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Inventiones mathematicae

Fixed point indices of iterated maps

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1. Introduction and statement of the main results

We determine all universal relations which hold between the fixed point indices $\{I(f^n)\}_{n=1,2...}$ of the iterates f^n of a map $f: V \to Y$, where Y is an ENR (=euclidean neighborhood retract; cf. [4], IV.8), $V \subset Y$ is an open subset, and the iterates $f^n: V_n \to Y$ are defined inductively by $f^1 = f$, $V_n = f^{-1}(V_{n-1})$, $f^n(v) = f^{n-1}(f(v))$ for $n > 1$. The index $I(f^n) \in \mathbb{Z}$ is defined if the fixed point set $\mathrm{Fix}(f^n) = \{v \in V_n \mid f^n(v) = v\}$ is compact; the points of $\mathrm{Fix}(f^n)$ are the n-periodic points of f. If m divides n then $\mathrm{Fix}(f^m)$ is a closed subset of $\mathrm{Fix}(f^n)$ – hence compact if $\mathrm{Fix}(f^n)$ is compact. If $n = p$ is a prime then Zabreĭko-Krasnosel'skii [13] and Steinlein [12] proved that p always divides $I(f^p) - I(f)$ – provided $\mathrm{Fix}(f^p)$ is compact. We generalise their result (for ENRs; compare 6.11) as follows.

(1.1) **Theorem.** *For any natural number $n > 1$, if $\mathrm{Fix}(f^n)$ is compact then n divides the number*

$$I_n(f) = \sum_{\tau \subset P(n)} (-1)^{|\tau|} I(f^{n:\tau}), \tag{1.2}$$

where $P(n)$ is the set of all primes which divide n, the sum extends over all subsets τ of $P(n)$, $|\tau|$ = cardinality of τ, and $n:\tau = n(\prod_{p \in \tau} p)^{-1} = n$ divided by all $p \in \tau$.

For instance, if $n = p$ is a prime then Theorem 1.1 becomes the Zabreĭko-Krasnsel'skii theorem. If $n = p^k$ is a power of a prime then [13] still asserts $p/I_{p^k}(f)$ whereas Theorem 1.1 gives $p^k/I_{p^k}(f)$. On the other hand, Theorem 1.1 easily reduces to the special case where n is a power of a prime.

The congruences $n/I_n(f)$ are the only relations which are satisfied by the fixed point indices $\{I(f^\nu)\}_{\nu=1,2...}$ of arbitrary maps f as above. Slightly more general,

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(1.3) **Theorem.** *If* s: $\mathbb{N}\to\mathbb{Z}$ *is a sequence of integers such that* n *divides the number*

$$M_s(n)=\sum_{\tau\subset P(n]}(-1)^{|\tau|}s(n:\tau) \tag{1.4}$$

for every natural number $n>1$ *then there is a (connected 2-dimensional) simplicial complex* Y *and a continuous map* f: $Y\to Y$ *such that* $\operatorname{Fix}(f^\nu)$ *is compact and* $I(f^\nu)=s(\nu)$ *for all* $\nu\geqq 1$. *In fact,* f *can be so chosen that, for every* ν, f *has exactly* $|M_s(\nu)|$ *points of smallest period* ν *each of index* $M_s(\nu)|M_s(\nu)|^{-1}=\pm 1$.

The notation on the right side of (1.4) is as in (1.2). On the left side we use M, as in *Möbius*, because (1.4) resp. (2.7) is in fact the Möbius inversion formula (cf. [1], App.).

If one wants the ENR Y in Theorem 1.3 to be *compact* then one has to impose additional finiteness conditions on the sequence s. A crude way to do so is indicated in 3.8. For a complete answer to this question one needs the *Lefshetz power series* $L(f;t)$ which is defined as follows.

(1.5) **Definition.** If f: $V\to Y$ is a continuous map as above (Y an ENR, $V\subset Y$ open) and $\operatorname{Fix}(f^n)$ is compact for all $n=1,2\ldots$ then we define the Lefshetz (formal) power series $L(f;t)=\sum_{n=0}^{\infty}L_n(f)t^n$ by the (Newton) recursion formula $L_0(f)=1$,

$$nL_n(f)=\sum_{j=1}^{n}(-1)^{j+1}L_{n-j}(f)I(f^j)\qquad\text{for } n>0. \tag{1.6}$$

Alternatively (compare 4.4′ and 4.4)

$$L(f;t)=\exp\left(-\sum_{\nu=1}^{\infty}\frac{I(f^\nu)}{\nu}t^\nu\right). \tag{1.6′}$$

We shall see (cf. 1.8) that $L(f;t)$ always has *integral* coefficients, $L_n(f)\in\mathbb{Z}$. Thus $L(f;t)\in\mathbb{Z}[[t]]$; in fact,

$$L(f;t)\in(1+t\mathbb{Z}[[t]]) \tag{1.7}$$

because $L_0(f)=1$. We can then reformulate 1.3 as follows.

(1.8) **Theorem.** *For a sequence of integers* s: $\mathbb{N}\to\mathbb{Z}$ *to be the sequence of indices* $I(f^\nu)$ *of a map* f *as above it is necessary and sufficient that the formal power series* $\zeta(t)=\exp\left(-\sum_{\nu=1}^{\infty}\frac{s(\nu)}{\nu}t^\nu\right)$ *has integral coefficients.*

And we can answer the question whether Y can be taken compact as follows

(1.9) **Theorem.** *For a sequence of integers* s: $\mathbb{N}\to\mathbb{Z}$ *to be the sequence of indices* $I(f^\nu)$ *of a map* f: $Y\to Y$ *with compact* ENR Y *it is necessary and sufficient that the formal power series* $\zeta(t)=\exp\left(-\sum_{\nu=1}^{\infty}\frac{s(\nu)}{\nu}t^\nu\right)$ *is an integral rational function.*

In this case, $L(f;t)=\exp\left(-\sum\frac{s(v)}{v}t^v\right)$ *coincides with* $\frac{\det(id-tf_+)}{\det(id-tf_-)}$, *where* f_+ *resp.* f_- *is the endomorphism which* f *induces on the even resp. odd homology of* Y *with rational coefficients* (or integral homology mod torsion), $H_{\text{even}}=\bigoplus_i H_{2i}$, $H_{\text{odd}}=\bigoplus_i H_{2i+1}$.

As pointed out to me by T. tom Dieck (cf. also [9]) the preceding establishes a strong connection between periodic point theory and the theory of λ-rings (cf. [8]). In particular, it suggests a geometric model $\mathfrak{P}$ (cf. 6.6) for the universal λ-ring Λ over $\mathbb{Z}$ (whose additive group is given by multiplication in $1+t\mathbb{Z}[[t]]$). Although I was unable to settle a basic problem in this context (cf. text after 6.8) I can at least offer a simple combinatorial model PER for the universal λ-ring, in the spirit of and related to Burnside rings (cf. 2.16 and 6.10).

As for the proofs, Theorem 1.1 has a simple combinatorial background (§2) to which it is essentially reduced (in §5) by a transversality argument (Prop. 5.7; cf. also [12]). The same tools and the Lefshetz-Hopf theorem are used to prove Theorem 1.8 in §4. The proof of Theorem 1.3 (in §3) uses only §2 and elementary fixed point theory; the proof of 1.9 (in §4) uses the Lefshetz-Hopf theorem. – The last §6 presents comments, examples, and problems.

2. Self-maps $f\colon Y\to Y$ of discrete sets Y

The fixed point index in this case is the cardinality of the fixed point set. We *assume* $\mathrm{Fix}(f^n)$ *to be finite* for all n, thus $I(f^n)=|\mathrm{Fix}(f^n)|$. We should also assume that Y is countable (to be ENR). In fact, for our purposes the map f matters only in the neighborhood of the fixed point sets; so we can (and shall) *assume that*

(2.1) $$Y=\bigcup_{n=1}^{\infty}\mathrm{Fix}(f^n),\qquad \text{i.e. } \textit{every point is periodic.}$$

Thus f is a permutation, Y decomposes into finite cycles, and for every n the number of n-cycles is finite. We call f a *permutation of finite type.* Let

(2.2) $$\mathrm{Fix}_n(f)=\{y\in Y\,|\,f^n(y)=y \text{ but } f^m(y)\neq y \text{ for } m<n\}$$

the set of points of period exactly n. Its cardinality satisfies

(2.3) $$|\mathrm{Fix}_n(f)|=\sum_{\tau\subset P(n)}(-1)^{|\tau|}I(f^{n:\tau})=I_n(f),$$

i.e. the Möbius number (1.2) coincides with the number of points of period exactly n, or

(2.4) $$\frac{1}{n}I_n(f)=\textit{number of } n\textit{-cycles of } f.$$

Theorem 1.1 therefore has a very simple (and well-known) explanation in this case. We give a short proof of (2.3) (compare [1], l.c.): Note first that

$\mathrm{Fix}(f^a)\cap\mathrm{Fix}(f^b)=\mathrm{Fix}(f^c)$ for all $a, b\in\mathbb{N}$, $c=gcd(a,b)$. Now, given $y\in\mathrm{Fix}(f^n)$ let $t\subset P(n)$ the largest set of primes for which $f^{n:t}(y)=y$. Then $y\in\mathrm{Fix}(f^{n:\tau})\Leftrightarrow\tau\subset t$. Therefore, y contributes to $I(f^{n:\tau})$ iff $\tau\subset t$. Therefore, the contribution of y to $I_n(f)$ is

(2.5) $$\sum_{\tau\subset t}(-1)^{|\tau|}=\sum_i(-1)^i\binom{|t|}{i}$$
$$=(1-1)^{|t|}=0 \quad \text{if } t\neq\emptyset, \text{ and } =1 \text{ if } t=\emptyset.$$

But $t=\emptyset$ iff $y\in\mathrm{Fix}_n(f)$. □

Every n-periodic point has a unique least period m, and m divides n. Therefore $\mathrm{Fix}(f^n)$ is the disjoint union of the sets $\mathrm{Fix}_m(f)$ with m/n, hence (2.3) implies

(2.6) $$I(f^n)=\sum_{m/n}I_m(f).$$

More generally, for every n-tupel $s=(s(1),s(2),\dots,s(n))$ of rational numbers Möbius inversion ([1], l.c.) asserts

(2.7) $$s(n)=\sum_{m/n}M_s(m).$$

(2.8) *Example.* Let $\zeta_n\colon Z_n\to Z_n$ be a cyclic permutation of length n. Then $I((\zeta_n)^k)=n$ if n/k, and $=0$ otherwise. And $I_k(\zeta_n)=n$ if $k=n$, and $=0$ otherwise. The Lefshetz power series is

$$L(\zeta_n;t)=1-t^n,$$

because $\log(1-t^n)=-\sum_i\frac{t^{in}}{i}=-\sum_\nu\frac{I((\zeta_n)^\nu)}{\nu}t^\nu.$

(2.9) **Definitions.** One can "add" (isomorphism classes of) permutations of finite type by taking the disjoint union, $f_1+f_2\colon Y_1\cup Y_2\to Y_1\cup Y_2$; thus, $f_1+f_2|Y_j=f_j$. The cycle decomposition of f_1+f_2 is the disjoint union of the cycle decompositions of f_1 and f_2. Every $f\colon Y\to Y$ is the (infinite) sum of its cycles, i.e. *every f can be written in the form*

(2.10) $$f=\sum_{\nu=1}^{\infty}i_\nu\zeta_\nu$$

where the $i_\nu\geqq 0$ are uniquely determined natural numbers. Under this addition the set of (isomorphism classes of) permutations of finite type becomes a commutative monoid which we denote by PER^+. The decomposition (2.10) shows that

(2.10′) $$\mathrm{PER}^+\cong\prod_{\nu=1}^{\infty}\mathbb{N},$$

a countable product of factors $\mathbb{N}=\{0,1,2,\dots\}$.

We can adjoin negatives to this monoid (i.e. form the Grothendieck group) and obtain an abelian group which we denote by PER. Every $\varphi\in\mathrm{PER}$ can be written in the form $\varphi=\varphi^+-\varphi^-$, where φ^+ and φ^- are in PER^+. And φ can (uniquely) be written in the form

$$(2.11)\qquad \varphi=\sum_{\nu=1}^{\infty} j_\nu \zeta_\nu \quad \text{with} \quad j_\nu\in\mathbb{Z},$$

thus $\mathrm{PER}\cong\prod_{\nu=1}^{\infty}\mathbb{Z}$.

If we assign to each $f\in\mathrm{PER}^+$ its Lefshetz power series we obtain a map

$$(2.12)\qquad L\colon \mathrm{PER}^+\to 1+t\mathbb{Z}[[t]], \quad f\mapsto L(f;t).$$

It is easy to see that $L(f_1+f_2;t)=L(f_1;t)\cdot L(f_2;t)$, i.e. L is homomorphic. In fact (using 2.8, 2.10, and looking at finite segments of the power series), one sees that

$$L\left(\sum_{\nu=1}^{\infty} i_\nu \zeta_\nu\right)=\prod_{\nu=1}^{\infty}(1-t^\nu)^{i_\nu}.$$

Therefore L extends to PER,

$$(2.13)\qquad L\colon \mathrm{PER}\to 1+t\mathbb{Z}[[t]], \quad L\left(\sum_{\nu=1}^{\infty} j_\nu \zeta_\nu\right)=\prod_{\nu=1}^{\infty}(1-t^\nu)^{j_\nu}.$$

Since every power series $q(t)\in(1+t\mathbb{Z}[[t]])$ has a unique decomposition $q(t)=\prod_{\nu=1}^{\infty}(1-t^\nu)^{j_\nu}$ with integral exponents $j_\nu\in\mathbb{Z}$ we see that

(2.14) **Proposition.** *L is an isomorphism of abelian groups,* $\mathrm{PER}\cong 1+t\mathbb{Z}[[t]]$. □

In fact, both sides have more structure: *Both are λ-rings.* For $1+t\mathbb{Z}[[t]]$ $(=\Lambda)$ the reader may consult [8], in particular 1.2; cf. also proof of 2.16. In PER^+ the multiplication is obvious: If $f_i\colon Y_i\to Y_i$ are in PER^+ for $i=1,2$ then $f_1\times f_2\colon Y_1\times Y_2\to Y_1\times Y_2$ is their product; this product is bilinear and therefore extends uniquely to PER. In order to define λ-operations in PER one defines symmetric powers s^k first. This is obvious in PER^+ where $s^k f\colon s^k Y\to s^k Y$ has the usual meaning: $s^k Y=Y^k/S(k)=$cartesian power divided by the action of the symmetric group; $s^0 Y=$ a point. By the usual trick these operations $\{s^k\}_{k\geq 0}$ can be combined into a single homomorphic operation $s_t=\sum_{k=0}^{\infty} s^k t^k$ (with indeterminate t), and therefore extended to PER. Finally, "exterior powers" $\lambda^n\varphi$ can be defined for $\varphi\in\mathrm{PER}$ by the recursive equation $\lambda^0\varphi=1$,

$$(2.15)\qquad \lambda^n\varphi=\sum_{i=1}^{n}(-1)^{i+1}(\lambda^{n-i}\varphi)\cdot(s^i\varphi), \qquad \text{for } n>0.$$

(2.16) **Theorem.** L: $\mathrm{PER}\to\Lambda=1+t\mathbb{Z}[[t]]$ *is an isomorphism of λ-rings.*

Sketch of proof. The ring-multiplication and the λ^i-resp. s^j-operations in Λ are so designed (cf. [8], 1.2) that the following identities (2.17) hold for characteristic polynomials $\psi(\alpha)=\det(1-t\alpha)\in\Lambda$ of square integral matrices α:

(2.17) $$\psi(\alpha)\psi(\beta)=\psi(\alpha\otimes\beta), \qquad \lambda^i\psi(\alpha)=\psi(\lambda^i\alpha), \qquad s^j\psi(\alpha)=\psi(s^j\alpha),$$

where $\lambda^i\alpha$ is the i-th exterior power of α, and $s^j\alpha=\overset{j}{\bigotimes}\alpha/S(j)$. On the other hand, if $f\colon Y\to Y$ is a permutation of a *finite* set then $L(f)$ is just the characteristic polynomial of the corresponding permutation matrix $\pi(f)\colon \mathbb{Z}Y\to\mathbb{Z}Y$, $L(f)=\psi(\pi(f))$, where $\mathbb{Z}Y$ has basis Y. And $\pi(f_1\times f_2)=(\pi f_1)\otimes(\pi f_2)$, $\pi(s^j f)=s^j(\pi(f))$. This proves that L commutes with products and s^j-operations (hence λ^i-operations) as long as we stay with finite permutations. The general case (permutations of finite type) then follows by an easy passage to $\varprojlim$. □

3. Proof of Theorem 1.3, and a related result (3.8)

If we don't care about Y being connected then the proof of 1.3 becomes very simple: Let $i_\nu=\frac{1}{\nu}M_s(\nu)$. If $i_\nu\geqq 0$ we take i_ν copies of ζ_ν (cf. 2.8). If $i_\nu<0$ we take $-i_\nu$ copies of $\zeta_\nu\times(2)$, where $(2)\colon \mathbb{R}\to\mathbb{R}$ is multiplication by 2; note that (2) and all of its iterates have one fixed point, 0, and index -1. It follows that $I((\zeta_\nu\times(2))^r)=-I((\zeta_\nu)^r)$ for all r, hence $I_\nu(\zeta_\nu\times(2))=-I_\nu(\zeta_\nu)=-\nu$ and $I_k(\zeta_\nu\times(2))=0$ for $k\neq\nu$. Now let

(3.1) $$f=\sum_{i_\nu\geqq 0} i_\nu\zeta_\nu+\sum_{i_\nu<0}(-i_\nu)(\zeta_\nu\times(2)),$$

the disjoint union. Since I_m is additive we obtain

$$I_m(f)=i_m I_m(\zeta_m)=i_m\cdot m=M_s(m)$$

for all m. Therefore, $I(f^n)=s(n)$ by (2.6) and (2.7). Also, it is clear that f has exactly $|I_m(f)|=|M_s(m)|$ points of period exactly m, each of them with index ± 1 depending on whether $i_m>0$ or $i_m<0$. This f therefore almost proves the theorem; its only defect is that it is defined on a disconnected space.

In order to correct this defect we multiply the ζ_ν with certain maps e of spheres $S^k=\mathbb{R}^k\cup\{\infty\}$, $k=1$ or 2, and connect the results in a wedge-like fashion. Define $e\colon S^k\to S^k$ by

(3.2) $$e(x)=\begin{cases}\dfrac{2x}{1-\|x\|^2} & \text{for } \|x\|<1\\ \infty & \text{for } \|x\|\geqq 1.\end{cases}$$

It has two fixed points, 0 and ∞, and the same holds for all iterates e^n. The fixed point ∞ (whose index is $+1$) will not matter later on (we shall get rid of it). At 0 the derivative $\dot{e}^n$ is $2^n id$, and the index equals the sign of $\det(\mathrm{id}-\dot{e}^n)$, i.e. $+1$ if $k=2$ and -1 if $k=1$, thus

(3.3) $$I(e^n|\mathbb{R}^k)=(-1)^k \quad \text{for all } n\geqq 1.$$

Define $S^k\zeta_\nu\colon S^kZ_\nu\to S^kZ_\nu$ as follows:

(3.4) $$S^kZ_\nu=\frac{S^k\times Z_\nu}{\{\infty\}\times Z_\nu}, \qquad S^k\zeta_\nu=\overline{e\times\zeta_\nu}=\text{induced by } e\times\zeta_\nu.$$

Thus $S^k Z$ is a wedge of v k-spheres, joined at the wedge-point ∞. The map $S^k \zeta_v$ permutes the spheres cyclically and at the same time pushes away from the zeros to the wedge point ∞. The periodic points of $S^k \zeta_v | S^k Z_v - \{\infty\}$ are the same (at $\{0\} \times Z_v$) as for ζ_v; the indices are also the same if $k=2$, and they are all opposite if $k=1$.

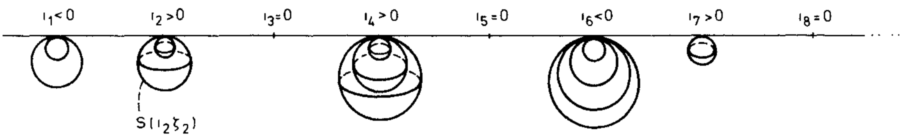
Slightly more general we can replace ζ_v by a multiple $i\zeta_v$, $i \in \mathbb{N}$. We obtain

$$(3.5) \qquad S^k(i\zeta_v)\colon S^k(iZ_v) \to S^k(iZ_v), \qquad S^k(iZ_v) = \frac{S^k \times (iZ_v)}{\{\infty\} \times (iZ_v)},$$

$$S^k(i\zeta_v) = \overline{e \times (i\zeta_v)} = \text{induced by } e \times (i\zeta_v).$$

Thus $S^k(iZ_v)$ is a wedge of iv k-spheres, joined at the wedgepoint ∞, and $S^k(i\zeta_v)|S^k(iZ_v) - \{\infty\}$ has the same periodic points as $i\zeta_v$, each with index $+1$ if $k=2$, with index -1 if $k=1$.

Now we prove Theorem 1.3 as follows: We put $i_v = \frac{1}{v} M_s(v)$ as above. We let $S(i_v\zeta_v) = S^2(i_v\zeta_v)$ if $i_v \geqq 0$, and $S(i_v\zeta_v) = S^1(-i_v\zeta_v)$ if $i_v < 0$. We attach $S(i_v Z_v)$ to the real line $\mathbb{R}$ by identifying $\infty \in S(i_v Z_v)$ with $v \in \mathbb{R}$. The resulting space Y looks like a long "washing line" $\mathbb{R}$ with $v|i_v|$ 2-spheres (if $i_v > 0$) resp. 1-spheres (if $i_v < 0$) hanging at $v \in \mathbb{R}$, for $v = 1, 2 \ldots$



(the picture isn't correct: the number of spheres attached at v should be divisible by v).

It is a connected ENR of dimension $\leqq 2$. We have a self-map of Y which is the identity on $\mathbb{R}$ and $S(i_v\zeta_v)$ on each $S(i_v Z_v)$, but this has too many fixed points. We therefore move $\mathbb{R}$ to the right and pull a neighborhood of $\mathbb{R}$ in $S(i_v Z_v)$ along. More precisely we define $f\colon Y \to Y$ as follows.

$$(3.6) \quad f(x) = \begin{cases} (x+1) \in \mathbb{R} & \text{if } x \in \mathbb{R}, \\ \left(v + 1 - \dfrac{1}{\|x\|}\right) \in \mathbb{R} & \text{if } x \in S(i_v Z_v) \text{ and } \|x\| \geqq 1, \\ (S(i_v\zeta_v))(x) \in S(i_v Z_v) & \text{if } x \in S(i_v Z_v) \text{ and } \|x\| \leqq 1. \end{cases}$$

In order to understand the norm $\|x\|$ in the second case the reader should remember that each $x \in S(i_v Z_v)$ is in one of the spheres $S^k = \mathbb{R}^k \cup \{\infty\}$, and $\|x\|$ is the norm of x in $\mathbb{R}^k$.

This mapping f has periodic points only in $\{0\}\times i_\nu Z_\nu$ (for $\nu=1,2\ldots$); and in the neighborhood $\|x\|<1$ of $\{0\}\times i_\nu Z_\nu$ the map f coincides with $S(i_\nu\zeta_\nu)$. The latter has the same indices as $i_\nu\zeta_\nu$ if $i_\nu\geqq 0$, and the negative indices of $(-i_\nu)\zeta_\nu$ if $i_\nu<0$. It follows (by additivity of the indices and 2.7) that

$$(3.7)\qquad I_m(f)=i_m I_m(\zeta_m)=i_m m=M_s(m)$$

for all m. Therefore, $I(f^n)=s(n)$ by (2.6) and (2.7). Also it is clear that f has the required number $|M_s(m)|$ of points of period exactly m (with multiplicity ± 1) because this number is the same as for $\pm i_m\zeta_m$. □

We already pointed out that Y cannot, in general, be chosen *compact* in Theorem 1. However, if we only prescribe finitely many of the values $I(f^\nu)$, or if almost all $I_n(f)=0$, then we can make Y compact.

(3.8) **Theorem.** *If $s=\{s(\nu)\}_{\nu=1,2\ldots N}$ is an N-tuple of integers ($N\in\mathbb{N}$) such that n divides $M_s(n)$ for every natural number n with $1<n\leqq N$ then there is a compact connected simplicial complex K (essentially a wedge of circles and 2-spheres) and a continuous map g: $K\to K$ such that $I(g^\nu)=s(\nu)$ for all $1\leqq\nu\leqq N$.*

Proof. We put $i_\nu=\frac{1}{\nu}M_s(\nu)$ as before if $1\leqq\nu\leqq N$, and $i_\nu=0$ for $\nu>N$. We construct Y and f: $Y\to Y$ as before (3.6). Since $i_\nu=0$ for $\nu>N$ nothing is attached to points $\nu\in\mathbb{R}$ with $\nu>N$. Let $\alpha\in(\mathbb{R}-\mathbb{Q})$ an irrational number and reduce mod α in $\mathbb{R}$, i.e. identify points x_1, $x_2\in\mathbb{R}$ if $\left(\frac{x_1-x_2}{\alpha}\right)\in\mathbb{Z}$. Let K be the quotient of Y obtained by this reduction, and let g: $K\to K$ be induced by f. The reduction creates no new periodic points (because $\alpha\notin\mathbb{Q}$) and leaves the map unchanged in the neighborhood of the periodic point set so that (3.7) and the rest of the above proof applies again. □

4. Proofs of Theorems 1.8 and 1.9

We first recall some linear algebra. If α is a square matrix over a commutative ring R let

$$(4.1)\qquad \Psi(t)=\Psi(\alpha;t)=\det(\mathrm{id}-t\alpha)$$

denote its characteristic polynomial. If λ_1, λ_2, ..., λ_r are the (non-zero) eigenvalues of α then $\Psi(t)=\prod_{i=1}^{r}(1-t\lambda_i)$, hence

$$(4.2)\qquad \frac{\Psi'(t)}{\Psi(t)}=-\sum_{i=1}^{r}\frac{\lambda_i}{1-t\lambda_i}=-\sum_{\nu=0}^{\infty}\left(\sum_{i=1}^{r}\lambda_i^{\nu+1}\right)t^\nu$$

$$=-\sum_{\nu=0}^{\infty}\operatorname{trace}(\alpha^{\nu+1})t^\nu,$$

where $\Psi'(t)=\frac{d}{dt}\Psi(t)$. Similarly, if $\alpha=(\alpha_+,\alpha_-)$ is a pair of square matrices (a 2-graded matrix) then

$$\Psi(t)=\Psi(\alpha;t)=\frac{\det(\mathrm{id}-t\alpha_+)}{\det(\mathrm{id}-t\alpha_-)}=\frac{\Psi(\alpha_+;t)}{\Psi(\alpha_-;t)} \tag{4.3}$$

is the *characteristic rational function* of α. Taking the logarithmic derivative of (4.3) we obtain by (4.2) that

$$\frac{\Psi'(\alpha;t)}{\Psi(\alpha;t)}=-\sum_{v=0}^{\infty}\Lambda(\alpha^{v+1})t^v, \tag{4.4}$$

where $\Lambda(\alpha^v)=\mathrm{trace}(\alpha_+^v)-\mathrm{trace}(\alpha_-^v)$ is the so-called *Lefshetz trace* of $\alpha^v=(\alpha_+^v,\alpha_-^v)$. Writing (4.4) as

$$\Psi'(t)=-\Psi(t)\sum_{v=0}^{\infty}\Lambda(\alpha^{v+1})t^v \tag{4.4'}$$

and comparing coefficients one obtains the Newton recursion formula relating the coefficients of $\Psi(\alpha;t)$ to the traces $\Lambda(\alpha^v)$.

Proof of 1.9. If $f\colon Y\to Y$ is a self-map of a compact ENR with $I(f^v)=s(v)$ then by the Lefshetz-Hopf theorem ([4], VII.6)

$$I(f^v)=\Lambda(f_+^v,f_-^v)=\text{Lefshetz trace of } (f_+^v,f_-^v). \tag{4.5}$$

We use integral homology mod torsion so that $\alpha=(f_+,f_-)$ is a pair of square *integral* matrices. Comparing 1.6′ and (4.4) then shows that

$$L(f;t)=\Psi(\alpha;t)=\frac{\det(\mathrm{id}-tf_+)}{\det(\mathrm{id}-tf_-)}$$

as asserted in 1.9; in particular, $\zeta(t)=L(f;t)$ is an integral rational function.

Conversely, if $\zeta(t)\in(1+t\mathbb{Z}[[t]])$ is a rational function, hence ([10], p. 511) $\zeta(t)=\frac{p(t)}{q(t)}$ with polynomials $p(t)=1+a_1t+\ldots+a_rt^r$, $q(t)=1+b_1t+\ldots+b_st^s$, and a_i, $b_j\in\mathbb{Z}$, then we choose integral $r\times r$-resp. $(s+1)\times(s+1)$-matrices α, β such that $\det(\mathrm{id}-t\alpha)=p(t)$, $\det(\mathrm{id}-t)=(1-t)q(t)$ - for instance, $\alpha(e_i)=e_{i+1}$ $(i<r)$, $\alpha(e_r)=-\sum_{i=1}^{r}a_ie_{r-i+1}$ on the standard basis $\{e_i\}$ of $\mathbb{Z}^r$. - We can realise α (cf. [4], V, 6.16; Exerc. 1) by a self-map f_α of a wedge $\bigvee^r S^2$ of r 2-spheres, so that f_α induces α on $H_2(V^rS^2)\cong\mathbb{Z}^r$. Similarly, we realise β by a self-map β of a wedge $\bigvee^{s+1}S^1$ of $s+1$ circles, so that f_β induces β on $H_1(\bigvee^{s+1}S^1)\cong\mathbb{Z}^{s+1}$. Then $f=f_\alpha\vee f_\beta$ is a self-map of the wedge $(\bigvee^rS^2)\vee(\bigvee^{s+1}S^1)$ with $\det(\mathrm{id}-tf_+)=(1-t)p(t)$, $\det(\mathrm{id}-tf_-)=(1-t)q(t)$, hence $L(f;t)=\frac{p(t)}{q(t)}=\zeta(t)$. Comparing $\zeta(t)$ with (1.6′) then shows that $s(v)=I(f^v)$. □

Proof of 1.8. If $s(v)=I(f^v)$ for some map then n divides $M_s(n)=I_n(f)$ by Theorem 1.1 (all v,n). For any fixed $N\in\mathbb{N}$ Theorem 3.8 gives a map $g\colon K\to K$

such that K is a compact ENR and $I(g^\nu)=s(\nu)=I(f^\nu)$ for all $1 \leqq \nu \leqq N$. If follows that $L(f;t)\equiv L(g;t) \bmod t^N$, i.e. this two power series have the same coefficients up to t^N. But $L(g;t)$ has integral coefficients (cf. 1.9), hence also $\zeta(t) = L(f;t)$.

Conversely, if $\zeta(t)$ has integral coefficients then approximating polynomials can be realised as $L(g;t)$ of some $g\colon K \to K$ with compact ENR K (cf. 1.9 or, the 2nd half of its proof); thus, for any given $N \in \mathbb{N}$ we can find g with $\zeta(t)\equiv L(g;t) \bmod t^N$, hence $s(\nu)=I(g^\nu)$ up to N. Therefore, we know by 1.1 that n divides $I_n(g)=M_s(n)$ for all n (up to N), hence Theorem 1.3 gives a map f: $Y \to Y$ such that $s(\nu)=I(f^\nu)$ for all ν. □

5. Proof of Theorem 1.1

Since Y is an ENR there is an open subset Q of some $\mathbb{R}^k$ and maps $Y \xrightarrow{i} Q \xrightarrow{r} Y$ with $ri=\mathrm{id}$. Then $f\colon Y \to Y$ has the same fixed point index as $g=ifr$: $r^{-1}V \to \mathbb{R}^k$ (cf. [4]; VII, 5.10). Furthermore $g^n = if^n r$, hence $I(g^n)=I(f^n)$. Therefore, it suffices to prove the theorem for g, i.e. *we can* (and shall) *assume that* $Y = \mathbb{R}^k$, $V \subset \mathbb{R}^k$ open, $f\colon V \to \mathbb{R}^k$ continuous, $\mathrm{Fix}(f^n)$ compact. We may also assume that $\bar{V}$ *is compact* and f can be continuously extended to $\bar{V}$ (hence f^ν to $\bar{V}_\nu$) without ν-periodic points on the "boundary" $\bar{V}_\nu - V_\nu$, for ν/n. If this assumption is not automatically satisfied then we choose a compact neighborhood K of $\mathrm{Fix}(f^n)$ in V_n, and replace f by $f|\mathring{K}$ (V by $\mathring{K}$); this restriction does not affect any of the indices $I(f^\nu)$ with ν/n.

We consider the following auxiliary map $f_n\colon V^n \to (\mathbb{R}^k)^n = \mathbb{R}^{kn}$

$$f_n(x^1,\dots,x^n)=(f(x^n),f(x^1),f(x^2),\dots,f(x^{n-1})), \tag{5.1}$$

where $x^j \in V$ for $j=1,2,\dots,n$. Its fixed points satisfy $x^{j+1}=f^j(x^1)$ for $j<n$ and $f^n(x^1)=x^1$. Thus $\mathrm{Fix}(f_n)$ is homeomorphic to $\mathrm{Fix}(f^n)$ under the projection $(x^1,\dots,x^n)\mapsto x^1$. In particular, $\mathrm{Fix}(f_n)$ is compact. The map f_n is also defined on the boundary $\bar{V}^n - V^n$ but has no fixed points there. Let η denote the minimum of $\|z-f_n(z)\|$ as z ranges over $\bar{V}^n - V$,

$$\eta = \mathrm{Min}\{\|z-f_n(z)\| \mid z \in (\bar{V}^n - V^n)\} > 0. \tag{5.2}$$

We shall consider ε-approximations $g\colon \bar{V} \to \mathbb{R}^k$ of f, where $\varepsilon < \frac{\eta}{\sqrt{n}}$. Then $f_n(x^1,\dots,x^n)$ and $g_n(x^1,\dots,x^n)=(g(x^n),g(x^1),\dots,g(x^{n-1}))$ differ by less than $\sqrt{n}\varepsilon < \eta$,

$$\begin{aligned} &\|f(x)-g(x)\| < \varepsilon \quad \text{for all } x \in \bar{V} \Rightarrow \\ &\|f_n(z)-g_n(z)\| < \eta \quad \text{for all } z \in \bar{V}^n. \end{aligned} \tag{5.3}$$

Moreover, in this case, the map $(1-t)f+tg$ differs from f by less than ε, for all $t \in [0,1]$, hence $(1-t)f_n + tg_n$ differs from f_n by less than η. It follows that $(1-t)f_n+tg_n$ has no fixed points in $\bar{V}^n - V^n$; all of its fixed points are in V^n, and the total fixed point set

$$(5.4)\qquad \{(z,t)\in V^n\times[0,1]\,|\,(1-t)f_n(z)+tg_n(z)=z\}$$

is compact. But this is homeomorphic (as above) to

$$(5.5)\qquad \{(x,t)\in V\times[0,1]\,|\,((1-t)f+tg)^n(x)=x\}$$

which is therefore compact. Homotopy invariance ([5], 2.9) of the fixed point index then shows $I(g^n)=I(f^n)$, and similarly $I(g^\nu)=I(f^\nu)$ for ν/n. Altogether,

$$(5.6)\qquad \begin{array}{ll} \|f(x)-g(x)\|<\varepsilon & \text{for all } x\in\bar{V}\Rightarrow \\ I(g^\nu)=I(f^\nu) & \text{for all } \nu \text{ which divide } n. \end{array}$$

For instance, we can approximate f by a smooth map g without changing the relevant fixed point indexes; i.e. *we can assume f to be smooth*. And in this case we shall prove the following transversality property (compare [7], p. 68/69).

(5.7) **Proposition.** *If $f\colon \bar{V}\to\mathbb{R}^k$ as above is smooth then there is a polynomial map $p\colon \mathbb{R}^k\to\mathbb{R}^k$ with the following properties.*

(i) *The components of p are polynomials of degree at most $2n-1$* [1].

(ii) $\|p(x)\|<\varepsilon$ *for all* $x\in\bar{V}$.

(iii) *The map g^n, where $g=f+p$, has only regular fixed points, i.e.* $\mathrm{Fix}(g^n)$ *is finite and*

$$(5.8)\qquad \det(\mathrm{id}-Dg^n(a))\neq 0$$

for all $a\in\mathrm{Fix}(g^n)$; D denotes the derivative.

Proof. We'll use the following well-known facts (A) and (B) from algebra resp. analysis.

(A) If $a^1, a^2, \ldots, a^\nu\in\mathbb{R}^k$ are distinct points, $b^1, b^2, \ldots, b^\nu\in\mathbb{R}^k$ are arbitrary points, and $\varphi_1, \ldots, \varphi_\nu\in\mathscr{L}(\mathbb{R}^k)$ are arbitrary linear maps $\mathbb{R}^k\to\mathbb{R}^k$ then there is a polynomial map $p\colon\mathbb{R}^k\to\mathbb{R}^k$ of degree $<2\nu$ such that $p(a^i)=b^i$ and $Dp(a^i)=\varphi_i$ for all $i=1,2,\ldots$.

(B) If $\pi\colon\mathbb{R}^l\to\mathbb{R}$ is a polynomial function, $\pi\neq 0$, then $\pi^{-1}(0)$ is (contained in) a finite union of smooth submanifolds $M\subset\mathbb{R}^l$ with dimension$(M)<l$. In fact,

$$(5.9)\qquad \pi^{-1}(0)\subset\bigcup_{(\rho,j)}\{x\in\mathbb{R}^l\,|\,\rho(x)=0 \text{ and } (D_j\rho)(x)\neq 0\}$$

where ρ ranges over all partial derivatives of π of order$(\rho)<$degree(π), and $j=1,2,\ldots,l$ (D_j = partial derivative).

Consider the following smooth maps (for ν/n)

$$(5.10)\qquad \begin{array}{l} \Phi=\Phi^{(\nu)}\colon V^{(\nu)}\times P\to(\mathbb{R}^k)^\nu\times\mathscr{L}(\mathbb{R}^k)^\nu=\mathbb{R}^{k\nu}\times\mathscr{L}(\mathbb{R}^{k\nu},\mathbb{R}^k) \\ \Phi(z,p)=(z-f_\nu(z)-p_\nu(z),\, Df_\nu(z)+Dp_\nu(z)), \end{array}$$

[1] A little more effort in the proof would show that p can be chosen of degree $<n$

where

$$(5.11)\qquad V^{(\nu)}=\{(x^1,x^2,\dots,x^\nu)\in V^\nu \mid x^i\neq x^j \text{ for } i\neq j\}=$$

set of distinct ν-tuples $z=(x^1,\dots,x^\nu)$ in V, P = vector space of polynomial maps $p\colon \mathbb{R}^k\to\mathbb{R}^k$ of degree$(p)<2n$, and $\mathscr{L}(\mathbb{R}^k)$ = space of linear maps $\mathbb{R}^k\to\mathbb{R}^k$. The maps f_ν are defined as in 5.1,

$$f_\nu(x^1,\dots,x^\nu)=(f(x^\nu), f(x^1), f(x^2),\dots, f(x^{\nu-1})).$$

We claim that Φ *is a submersion*, i.e. of rank $k\nu+k^2\nu$ at every point $(z,p)\in V^{(\nu)}\times P$. In fact, already the partial derivative $D_p\Phi$ with respect to the variable p is surjective: As a function of p, Φ is affine, hence

$$D_p\Phi(z,p_0)\colon P\to(\mathbb{R}^k)^\nu\times\mathscr{L}(\mathbb{R}^k)^\nu$$

is the linear map which takes $p\in P$ into

$$(-p_\nu(z), Dp_\nu(z))\sim(\{-p(x^i)\},\{Dp(x^i)\})_{i=1,2,\dots,\nu},$$

and this map is surjective by (A).

Consider then the polynomial function $\pi\colon\mathscr{L}(\mathbb{R}^k)^\nu\to\mathbb{R}$, where

$$(5.12)\qquad \pi(\varphi_1,\dots,\varphi_\nu)=\det\left(\mathrm{id}-(\varphi_1\circ\varphi_2\circ\dots\circ\varphi_\nu)^{\frac{n}{\nu}}\right).$$

The set $\pi^{-1}(0)$ is covered by finitely many manifolds $M\subset\mathscr{L}(\mathbb{R}^k)^\nu$ of codimension >0, by (B). Since Φ is a submersion, $\Phi^{-1}(\{0\}\times M)\subset V^{(\nu)}\times P$ is a manifold of codimension $>k\nu$, i.e., $\dim(\Phi^{-1}(\{0\}\times M))<\dim P$. Therefore the projection of $\Phi^{-1}(\{0\}\times M)$ in P has measure zero (by the easy Sard theorem), and the finite union of all of these projections is still a set of measure zero, say $N_\nu\subset P$. For $p\in(P-N_\nu)$ we have $\Phi(z,p)\notin\{0\}\times\pi^{-1}(0)$ for all $z\in V^{(\nu)}$. By definition of $\Phi^{(\nu)}$ and π this means: If $z=f_\nu(z)+p_\nu(z)$ then $\pi D(f+p)_\nu(z)\neq 0$. But $(f+p)_\nu(z)=z$ means that $z=(x^1,x^2,\dots,x^\nu)$ is of the form $x^j=(f+p)^j(x)$ and x has period exactly ν. And $\pi D(f+p)_\nu(z)\neq 0$ then means

$$\det[\mathrm{id}-(D((f+p)^\nu)(x))^{\frac{n}{\nu}}]\neq 0,\quad\text{or}\quad \det[\mathrm{id}-D((f+p)^n)(x)]\neq 0$$

for all $x\in\mathrm{Fix}((f+p)^n)$ with period exactly ν. And if we take $p\in(P-N)$ where $N=\bigcup_{\nu/n}N_\nu$ then $\det[\mathrm{id}-D((f+p)^n)(x)]\neq 0$ for all $x\in\mathrm{Fix}(f+p)^n$. But then all fixed points of $(f+p)^n$ are regular, in particular isolated; hence $\mathrm{Fix}((f+p)^n)$ is discrete in V, for all $p\in(P-N)$.

Since N has measure zero $P-N$ is dense in P. In particular, we can choose $p\in(P-N)$ arbitrarily close to 0, e.g. such that $\|p(v)\|<\varepsilon$ for all v in the compact set $\bar V$ $\left(\text{where } \varepsilon<\frac{\eta}{\sqrt{n}} \text{ as above}\right)$. Then (by 5.4, 5.5) $g=f+p$ has no n-periodic points in $\bar V-V$, and $\mathrm{Fix}(g^n)\subset V$ is compact. Since it is also discrete it is finite. □

We now come to the proof of 1.1 proper. By 5.6 it suffices to show that n divides $I_n(g)$ where g is as in Prop. 5.7. Let $a \in \mathrm{Fix}(g^n)$ and ν its precise period so that the g-orbit of a consists of ν points $\{g^i(a)\}_{i=1,2,\ldots,\nu}$. The set $\mathrm{Fix}(g^n)$ decomposes into finitely many g-orbits. We can separate them by disjoint open neighborhoods, and by additivity of the fixed point index (cf. [4], VII, 5.6 or 5.13) we can prove our assertion for each neighborhood separately. In other words, *we can* (and shall) *assume that* $\{g^i(a)\}_{i=1,2,\ldots,\nu}$ *are the only fixed points of* g^n hence

$$\mathrm{Fix}(g^{q\nu}) = \mathrm{Fix}(g^\nu) = \{a, g(a), \ldots, g^{\nu-1}(a)\} \tag{5.13}$$

for all q with $q\nu/n$. Moreover, the fixed point index $I(g^{q\nu}$ near $g^i(a))$ is the same at all points $g^i(a)$, by commutativity of the index (cf. [4], VII, 5.9 or 5.16) applied to the composition $g^{q\nu-i} \circ g^i$. More precisely, it is $+1$ at each point or -1 at each point because the fixed points are regular. It coincides with the sign of $\det(\mathrm{id} - D(g^{q\nu})(a))$, i.e. it is $(-1)^\alpha$, where α is the number of real eigenvalues (=EV) of $D(g^{q\nu})(a)$ which are >1 (compare [4], VII, 5.17 Exerc. 4 and IV, 5.13 Exerc. 3). If we write $D(g^{q\nu})(a) = (D(g^\nu)(a))^q$ this becomes

$$\begin{aligned} &I(g^{q\nu} \text{ near } a) = (-1)^\alpha, \\ &\text{where } \alpha = |\{\lambda \mid \lambda \text{ is EV of } D(g^\nu)(a) \text{ and } \lambda^q > 1\}|. \end{aligned} \tag{5.14}$$

The non-real EVs occur in conjugate pairs and therefore do not contribute to this sign, hence

$$I(g^{q\nu} \text{ near } a) = I(g^\nu \text{ near } a) \quad \text{if } q \text{ is odd.} \tag{5.15}$$

$$I(g^{q\nu} \text{ near } a) = I(g^{2\nu} \text{ near } a) \quad \text{if } q \text{ is even.} \tag{5.16}$$

Using this we can now calculate $I_n(g)$ essentially as we did in Sect. 2 for finite sets (namely for the finite set $\mathrm{Fix}(g^n)$). We distinguish three cases (i)–(iii).

(i) If $\frac{n}{\nu}$ is odd then

$$\pm I_n(g) = I_n(g \mid \mathrm{Fix}(g^n)) = \begin{cases} 0 & \text{if } \nu < n \\ n & \text{if } \nu = n. \end{cases}$$

(ii) If 4 divides $\frac{n}{\nu}$ then all exponents $n:\tau$ which occur in the Definition (1.2) of $I_n(g)$ are even multiples of ν, hence all fixed point indexes which occur are those of $g^{2\nu}$ by (5.16), hence

$$\pm I_n(g) = I_n(g \mid \mathrm{Fix}(g^n)) = 0, \qquad \text{as above (NB. } 2\nu < n).$$

(iii) Suppose 2 divides $\frac{n}{\nu}$ but 4 doesn't. In the Definition (1.2) of $I_n(g)$ we have $I(g^{n:\tau}) = 0$ unless $\tau \subset P\left(\frac{n}{\nu}\right)$; we therefore sum over these τ only, and we distinguish $2 \in \tau$ and $2 \notin \tau$. Thus

$$I_n(g) = \sum_{2 \in \tau} (-1)^{|\tau|} I(g^{n:\tau}) + \sum_{2 \notin \tau} (-1)^{|\tau|} I(g^{n:\tau}). \tag{5.17}$$

The summands in each sum now essentially correspond to subsets of $P\binom{n}{v}-\{2\}$, but with opposite signs. The contribution of a (or $g^i(a)$) to (5.17) is therefore (using 5.15 and 5.16)

$$\text{(5.18)} \qquad \sum_k \binom{r}{k}(-1)^{k+1} I(g^v \text{ near } a) + \sum_k \binom{r}{k}(-1)^k I(g^{2v} \text{ near } a),$$

where $r = \left|P\binom{n}{v}\right| - 1$. Both sums are zero ($= \pm(1-1)^r$) if $r \neq 0$, i.e. if $v \neq \frac{n}{2}$. If $2v = n$ we get $I(g^n \text{ near } a) - I(g^{n/2} \text{ near } a)$. Doing this for all $g^i(a)$ we obtain

$$\text{(5.19)} \qquad I_n(g) = 0 \quad \text{if } v < \frac{n}{2},$$

$$I_n(g) = I(g^n) - I(g^{n/2}) = \pm I(g^{n/2}) - I(g^{n/2})$$

$$= 0 \quad \text{or} \quad \pm n \quad \text{if } v = \frac{n}{2}.$$

Thus, in all three cases (i)–(iii), $I_n(g) = 0$ or $= \pm n$. □

6. Comments, examples, problems

(6.1) *The number of regular points of period exactly n*

If $f: Y \to Y$ is a self-map of a discrete space then this number coincides with $I_n(f) = \sum_{\tau \subset P(n)} (-1)^{|\tau|} I(f^{n:\tau})$, by (2.3). This is in fact the combinatorial background of Theorem 1.1 and its proof. It is natural to ask what the geometric significance of $I_n(f)$ is in the non-discrete case $f: V \to Y$ – at least when f is smooth and all fixed points of f^n are regular. Is it still true that $I_n(f)$ is the number of points of period exactly n, each point counted with its multiplicity ± 1? The answer is yes if n is odd, but no in general for n even. The explanation can be found in the proof of 1.1 (after the proof of 5.7): Each point of period exactly n contributes with its multiplicity ± 1 to $I_n(f)$, but in addition some points of period exactly $\frac{n}{2}$ contribute -2 times their multiplicity. These points $a \in \operatorname{Fix}(f^{n/2})$ – being regular for f^n – are characterized by $I(f^n \text{ near } a) = -I(f^{n/2} \text{ near } a)$; we call them *inverting*. Thus $a \in \operatorname{Fix}(f^{n/2})$ is inverting iff the derivative $(Df^{n/2})(a)$ has an odd number of real eigenvalues < -1.

(6.2) **Proposition.** *If all fixed points of f^n are regular then $I_n(f)$ is the number of points of period exactly n minus twice the number of inverting points of period exactly $\frac{n}{2}$, each point counted with its multiplicity ± 1.*

The proof is contained in the proof of 1.1 where the contribution to $I_n(f)$ of each f-orbit in $\operatorname{Fix}(f^n)$ is explicitely calculated. □

(6.3) *Example.* The map $f: \mathbb{R} \to \mathbb{R}$, $f(x) = -x^3$, has one fixed point, and $I(f) = 1$. There are three points (0 resp. ± 1) of period two; their indices are $+1$ resp. -1, and $I(f^2) = -1$. Thus $I_2(f) = I(f^2) - I(f) = -2 =$ number of points of period exactly two. The linear map $g: \mathbb{R} \to \mathbb{R}$, $g(x) = -2x$ has only one periodic point (0), and $I(g) = -I(g^2) = 1$. Thus $I_2(g) = I(g^2) - I(g) = -2$ although there are no points of period exactly $n = 2$. But there is an inverting point of period $\frac{n}{2} = 1$ and index 1; it accounts for the -2 in the sense of 6.2.

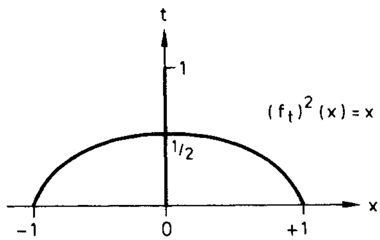
The linear deformation of f into g,

(6.4) $$f_t: \mathbb{R} \to \mathbb{R}, \quad f_t(x) = (t-1)x^3 - 2tx; \; 0 \leqq t \leqq 1$$

has no periodic points outside $[-1, 1]$ (because $\|x\| > 1 \Rightarrow \|f_t(x)\| > \|x\|$), hence $f = f_0$ and $g = f_1$ are equivalent in periodic point theory (in the sense of 6.7). The deformation shows how an inverting fixed point P can (in the course of the deformation) split into a fixed point (of opposite index) and a pair of 2-periodic points (of the same index as P). In fact, it is not hard to see that

(6.5) $$(f_t)^2(x) = x \Leftrightarrow x = 0 \quad \text{or} \quad (1-t)x^2 = 1 - 2t$$

so that $\{(x, t) | (f_t)^2(x) = x\}$ looks approximately as follows



As t decreases and passes through $\frac{1}{2}$ the fixed point 0 splits into three points in $\mathrm{Fix}(f_t^2)$.

(6.6) *The homotop ring $\mathfrak{P}$ of periodic point theory*

It is natural to ask for all possible periodic-point invariants, and for the relations among them. For instance, the indices $I(f^\nu)$ are periodic-point invariants, and Theorem 1.1 describes all relations among them (by 1.3). Are there further invariants which are not functions of $\{I(f^\nu)\}_{\nu \in \mathbb{N}}$? The question can be made more precise and explicit by introducing the "periodic point ring" $\mathfrak{P}$, as follows (analoguous to FIX_B in [5]).

(6.7) **Definition.** Consider continuous maps $f: V \to Y$, where Y is an ENR, $V \subset Y$ is open, and $\mathrm{Fix}(f^\nu)$ is compact for all $\nu \in \mathbb{N}$. Two such maps f_0, f_1 are said to be $\mathfrak{P}$-equivalent if there is a third such map $F: W \to Z$ which lies over $[0, 1]$ (i.e. $p: Z \to [0, 1]$ is $\mathrm{ENR}_{[0,1]}$ in the sense of [5], and $pF = p|W$) and whose parts over 0 resp. 1 are homeomorphic to f_0 resp. f_1. Let $\mathfrak{P}$ denote the set of equivalence classes $[f]$ of such maps f. Geometric addition (topological sum) and multiplication (cartesian product) are compatible with $\mathfrak{P}$-equivalence; they define a commutative ring structure in $\mathfrak{P}$, with $0=[\emptyset]$, $1=[\text{point}\to\text{point}]$, $-1=[\mathbb{R}\xrightarrow{2}\mathbb{R}]$. The indices define additive homomorphisms $\mathfrak{P}\to\mathbb{Z}$, $[f]\mapsto I(f^\nu)$. More generally, the Lefshetz power series defines a ring homomorphism

$$(6.8)\qquad L\colon \mathfrak{P}\to\Lambda=(1+t\mathbb{Z}[[t]]),\qquad [f]\mapsto L(f;t);$$

cf. 1.5 resp. 1.8. The fundamental question (suggested by T. tom Dieck) is whether (6.8) is an isomorphism.

Considering discrete spaces Y and permutations $f\colon Y\to Y$ of finite type as in (2.1) one obtains a ring homomorphism $\iota\colon \mathrm{PER}\to\mathfrak{P}$, $\iota(f)=[f]$, and we have seen in 2.14 resp. 2.16 that the composition

$$(6.9)\qquad \mathrm{PER}\xrightarrow{\iota}\mathfrak{P}\xrightarrow{L}\Lambda=(1+t\mathbb{Z}[[t]]$$

is isomorphic. In particular, L is epimorphic. What about the kernel(L) – is it zero? A simple candidate for a non-zero element is as follows: Let $q\colon S^1\to S^1$ denote squaring, $q(z)=z^2$. Then $[q]-\iota(L\iota)^{-1}L[q]$ is in $\ker(L)$: is it zero? In other words, is q $\mathfrak{P}$-equivalent (up to sign) to a discrete example?

The ring $\mathfrak{P}$ also admits symmetric powers (geometrically), hence exterior powers (cf. (2.15)), and L commutes with these operations (compare proof of 2.16). I haven't verified whether $\mathfrak{P}$ really is a λ-ring (i.e. whether 2) and 3) on p. 13 of [8] are satisfied.) – partly because I shunned the labor and partly because it would be superfluous if L were isomorphic.

(6.10) *Periodic point rings and Burnside rings*

One has a "multiplicative filtration" of $\mathfrak{P}$ by ring epimorphisms $\pi_n\colon \mathfrak{P}\to\mathfrak{P}_n$, where $\mathfrak{P}_n$ is defined by paying attention to n-periodic points only. Thus, elements of $\mathfrak{P}_n$ (with $n\in\mathbb{N}$ fixed) are represented by maps $f\colon V\to Y$ with $\mathrm{Fix}(f^n)$ compact; and f_0, f_1 represent the same element of $\mathfrak{P}_n$ iff there is such a map $F\colon W\to Z$ over $[0,1]$ which connects f_0, f_1 (as in 6.7). Ignoring periodic points outside $\mathrm{Fix}(f^n)$ defines π_n; thus $\pi_n[f]=[f]_n$. The ring $\mathfrak{P}_n$ is in fact isomorphic to the Burnside ring $A(\mathbb{Z}/n\mathbb{Z})$ of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Generators of $A(\mathbb{Z}/n\mathbb{Z})$ are represented by permutations $\varphi\colon X\to X$ of finite sets with $\varphi^n=\mathrm{id}$, and the isomorphism $A(\mathbb{Z}/n\mathbb{Z})\cong\mathfrak{P}_n$ takes φ into $[\varphi]_n\in\mathfrak{P}_n$.

If m divides n one has $\pi_{n,m}\colon \mathfrak{P}_n\to\mathfrak{P}_m$ by ignoring periodic points outside $\mathrm{Fix}(f^m)$; thus $\pi_{n,m}([f]_n)=[f]_m$. For Burnside rings this corresponds to $\varphi\mapsto\varphi|\mathrm{Fix}(\varphi^m)$. The fundamental question (after 6.8) then amounts to the question whether the filtration $\{\pi_n\colon \mathfrak{P}\to\mathfrak{P}_n\}_{n\in\mathbb{N}}$ is Hausdorff, or $\mathfrak{P}\cong\varprojlim\mathfrak{P}_n$.

(6.11) *Infinite-dimensional spaces*

The Zabreĭko-Krasnosel'skii theorem deals with maps of ANRs, not just ENRs. It doesn't seem to be difficult to extend our Theorem 1.1 (by suitable finite-dimensional approximation) to (partially defined) self-maps of ANRs Y provided f is compact. Whether, or to what extent, it suffices to assume iterates of f compact is, of course, a difficult question – to which I cannot contribute.

(6.12) *Parametrised periodic point theory*

One can formulate the basic notions of this paper for fibre-preserving maps (of ENR_B-spaces; cf. [5]) over a parameter space B. Some of the questions which arise are as follows: What are the relations between the indices $I(f^n) \in \pi^0_{st}(B \oplus pt)$, $n = 1, 2, \ldots$ of fibre-preserving maps over B, in the sense of [5]? Can one still define a Lefshetz power series with coefficients in $\pi^0_{st}(B \oplus pt)$? Is it still true that $\mathfrak{P}_n(B) \cong A(\mathbb{Z}/n\mathbb{Z}; B)$, in analogy to 6.10? What are the relations between the transfer- or trace-maps (cf. [6]) of the iterates f^n, as $n = 1, 2, \ldots$?

(6.13) (added after referee's report). *The case of just one smooth periodic point* is treated in [2]. These authors consider C^1-maps $f: \mathbb{R}^n \to \mathbb{R}^n$ such that $0 \in \mathbb{R}^n$ is an isolated fixed point of f^m for all $m = 1, 2, \ldots$. Roughly speaking they show (their Theorem 2.2) that $[f$ near $0]$, as an element of the periodic point ring $\mathfrak{P}$ (cf. 6.7 above), is an integral linear combination of ν-cycles $[\xi_\nu]$, or of $[\xi_\nu] - [\xi_{2\nu}]$ in certain cases, where ν ranges over the minimal periods of the derivative $Df(0)$ ($\exists\, y \in \mathbb{R}^n$ such that $Df(0)^\nu(y) = y$ and $Df(0)^k(y) \neq y$ for $1 \leqq k < \nu$). This refines the finiteness result in [11]. A simple example in [11], namely $z \mapsto 2z^2 \|z\|^{-1}$ in $\mathbb{C} = \mathbb{R}^2$, shows that differentiability C^1 cannot be replaced by mere continuity C^0.

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