sup_{1\leqq k\leqq n} k^\alpha e_k(u) \leqq a(\alpha) \sup_{1\leqq k\leqq n} k^\alpha c_k(u) \tag{1.10}$$

for $n=1,2,\ldots$, where $a(\alpha)$ depends only on α. The relation (1.10) is also valid for Kolmogorov numbers instead of Gelfand numbers. The following lemma that we shall need is an immediate consequence of this result.

Lemma 1.3. *Let X and Y be Banach spaces and let $u: X\to Y$ be an operator. Let $\alpha>0, \beta>0, \gamma>0$ and let $\{s_i\}$ stand either for Gelfand numbers $\{c_i\}$ or Kolmogorov numbers $\{d_i\}$. Then*

$$\sup_{1\leqq k\leqq n} k^\alpha \log^{-\beta}(1+\gamma/k)\, e_k(u) \leqq a(\alpha,\beta) \sup_{1\leqq k\leqq n} k^\alpha \log^{-\beta}(1+\gamma/k)\, s_k(u) \tag{1.11}$$

for $n=1,2,\ldots$ where $a(\alpha,\beta)$ depends only on α and β.

Proof. Let $1\leqq m\leqq n$. From (1.10) we deduce that

$$m^{\alpha+\beta} e_m(u) \leqq a(\alpha+\beta) \sup_{1\leqq k\leqq m} (k\log(1+\gamma/k))^\beta \sup_{1\leqq k\leqq m} k^\alpha \log^{-\beta}(1+\gamma/k)\, s_k(u).$$

Now observe that $k\log(1+\gamma/k)$ is an increasing function of k. Hence

$$m^\alpha \log^{-\beta}(1+\gamma/k)\, e_m(u) \leqq a(\alpha+\beta) \sup_{1\leqq k\leqq m} k^\alpha \log^{-\beta}(1+\gamma/k)\, s_k(u).$$

Taking the supremum on $m\leqq n$, we get (1.11).

Corollary 1.4. *Let $f_1,\ldots,f_m$ be an orthonormal basis of l_2^m.*

(a) *Let $u: X\to l_2^m$ be a rank n operator. Then*

$$k^{1/2} e_k(u) \leqq d\left(\operatorname*{Average}_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^m \varepsilon_i u^*(f_i)\right\|^2\right)^{1/2} (\log(1+n/k))^{1/2}$$

for $1\leqq k\leqq n\leqq m, m=1,2,\ldots$ where d is a universal constant.

(b) *Let $v: l_2^m\to Y$ be a* rank *n operator. Then*

$$k^{1/2} e_k(v) \leqq d\left(\operatorname*{Average}_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^m \varepsilon_i v(f_i)\right\|^2\right)^{1/2} (\log(1+n/k))^{1/2}$$

for $1\leqq k\leqq n\leqq m$, $m=1,2\ldots$ where d is a universal constant.

Proof. The first inequality follows immediately from Theorem 1.2 and from (1.11) in Lemma 1.3. The second inequality can be checked along the same line by using in addition the relation $c_k(v^*)=d_k(v)$.

Remark. The estimates of Corollary 1.4 are sharp (cf. remark to Corollary 2.4).

The part (a) of corollary 1.4 is a Rademacher version of the Sudakov inequality. To make it clear we restate this inequality.

Corollary 1.5. Let $T \subset \mathbb{R}^n$ *and let* $N_2(T, \varepsilon)$ *be the minimal cardinality of an* ε*-net of* T *in the euclidean metric. Then*

$$\underset{\varepsilon_i=\pm 1}{\text{Average}} \; \underset{(t_i)_{i=1}^{i=n} \in T}{\text{Sup}} \left| \sum_{i=1}^{n} \varepsilon_i t_i \right| \geq a\, \varepsilon (\log N_2(T, \varepsilon))^{1/2} (\log(2 + n/\log N_2(T, \varepsilon)))^{-1/2}$$

where $a > 0$ *is a universal constant.*

Remark. (1) The Rademacher norm $r(\cdot)$ is usually defined as an L_2-norm. From Kahane's inequalities, Corollary 1.5 is equivalent to Corollary 1.4 part (a).

(2) Rademacher processes play an important role in the study of empirical processes (see [Gi-Z], [T]). Within this framework, the inequality in Corollary 1.5 gives new information.

2. Gelfand numbers of operators from l_1^n into a Hilbert space

The main aim of this section is to give an extension of the Garnaev-Gluskin result (1.3) to arbitrary operators from l_1^n into a Hilbert space H.

For doing this we need some more notions. We will say that an operator u from a Banach space X into a Banach space Y is p-summing, $0 < p < \infty$, if there is a constant $C \geq 0$ such that, for all finite families $x_1, \ldots, x_n \in X$ we have

$$\left(\sum_{i=1}^{n} \|u(x_i)\|^p \right)^{1/p} \leq C \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, a \rangle|^p \right)^{1/p} : \|a\| \leq 1 \right\}.$$

The smallest constant C satisfying this inequality is denoted by $\pi_p(u)$. Sometimes we use the following easy characterization of $\pi_p(u)$, $\frac{1}{p'} = 1 - \frac{1}{p}$,

$$\pi_p(u) = \sup\{\pi_p(uv) : \|v : l_{p'}^n \to X\| \leq 1, n = 1, 2, \ldots\}.$$

Furthermore, we say that a Banach space X is of (Rademacher) type p, $1 < p \leq 2$, if there is a constant $C \geq 0$ such that for all finite families $x_1, \ldots, x_n \in X$ the inequality

$$\left(\underset{\varepsilon_i = \pm 1}{\text{Average}} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^2 \right)^{1/2} \leq C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}$$

is valid. The (Rademacher) type p constant of X is defined by $\tau_p(X) = \inf C$. As an example let us mention that the function spaces L_r, $1 \leq r < \infty$, over arbitrary σ-finite measure space are of type $\min(r, 2)$ (see [M-Sche]). For the Rademacher type constant one has $\tau_{\min(r,2)}(L_r) \leqq \sqrt{r}$ (cf. [Ma-P]).

First of all we supplement the statement of Theorem 1.1 by the following conclusion.

Lemma 2.1. *Let X and Y be Banach spaces such that X^* is of type 2. Then for any 2-summing operator u from X into Y the inequality*

$$\sup_{k \geqq 1} k^{1/2} c_k(u) \leqq a \, \tau_2(X^*) \, \pi_2(u) \tag{2.1}$$

is valid with the universal constant a of Theorem 1.1.

Proof. First we suppose $Y = H$ to be a Hilbert space. For a 2-summing operator v from X into H we have

$$l(v^*) \leqq \tau_2(X^*) \, \pi_2(v^{**})$$

by [D-M-To]. A result of Pietsch tells us that $\pi_2(v^{**}) = \pi_2(v)$ (cf. [Pie]), so that in consequence of the inequality (1.1) of Theorem 1.1. the inequality

$$\sup_{k \geqq 1} k^{1/2} c_k(v) \leqq a \, \tau_2(X^*) \, \pi_2(v)$$

comes out to be true. In the general case if u is a 2-summing operator from X into Y we employ the well-known factorization theorem for 2-summing operators which states that $u = wv$ with a 2-summing operator v from X into H and an operator w from H into Y, where H is a Hilbert space and

$$\pi_2(u) = \|w\| \, \pi_2(v)$$

(cf. [Pie]). This yields

$$\sup_{k \geqq 1} k^{1/2} c_k(u) \leqq \|w\| \sup_{k \geqq 1} k^{1/2} c_k(v)$$

and thus finally

$$\sup_{k \geqq 1} k^{1/2} c_k(u) \leqq a \, \tau_2(X^*) \, \pi_2(u). \quad \square$$

Now we are in a position to prove the main inequality of this section.

Theorem 2.2. *Let u be an operator from l_1^n into a Hilbert space H. Then*

$$c_k(u) \leqq C \left(\frac{\log\left(\frac{n}{k} + 1\right)}{k} \right)^{1/2} \|u\| \tag{2.2}$$

for $1 \leqq k \leqq n$, $n = 1, 2, \ldots$, where $C > 0$ is a universal constant.

Proof. To provide this inequality we first derive a corresponding estimate to (1.3) for identity operators from l_1^n into l_p^n, $1 < p < 2$, and combine this estimates with the inequality (2.1) for arbitrary operators from l_p^n into a Hilbert space

H by choosing an appropriate p. Indeed, we factorize an arbitrary operator $u: l_1^n \to H$ by

$$u = u_p\, i_{1,p},$$

where $i_{1,p}: l_1^n \to l_p^n$ is the identity operator from l_1^n into l_p^n and $u_p: l_p^n \to H$ the operator from l_p^n into H induced by u. The multiplicativity of the Gelfand numbers then yields

$$c_{2k}(u) \leqq c_k(u_p)\, c_k(i_{1,p}). \tag{2.3}$$

The behaviour of the Gelfand numbers $c_k(i_{1,p})$ can be derived from the behaviour of the Gelfand numbers $c_k(i: l_1^n \to l_2^n)$ by the inequality

$$c_k(i_{1,p}) \leqq c_k(i)^{\frac{2}{p'}}, \qquad \frac{1}{p'} = 1 - \frac{1}{p}, \qquad 1 < p < 2,$$

which is an immediate consequence from Hölder's inequality

$$\|x\|_p \leqq \|x\|_2^{\frac{2}{p'}} \|x\|_1^{1-\frac{2}{p'}} \qquad \text{for } x \in l_1^n.$$

Now from (1.3) we obtain

$$c_k(i_{1,p}) \leqq c^{\frac{2}{p'}} \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{\frac{1}{p'}} \leqq c \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{\frac{1}{p'}}, \tag{2.4}$$

$1 < p < 2$. It remains to estimate $c_k(u_p)$. Since the dual $l_{p'}^n$ of l_p^n has got a (Rademacher) type 2 constant $\tau_2(l_{p'}^n)$ with

$$\tau_2(l_{p'}^n) \leqq \sqrt{p'}$$

(cf. [Ma-P]), Lemma 2.1 tells us that

$$c_k(u_p) \leqq a \sqrt{p'}\, k^{-1/2}\, \pi_2(u_p).$$

In this situation we again go back to the original operator $u: l_1^n \to H$ by putting

$$u_p = u\, i_{p,1}$$

with $i_{p,1}: l_p^n \to l_1^n$ as the identity operator from l_p^n into l_1^n. Because of

$$\pi_2(u_p) \leqq \pi_2(u)\, \|i_{p,1}\| \leqq n^{\frac{1}{p'}}\, \pi_2(u)$$

the problem of estimating $c_k(u_p)$ finally reduced to the problem of estimating $\pi_2(u)$. Grothendieck's inequality

$$\pi_2(u) \leqq C_G \|u\|$$

serves for this purpose, C_G being a universal constant (cf. [L-P]). The result is

$$c_k(u_p) \leqq a\, C_G \sqrt{p'}\, n^{\frac{1}{p'}}\, k^{-\frac{1}{2}} \|u\|. \tag{2.5}$$

By the aid of (2.5) and (2.4) we now actually can estimate $c_{2k}(u)$, namely

$$c_{2k}(u) \leqq a\, C\, C_G \sqrt{p'} \left(\log\left(\frac{n}{k}+1\right)\right)^{\frac{1}{p'}} \left(\frac{n}{k}\right)^{\frac{1}{p'}} k^{-\frac{1}{2}} \|u\| \tag{2.6}$$

according to (2.3). We still enlarge

$$\left(\log\left(\frac{n}{k}+1\right)\right)^{\frac{1}{p'}} \left(\frac{n}{k}\right)^{\frac{1}{p'}}$$

on the right-hand side of (2.6) into

$$\left(\frac{n}{k}+1\right)^{\frac{2}{p'}} \geqq \left(\log\left(\frac{n}{k}+1\right)\right)^{\frac{1}{p'}} \left(\frac{n}{k}\right)^{\frac{1}{p'}}$$

thus simplifying the inequality (2.6) to

$$c_{2k}(u) \leqq a\, C\, C_G \sqrt{p'} \left(\frac{n}{k}+1\right)^{\frac{2}{p'}} k^{-1/2} \|u\|. \tag{2.7}$$

So far we have not disposed of p and p', respectively. Now we do by setting

$$p' = 4 \log\left(\frac{n}{k}+1\right) > 2 \tag{2.8}$$

which ensures $1<p<2$ as required in connection with (2.4). The statement (2.8) implies that the factor $\left(\frac{n}{k}+1\right)^{\frac{2}{p'}}$ on the right-hand side of (2.7) appears as the universal constant

$$\sqrt{e} = e^{\frac{2}{p'} \log\left(\frac{n}{k}+1\right)} = \left(\frac{n}{k}+1\right)^{\frac{2}{p'}}$$

while $\sqrt{p'}$ gives rise to the desired logarithmic term

$$\sqrt{p'} = 2\left(\log\left(\frac{n}{k}+1\right)\right)^{1/2}.$$

This way we reach the stage

$$c_{2k}(u) \leqq 2\sqrt{e}\, a\, C\, C_g \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} \|u\|, \tag{2.9}$$

which at once implies the desired inequality by passing from $2k$ to k. □

The inequality (2.2) of Theorem 2.2 possesses already in germ a refinement which we state in the next theorem.

Theorem 2.3. (i) *Let u be an operator from l_1^n into a Hilbert space H. Then*

$$c_{k+m-1}(u) \leqq C \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} d_m(u) \tag{2.10}$$

for $1 \leqq k, m \leqq n$, $n=1,2,\ldots$, where $C>0$ is a universal constant.

(ii) *Let v be an operator from a Hilbert space H into l_∞^n. Then*

$$d_{k+m-1}(v) \leqq C \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} c_m(v) \tag{2.11}$$

for $1 \leqq k, m \leqq n$, $n=1,2,\ldots$, where $C>0$ is a universal constant.

Proof. To check the inequality (i) we assume that $w: l_1^n \to H$ is an operator with rank $w<m$. Then

$$c_{k+m-1}(u) \leqq c_k(u-w) + c_m(w) = c_k(u-w),$$

since $c_m(w)=0$. Applying Theorem 2.2 to the operator $u-w$ we arrive at

$$c_{k+m-1}(u) \leqq c_k(u-w) \leqq C \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} \|u-w\|$$

for any $w: l_1^n \to H$ with rank $(w)<m$. This already implies the first assertion since for operators on l_1^n the Kolmogorov numbers and approximation numbers coincide,

$$d_m(u) = a_m(u) = \inf\{\|u-w\| : \operatorname{rank}(w) < m\},$$

(cf. [Pie]). The second inequality of the theorem can be immediately checked from the first one with the aid of the duality relations

$$d_{k+m-1}(v) = c_{k+m-1}(v^*) \quad \text{and} \quad c_m(v) = d_m(v^*)$$

which are consequences of Lindenstrauss-Rosenthal's principle of local reflexivity (cf. [Pie]). □

Concerning entropy numbers we may give the following inequalities.

Corollary 2.4. (i) *Let u be an operator from l_1^n into a Hilbert space H. Then*

$$e_k(u) \leqq C \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} \|u\| \tag{2.12}$$

for $1 \leqq k \leqq n$, $n = 1, 2, \ldots$, *where* $C > 0$ *is a universal constant.*

(ii) *Let v be an operator from a Hilbert space H into l_∞^n. Then*

$$e_k(v) \leqq C \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} \|v\|, \tag{2.13}$$

for $1 \leqq k \leqq n$, $n = 1, 2, \ldots$, *where* $C > 0$ *is a universal constant.*

Proof. The inequalities follow immediately from the corresponding inequalities of Theorem 2.3 by applying Lemma 1.3. □

Remark. 1) The inequality (i) of Corollary 2.4 appears as a special case of an inequality in [C.2] for operators from l_1^n with values in a Banach space of type p.

2) A recent result of Tomczak-Jaegermann [To] on duality of entropy numbers states that actually (i) and (ii) of Corollary 2.4 are equivalent.

3) The estimates of Corollary 2.4 are sharp as can be seen from the following result of Schütt [S] which states that for the identity operator i from l_1^n into l_2^n the expression

$$\min \left\{ 1; \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} \right\}$$

gives an exact description for the behaviour of

$$e_k(i: l_1^n \to l_2^n) \quad \text{for } 1 \leqq k \leqq n, \quad n = 1, 2, \ldots.$$

Finally, in the remaining part of this section we shall show that the estimate in Theorem 2.2 is the best possible for identity operators from l_1^n into l_2^n. We show that the previous expression for entropy numbers even gives an exact description for the behaviour of the Gelfand numbers $c_k(i\colon l_1^n \to l_2^n)$. This fact has been proved by Garnaev and Gluskin [G-G]. One side of the inequality already appeared in (1.3). Now our proof of the opposite inequality is different.

Corollary 2.6. (Garnaev/Gluskin). *Let* $i\colon l_1^n \to l_2^n$ *be the identity operator from* l_1^n *into* l_n^2. *Then*

$$C_0 \min \left\{ 1; \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} \right\} \leqq c_k(i) \leqq C_1 \min \left\{ 1; \left(\frac{\log\left(\frac{n}{k}+1\right)}{k} \right)^{1/2} \right\}$$

for $1 \leqq k \leqq n$, $n = 1, 2, \ldots$, C_0, $C_1 > 0$ *are universal constants.*

Proof. From (1.3) and the obvious estimate $c_k(i) \leqq \|i\| \leqq 1$ we at once conclude the estimate from above. To provide the estimate from below we insert the result of Schütt [S],

$$\rho_0 \min\left\{1; \left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\right)^{1/2}\right\} \leqq e_k(i) \tag{2.14}$$

for $1 \leqq k \leqq n$, $n=1,2,\ldots$, where $\rho_0>0$ is a universal constant. In particular we have

$$\rho_1 \left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\right)^{1/2} \leqq e_k(i) \quad \text{for } \log n \leqq k \leqq n, \tag{2.15}$$

$n=1,2,\ldots$, where $\rho_1>0$ is a universal constant. Now we apply (1.10) and obtain

$$\rho_1\left(k \log\left(\frac{n}{k}+1\right)\right)^{1/2} \leqq k\, e_k(i) \leqq \rho_2 \sup_{1 \leqq j \leqq k} j\, c_j(i). \tag{2.16}$$

Hereafter we introduce a bound $\frac{k}{\lambda}$ for j, dividing the supremum with respect to j into two items

$$\sup_{1 \leqq j \leqq k} j\, c_j(i) \leqq \sup_{1 \leqq j \leqq \frac{k}{\lambda}} j\, c_j(i) + \sup_{\frac{k}{\lambda} < j \leqq k} j\, c_j(i). \tag{2.17}$$

The first item on the right-hand side of this inequality is estimated by the aid of (1.3), namely

$$\sup_{1 \leqq j \leqq \frac{k}{\lambda}} j\, c_j(i) \leqq C \sup_{1 \leqq j \leqq \frac{k}{\lambda}} \left(j \log\left(\frac{n}{j}+1\right)\right)^{1/2}$$
$$\leqq C\left(\frac{k}{\lambda} \log\left(\frac{\lambda n}{k}+1\right)\right)^{1/2}$$

since $x \log\left(\frac{n}{x}+1\right)$ is monotonously increasing for $1 \leqq x \leqq n$. Moreover, because of $\lambda \geqq 1$ and

$$\frac{\lambda n}{k}+1 \leqq \left(\frac{n}{k}+1\right)^{1+2\log\lambda}$$

we have

$$\sup_{1 \leqq j \leqq \frac{k}{\lambda}} j\, c_j(i) \leqq C\left(\frac{1+2\log\lambda}{\lambda}\right)^{1/2} \left(k \log\left(\frac{n}{k}+1\right)\right)^{1/2}.$$

The second item on the right-hand side of (2.17) is enlarged by passing to

$$\sup_{\frac{k}{\lambda}\leqq j\leqq k} j\, c_j(i)\leqq k\, c_{\left[\frac{k}{\lambda}\right]}(i).$$

Hence it follows

$$\left(\rho_1 - C\,\rho_2\left(\frac{1+2\log\lambda}{\lambda}\right)^{1/2}\right)\left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\right)^{1/2}\leqq \rho_2\, c_{\left[\frac{k}{\lambda}\right]}(i) \tag{2.18}$$

from (2.16). So far we have not yet disposed of $\lambda\geqq 1$. Now we do by demanding

$$C\,\rho_2\left(\frac{1+2\log\lambda}{\lambda}\right)^{1/2}\leqq\frac{\rho_1}{2}. \tag{2.19}$$

Owing to $\lim_{\lambda\to\infty}\frac{1+2\log\lambda}{\lambda}=0$ the condition (2.19) can be satisfied for some $\lambda=\lambda_0\geqq 1$. Hence (2.18) enables us to state

$$c_{\left[\frac{h}{\lambda_0}\right]}(i)\geqq\frac{\rho_1}{2\rho_2}\left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\right)^{1/2}\qquad\text{for }\log n\leqq k\leqq n.$$

Changing from $\left[\frac{k}{\lambda_0}\right]$ to k we may achieve with some new universal constant $\rho_3>0$ the estimate

$$c_k(i)\geqq\rho_3\left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\right)^{1/2}\qquad\text{for }\log n\leqq k\leqq n.$$

Obviously, with some universal constant $\rho_4>0$, we have

$$c_k(i)\geqq\rho_4\qquad\text{for }1\leqq k\leqq\log n.$$

Altogether we arrive at the desired estimate from below,

$$c_k(i)\geqq C_0\min\left\{1;\left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\right)^{1/2}\right\}.\quad\square$$

Remark. In [C-D] has been developed a general but elementary concept for proving new inequalities in the theory of absolutely summing operators. Actually by this concept one can show that the estimates from Theorem 2.2 as well as from Corollary 2.4 are even equivalent to the famous Grothendieck-inequality in the metric theory of tensor products.

Recently, Theorem 2.2 has been successfully applied in [C-H-K] to give new insights into the theory of integral operators with values in $C(X)$ of all continuous functions over a compact metric space X. There one can find an interplay between entropy properties of the underlying compact metric space X and eigenvalues, approximation and entropy quantities of the integral operator.

3. Large subspaces of l_∞^N and l_p^N

In this section we study some properties of "large" subspace of l_∞^N and l_p^N expressed in terms of volume ratio, Banach-Mazur distance and projection constants by using the result in Sect. 2. In particular we improve some results of Figiel and Johnson [F-J] and complement results of König [Kö].

First of all we recall some definitions. Let X and Y be two Banach spaces. The Banach-Mazur distance is defined by

$$d(X, Y)=\inf\{\|u\|\,\|u^{-1}\|\}$$

where the infimum runs over all isomorphism $u\colon X\to Y$.

Moreover, let X be an n-dimensional Banach space and let $\mathrm{vol}(\cdot)$ be any volume measure on X. Then the volume ratio $vr(X)$ is defined by

$$vr(X)=\inf(\mathrm{vol}(B_X)/\mathrm{vol}(\mathscr{E}))^{1/n}$$

where the infimum runs over all ellipsoids $\mathscr{E}$ contained in B_X.

We start our considerations with a sharp estimate for entropy numbers of operators from l_1^N into a Banach space of type p which is adapted from [C2, Proposition 1].

Lemma 3.1. *Let $N=1,2,\dots$ and v be an operator from l_1^N into a Banach space X of type p, then*

$$e_k(v)\leqq c\,\tau_p(X)\,\|v\|\left(\frac{\log\left(1+\frac{N}{k}\right)}{k}\right)^{1-1/p} \tag{3.1}$$

for $k=1,2,\dots,N$ where c is a universal constant.

Now the following result on Banach-Mazur distances will be an immediate consequence of this entropy estimate.

Theorem 3.2. *Let X be an n-dimensional subspace of l_∞^N and E an n-dimensional Banach space. Then*

$$d(X, E)\,\tau_p(E^*)\geqq c\,n^{1-1/p}\left(\log\left(1+\frac{N}{n}\right)\right)^{-1+1/p} \tag{3.2}$$

where $c>0$ is a universal constant.

Proof. Let $v: E \to X$ be any invertible operator and let i be the natural embedding from X into l_∞^N. Then owing to

$$e_k(\mathrm{id}_{X^*}) = e_k(i^*) = e_k(v^{*-1} v^* i^*) \leqq \|v^{*-1}\| e_k(v^* i^*)$$

we gain from the easy inequality

$$2^{-k/n} \leqq e_k(\mathrm{id}_{X^*})$$

and Lemma 3.1, the estimate

$$2^{-k/n} \leqq c \|v^{*-1}\| \tau_p(E^*) \|v^* i^*\| \left(\frac{\log\left(1+\frac{N}{k}\right)}{k}\right)^{1-1/p}$$

or

$$\|v\| \|v^{-1}\| \tau_p(E^*) \geqq \frac{1}{c} 2^{-k/n} \left(\frac{k}{\log\left(1+\frac{N}{k}\right)}\right)^{1-1/p}.$$

Putting $k=n$ and taking the infimum over all invertible operators on the left hand side of the last inequality, we conclude the desired estimates for the Banach-Mazur distance.

The following theorem gives a sharp estimate on the volume ratio of large euclidean section of l_∞^N which improves a result from [F–J].

Theorem 3.3. *Let X be an n-dimensional subspace of l_∞^N. Then*

$$d(X, l_2^n) \geqq vr(X) \geqq a\, n^{1/2} (\log(1+N/n))^{-1/2} \tag{3.3}$$

where a is a universal constant.

Proof. Of course the inequality $d(X, l_2^n) \geqq vr(X)$ is valid for every n-dimensional Banach space. So we prove the second inequality. Let $v: l_2^n \to X$ be an operator with $\|v\| \leqq 1$ and let $i: X \to l_\infty^N$ be the embedding map. From corollary 2.4 we get that

$$e_n(v) \leqq 2\, e_n(iv) \leqq 2\, a\, n^{-1/2} (\log(1+N/n))^{1/2}.$$

Now we clearly have

$$(\mathrm{vol}(v(B_2^n))/\mathrm{vol}(B_X))^{1/n} \leqq 2\, e_n(v).$$

Therefore we obtain

$$(\mathrm{vol}(B_X)/\mathrm{vol}(\mathscr{E}))^{1/n} \geqq (1/4a)\, n^{1/2} (\log(1+N/n)^{-1/2}$$

for every ellipsoid $\mathscr{E}$ contained in B_X, which proves (3.3).

Remarks. (1) The following example from [F-J] shows that (3.3) is sharp. Let $k \geqq 1$, $m \geqq 1$ and let E be an m-dimensional subspace of $l_\infty^{5^m}$ which is 2-isomorphic to l_2^m (it is well-known that there exists such a space). Moreover let $X = E \oplus_\infty \ldots \oplus_\infty E$ be the l_∞ direct sum of k copies of E. Let $n = mk = \dim X$ and let $N = k\, 5^m$ so that $X \subset l_\infty^N$. Since $d(E, l_2^m) \leqq 2$ we have $d(X, l_2^n) \leqq 2\, k^{1/2}$ which show that the inequality (3.3) is sharp for $n \geqq \log N$.

(2) The inequality

$$d(X, l_2^n) \geqq a\, n^{1/2} (\log(1 + N/n))^{-1/2}$$

in Theorem 3.3 was also recently obtained in [B-L-M] and by E.D. Gluskin.

(3) As it is well-known from Santalo's inequality $vr(X)\; vr(X^*) \leqq d(X, l_2^n) \leqq n^{1/2}$. Therefore if X is an n-dimensional subspace of l_∞^N, then

$$vr(X^*) \leqq a (\log(1 + N/n))^{1/2}$$

where a is a universal constant.

(4) The inequality (3.3) is in some sense optimal for "random" subspaces of l_∞^N. Indeed from a result of Kashin [K], we have that for N^α-dimensional "random" subspaces E of $\mathbb{R}^N$, $0 < \alpha < 1$,

$$c \sqrt{\frac{N}{n}}\, E \cap B_2^N \subset E \cap B_\infty^N \subset c(\alpha) \sqrt{\frac{N}{\log N}}\, E \cap B_2^N$$

for some constants $c > 0$ and $c(\alpha)$.

(5) Let X be an n-dimensional Banach space and let A be a set of N points in the unit ball B_X of X. Then Lemma 3.1 immediately implies the following estimates:

$$(\mathrm{vol}(\mathrm{conv}(A))/\mathrm{vol}(B_X))^{1/n} \leqq c\, \tau_p(X) \left(\log\left(1 + \frac{N}{n}\right) \Big/ n\right)^{1-1/p}$$

where c is a universal constant. It may be checked, that when $X = l_q^n$, by choosing a suitable p, the latter estimate on the volume of the convex hull of A is optimal.

Theorem 3.4. *Let X be an n-dimensional subspace of l_p^n, $2 < p < \infty$. Then*

$$d(X, l_2^n) \geqq vr(X) \geqq \frac{c}{\sqrt{p}}\, n^{1/2 - 1/p} \left(\frac{n}{N}\right)^{1/p}$$

where $c > 0$ is a universal constant.

Proof. Let u: $l_2^n \to Y$ be any operator. Then from Lemma 2.1 and from (1.10) we get

$$k^{1/2}\, e_k(u) \leqq c\, \tau_2(Y)\, \pi_2(u^*)$$

Now let X be any n-dimensional subspace of l_p^N, v: $l_2^n \to X$ an invertible operator and i: $X \to l_p^N$ the embedding map. Then

$$n^{1/2}\, e_n(v) \leqq 2 n^{1/2}\, e_n(iv) \leqq 2\, c\, \tau_2(l_p^N)\, \pi_2(v^*\, i^*).$$

From Grothendieck's inequality we check

$$\pi_2(v^* i^*) \leqq \pi_2(v^* i^* \mathrm{id}_{1,p'}) \|\mathrm{id}_{p',1}\| \\ \leqq c' N^{1/p} \|v^* i^* \mathrm{id}_{1,p'}\| \leqq c' N^{1/p} \|v\|.$$

Consequently, owing to $\tau_2(l_p^N) \leqq \sqrt{p}$, we obtain

$$e_n(v) \leqq 2cc' \sqrt{p}\, n^{-1/2} N^{1/p} \|v\|.$$

Similarly as in Theorem 3.3 we immediately get the assertion of the theorem.

Remark. (1) We mention that the inequality in Theorem 3.4 is in a certain sense optimal. Indeed, by [F-L-M] there exists an m-dimensional subspace E of $l_p^{m^{p/2}}$ which is a-isomorphic to l_2^m. Moreover, let $X = E \oplus_p \ldots \oplus_p E$ be the l_p direct sum of k copies of E. Then $\dim X = mk = n$. Setting $N = k\,m^{p/2}$ we have $X \subset l_p^N$. Since $d(E, l_2^m) \leqq a$ we gain

$$\frac{c}{\sqrt{p}} n^{1/2} N^{-1/p} \leqq d(X, l_2^n) \leqq a\, k^{1/2-1/p} = a\, n^{1/2} N^{-1/p}.$$

(2) A result of Lewis [Le] states that for an n-dimensional subspace X of l_p^N we always have

$$d(X, l_2^n) \leqq n^{1/2-1/p}.$$

If we take δ with $0 < \delta < 1$ and let $n = [\delta N]$, then we have by Theorem 3.4

$$\frac{c}{\sqrt{p}} \delta^{1/p} n^{1/2-1/p} \leqq d(X, l_2^n) \leqq n^{1/2-1/p}.$$

Our next theorem uses a result from [Pa] based on a combinatorial lemma of Sauer and Vapnik and Cervonenkis. First of all let us recall that the Minkowski sum of two bodies K_1 and K_2 in $\mathbb{R}^n$ is defined by

$$K_1 + K_2 = \{x_1 + x_2;\, x_1 \in K_1,\, x_2 \in K_2\}.$$

Lemma 3.5 [Pa]. *Let $K \subset \mathbb{R}^N$ be a convex compact set satisfying*

$$K \subset [-1,1]^N \quad \textit{and} \quad \mathrm{vol}(K + t[-1,1]^N) \geqq 2^N (t+a)^N$$

for some numbers $t > 0$ and $0 < a \leqq 1$. Then, for every $\varepsilon > 0$, there exists a subset I of $\{1, 2, \ldots, N\}$ with cardinality larger than $c(t, a, \varepsilon) N$ such that the canonical projection of K onto $\mathbb{R}^I$ contains the ball *$a(1-\varepsilon)[-1,1]^I$ where $c(t, a, \varepsilon) > 0$ depends only on the numbers t, a, ε.*

Remark. Lemma 3.5 is in some sense sharp and is a refinement of a result of J. Elton [EL].

Theorem 3.6. *Let X be an n-dimensional subspace of l_∞^N and set $\delta = N/n$. Then there exists a subset I of $\{1, 2, \ldots, N\}$ with cardinality larger than $c(\delta)\, n$ such* *that the quotient space X/E is $d(\delta)$-isomorphic to $l_\infty^{\text{card} I}$ where E is the space $\{x=(x_1, \ldots, x_n\} \in X$: $x_i=0$ for all $i \notin I\}$ and where $c(\delta)>0$ and $d(\delta)$ depend only on δ.*

Proof. Let $K=B_X \subset [-1,1]^N$ and let i: $X \to l_\infty^N$ be the embedding map. Set $t = e_{N+1}(i)/4$. This is easy to check that $a_1^{-\delta} \geqq t \geqq a_2^{-\delta}$ for some absolute constants a_1, $a_2>1$. Moreover let $\Lambda \subset K$ be a maximal subset of points which are $2t$ – distant in l_∞^N. Since it is a $2t$ – net in l_∞^N with $2t<e_{N+1}(i)$, the cardinality of Λ is larger than 2^N. Now the l_∞^N balls of radius t, centered on the points of Λ are mutually disjoint, so that

$$\operatorname{vol}(K+t[-1,1]^N) \geqq 2^N (2t)^N.$$

Applying Lemma 3.5 with $t=a$ and $\varepsilon=1/2$, say, we obtain a subset I of $\{1,2,\ldots,N\}$ with cardinality larger than $c(t,t,1/2)\, n \geqq C(t,t,1/2)\, \delta n = c(\delta) n$ such that the canonical projection of K onto $\mathbb{R}^I$ contains the ball $(t/2)[-1,1]^I$. Set $E=\{x=x_1, \ldots, x_N) \in X$: $x_i=0$ for all $i \notin I\}$ then X/E is $(t/2)$-isomorphic to $l_\infty^{\text{card} I}$. This accomplishes the proof because of $a_1^{-\delta} \geqq t \geqq a_2^{-\delta}$.

Remark. (1) From Theorem 3.3. we deduce that

$$\tau_2(X^*) \geqq vr(K) \geqq a\, n^{1/2} (\log(1+\delta))^{-1/2}.$$

Thereform from ([Pa], Theorem 3.12) there exists a quotient space X/E with $\dim(X/E) > C(\delta) n$ and which is $d(\delta)$-isomorphic to l_∞^m where $m=\dim(X/E)$. Theorem 3.6 is more precise concerning the position of E.

(2) The dual statement to Theorem 3.6 may be stated as follows: let $E \subset \mathbb{R}^N$ be an n-dimensional subspace and consider the orthogonal projection P onto E as acting on l_1^N. Moreover denote by $e_1, \ldots, e_N$ the canonical basis of l_1^N and let $Y=l_1^N/\operatorname{Ker} P$. Then there exists a subset I of $\{1,2,\ldots,N\}$ with cardinality larger than $c(\delta) n$ $(\delta=N/n)$ such that the basis $(Pei)_{i \in I}$ in Y is $d(\delta)$-isomorphic to the canonical basis of $l_1^{\text{card} I}$.

(3) Observe that from [F-J] the large subspaces of l_∞^N say for instance $\dim X \geqq N/2$, may not contain isomorphically l_∞^m for $m \geqq C N^{1/2}$ where C is some universal constant. This is in some sense optimal (see [B]).

We now give a result on the existence of badly complemented subspaces in large subspaces of l_∞^N.

Theorem 3.7. *Let X be a n-dimensional subspace of l_∞^N. Then there exists a subspace E of X with $\dim E=[n/2]$ such that every projection of X onto E has norm larger than $C n^{1/2} (\log(1+N/n))^{-1/2}$ where $c>0$ is a universal constant.*

Proof. Let v: $l_2^n \to X$ be a John mapping, that is $\|v\| \leqq 1$ and $\pi_2(v^{-1}) \leqq n^{1/2}$. Let $1 \leqq k \leqq n$ and let $E \subset l_2^n$ be a subspace with $\operatorname{codim} E=k$. Then

$$(n-k)^{1/2} \leqq (\dim E)^{1/2} = \pi_2(\mathrm{id}_E) \leqq \|v|_E\|\, \pi_2(v^{-1}) \leqq \|v|_E\|\, n^{1/2}.$$

Hence

$$(1-k/n)^{1/2} \leqq c_{k+1}(v). \tag{3.4}$$

For a subspace F of X, with $\dim F = k$, define $\lambda(F, X)$ to be the infimum $\|P\|$ over all projection P of X onto F. Moreover set

$$\lambda = \sup\{\lambda(F, X): F \subset X, \dim F = k\}.$$

It is easy to check that

$$c_{k+1}(v) \leqq (1+\lambda)\, d_{k+1}(v). \tag{3.5}$$

Now a result from ([M-P.2]) appendix) states that

$$c_k(v^*) \leqq a(n/k)^{1/2}\, e_{\delta k}(v^*) \tag{3.6}$$

for some universal constants a and $\delta > 0$.

Therefore

$$d_k(v) = c_k(v^*) \leqq a(n/k)^{1/2}\, e_{\delta k}(v^*\, i^*)$$

where $i: X \to l_\infty^N$ is the embedding operator.

Applying Corollary 2.4 and the Relation (3.4) and (3.5) we get

$$(1-k/n)^{1/2} \leqq c(1+\lambda)(n/k)^{1/2}\, k^{-1/2}(\log(1+N/k))^{1/2}.$$

Finally we choose k to conclude the proof.

Remark. Let X be an n-dimensional subspace of l_∞^N and let $i: X \to l_\infty^N$ be the embedding map. Then from Theorem 2.3 we have the following "extremal" property: for every operator $u: l_2 \to X$ we have

$$d_{2k}(iu) \leqq a\, k^{-1/2}(\log(1+N/k))^{1/2}\, c_k(iu),$$

$k = 1, 2, \ldots, n$ where a is a universal constant.

"Extremal" comes from the fact that $d_k(v) \geqq k^{-1/2}\, c_k(v)$ is valid for any operator.

Now we carry over to large projections in l_p^N spaces.

Theorem 3.8. *Let X be an n-dimensional subspace of l_p^N, $2 < p < \infty$. Then there exists a subspace E of X with $\dim E = [n/2]$ such that every projection of X onto E has norm larger than*

$$\frac{c}{\sqrt{p}}\, n^{1/2-1/p} \left(\frac{n}{N}\right)^{1/p},$$

where $c > 0$ is a universal constant.

Proof. Let $v: l_2^n \to X$ be a John mapping, that means $\|v\| \leqq 1$ and $\pi_2(v^{-1}) = n^{1/2}$. As in the proof of Theorem 3.7 we get

$$\begin{aligned}(1-k/n)^{1/2} &\leqq c_{k+1}(v) \leqq (1+\lambda)\, d_{k+1}(v)\\ &\leqq (1+\lambda)\, c_{k+1}(v)^* \leqq (1+\lambda)\, a(n/k)^{1/2}\, e_{\delta k}(v^*\, i^*)\end{aligned}$$

where i is the embedding map from X into l_p^N and λ is defined as in the proof of Theorem 3.7, a and δ coming from (3.6).

By Lemma 2.1 and (1.10) we get

$$k^{1/2}\, e_k(v^*\, i^*) \leqq c\, \tau_2\left(l_p^N\right) \pi_2(v^*\, i^*)$$
$$\leqq c\sqrt{p}\, \pi_2(v^*\, i^*).$$

As in the proof of Theorem 3.4 we have

$$\pi_2(v^*\, i^*) \leqq c'\, N^{1/p} \|v\| \leqq c'\, N^{1/p}.$$

Combining the previous estimates we arrive at

$$(1+\lambda) \geqq \frac{1}{a c c' \sqrt{p}} \left(1-\frac{k}{n}\right)^{1/2} \left(\frac{k}{n}\right)^{1/2} k^{1/2-1/p} \left(\frac{k}{n}\right)^{1/p}.$$

Putting $k=[n/2]$ we get the desired assertion.

Remark. Let X be an n-dimensional subspace of l_p^N, $2<p<\infty$, then from a result of Lewis [Le] we have that for all subspaces E of X with the dimension $[n/2]$ there exists a projection P in X onto E such that

$$\|P\| \leqq (n/2)^{1/2-1/p}$$

Remark. (1) In fact we may even give probabilistic versions of Theorem 3.7 and 3.8. Namely, for all subspaces X of l_p^N, $2<p\leqq\infty$, with dimension [N/2] we have that with a "high probability" the projection in X of rank [N/4], say, have norm larger than $c\sqrt{N}$, where $c>0$ is a universal constant. This fact may be obtained by using a probabilistic version of inequality (3.6).

(2) In contrast to the last remark Szarek [Sz] proved that there exists an n-dimensional Banach space X such that all projections in X of rank $[an]$ have norm larger than $c\sqrt{n}$, where $a>0$ and $c>0$ are universal constants.

4. Large euclidean sections

The problem of constructing large euclidean sections in finite dimensional Banach spaces was investigated by V. Milman in ([M.1], [M.2]). In this framework we give new estimates for spaces with an unconditional basis and for finite dimensional subspaces of L^1.

We first of all recall some definitions. Let X be an n-dimensional Banach space. The Rademacher cotype 2 constant of X, that we denote by $C_2(X)$ is defined as the smallest constant C such that

$$\Big(\sum_{i\geqq 1} \|x_i\|^2\Big)^{1/2} \leqq C \Big(\operatorname*{Average}_{\varepsilon_i=\pm 1} \Big\|\sum_{i\geqq 1} \varepsilon_i\, x_i\Big\|^2\Big)^{1/2}$$

holds for every finite sequence $(x_i)_{i\geq 1}\subset X$. A basis $(e_i)_{i=1,2,\dots,n}$ of X is called 1-unconditional if

$$\left\|\sum_{i=1}^{n} \varepsilon_i a_i e_i\right\| = \left\|\sum_{i=1}^{n} a_i e_i\right\|$$

for every choices of signs $\varepsilon_i = \pm 1$ and every scalars $(a_i)_{i=1,2,\dots,n}$.

When a Banach space has an unconditional basis, Theorem 1.2 will enable us to improve the known estimate on the distance of "large sections" of X to euclidean spaces.

Theorem 4.1. *Let X be an n-dimensional Banach space with a 1-unconditional basis. Then for every $k=1,2,\dots,n$ there exists a subspace E of X with* codim $E=k$ *such that*

$$d(E, l_2^{n-k}) \leqq a\, C_2(X)(n/k)^{1/2}(\log(1+n/k))^{1/2}$$

where a is a universal constant.

Proof. To estimate the distance to Hilbert space, we first determine a "good" operator $u\colon X\to l_2^n$. As in [Sz-To] this will be obtained by the following renorming result from [F]. There exists a norm, say $\|\cdot\|_1$, on X such that dual norm $\|\cdot\|_1^*$ is 2-convex (see [F]) and such that

$$\|x\| \leqq \|x\|_1 \leqq a\, C_2(X)\, \|x\| \qquad \text{for every } x \quad \text{in} \quad X, \tag{4.1}$$

where a is a universal constant. Let X_1 be the space X equipped with the norm $\|\cdot\|_1$. From a result of [Sz-To], there exists a 1-unconditional basis $(e_1, e_2, \dots, e_n)$ of X_1 such that

$$\sum_{i=1}^{n} |a_i| \leqq \left\|\sum_{i=1}^{n} a_i e_i\right\|_1 \leqq n^{1/2}\left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2} \tag{4.2}$$

for every scalars $a_1, \dots, a_n$. Let $f_1 \dots f_n$ be an orthonormal basis of l_2^n and define $u\colon X\to l_2^n$ by setting $u(e_i)=f_i$, $i=1,2,\dots n$. Clearly from (4.2) follows $\|u^{-1}\| \leqq n^{1/2}$ and

$$\operatorname*{Average}_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{n} \varepsilon_i u^*(f_i)\right\|^2 \leqq 1.$$

Applying Theorem 1.2, we deduce that there exists a subspace E of X_1 with codim $E=k$ such that

$$d(E, l_2^{n-k}) \leqq b(n/k)^{1/2}(\log(1+n/k))^{1/2}.$$

To conclude, recall that from (4.1), X and X_1 are $a\,C_2(X)$ -isomorphic.

Remark. Estimates for large euclidean sections were also obtained in [Pa-To.1] and [M-P.2] in terms of the Gaussian cotype 2 constant. Observe that in the statement of Theorem 4.1. the Rademacher cotype 2 constant cannot be replaced by the Gaussian one. This can be seen by taking $X=l_\infty^n$. We also note a result in [M-P.2]. Namely, if for every $k=1,2,\ldots,n$, there exists a subspace E of X, with $\operatorname{codim} E=k$ such that $d(E, l_2^{n-k}) \leqq C(n/k)^\alpha$ for some constant $\alpha>0$, then $vr(X) \leqq a(\alpha)\, C$ where $a(\alpha)$ depends only on α.

In the next theorem we show that the estimate in Theorem 4.1 is also valid for subspaces of L^1. The method of the proof is similar to that used for proving Theorem 2.2 and moreover involves a result from [B-L-M].

Theorem 4.2. *Let (Ω, μ) be a probability space and let X be an n-dimensional subspace of $L_1(\Omega, \mu)$. Then for every $k=1,2,\ldots,n$ there exists a subspace E of X with* codim $E=k$ such that

$$d(E, l_2^{n-k}) \leqq a(n/k)^{1/2}(\log(1+n/k))^{1/2} \tag{4.3}$$

where a is a universal constant.

Proof. To prove the estimate (4.3) we may of course work with an isometric copy of X. For this purpose we recall a result of Lewis [Le] which states that there exists a basis $\varphi_1, \ldots, \varphi_n$ of X such that, for every scalars $a_1, \ldots, a_n$,

$$\sum_{i=1}^{n} a_i^2/n = \int_\Omega |\sum a_i\, \varphi_i|^2/\Phi\, d\mu$$

where

$$\Phi = \left(\sum_{i=1}^{n} \varphi_i^2\right)^{1/2} \quad \text{and} \quad \|\Phi\|_{L_1(\mu)}=1.$$

As in [B-L-M] we define a probability v on Ω by $dv=\Phi\, d\mu$. Then the mapping $f \to \Phi^{-1} f$ defines an isometry from X onto a subspace X_1 of $L_1(\Omega, \psi)$. Now it is sufficient to prove the statement for the isometric copy X_1. For this let $1 \leqq p \leqq 2$ and let X_p be the space X_1 equipped with the norm induced by $L_p\ (\Omega, v)$. Moreover, let $i_p\colon X_1 \to X_p$ be the identity operator from X_1 into X_p. As shown in [B-L-M] (Lemma 4.5) the 2-summing norm of $i_2\colon X_1 \to X_2$ is subject to the following inequality

$$\pi_2(i_2) \leqq a_1\, n^{1/2} \tag{4.4}$$

where a_1 is a universal constant.

Our aim is to prove the estimate

$$c_k(i_2) \leqq a(n/k)^{1/2}(\log(1+n/k))^{1/2}, \qquad k=1,2,\ldots,n, \tag{4.5}$$

where a is a universal constant.

This clearly will imply (4.3) since $\|i_2^{-1}\| \leqq 1$. In view of (4.5) we start with a rough estimate:

$$c_k(i_2) \leqq a_2(n/k)^\alpha, \qquad k=1,2,\ldots,n, \tag{4.6}$$

for some universal constant a_2 and some $\alpha>0$.

Indeed, since $X \subset L^1(\Omega, v)$ has a bounded cotype 2 constant, from a result in [M-P-2],

$$k^{1/2}\, c_k(i_2) \leqq a_2(n/k)(\log(1+n/k))\, \pi_2(i_2),$$

$k=1,2,\ldots,n$. Therefore using (4.4) we see that (4.6) holds with $\alpha=2$, say. Let $1<p\leqq 2$ and set $p'=p/p-1$. By Hölder's inequality, we get

$$c_k(i_p)\leqq a_2^{2/p'}(n/k)^{2\alpha/p'}\leqq a_2(n/k)^{2\alpha/p'},$$

$k=1,2,\ldots,n$. Now from [P.1], $\tau_2(X_p^*)\leqq(p')^{1/2}$.

Now we are in a position to reproduce the line of the proof of Theorem 2.2. Indeed, using Theorem 1.1 and the Relations (2.3), (2.1) we get

$$\begin{aligned} c_{2k}(i_2)&\leqq c_k(i_p)\,c_k(i_2\,i_p^{-1})\\ &\leqq a_3(n/k)^{2\alpha/p'}\,\tau_2(X_p^*)\,\pi_2(i_2\,i_p^{-1})\,k^{-1/2}\\ &\leqq a_4(p')^{1/2}(n/k)^{1/2+2\alpha/p'}, \end{aligned}$$

where $a_2, a_3, \ldots$ denote universal constants.

Finally choosing $p'=4\log(1+n/k)$ we arrive at the desired estimate (4.5) which accomplishes the proof.

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