

Minimal Singularities in GL_n

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Introduction

In a celebrated paper [1] Brieskorn describes a beautiful relation between simple complex Lie groups and simple surface singularities (cf. also [11]). His result is the following.

Let G be a complex Lie group of type A_n , D_n , E_6 , E_7 , E_8 and U the subvariety of unipotent elements of G . U is the closure \bar{C} of the *regular unipotent class* C and we have:

- i) There is a unique conjugacy class C' , of codimension 2 in U with $\bar{C}' = U - C$ (C' is called the *subregular class*).
- ii) The singularity of U in C' is a *simple surface singularity* of corresponding type.

As for the variety U itself Kostant has shown [7] that it is always normal and Cohen-Macaulay. In fact it has even rational singularities ([4]; for the definition see [6]).

In [8, 10] we proved these last results for the closure of any conjugacy class in $GL_n(\mathbb{C})$. In this paper we will generalize the theorem of Brieskorn, always in the case of $GL_n(\mathbb{C})$ (Theorem 3.2; cf. [3] for related results).

Our setting is the following: Let C be a conjugacy class in $GL_n(\mathbb{C})$ and \bar{C} its closure. In general the complement $D = \bar{C} - C$, of C in \bar{C} , contains more than one single open conjugacy class (unlike the case of the regular unipotent class). Nevertheless if C_i is an open conjugacy class in D (a *minimal degeneration* of C), we are able to describe the singularity of \bar{C} in C_i up to smooth equivalence.

If C_i has codimension 2 in \bar{C} this singularity is still a *simple singularity* of type A_j , where j is determined by the Young-diagrams of the conjugacy classes. If C_i has codimension >2 the singularity is equivalent to the *singularity of a minimal class* in the unipotent variety of some $GL_m(\mathbb{C})$. This singularity can also be described as the *collapsing of the cotangent bundle* of the projective space \mathbb{P}^{m-1} (2.4). In this case the singularity is determined only by its codimension which can also be read off from the diagrams. There is furthermore a *formal duality* between the two types (Remark 3.3).

The theory exposed here can be completely developed for conjugacy classes in classical groups and in fact we were motivated by the study of the normality problem in these cases. These results are contained in [9]; it seemed to us worthwhile to expose independently the results in the simpler case of GL_n in order to complete the study carried out in [8] and to pave the road for the somewhat complicated analysis in [9].

One final important remark: By the general theory of group actions on a variety it easily follows that all problems concerning singularities of closures of conjugacy classes can be reduced to the case of *unipotent classes* [5]. In turn, in characteristic 0, the variety of unipotent elements of a group G is isomorphic (in a G -equivariant way) to the variety of nilpotent elements in the Lie algebra $\mathfrak{g} = \text{Lie } G$. Thus our analysis is always restricted to *nilpotent elements* in \mathfrak{g} (in the present paper nilpotent matrices).

We advise the reader to consult the tables at the end, to visualize the results for GL_n with $n \leq 9$.

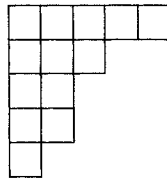
We always work over an algebraically closed field k of characteristic 0; some results easily extend to positive characteristic.

1. Degeneration of Conjugacy Classes

1.1. Let us fix some notations. Any nilpotent $n \times n$ -matrix is conjugate to one in normal Jordan block form:

$$\begin{bmatrix} J_{p_1} & & & & \\ & J_{p_2} & & & \\ & & \ddots & & \\ & & & J_{p_s} & \\ & & & & \end{bmatrix}, \quad J_t := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \text{ a } t \times t\text{-block.}$$

We can assume $p_1 \geq p_2 \geq \dots \geq p_s$; this decreasing sequence $\eta = (p_1, p_2, \dots, p_s)$ is a *partition* of n and it is convenient to represent it geometrically as a *Young-diagram* with rows consisting of p_1, p_2, \dots, p_s boxes respectively: e.g. the diagram



corresponds to the partition $(5, 3, 2, 2, 1)$ of 13.

The *dual partition* $\hat{\eta} = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_i)$ is defined setting \hat{p}_i equal to the length of the i -th column of the diagram η : $\hat{p}_i := \#\{j | p_j \geq i\}$. In case of a partition η associated to the normal Jordan form of a nilpotent matrix A , the dual partition $\hat{\eta}$ has the following interpretation:

$$\dim \ker A^j = \sum_{i=1}^j \hat{p}_i$$

or equivalently

$$\text{rk } A^j = \sum_{i>j} \hat{p}_i.$$

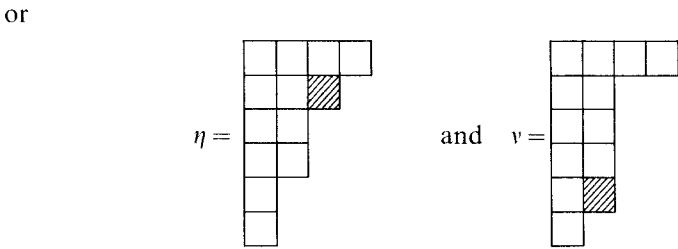
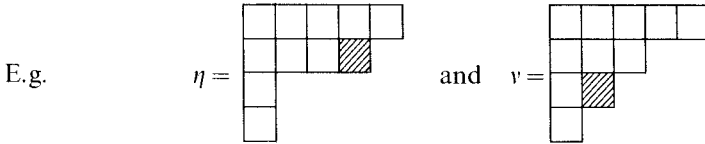
It is clear now that we have a *one to one correspondence* between nilpotent conjugacy classes of $n \times n$ -matrices and partitions of n . If η is a partition the corresponding conjugacy class will be denoted by C_η .

1.2. Given two partitions $\eta = (p_1, p_2, \dots, p_s)$ and $\nu = (q_1, q_2, \dots, q_t)$ of n , we say $\eta \geq \nu$ if we have

$$\sum_{i=1}^j p_i \geq \sum_{i=1}^j q_i \quad \text{for all } j.$$

This is equivalent to $\sum_{k>j} \hat{p}_k \geq \sum_{k>j} \hat{q}_k$ for all j .

Lemma ([3], Proposition 3.9). *If $\eta > \nu$ and no partition is in between them (i.e. η and ν are adjacent in the ordering), then the diagram ν is obtained from η by moving one box down either to the next row or to the next column.*



More formally we have the following two possibilities for an adjacent pair $\eta > \nu$, $\eta = (p_1, p_2, \dots, p_s)$, $\nu = (q_1, q_2, \dots, q_t)$:

- I) There is an $i \in \mathbb{N}^+$ such that $p_k = q_k$ for $k \neq i, i+1$ and $q_i = p_i - 1 \geq q_{i+1} = p_{i+1} + 1$.
- II) There are $i, j \in \mathbb{N}^+$, $i < j$, such that $p_k = q_k$ for $k \neq i, j$ and $q_i = p_i - 1 = q_j = p_j + 1$.

We remark that if $\eta > \nu$ are adjacent of type I (of type II) then $\hat{\nu} > \hat{\eta}$ are adjacent of type II (of type I).

1.3. The following is the basic result on degenerations of nilpotent conjugacy classes and on their dimension (cf. [3] Theorem 3.10 and Corollary 3.8(a)).

Proposition. a) Given two partitions η and ν of n , we have $\eta \geq \nu$ if and only if $\bar{C}_\eta \supseteq C_\nu$.

b) If $\eta = (p_1, \dots, p_s)$ is a partition of n and $\hat{\eta} = (\hat{p}_1, \dots, \hat{p}_t)$ the dual partition, we have

$$\dim C_\eta = n^2 - \sum_{i,j} \min(p_i, p_j) = n^2 - \sum_i \hat{p}_i^2 = 2 \sum_{i < j} \hat{p}_i \hat{p}_j.$$

1.4. Let us call a degeneration $C_\nu \subseteq \bar{C}_\eta$ minimal if C_ν is open in $\bar{C}_\eta - C_\eta$, i.e. if $C_\nu \not\subseteq C_\eta$ and there is no conjugacy class C such that $\bar{C}_\nu \subsetneq \bar{C} \subsetneq \bar{C}_\eta$. By Proposition 1.3 this means that $\nu < \eta$ are adjacent. From 1.2 we get the following result.

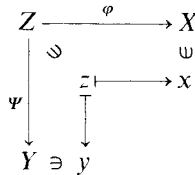
Corollary. Let $C_\nu \subset \bar{C}_\eta$ be a minimal degeneration. Then we have one of the following two cases:

- I) $\text{codim}_{\bar{C}_\eta} C_\nu = 2$ and the diagram ν is obtained from η by moving one box down to the next row,
- II) $\text{codim}_{\bar{C}_\eta} C_\nu = 2r$ and the diagram ν is obtained from η by moving one box down to the next column; if the box is moved from the i -th row to the j -th row then $r = j - i$.

We will refer to this by saying that $C_\nu \subseteq \bar{C}_\eta$ is a minimal degeneration of type I or of type II respectively.

2. Subregular and Minimal Singularities

2.1. Definition. If X, Y are varieties and $x \in X, y \in Y$ two points, we say that the singularity of X in x is smoothly equivalent to the singularity of Y in y , if there is a variety Z , a point $z \in Z$ and two maps $\varphi: Z \rightarrow X, \psi: Z \rightarrow Y$ such that $\varphi(z) = x, \psi(z) = y$ and φ and ψ are smooth in z .



It is easily seen that this defines an equivalence relation; the equivalence classes will be denoted by $\text{Sing}(X, x)$.

Remark. If $\dim_x X = \dim_y Y + r$ with $r \in \mathbb{N}$ one can show that $\text{Sing}(X, x) = \text{Sing}(Y, y)$ if and only if $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \hat{\mathcal{O}}_{Y,y} [[T_1, \dots, T_r]]$ (cf. [11] III 5.1 Hilsf-lemma; $\hat{\mathcal{O}}_{X,x}$ denotes the completion of the local ring $\mathcal{O}_{X,x}$).

Assume that an algebraic group G acts on a variety X . If $O \subset X$ is an orbit under G we have $\text{Sing}(X, x) = \text{Sing}(X, y)$ for all $x, y \in O$. In this case we also write $\text{Sing}(X, O)$ for the corresponding equivalence class.

we obtain the singularity \bar{C}_{\min} by “collapsing” the cotangent bundle of \mathbb{P}_k^{n-1} ([6]). We will denote this singularity by a_{n-1} :

$$\text{Sing}(\bar{C}_{\min}, 0) = a_{n-1}.$$

Of course we have $a_1 = A_1$.

3. The Main Theorem

3.1. Our main theorem on “minimal singularities” will be an easy consequence of the following general “reduction result”.

Proposition. *Let $C_v \subseteq \bar{C}_\eta$ be a degeneration of nilpotent conjugacy classes in $M_n(k)$ and assume that the first r rows and the first s columns of η and v coincide. Denote by η' and v' the Young-diagrams obtained from η and v erasing these rows and columns. Then $C_{v'} \subseteq \bar{C}_{\eta'}$,*

$$\text{codim}_{C_{\eta'}} C_{v'} = \text{codim}_{C_\eta} C_v \quad \text{and} \quad \text{Sing}(\bar{C}_{\eta'}, C_{v'}) = \text{Sing}(\bar{C}_\eta, C_v).$$

The proof will be given in the following two sections, where we will treat the two cases “erasing rows” (Proposition 4.4) and “erasing columns” (Proposition 5.4) separately. The first case will be handled by a cross section argument using the normality of the closure \bar{C}_η ([8]); in the second we need a careful analysis of the “induction Lemma” of [8] given by the first fundamental theorem of invariant theory.

Remark. The closure of any nilpotent conjugacy class in $M_n(k)$ has a natural desingularization analogue to the one described in 2.4 for \bar{C}_{\min} . N. Spaltenstein has informed us that the exceptional fibres of an element in C_v and an element in $C_{v'}$ (notations of the proposition above) turn out to be isomorphic.

3.2. Now we can prove our main result.

Theorem. *Let $C' \subseteq \bar{C}$ be a minimal degeneration of nilpotent conjugacy classes in $M_n(k)$ (i.e. C' is open in $\bar{C} - C$). Then the singularity of \bar{C} in C' is either simple of type A_m or it is of type a_m for some $m < n$. More precisely*

$$\text{Sing}(\bar{C}, C') = \begin{cases} A_m & \text{for some } m < n \text{ if } \text{codim}_C C' = 2 \\ a_m & \text{if } \text{codim}_C C' = 2m > 2. \end{cases}$$

Proof. Let η and v be the associated partitions to C and C' , $\eta = (p_1, p_2, \dots, p_s)$. If $C' \subseteq \bar{C}$ is a minimal degeneration of type I (1.4) then $v = (p_1, \dots, p_{i-1}, p_i - 1, p_{i+1} + 1, p_{i+2}, \dots, p_s)$ for some $i \in \mathbb{N}^+$ (1.2). The Proposition 3.1 implies (erasing the first $i - 1$ rows and the first p_{i+1} columns

$$\text{Sing}(\bar{C}, C') = \text{Sing}(\bar{C}_{(t)}, C_{(t-1, 1)}) = A_{t-1}$$

with $t = p_i - p_{i+1} \leq n$, and $\text{codim}_C C' = 2$.

If $C' \subseteq \bar{C}$ is a minimal degeneration of type II we find from 1.2

$$v = (p_1, \dots, p_{i-1}, p_i - 1, p_{i+1}, \dots, p_j + 1, p_{j+1}, \dots, p_s)$$

for some $i < j$ with $p_i - 1 = p_{i+1} = \dots = p_j + 1$. Hence again from Proposition 3.1 we get (erasing the first $i - 1$ rows and the first p_j columns)

$$\text{Sing}(\bar{C}, C') = \text{Sing}(\bar{C}_{(2, 1, 1, \dots, 1)}, 0) = a_{j-i}$$

and $\text{codim}_C C' = 2(j - i)$. q.e.d.

3.3. Remark. If C is a nilpotent conjugacy class with associated partition η we denote by \hat{C} the class associated to the dual partition $\hat{\eta}$. The map $C \mapsto \hat{C}$ is an ‘‘order reversing’’ bijection i.e. $C' \subseteq \bar{C}$ is equivalent to $\hat{C} \subseteq \bar{\hat{C}'}$. The proof above immediately implies the following *duality result*:

If $C' \subseteq \bar{C}$ is a minimal degeneration with $\text{Sing}(\bar{C}, C') = A_m$, then $\hat{C} \subseteq \bar{\hat{C}'}$ is a minimal degeneration with $\text{Sing}(\bar{\hat{C}'}, \hat{C}) = A_m$ and vice versa.

4. First Reduction by Cross Sections

4.1. We shortly recall the notion of *cross sections* (cf. [3] 1.7). Let G be an algebraic group and V a G -module, i.e. a finite dimensional representation of G . The Lie algebra $\mathfrak{g} := \text{Lie } G$ also acts on V ; we write $(X, v) \mapsto Xv$ for this action. Let Gx be the orbit of $x \in V$ and $y \in \overline{Gx}$. Choose a complement N of the tangent space $T_y(Gy) = \mathfrak{g}y$ in V and define $S := (N + y) \cap \overline{Gx}$. The variety S is called a *cross section of \overline{Gx} in y* .

Lemma. *With the notations above one has:*

- i) S (as a schematic intersection) is reduced in a neighbourhood of y ,
- ii) $\dim_y S = \text{codim}_{\overline{Gx}} Gy$ and $\text{Sing}(S, y) = \text{Sing}(\overline{Gx}, Gy)$,
- iii) \overline{Gx} is normal in y if and only if S is normal in y .

Proof. The map $\varphi: G \times (N + y) \rightarrow V$ given by the operation of G on V is smooth in $(1_G, y)$. One easily checks that $\varphi^{-1}(\overline{Gx}) = G \times S$, proving i). Since the projection $G \times S \rightarrow S$ is smooth too, ii) follows. Finally iii) is a consequence of the previous statements and the fact that normality is preserved under smooth maps ([2] IV 17.5.8). q.e.d.

4.2. Proposition. *Let G be an algebraic group, $H \subseteq G$ a closed subgroup, V a G -module and $U \subset V$ an H -submodule. Consider elements $x \in U$ and $y \in \overline{Hx}$ and assume*

- i) there is a complement \mathfrak{m} of $\text{Lie } H$ in $\text{Lie } G$ such that $\mathfrak{m}y \cap U = \{0\}$,
- ii) $\text{codim}_{\overline{Hx}} Hy = \text{codim}_{\overline{Gx}} Gy$,
- iii) \overline{Gx} is normal in y .

Then $\text{Sing}(\overline{Gx}, Gy) = \text{Sing}(\overline{Hx}, Hy)$.

Proof. Put $\mathfrak{g} := \text{Lie } G \supset \mathfrak{h} := \text{Lie } H$ and let U' be a complement of the tangent space $T_y(Hy) = \mathfrak{h}y$ in U : $U = \mathfrak{h}y \oplus U'$. Then

$$T_y(Gy) = \mathfrak{g}y = \mathfrak{h}y + \mathfrak{m}y$$

and by i) $\mathfrak{h}y + \mathfrak{m}y + U'$ is a direct sum in V . Choose a complement U'' of this sum in V :

$$V = \mathfrak{h}y \oplus U' \oplus \mathfrak{m}y \oplus U'' = \mathfrak{g}y \oplus V', \quad V' := U' \oplus U''.$$

Then $S := (U' + y) \cap \overline{HX}$ is a cross section of \overline{HX} in y and $T := (V' + y) \cap \overline{GX}$ is a cross section of \overline{GX} in y (4.1). Furthermore $T \supset S$ and $\dim_y T = \dim_y S$ by assumption ii). Since T is normal in y by iii) and Lemma 4.1 iii), this implies that S and T coincide in a neighbourhood of y , hence $\text{Sing}(\overline{HX}, Hy) = \text{Sing}(S, y) = \text{Sing}(T, y) = \text{Sing}(\overline{GX}, Gy)$. q.e.d.

4.3. Remark. It follows from the proof above that we can replace the assumption iii) by

iii)' \overline{GX} is unibranch in y .

(In fact this assumption implies that the cross section T is irreducible in y , which suffices to deduce that S and T coincide in a neighbourhood of y .)

4.4. Proposition. Let $C_\nu \subset \overline{C}_\eta$ be a degeneration of nilpotent conjugacy classes in $M_n(k)$. Assume that the first r rows of the diagrams η and ν coincide and denote by η' and ν' the diagrams obtained by erasing these rows. Then

$$\text{codim}_{C_\eta} C_\nu = \text{codim}_{C_{\eta'}} C_{\nu'} \quad \text{and} \quad \text{Sing}(\overline{C}_\eta, C_\nu) = \text{Sing}(\overline{C}_{\eta'}, C_{\nu'}).$$

Proof. It is enough to treat the case $r = 1$. Let m be the length of the first row of η and ν : $\eta = (m, p_2, p_3, \dots, p_s)$, $\nu = (m, q_2, \dots, q_t)$, $\eta' = (p_2, \dots, p_s)$, $\nu' = (q_2, \dots, q_t)$. Consider $G := GL_n(k) \supseteq H := GL_m(k) \times GL_{n-m}(k)$ and $M_n(k) = \text{Lie } G \supseteq \text{Lie } H = M_m(k) \oplus M_{n-m}(k)$ with the usual embeddings. We can find elements $X = (Z, X')$, $Y = (Z, Y') \in \text{Lie } H$ such that $X \in C_\eta$, $Y \in C_\nu$, $X' \in C_{\eta'}$, $Y' \in C_{\nu'}$, $Z \in C_{(m)}$. Since $\overline{HX} = \overline{C}_{(m)} \times \overline{C}_{\eta'}$ and $HY = C_{(m)} \times C_{\nu'}$ we have

$$\text{codim}_{C_\eta} C_\nu = \text{codim}_{\overline{HX}} HY \tag{*}$$

and $\text{Sing}(\overline{HX}, HY) = \text{Sing}(\overline{C}_{\eta'}, C_{\nu'})$. In order to prove $\text{Sing}(\overline{C}_\eta, C_\nu) = \text{Sing}(\overline{HX}, HY)$ we apply Proposition 4.2 for $V := \text{Lie } G \supseteq U := \text{Lie } H$. Assumption i) follows from the fact that we can find an H -stable complement \mathfrak{m} of $\text{Lie } H$ in $\text{Lie } G$ (since H is reductive) for which we even have $[\mathfrak{m}, \text{Lie } H] \subseteq \mathfrak{m}$. Assumption ii) is an immediate consequence of the dimension formula 1.3 b) and (*), and iii) is the normality result of [8]. q.e.d.

5. Second Reduction by Classical Invariant Theory

5.1. We first recall the main construction of [8]. Let X be a nilpotent element of $M_n(k)$ with Young-diagram $\eta = (p_1, \dots, p_s)$ and $\text{rk } X =: m$. Then the first column of η has length $\hat{p}_1 = n - m$ and the diagram η' obtained from η by removing the

first column is exactly the Young-diagram of the nilpotent endomorphism $X' := X|_{\text{im } X}: \text{im } X \rightarrow \text{im } X$. Consider the following two maps:

$$\begin{array}{ccc} L_{n,m} := M_{n,m}(k) \times M_{m,n}(k) & \xrightarrow{\pi} & M_m(k) \\ \downarrow \rho & & \\ M_n(k) & & \end{array}$$

given by $\pi(A, B) := BA$, $\rho(A, B) := AB$.

The group $GL_n(k) \times GL_m(k)$ acts on $L_{n,m}$ in the obvious way, $(g, h)(A, B) := (gAh^{-1}, hBg^{-1})$, and π and ρ are *quotient maps* (with respect to $GL_n(k)$ and $GL_m(k)$ respectively). π is surjective and the image of ρ is the determinantal variety of matrices of rank $\leq m$ (cf. [8] Theorem 2.2).

5.2. Define $L'_{n,m} := \{(A, B) \in L_{n,m} \mid \text{rk } A = \text{rk } B = m\}$.

Lemma. a) *The map π is smooth in $L'_{n,m}$ with image*

$$\pi(L'_{n,m}) = \{Y \in M_m(k) \mid \text{rk } Y \geq 2m - n\}.$$

b) $\rho(L'_{n,m}) = \{X \in M_n(k) \mid \text{rk } X = m\}$ and the induced map $\rho': L'_{n,m} \rightarrow \rho(L'_{n,m})$ is a fibration with typical fibre $GL_m(k)$. (Here fibration means locally trivial in the étale topology.)

Proof. a) (cf. [8] Proposition 3.5). The tangent map $d\pi_{(A,B)}$ is given by $(P, Q) \mapsto BP + QA$, hence is surjective if $(A, B) \in L'_{n,m}$. Furthermore it is easy to see that a matrix $Y \in M_m(k)$ can be written in the form $Y = BA$ with $(A, B) \in L'_{n,m}$ if and only if $\text{rk } Y \geq 2m - n$.

b) Consider the action of $G := GL_n(k) \times GL_m(k) \times GL_n(k)$ on $L_{n,m}$ given by $(g', h, g)(A, B) = (g'Ah^{-1}, hBg^{-1})$. Under this action $L'_{n,m}$ is an orbit and the map ρ' is of the form $pr: G/H \rightarrow G/H'$ with two subgroups $H \subseteq H'$ of G , hence ρ' is locally trivial in the étale topology. Furthermore for each $X \in M_n(k)$ of rank m the fibre $\rho^{-1}(X) = \{(A, B) \in L_{n,m} \mid AB = X\}$ is a single orbit under $GL_m(k)$, contained in $L'_{n,m}$, and the stabilizer of any element in $\rho^{-1}(X)$ is trivial. *q.e.d.*

5.3. Lemma (Notations of 5.1): Put $N_\eta := \pi^{-1}(\bar{C}_\eta)$.

- a) $\rho(N_\eta) = \bar{C}_\eta$,
- b) $\rho^{-1}(C_\eta)$ is an orbit under $GL_n(k) \times GL_m(k)$ contained in $L'_{n,m} \cap N_\eta$,
- c) $\pi(\rho^{-1}(C_\eta)) = C_\eta$.

$$\begin{array}{ccc} N_\eta = \pi^{-1}(\bar{C}_\eta) & \xrightarrow{\pi} & \bar{C}_\eta \\ \downarrow \rho & & \\ C_\eta & & \end{array}$$

Proof. a) is proven in [8] (Lemma 2.3). For b) and c) consider a matrix $X \in C_\eta$. We already remarked in the proof of Lemma 5.2b) that $\rho^{-1}(X)$ is an orbit under $GL_m(k)$ contained in $L'_{n,m}$. Furthermore for any $(A, B) \in \rho^{-1}(X)$ the matrix BA corresponds to the endomorphism $X' = X|_{\text{im } X}$ (cf. 5.1), i.e. $\pi(A, B) \in C_{\eta'}$. This implies first that $\rho^{-1}(C_\eta)$ is an orbit under $GL_n(k) \times GL_m(k)$ contained in $L'_{n,m}$ and then that $\pi(\rho^{-1}(C_\eta)) = C_{\eta'}$, hence also $\rho^{-1}(C_\eta) \subset N_\eta$. *q.e.d.*

Remark. The construction above depends only on the rank of the elements of C_η , i.e. only on the length of the first column of η . In particular, if $C_v \subseteq \bar{C}_\eta$ is a degeneration such that the first column of v and η coincide and if v' is obtained from v by erasing the first column, we get

- 1) $C_{v'} \subseteq \bar{C}_{\eta'}$, (1.3a), hence $N_{v'} := \pi^{-1}(\bar{C}_{v'}) \subseteq N_{\eta'}$,
- 2) $\rho^{-1}(C_v) \subset N_v \cap L'_{n,m}$ and $\pi(\rho^{-1}(C_v)) = C_{v'}$.

5.4. Proposition. *Let $C_v \subseteq \bar{C}_\eta$ be a degeneration of nilpotent conjugacy classes in $M_n(k)$. Assume that the first s columns of η and v coincide and denote by η' and v' the diagrams obtained from η and v by erasing these columns. Then*

$$\text{codim}_{C_\eta} C_v = \text{codim}_{C_{\eta'}} C_{v'} \quad \text{and} \quad \text{Sing}(\bar{C}_\eta, C_v) = \text{Sing}(\bar{C}_{\eta'}, C_{v'}).$$

Proof. It is enough to treat the case $s=1$. The codimension formula follows immediately from 1.3b). Let $m := \dim \ker X$ for some $X \in C_\eta$. Using the notations introduced above we get the following diagram of maps (cf. 5.3):

$$\begin{array}{ccc} N_\eta := \pi^{-1}(\bar{C}_\eta) & \xrightarrow{\bar{\pi}} & \bar{C}_\eta \\ & \downarrow \bar{\rho} & \\ & & \bar{C}_{\eta'} \end{array}$$

Now the second claim follows if we can find an element $z \in N_\eta$ satisfying the following condition (cf. Definition 2.1):

$$\bar{\pi}(z) \in C_{v'}, \quad \bar{\rho}(z) \in C_v \quad \text{and} \quad \bar{\pi} \text{ and } \bar{\rho} \text{ are smooth in } z. \tag{*}$$

Consider the open subset $N'_\eta := N_\eta \cap L'_{n,m}$ of N_η . From Remark 5.3 we get $\rho^{-1}(C_v) \subseteq N_v \cap L'_{n,m} \subseteq N'_\eta$ and $\pi(\rho^{-1}(C_v)) = C_{v'}$. Furthermore $\bar{\pi}$ is smooth in N'_η by 5.2a) and $\bar{\rho}|_{N'_\eta}: N'_\eta \rightarrow \bar{\rho}(N'_\eta)$ is a fibration by 5.2b) (since N'_η is a locally closed subvariety of $L'_{n,m}$ stable under $GL_m(k)$), hence $\bar{\rho}$ is smooth on N'_η too. In particular each element $z \in \rho^{-1}(C_v)$ satisfies the above condition (*). *q.e.d.*

6. Tables

In this paragraph we draw tables representing conjugacy classes in GL_n , $n \leq 9$. The tables are constructed (following Hesselink [3]) as follows:

Each conjugacy class is represented by a dot, its corresponding partition and dimension (taken from [3]) is indicated at its right. For every minimal de-

generation of classes we draw an edge and we place the dots from top to bottom according to the containment \supseteq of closures.

On each edge we write the type A_j or a_j of the corresponding singularity. We recall that:

- A_j is a simple surface singularity,
- a_j is the collapsing of the cotangent bundle of \mathbb{P}^j .

There is a simple rule for determining the type of the singularity once one has the diagram and the codimensions. For each codimension $2r$ with $r > 1$ the type is a_r ; once all these degenerations have been determined the types A_r can be read off by the duality in the diagram (Remark 3.3). Degenerations which are

GL_2	η	dim
A_1	(2)	2
	(1, 1)	0

GL_3	η	dim
A_2	(3)	6
	(2, 1)	4
	(1, 1, 1)	0

GL_4	η	dim
A_3	(4)	12
	(3, 1)	10
	(2, 2)	8
	(2, 1, 1)	6
a_3	(1 ⁴)	0

GL_5	η	dim
A_4	(5)	20
	(4, 1)	18
	(3, 2)	16
	(3, 1, 1)	14
	(2, 2, 1)	12
	(2, 1 ³)	8
	(2, 1 ⁴)	0

GL_6	η	dim
	(6)	30
	(5, 1)	28
	(4, 2)	26
	(4, 1, 1)	24
	(3, 3)	24
	(3, 2, 1)	22
	(2, 2, 2)	18
	(3, 1, 1, 1)	18
	(2, 2, 1, 1)	16
	(2, 1 ⁴)	10
	(1 ⁶)	0

GL_7	η	dim
	(7)	42
	(6, 1)	40
	(5, 2)	38
	(5, 1, 1)	36
	(4, 3)	36
	(4, 2, 1)	34
	(3, 3, 1)	32
	(4, 1 ³)	30
	(3, 2, 2)	30
	(3, 2, 1, 1)	28
	(2, 2, 2, 1)	24
	(3, 1 ⁴)	22
	(2, 2, 1 ³)	20
	(2, 1 ⁵)	12
(1 ⁷)	0	

GL_8	η	dim
	(8)	56
	(7, 1)	54
	(6, 2)	52
	(6, 1, 1)	50
	(5, 3)	50
	(5, 2, 1)	48
	(5, 1 ³)	48
	(4, 4)	48
	(4, 3, 1)	46
	(4, 2, 2)	44
	(4, 2, 1, 1)	42
	(3, 3, 2)	42
	(3, 3, 1, 1)	40
	(3, 2, 2, 1)	38
	(2, 2, 2, 2)	32
	(4, 1 ⁴)	36
	(3, 2, 1 ³)	34
	(2, 2, 2, 1 ²)	30
	(3, 1 ⁵)	26
	(2, 2, 1 ⁴)	24
	(2, 1 ⁶)	14
	(1 ⁸)	0

GL_9	η	dim
	(9)	72
	(8, 1)	70
	(7, 2)	68
	(7, 1, 1)	66
	(6, 3)	66
	(6, 2, 1)	64
	(6, 1, 1, 1)	60
	(5, 4)	64
	(5, 3, 1)	62
	(5, 2, 2)	60
	(5, 2, 1, 1)	58
	(4, 4, 1)	60
	(4, 3, 2)	58
	(4, 3, 1, 1)	56
	(5, 1 ⁴)	52
	(3, 3, 3)	54
	(4, 2, 2, 1)	54
	(3, 3, 2, 1)	52
	(3, 2, 2, 2)	48
	(4, 2, 1 ³)	50
	(3, 3, 1 ³)	48
	(3, 2, 2, 1 ²)	46
	(2, 2, 2, 2, 1)	40
	(4, 1 ⁵)	42
	(3, 2, 1 ⁴)	40
	(2, 2, 2, 1 ³)	36
	(3, 1 ⁶)	38
	(2, 2, 1 ⁵)	28
	(2, 1 ⁷)	16
	(1 ⁹)	0

self dual are necessarily of type $A_1 = a_1$. Further symmetries can be discovered considering the additive behaviour of codimensions in a string.

Notice also that the singularities of type a_j accumulate towards the bottom as is intuitively clear if one observes that approaching the zero class the diagrams tend to have few but very long columns.

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