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A characterization of non-hyperelliptic Jacobi varieties

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In this paper we give a proof of the following result:

Theorem. *Let* (X, Θ) *be a complex principally polarized abelian variety of dimension g. Assume:*

(i) $\dim(\operatorname{Sing}\Theta) \leqq g-4$;

(ii) *there exists a one-dimensional subset* $U \subset X$ *such that for generic* $u \in U$ *one has:*

$$\Theta \cap \Theta_u \subset \Theta_x \cup \Theta_y$$

for some $x, y \in X$ *with* $\{0, u\} \cap \{x, y\} = \emptyset$.

Then (X, Θ) *is the polarized jacobian of a (smooth irreducible non-hyperelliptic) curve.*

This gives a positive answer to a – stronger version of a – problem proposed by D. Mumford ([5], p. 81), under the additional hypothesis (i) above. In particular, by work of A. Beauville [1], it implies:

Corollary. *Let* (X, Θ) *be an irreducible complex principally polarized abelian variety of dimension* $\leqq 5$. *The following are equivalent:*

a) *There is a one-dimensional subset* $U \subset X$ *satisfying property* (ii) *above.*

b) (X, Θ) *is the polarized jacobian of a curve.*

This improves the main result of Z. Ran's paper [6], and Theorem 7, p. 476, of his earlier paper [7]. (We remark that, since any reducible principally polarized abelian variety is a – trivial – solution to the hypotheses in Mumford's problem, the main statement in [6] is not entirely correct.) Both works are the main source of inspiration for the present one. In contrast with Z. Ran's techniques, we use general facts about line bundles on abelian varieties. Secondly, an eventual use of R.C. Gunning's recent result [2] brings the proof to a quick end.

Proof. The Corollary is a consequence of the following facts, proved in [1]: If $g \leqq 5$ then: $\dim(\operatorname{Sing}\Theta) \leqq g-3$ is equivalent with (X, Θ) being irreducible, and $\dim(\operatorname{Sing}\Theta) \leqq g-4$ is equivalent with (X, Θ) being irreducible and not a hyperelliptic jacobian.

To prove the Theorem, we may assume Θ to be a symmetric theta divisor, and U irreducible. Let Y be a desingularization of Θ, fixed once for all in this proof. We shall need the following two propositions. (The condition $\dim(\operatorname{Sing}\Theta)\leq g-4$ is used in the first proposition, while the second one uses only $\dim(\operatorname{Sing}\Theta)\leq g-3$.)

Proposition 1 (Z. Ran [6], Corollary (3.3)). Restriction of line bundles gives an isomorphism $\operatorname{Pic}^0(X)\xrightarrow{\simeq}\operatorname{Pic}^0(Y)$.

Proposition 2. Let $a\in X$, $a\neq 0$. Restriction of sections gives an isomorphism $H^0\,\mathcal{O}_X(\Theta_a)\xrightarrow{\simeq} H^0\,\mathcal{O}_Y(\Theta_a)$.

Proof. Let $j\colon Y\to X$ be the obvious map. Since Θ is a normal variety, one has an exact sequence

$$0\to\mathcal{O}_X(-\Theta)\to\mathcal{O}_X\to j_*\mathcal{O}_Y\to 0.$$

By [4], p. 76, the sheaf $\mathcal{O}_X(\Theta_a-\Theta)$ has zero cohomology if $a\neq 0$. Hence the result follows by tensoring the above sequence with $\mathcal{O}_X(\Theta_a)$, q.e.d.

For $a\in X$, $a\neq 0$, we shall denote $\bar{\Theta}_a$ the divisor of Y which Θ_a defines by pullback. We choose also any divisor $\bar{\Theta}$ of the system $|\mathcal{O}_Y(\Theta)|$, fixed from now on.

i) The algebraic system $\{\bar{\Theta}_u\}_{u\in U}$ on Y is given by a certain divisor on $U\times Y$. Let D be an irreducible component of this divisor, such that $D(u)$, $u\in U$ is variable. Write R the residual divisor, thus $\bar{\Theta}_u=D(u)+R(u)$ for all $u\in U$. By the hypotheses in the Theorem, there exists an irreducible finite cover $\pi\colon\tilde{U}\to U$ and a morphism $x\colon\tilde{U}\to X$ such that, for all $\tilde{u}\in\tilde{U}$: $D(\pi(\tilde{u}))\leq\bar{\Theta}_{x(\tilde{u})}$ and $x(\tilde{u})\notin\{0,\pi(\tilde{u})\}$. For simplicity, we shall write $D(\tilde{u})$ instead of $D(\pi(\tilde{u}))$.

ii) *Claim. Without loss of generality, we may assume that $R(u)$ is fixed, for all $u\in U$.*

To see this, suppose that $R(u)$ is not fixed. For any $\tilde{u}$, $\tilde{v}\in\tilde{U}$ we have, by Proposition 1:

$$D(\tilde{v})-D(\tilde{u})\equiv\bar{\Theta}_{z(\tilde{u},\tilde{v})}-\bar{\Theta},$$

where $z\colon\tilde{U}\times\tilde{U}\to X$ is a suitable morphism. By the Theorem of the Square ([4], p. 59) this implies

$$D(\tilde{v})+R(\tilde{u})\equiv\bar{\Theta}_{z(\tilde{u},\tilde{v})+\pi(\tilde{u})}. \tag{$*$}$$

Now, we can fix an $\tilde{u}\in\tilde{U}$ such that the set

$$U'=\{z(\tilde{u},\tilde{v})+\pi(\tilde{u})\mid\tilde{v}\in\tilde{U}\}$$

is one-dimensional, and such that $x(\tilde{v})\neq z(\tilde{u},\tilde{v})+\pi(\tilde{u})$ for general $\tilde{v}$. In fact, if $x(\tilde{v})=z(\tilde{u},\tilde{v})+\pi(\tilde{u})$ for all $\tilde{u}$, $\tilde{v}\in\tilde{U}$, we would have $D(\tilde{v})+R(\tilde{u})\equiv\bar{\Theta}_{x(\tilde{v})}$, and $[R(\tilde{u})]$ would be constant in $\operatorname{Pic}(Y)$. Since $\bar{\Theta}_{\pi(\tilde{u})}$ is linearly isolated in Y, so is $R(\tilde{u})$, and $R(\tilde{u})$ ought to be fixed, against our assumption. As for the first condition, if $z(\tilde{u},\tilde{v})+\pi(\tilde{u})$ were independent from $\tilde{v}\in\tilde{U}$, so would be $D(\tilde{v})$, contradicting our assumptions, too.

By Proposition 2 and the equivalence $(*)$, this means that U' satisfies the initial requirements for U, namely

$$\Theta \cap \Theta_{z(\tilde{u},\tilde{v})+\pi(\tilde{u})} \subset \Theta_{x(\tilde{v})} \cup \Theta_y$$

for some fixed $y \in X$, $y \neq 0$. We aim to replace U by U'.

Since the map $\tilde{U} \to U'$ sending $\tilde{v}$ into $z(\tilde{u},\tilde{v})+\pi(\tilde{u})$ factors through U (use Proposition 1 and (*) to see that $\pi(\tilde{v}_1)=\pi(\tilde{v}_2)$ implies $z(\tilde{u},\tilde{v}_1)=z(\tilde{u},\tilde{v}_2)$), we infer that the divisor

$$\bigcup_{\tilde{v}\in\tilde{U}} \{z(\tilde{u},\tilde{v})+\pi(\tilde{u})\} \times D(\tilde{v})$$

of $U' \times Y$ is irreducible. Together with the equivalence (*) – an identity, in fact –, where now $R(\tilde{u})$ is kept fixed, this proves Claim ii).

Thus we may suppose that, for all $\tilde{u}\in\tilde{U}$: $\bar{\Theta}_{\pi(\tilde{u})}=D(\tilde{u})+R_0$, with R_0 a fixed component, the system $\{D(u)\}_{u\in U}$ irreducible, and $D(\tilde{u}) \leqq \bar{\Theta}_{x(\tilde{u})}$, $x(\tilde{u}) \notin \{0, \pi(\tilde{u})\}$.

iii) *Claim. There exists an irreducible subvariety $M \subset X$ of codimension 2 such that, for all $\tilde{u}\in\tilde{U}$, the image of $D(\tilde{u})$ in X equals $M_{\pi(\tilde{u})}$.*

In fact, writing $\bar{\Theta}_{x(\tilde{u})}=D(\tilde{u})+S(\tilde{u})$ in Y one has, for $\tilde{u}, \tilde{v}\in\tilde{U}$:

$$\begin{aligned} S(\tilde{v})+D(\tilde{u}) &\equiv \bar{\Theta}_{x(\tilde{v})}-D(\tilde{v})+D(\tilde{u}) \equiv \bar{\Theta}_{x(\tilde{v})}+\bar{\Theta}_{\pi(\tilde{u})}-\bar{\Theta}_{\pi(\tilde{v})} \\ &\equiv \bar{\Theta}_{(x(\tilde{v})-\pi(\tilde{v}))+\pi(\tilde{u})}. \end{aligned}$$

Fixing any $\tilde{v}\in\tilde{U}$, we get, by Proposition 2:

$$S(\tilde{v})+D(\tilde{u})=\bar{\Theta}_{(x(\tilde{v})-\pi(\tilde{v}))+\pi(\tilde{u})},$$

for (almost) all $\tilde{u}\in\tilde{U}$. Writing for a moment $M(\tilde{u})$ the image of $D(\tilde{u})$ in X, this implies $M(\tilde{u})_{-\pi(\tilde{u})} \subset \Theta_{x(\tilde{v})-\pi(\tilde{v})} \cap \Theta$, hence that $M(\tilde{u})_{-\pi(\tilde{u})}$ is constant, thereby proving our claim.

Thus, if N denotes the image of R_0 in X, we obtain in this way, for all $u\in U$: $\Theta \cdot \Theta_u = M_u + N$.

iv) *Claim.* $N=(-M)$.

Applying the symmetry of X to the equality $\Theta \cdot \Theta_u = M_u + N$ and translating by u we obtain $\Theta \cdot \Theta_u = (-M)+(-N)_u$. Comparing both expressions we deduce that $(-N)=M+F$ with $F_u=(-F)$ for all $u\in U$. We show that $F=0$. Suppose that $F \neq 0$. Fixing any $u'\in U$ and writing $V=U_{-u'}$, the curve V generates a proper abelian subvariety $A \subset X$, since $F_a=F$ for all $a\in A$. On the other hand, if $C=\bar{U}$ is the Zariski closure of U in X, the Pontrjagin product $[M]*[C]$ yields a non-zero multiple of $[\Theta]$ (since M_u is contained in Θ for all $u\in U$, and is variable), and $[F]*[C]=0$. Then, from the equality $\Theta \cdot \Theta_u = M_u + (-M)+(-F)$ we deduce that $[\Theta^2]*[C]=\lambda[\Theta]$ with $\lambda \neq 0$, hence that $[C]$ is a multiple of $[\Theta^{g-1}]/(g-1)!$. This contradicts the degeneracy of V.

Thus, summarizing: $\Theta \cdot \Theta_u = M_u + (-M)$ for all $u\in U$; moreover, M is irreducible and $C \subset X$ is non-degenerate.

It follows that, for all $u, u'\in U$, $u \neq u'$:

$$\Theta \cdot \Theta_{u-u'} = M_u + (-M)_{-u'}$$

(notice that $[M]=[\Theta^2]/2$ and that the preceding equality implies

$$M_u + (-M)_{-u'} \subset \Theta \cdot \Theta_{u-u'}).$$

v) *We finish this proof by applying Gunning's results* [2]. It follows from the preceding conclusion that, for all $u_1, u_2, u_3, u_4 \in C$, the divisors

$$\Theta_{u_1-u_2}+\Theta_{u_3-u_4}, \quad \Theta+\Theta_{u_1+u_3-u_2-u_4}, \quad \Theta_{u_1-u_4}+\Theta_{u_3-u_2}$$

belong to a linear pencil. (Use the Theorem of the Square and the exact sequence

$$0\to\mathcal{O}_X(\Theta_s)\to\mathcal{O}_X(\Theta+\Theta_s)\to j_*\,\mathcal{O}_Y(\Theta+\Theta_s)\to 0,$$

with $s=u_1+u_3-u_2-u_4$). Consider now the map $X\to\mathbb{P}^{2^g-1}$ defined by the linear system $|2\Theta|$; it is a standard fact that this map can be identified with the morphism

$$\psi\colon X\to|2\Theta|, \quad \psi(t)=\Theta_t+\Theta_{-t}.$$

What we have just seen implies: for all $a, b, c\in C$ and all $\zeta\in X$ with $2\zeta\in C-a-b-c$, the points

$$\psi(\zeta+a), \quad \psi(\zeta+b), \quad \psi(\zeta+c)$$

are collinear in $\mathbb{P}^{2^g-1}$. Applying now [2], Lemma 4, p. 385, to this situation we get the following: Let $\alpha\colon X\to X$ denote the endomorphism attached to the one-cycle C of X and the theta divisor Θ ([3], p. 415). For $a, b, c\in C$ generally chosen, the map $\alpha-I$ (I = identity) is constant on the set $\{a, b, c\}$.

Thus $\alpha-I$ is constant on C. Since C is non-degenerate, it follows that $\alpha=I$, hence, by the Matsusaka-Hoyt Criterion [3], p. 416, (X, Θ) is the jacobian of the (smooth, irreducible, non-hyperelliptic) curve C, q.e.d.

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