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Modules of finite virtual projective dimension

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Introduction

The remarkable success of the homological study of modules with finite free resolutions has over a number of years largely determined the focus of attention in commutative algebra. It is the purpose of this paper to show that some fundamental results on modules of finite projective dimension over a noetherian local ring R are in fact special cases of relations which are valid much more generally. The new invariants brought into consideration are derived from the asymptotic behaviour of the ranks of the free modules in minimal free resolutions.

Specifically, imitating Alperin and Evans [AE], we say the finitely generated R-module M has complexity, $\operatorname{cx}_R M$, equal to d, if $d-1$ is the smallest degree of a polynomial in n bounding the sequence of Betti numbers $b_n^R(M)$. Furthermore, we say M has virtual projective dimension, $\operatorname{vpd}_R M$, equal to v, if v is the smallest projective dimension which occurs when M is viewed as a Q-module and Q ranges over all deformations of R over regular bases (cf. (3.3) for the precise definition). The following statement provides a good illustration both of the way structural properties of modules are reflected in their homological invariants, and of the form well known relations may acquire when the assumption of finite projective dimension is relaxed.

(3.5) **Theorem.** *When $M \neq 0$ is a finitely generated R-module of finite virtual projective dimension, there is equality:* $\operatorname{vpd}_R M = \operatorname{depth} R - \operatorname{depth} M + \operatorname{cx}_R M$.

In particular, this applies to all non-trivial M when R is a local complete intersection. In an earlier version of this paper, it was conjectured that the equality holds for all $M \neq 0$ over any ring, or in other words, that finite complexity implies finite virtual projective dimension. This is proved in [Av_2] for rings of small embedding codepth, but fails in general, as recently demonstrated in [AGP].

The technique used in this paper is of interest in its own right. Based on a construction of Gulliksen [Gu] and Eisenbud [Ei], reworked in the first two sections, we proceed to associate in Sect. 3 an algebraic variety $V(Q, \mathbf{x}, M)$, depending on a given presentation $R = Q/(\mathbf{x})$, with $\mathbf{x}$ a Q-regular sequence. This provides an extremely convenient cohomological portrait of those R-modules which have finite projective dimension over Q: the relations between various invariants of M, obtained in Sects. 3 and 4, often are numerical expressions of the structure of $V(Q, \mathbf{x}, M)$. In Sect. 4 we also extend Eisenbud's result [Ei] on the periodicity of modules with bounded Betti numbers over complete intersections to modules with bounded Betti numbers and finite virtual projective dimension over arbitrary rings. A similar approach can be used to study invariants of injective resolutions: this is done in Sect. 5.

The idea to study a finite dimensional k-linear representation, M, of a finite group, G, by means of an associated cohomologically defined variety $V_G(M)$, was pioneered in the fundamental work of Quillen [Qu]. Various authors have contributed a number of deep results and perfected the technique. If one tries to mimic it for local rings, a serious obstacle arises: the k-algebra $\mathrm{Ext}_R^*(k, k)$, which is the "obvious" substitute for $H^*(G, k)$, is in general neither commutative nor noetherian. A way out is provided by the point of view of [Av_1], which centers on the homotopy Lie algebra $\pi^*(R)$ canonically associated to R. The universal enveloping algebra of its 2-dimensional central elements provides a polynomial k-subalgebra $\mathscr{R}$ of $\mathrm{Ext}_R^*(k, k)$. One now takes $V_R^*(M)$ to be the variety defined by the annihilator of $\mathrm{Ext}_R^*(M, k)$ in $\mathscr{R}$, for the action by Yoneda products. When also $V(Q, \mathbf{x}, M)$ is defined, there is a morphism $V_R^*(M) \to V(Q, \mathbf{x}, M)$, which is finite onto if $\mathrm{pd}_Q M < \infty$. These structures are explored in Sect. 6.

The next – and last – section deals with group cohomology: group algebras of finite abelian p-groups over a field k of characteristic $p>0$ are artinian complete intersections of a very special kind, which occupy a privileged place in the theory due to the existence of Quillen stratifications [Qu], [AS]. By specializing some of the preceding propositions, we are able to obtain independent proofs and sometimes sharper formulations of several key properties. Thus, the results of this paper extend to relative complete intersections of arbitrary dimension theorems whose earlier proofs made use of the full panoply of special techniques, available to group cohomology.

The efficiency of methods from (homological) local algebra in the study of group cohomology was initially demonstrated by Eisenbud in the important paper [Ei]. Along with the two papers [Ca_2], [Ca_3] of Carlson and that of Avrunin and Scott [AS], it has been a major source of insight and problems. Finally, I should like to thank David Eisenbud for some useful conversations.

1. Eisenbud's operators and their analogues for injective complexes

(1.1) *Notation.* In this paper a graded R-module, $\mathbb{A}$, is identified with the disjoint union of its homogeneous components, A_n (and not, as is often done, with their direct sum). We write $\mathbb{A} = \{A_n \mid n \in \mathbb{Z}\}$ and note that every non-zero element of $\mathbb{A}$ is unambiguously assigned a degree: $|a| = n \Leftrightarrow a \in A_n$. The *same* module is sometimes written with an upper grading: $\mathbb{A} = \{A^n \mid n \in \mathbb{Z}\}$, which always means $A^n = A_{-n}$; when we need to emphasize the kind of grading used, we write $\mathbb{A}_*$ or $\mathbb{A}^*$ instead of $\mathbb{A}$. A degree n homomorphism of graded modules $f: \mathbb{A} \to \mathbb{B}$ is a collection $\{f_i \in \mathrm{Hom}_R(A_i, B_{i+n}) \mid i \in \mathbb{Z}\}$. The set of all such homomorphisms is denoted $\mathrm{Hom}_R(\mathbb{A}, \mathbb{B})_n$: this is an R-module, and $\mathbb{H}\mathrm{om}_R(\mathbb{A}, \mathbb{B})_* = \{\mathrm{Hom}_R(\mathbb{A}, \mathbb{B})_n \mid n \in \mathbb{Z}\}$ is a graded R-module.

A complex (or differential graded module) is an $\mathbb{A}_*$ as above equipped with an endomorphism $\partial = \partial_A$ of degree -1, such that $\partial^2 = 0$; note that when we write $\mathbb{A}$ in the form $\mathbb{A}^*$, ∂ acquires degree $+1$. When $\mathbb{A}$ and $\mathbb{B}$ are DG R-modules, so is $\mathbb{H}\mathrm{om}_R(\mathbb{A}, \mathbb{B})$, for the differential $\partial f = \partial_B f - (-1)^{|f|} f \partial_A$. Then the cycles $Z_n \mathbb{H}\mathrm{om}_R(\mathbb{A}, \mathbb{B})$ consist of the degree n maps of complexes $\mathbb{A} \to \mathbb{B}$, and two degree n homomorphisms f, g are homotopic, i.e. $f - g = \partial_B s + (-1)^n s \partial_A$ for some $s \in \mathbb{H}\mathrm{om}_R(\mathbb{A}, \mathbb{B})_{n+1}$, if and only if they differ by a boundary. Thus, the elements of $H_n \mathbb{H}\mathrm{om}_R(\mathbb{A}, \mathbb{B})$ can (and will) be identified with the homotopy classes of degree n maps of complexes $\mathbb{A} \to \mathbb{B}$.

In this section we consider a ring R of the form $Q/(\mathbf{x})$, where $\mathbf{x} = x_1, \ldots, x_c$ is a Q-regular sequence, and work with the exact sequence

$$(1.1.1) \qquad Q^c \xrightarrow{\alpha} Q \xrightarrow{\pi} R \to 0, \qquad \alpha(y_1, \ldots, y_c) = x_1 y_1 + \ldots + x_c y_c.$$

The fact that R is local and noetherian plays no role.

$(1.2)_*$ [Ei, Sect. 1]. Let $\mathbb{F}_*$ be a complex of free R-modules. Choose a free graded Q-module $\tilde{\mathbb{F}}_*$ such that $\mathbb{F} = R \otimes_Q \tilde{\mathbb{F}}$. Applying $- \otimes_Q \tilde{\mathbb{F}}$ to the exact sequence (1.1.1) one obtains an exact sequence of graded Q-modules

$$\tilde{\mathbb{F}}^c \xrightarrow{\tilde{\alpha}} \tilde{\mathbb{F}} \xrightarrow{\tilde{\pi}} \mathbb{F} \to 0.$$

Because of the projectivity of $\tilde{\mathbb{F}}$ there exists a degree -1 endomorphism $\tilde{\partial}$ of $\tilde{\mathbb{F}}$, such that $\partial \tilde{\pi} = \tilde{\pi} \tilde{\partial}$. Since $R \otimes \tilde{\partial}^2 = (R \otimes \tilde{\partial})^2 = \partial^2 = 0$, the projectivity of $\tilde{\mathbb{F}}$ provides a factorization $\tilde{\partial}^2 = \tilde{\alpha} \tilde{t}$, for some degree -2 homomorphism $\tilde{t}: \tilde{\mathbb{F}} \to \tilde{\mathbb{F}}^c$. Now set $\tilde{t}_j = \tilde{t}_j(Q, \mathbf{x}, \tilde{\mathbb{F}})$ to denote the composition (j'th projection: $\tilde{\mathbb{F}}^c \to \tilde{\mathbb{F}}) \circ \tilde{t}$, and write $t_j = t_j(Q, \mathbf{x}, \mathbb{F})$ for $R \otimes \tilde{t}_j$: these are degree -2 endomorphisms of the graded R-module $\mathbb{F}$.

If $\tilde{t}'$ also is a lifting of $\tilde{\partial}^2$, then $\mathrm{Im}(\tilde{t} - \tilde{t}')$ is contained in Ker $\tilde{\alpha}$. By the exactness of the Koszul complex and the Q-freeness of $\tilde{\mathbb{F}}$, this module coincides with the Q-submodule of $\tilde{\mathbb{F}}^c$, generated by all elements of the form $(0, \ldots, 0, x_i y, 0, \ldots, 0, -x_j y, 0, \ldots, 0)$, where $y \in \tilde{\mathbb{F}}$, the first non-zero coordinate is in j'th place, and the second one – in i'th place. Thus $\mathrm{Im}(R \otimes \tilde{t} - R \otimes \tilde{t}') \subset R \otimes_Q \mathrm{Im}(\tilde{t} - \tilde{t}') = 0$, so that t_j does not depend on the choice of the lifting $\tilde{t}$.

$(1.2)^*$ Denote by $E_Q(-)$ the injective envelope of $(-)$ over Q. If I is R-injective, then applying $\mathrm{Hom}_Q(R, -)$ to the inclusion $I \to E_Q(I)$, one obtains an isomorphism $I \xrightarrow{\cong} \mathrm{Hom}_Q(R, E_Q(I))$. Indeed, the right-hand side is identified with $I' = \{z \in E_Q(I) \mid (\mathbf{x}) z = 0\}$, hence contains I as a submodule. Being injective, I splits off: $I' = I \oplus J$ (over R, hence over Q as well). However, this occurs inside the injective hull of I, which forces J to be zero, as claimed.

Let $\mathbb{I}^*$ be a complex of injective R-modules. Consider the graded Q-module $\tilde{\mathbb{I}}^*$, with $\tilde{I}^n = E_Q(I^n)$. Applying $\operatorname{Hom}_Q(-, \tilde{\mathbb{I}})$ to the exact sequence (1.1.1) we obtain, in view of the preceding remarks, the exact sequence of graded modules:

$$0 \to \mathbb{I} \xrightarrow{\tilde{\imath}} \tilde{\mathbb{I}} \xrightarrow{\tilde{\beta}} \tilde{\mathbb{I}}^c, \qquad \beta(z) = (x_1 z, \ldots, x_c z).$$

Because of the injectivity of $\tilde{\mathbb{I}}$ there exists a degree $+1$ endomorphism $\tilde{\partial}$ of $\tilde{\mathbb{I}}^*$ (upper degrees!), such that $\tilde{\imath}\partial = \tilde{\partial}\tilde{\imath}$. Since $\tilde{\partial}^2 \tilde{\imath} = \tilde{\imath}\partial^2 = 0$, there is a natural homomorphism $\operatorname{Coker} \tilde{\imath} \to \tilde{\mathbb{I}}$, which by the injectivity of $\tilde{\mathbb{I}}$ extends to a homomorphism $\tilde{u} = \tilde{u}(Q, \mathbf{x}, \mathbb{I})$: $\tilde{\mathbb{I}}^c \to \tilde{\mathbb{I}}$, such that $\tilde{\partial}^2 = \tilde{u}\tilde{\beta}$. Now set $u_j = u_j(Q, \mathbf{x}, \mathbb{I}) = \tilde{u}_j \tilde{\imath}$: these are degree $+2$ endomorphisms of the graded R-module $\mathbb{I}^*$.

If $\tilde{u}'$ also is an extension of $\tilde{\partial}^2$, then $\tilde{u}$ and $\tilde{u}'$ agree on $\operatorname{Im} \tilde{\beta}$. By the exactness of the Koszul complex and the injectivity of $\tilde{\mathbb{I}}$, this module consists of those $(z_1, \ldots, z_c) \in \tilde{\mathbb{I}}^c$, for which $x_i z_j = x_j z_i$ for all $1 \leqq i, j \leqq c$. On the other hand, the image of $\tilde{\imath}(\mathbb{I})$ under the j-th injection of $\tilde{\mathbb{I}}$ in $\tilde{\mathbb{I}}^c$ consists of those $(0, \ldots, 0, z, 0, \ldots, 0)$ with $x_i z = 0$ for $1 \leqq i \leqq c$, so is contained in $\operatorname{Im} \beta$. Thus u_j does not depend on the choice of the extension $\tilde{u}$.

(1.3) **Proposition.** *With the previous notation one has, for* $1 \leqq j \leqq c$:

(1) t_j *and* u_j *are homomorphisms of complexes,* i.e. $\partial t_j = t_j \partial$, *and* $\partial u_j = u_j \partial$;

(2) *if* $\mathbb{G}_*$ *(resp.* $\mathbb{J}^*$*) is a complex of free (resp. injective) Q-modules, and* f: $\mathbb{F}_* \to \mathbb{G}_*$ (resp. f: $\mathbb{I}^* \to \mathbb{J}^*$) *is a homomorphism of complexes, then* $f t_j(Q, \mathbf{x}, \mathbb{F})$ *and* $t_j(Q, \mathbf{x}, \mathbb{G}) f$ *(resp.* $f u_j(Q, \mathbf{x}, \mathbb{I})$ *and* $u_j(Q, \mathbf{x}, \mathbb{J}) f$*) are homotopic; in particular,* t_j *and* u_j *are uniquely determined up to homotopy;*

(3) *for any* $i (1 \leqq i \leqq c)$, $t_i t_j$ *is homotopic to* $t_j t_i$, *and* $u_i u_j$ *is homotopic to* $u_j u_i$;

(4) *consider a commutative diagram of homomorphisms of commutative rings:*

$$\begin{array}{ccc} Q' & \xrightarrow{\phi'} & Q \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} \\ R' & \xrightarrow{\phi} & R = R' \otimes_{Q'} Q \end{array}$$

in which $R' = Q'/(\mathbf{x}')$ *for some Q'-regular sequence* $\mathbf{x}' = x'_1, \ldots, x'_{c'}$; *write*

$$\phi'(x'_i) = \sum_{j=1}^{c} a_{ij} x_j, \qquad 1 \leqq i \leqq c' \quad \text{and} \quad a_{ij} \in Q,$$

and let $\mathbb{F}'$ *(resp.* $\mathbb{I}'$*) be a complex of free (resp. injective) R'-modules; then the homomorphisms of complexes*

$$t_j(Q, \mathbf{x}, R \otimes_R, \mathbb{F}') \quad \text{and} \quad \sum_{i=1}^{c'} a_{ij}(R \otimes t_i(Q', \mathbf{x}', \mathbb{F}'))$$

$\Big($*resp.*

$$u_j(Q,\,\mathbf{x},\,\mathbb{H}\mathrm{om}_{R'}(R,\,\mathbb{I}'))\ \text{and}\ \sum_{i=1}^{c'} a_{ij}\,\mathbb{H}\mathrm{om}_{R'}(R,\,u_i(Q',\,\mathbf{x},\,\mathbb{I}'))\Big)$$

are homotopic.

The assertions concerning the t_j's are copied from [Ei, Sect. 1]. The proofs essentially boil down to direct computations, and these "dualize" without problem to produce the corresponding properties of the u_j's. □

(1.4) **Proposition.** *Let* $\varepsilon\colon \mathbb{F}_* \to M$ *be a free resolution of the R-module M, and let* $\eta\colon N \to \mathbb{I}^*$ *be an injective resolution of the R-module N. The following diagram is then commutative:*

$$\begin{array}{ccccc}
H^*\mathbb{H}\mathrm{om}_R(\mathbb{F},N) & \xrightarrow{\eta_*} & H^*\mathbb{H}\mathrm{om}_R(\mathbb{F},\mathbb{I}) & \xleftarrow{\varepsilon^*} & H^*\mathbb{H}\mathrm{om}_R(M,\mathbb{I})\\
\downarrow{\scriptstyle H^*\mathbb{H}\mathrm{om}(t_j(Q,\mathbf{x},\mathbb{F}),N)} & & \downarrow{\scriptstyle H^*u_j(Q,\mathbf{x},\mathbb{H}\mathrm{om}_R(\mathbb{F},\mathbb{I}))} & & \downarrow{\scriptstyle H^*\mathbb{H}\mathrm{om}(M,u_j(Q,\mathbf{x},\mathbb{I}))}\\
H^*\mathbb{H}\mathrm{om}_R(\mathbb{F},N) & \xrightarrow[\eta_*]{} & H^*\mathbb{H}\mathrm{om}_R(\mathbb{F},\mathbb{I}) & \xleftarrow[\varepsilon^*]{} & H^*\mathbb{H}\mathrm{om}_R(M,\mathbb{I})
\end{array}$$

The proof proceeds along the lines of [Ei, Proposition 1.6]: one constructs $(\tilde{\mathbb{F}},\tilde{\partial})$ and $(\tilde{\mathbb{I}},\tilde{\partial})$ as above, then chooses $\tilde{t}_j$ and $\tilde{u}_j$ for them, and finally checks that $\mathbb{H}\mathrm{om}(\tilde{t}_j,\tilde{\mathbb{I}})+\mathbb{H}\mathrm{om}(\tilde{\mathbb{F}},\tilde{u}_j)$ can be used as $\tilde{u}_j(Q,\mathbf{x},\mathbb{H}\mathrm{om}_R(\tilde{\mathbb{F}},\tilde{\mathbb{I}}))$ for the complex of injective modules $\mathbb{H}\mathrm{om}_R(\tilde{\mathbb{F}},\tilde{\mathbb{I}})$. □

(1.5) *Conclusion.* Let $\mathbb{F}_* \to M$ be an R-free resolution of M and identify $\mathrm{Ext}_R^*(M,N)$ with $H^*\mathbb{H}\mathrm{om}_R(\mathbb{F},N)$. Then by (1.3.1) $\chi_j = H^*\mathbb{H}\mathrm{om}(t_j,N)$ is an endomorphism of $\mathrm{Ext}_R^*(M,N)$ of upper degree 2, which by (1.3.2) is independent of the choice of $\mathbb{F}$. If one chooses to identify $\mathrm{Ext}_R^*(M,N)$ with $H^*\mathbb{H}\mathrm{om}_R(M,\mathbb{I})$ for some injective resolution $N\to\mathbb{I}^*$, for the same reasons one obtains an endomorphism $\chi_j = H^*\mathbb{H}\mathrm{om}(M,u_j)$, which does not depend on the choice of $\mathbb{I}$. That both constructions of χ_j agree is the claim of (1.4). Furthermore, $\chi_i\chi_j=\chi_j\chi_i$ for $1\leqq i,j\leqq c$ by (1.3.3).

Thus, $\mathrm{Ext}_R^*(M,N)$ has a well defined structure of graded module over the polynomial ring $R[\chi_1,\ldots,\chi_c]$, whose grading is determined by the requirement $|\chi_j|=2$ (upper degree) for $1\leqq j\leqq c$.

The functoriality of this structure results from (1.3.2) and (1.3.4) and may be described as follows. Let a commutative square of ring homomorphisms, satisfying the conditions of (1.3.4), be given. Let $\mu\colon M'\to M$ and $\nu\colon N\to N'$ be homomorphisms of R-modules, and consider the canonical homomorphism

$$\Psi=\mathrm{Ext}_\Phi^*(\mu,\nu)\colon \mathrm{Ext}_R^*(M,N)\to \mathrm{Ext}_{R'}^*(M',N').$$

It is well known to be R-linear for the structures, induced from scalar multiplication on either of the module arguments. Furthermore, it has the property that

$\Psi(qe)=\Phi(q)\Psi(e)$, for $e\in\operatorname{Ext}_R^*(M,N)$, $q\in R[\chi_1,\ldots,\chi_c]$, and Φ denoting the R-algebra homomorphism $R[\chi_1,\ldots,\chi_c]\to R[\chi'_1,\ldots,\chi'_{c'}]$, defined by $\Phi(\chi_j) = \sum_{i=1}^{c'} a_{ij}\chi'_i (1\leqq j\leqq c)$.

2. Gulliksen's finiteness theorem

The title refers to the following statement, proved as the main result of [Gu]. We use pd_R (resp. id_R) to denote projective (resp. injective) dimension over R.

(2.1) **Theorem.** *Let $\mathbf{x}=x_1,\ldots,x_c$ be a regular sequence in the commutative noetherian ring Q, and set $R=Q/(\mathbf{x})$. Let M and N be finitely generated R-modules.*

If either $\operatorname{pd}_Q M<\infty$ or $\operatorname{id}_Q N<\infty$, then $\operatorname{Ext}_R^*(M,N)$ is a finitely generated module over the graded ring $R[\chi_1,\ldots,\chi_c]$ defined in (1.5).

(2.2) *Remarks.* Strictly speaking, Gulliksen uses in [Gu] a different set of operators, defined in a somewhat indirect way, and less suitable for our purposes later in this paper. It is reported by Eisenbud [Ei, p. 42] that Mehta has proved in his thesis (Berkeley, 1976) the coincidence of Gulliksen's operators with the χ_j. Furthermore, Eisenbud remarks [Ei, p. 44] that the finiteness over $R[\chi_1,\ldots,\chi_c]$ can be proved by means of a specific construction of resolutions, due to Shamash [Sh] and Eisenbud. Such an argument is given below, because its ingredients are used in the sequel.

The Theorem is easily deduced from the following proposition, whose proof requires some preparation.

(2.3) **Proposition.** *Let $\mathbf{x}=x_1,\ldots,x_c$ be a regular sequence in the commutative ring Q, and let ϕ denote the canonical map $R':=Q/(x_1,\ldots,x_{c-1})\to Q/(x_1,\ldots,x_c)=:R$. Let M and N be arbitrary R-modules.*

Then there exists an exact triangle of graded $R[\chi_1,\ldots,\chi_c]$-modules:

$$\begin{array}{ccc} \operatorname{Ext}_R^*(M,N) & \xrightarrow{\chi_c} & \operatorname{Ext}_R^*(M,N) \\ {\scriptstyle \Xi}\nwarrow & & \swarrow{\scriptstyle \Psi} \\ & \operatorname{Ext}_{R'}^*(M,N) & \end{array}$$

where the horizontal map is multiplication by χ_c, $\Psi=\operatorname{Ext}_\phi^*(M,N)$, and Ξ is a homomorphism of upper degree -1.

Proof of Theorem (2.1). Since each $\operatorname{Ext}_R^n(M,N)$ is finitely generated over R, there is nothing to prove when $c=0$. Assume by induction that $c\geqq 1$ and the statement holds for $E'=\operatorname{Ext}_{R'}^*(M,N)$, considered as a module over $R[\chi'_1,\ldots,\chi'_{c-1}]$. By (1.5) the homomorphism Ψ is compatible with the R-algebra map $\Phi\colon R[\chi_1,\ldots,\chi_c]\to R[\chi'_1,\ldots,\chi'_{c-1}]$ which sends χ_i to χ'_i for $1\leqq i\leqq c-1$ and χ_c to zero, hence $\operatorname{Im}\Psi$ is a submodule of the noetherian graded module E'. Take $e_1,\ldots,e_m$ in $E=\operatorname{Ext}_R^*(M,N)$, such that $\Psi(e_1),\ldots,\Psi(e_m)$ generate $\operatorname{Im}\Psi$ over

$R[\chi'_1, \ldots, \chi'_{c-1}]$. Thus, if G denotes the $R[\chi_1, \ldots, \chi_c]$-submodule of E generated by the e_i's, we have $E=G+\chi_c E$. Iterating, one obtains equalities $E=\sum_{i=0}^{n}(\chi_c)^n G + (\chi_c)^{n+1}E$ for every $n \geqq 0$, which implies $E=(R[\chi_c])G$, since $\bigcap_{n\geqq 0}(\chi_c)^n E=0$ due to degree reasons. □

The proposition is proved by using a construction of Shamash [Sh, Sect. 3], which has subsequently been generalized to sequences with more than one element by Eisenbud [Ei, Sect. 7]. A streamlining of the remarkably straightforward original approach is presented below.

(2.4) *Construction* [Sh]. Let x' be a non zero divisor in the commutative ring R', and set $R=R'/(x')$. Let $(\mathbb{F}', \partial')$ be an R'-free resolution of the R-module M. Then there exists a family of endomorphisms $\{s_n\}_{n\geqq 0}$ of the graded R'-module $\mathbb{F}'$ such that:

(i) $|s_n|=2n-1$;

(ii) $s_0=\partial'$;

(iii) $s_0 s_1 + s_1 s_0 = x'$;

(iv) $\sum_{i=0}^{n} s_i s_{n-i}=0$ for $n \geqq 2$.

Indeed, (ii) fixes s_0. Since $\mathbb{F}' \xrightarrow{x'} \mathbb{F}'$ is a map of resolutions over the zero map $M \xrightarrow{x'} M$, it is homotopic to zero; taking s_1 to be one such homotopy, (iii) holds by definition. Assume by inductions s_i's have been defined for $i<n (n\geqq 2)$, such that (i) through (iv) hold. Setting $b_1=x'$, $b_i=-\sum_{j=1}^{i-1} s_j s_{i-j}$ for $i\geqq 2$, the last condition reads $s_0 s_1 = b_i - s_i s_0$, which yields equalities $s_0 s_i s_{n-i} = b_i s_{n-i} - s_i b_{n-i} + s_i s_{n-i} s_0$ for $i=1, \ldots, n-1$. Summing up one sees that $s_0 b_n = b_n s_0$, which in view of (ii) means b_n is a degree $2n-2$ cycle in the complex $\mathbb{H}\mathrm{om}_{R'}(\mathbb{F}', \mathbb{F}')_*$. By the comparison theorem for resolutions, the augmentation $\varepsilon\colon \mathbb{F}' \to M$ induces an isomorphism of its homology with that of $\mathbb{H}\mathrm{om}_{R'}(\mathbb{F}', M)_*$, and the latter complex is trivial in all positive degrees. Thus, b_n is a boundary, i.e. there exists a s_n which satisfies (i) and also the equality $s_0 s_n + s_n s_0 = b_n$ (cf. (1.1)); this is nothing but (iv).

Define now $\tilde{\mathbb{F}}$ to be the graded R'-module $\mathbb{H}\mathrm{om}_{R'}(R'[\chi'], \mathbb{F}')_*$, where the polynomial ring $R'[\chi']$ is graded by assigning to χ' lower degree -2; thus, by our conventions (1.1), $\tilde{\mathbb{F}}_i=0$ for $i<0$. It has an endomorphism $\tilde{\partial}$ of degree -1 given by the formula

$$(\tilde{\partial} f')(y)=\sum_{i\geqq 0} s_i f'(\chi'^i y),$$

which makes sense since $f'(\chi'^i y)=0$ for every homogeneous $y \in R'[\chi']$, as soon as $2i > |f'| + |y|$.

(2.5) **Proposition** [Sh], [Ei]. *Setting* $\mathbb{F} = R \otimes_{R'} \tilde{\mathbb{F}}'$ *and* $\partial = R \otimes_{R'} \tilde{\partial}$, *one obtains an R-free resolution of M. For it the map* $t = t(R', x', \mathbb{F})$ *defined in* $(1.2)_*$ *is induced by the map* $\tilde{t} = \tilde{t}(R', x', \tilde{\mathbb{F}})$, *given by multiplication by* χ': $(\tilde{t}f')(y) = f'(\chi' y)$.

Proof. Applying $\mathbb{H}\mathrm{om}_{R'}(-, \mathbb{F}')$ to the exact sequence $0 \to R'[\chi'] \xrightarrow{\chi'} R'[\chi'] \to R' \to 0$, one obtains an exact sequence of graded Q-modules $0 \to \mathbb{F}' \to \tilde{\mathbb{F}}' \xrightarrow{\tilde{t}} \tilde{F}' \to 0$. After tensorization with R it yields the exact sequence of *complexes* (since $\partial^2 = 0$ in view of conditions (2.4.ii) to (2.4.iv)):

$$0 \to R \otimes_{R'} \mathbb{F}' \to \mathbb{F} \xrightarrow{t} \mathbb{F} \to 0. \tag{2.5.1}$$

Note that $H_q(R \otimes_{R'} \mathbb{F}') = \mathrm{Tor}_q^{R'}(R, M) \cong M$ for $q = 0, 1$, and that this group is trivial otherwise. Taking homology one obtains a long exact sequence

$$\ldots \to H_q(R \otimes_{R'} \mathbb{F}') \to H_q(\mathbb{F}) \to H_{q-2}(\mathbb{F}) \to H_{q-1}(R \otimes_{R'} \mathbb{F}') \to \ldots$$

which immediately shows that $H_0(\mathbb{F}) = M$, and $H_q(\mathbb{F}) = 0$ for $q \neq 0$. □

Proof of Proposition (2.3). Apply $\mathbb{H}\mathrm{om}_R(-, N)$ to the exact sequence (2.5.1), take homology, and note that by (1.5) multiplication by χ' and by χ_c yield the same operator. □

In order to deduce numerical information from Theorem (2.1), it should be noted that $\mathrm{Ext}_R^*(M, N)$ is annihilated by the ideal $\mathrm{Ann}(M, N) = \mathrm{Ann}\, M + \mathrm{Ann}\, N \subset R$, hence it is in fact a module over $(R/\mathrm{Ann}(M, N))[\chi_1, \ldots, \chi_c]$. Suppose now the ring $R/\mathrm{Ann}(M, N)$ is artinian. Then each $\mathrm{Ext}_R^n(M, N)$ has finite length over R, hence – by a classical result on finitely-generated modules over graded polynomial rings [Na], [Se] – the theorem yields:

(2.6) **Corollary** [Gu]. *With R, M, and N as in* (2.1), *assume furthermore* $R/\mathrm{Ann}(M, N)$ *is artinian. Then* $(1-t^2)^c \cdot \sum_{n \geq 0} \mathrm{length}_R \mathrm{Ext}_R^n(M, N)\, t^n$ *is a polynomial in t with integer coefficients.* □

(2.7) *Notation.* Let R be local with residue field k. The integer $b_n^R(M) = \dim_k \mathrm{Ext}_R^n(M, k)$ is called the n-th Betti number of M: it is equal to the rank of F_n in a *minimal free resolution* $\mathbb{F}$ of M. The generating function $P_M^R(t) = \sum_{n \geq 0} b_n^R(M)\, t^n$ is called the Poincaré series of M (over R).

The integer $\mu_R^n(M) = \dim_k \mathrm{Ext}_R^n(k, M)$ is called the n-th Bass number of M: it is equal to the number of copies of the injective envelope $E_R(k)$ in a *minimal injective resolution* $\mathbb{I}$ of M. The generating function $I_R^M(t) = \sum_{n \geq 0} \mu_R^n(M)\, t^n$ is called the Bass series of M (over R).

These formal power series can be used effectively to explore the action of χ' on Ext* under specific assumptions. We illustrate this in a case which will find applications later on:

(2.8) **Proposition.** *Let M be a finitely generated module over the ring $R = R'/(x')$, where R' is local noetherian and x' is a non zero divisor, contained in its maximal ideal $\mathfrak{m}'$. In the notation of* (2.3), *we then have*:

(1) *If* $x' \notin (\mathfrak{m}')^2$, *then* $\chi' . \operatorname{Ext}_R^*(M, k) = 0$.

(2) *If* $x' \in \mathfrak{m}' . \operatorname{Ann}_{R'}(M)$, *then* χ' *is a non zero divisor on* $\operatorname{Ext}_R^*(M, k)$.

Proof. The preceding statements are essentially reformulations – via the exact sequence (2.3) – of results due respectively to Nagata and Shamash.

For (1), [Na (27.3)] readily yields the equality

$$P_M^{R'}(t) = (1+t)\, P_M^R(t) \tag{2.8.1}$$

(cf. also [Sh, Sect. 2, Corollary 1]). In view of (2.3), this is possible if and only if multiplication by χ' annihilates $\operatorname{Ext}_R(M, k)$.

For (2), [Sh, Sect. 3, Corollary 1] asserts the equality of Poincaré series

$$P_M^{R'}(t) = (1-t^2)\, P_M^R(t): \tag{2.8.2}$$

this is a consequence of the more precise result, that in (2.4) one can take s_i with $\operatorname{Im} s_i \subset (\mathfrak{m}')^i \mathbb{F}'$ [Idem, Lemma 3], which shows the resolution constructed in (2.5) is minimal. Once again, the claim follows by interpreting this equality as a statement on the kernel of the multiplication by χ', using the exact triangle (2.3). □

3. Complexity and virtual projective dimension

From now on we write $(R, \mathfrak{m}, k)$ in order to denote a noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The local ring $(Q, \mathfrak{n}, k)$ is a (codimension c) *deformation* of R, if a surjective homomorphism $\rho: Q \to R$ is given, with $\operatorname{Ker} \rho$ generated by a Q-regular sequence (of length c). [It should be noted, that when Q is a k-algebra, this notion coincides with that of "deformation with *regular* basis" in the deformation theory of commutative algebras. However, we shall use this terminology for arbitrary rings, and never will consider any other kind of deformation.] The deformation is called *embedded* if $\operatorname{Ker} \rho \subset \mathfrak{n}^2$. Given a deformation of R, we view every R-module as a Q-module via ρ, and (usually) suppress the homomorphism from the notation.

(3.1) **Definition.** Let d be a non-negative integer. We say the finitely generated module M over $(R, \mathfrak{m}, k)$ has *complexity* d, and write $\operatorname{cx}_R M = d$, if there exists a positive real number γ, such that the inequality (cf. (2.7))

$$b_n^R(M) \leqq \gamma n^{d-1}$$

holds for all sufficiently large n, and d is the smallest non-negative integer with this property. If no such d exists, we say M has infinite complexity, and write $\operatorname{cx}_R M = \infty$.

(3.2) *Remarks.* (1) The notion of complexity was introduced in somewhat different terms by Alperin and Evans [AE], with M a finite-dimensional k-representation of the finite group G, and $R=k[G]$. Both cases are covered by a general definition, which makes sense for any module over an arbitrary ring, and has some satisfactory formal properties: cf. [Av_2].

(2) $\mathrm{cx}_R M \geqq 0$ with equality if and only if $\mathrm{pd}_R M < \infty$.

(3) If Q is a codimension c deformation of R, then

$$\mathrm{cx}_Q M \leqq cx_R M \leqq \mathrm{cx}_Q M + c.$$

Indeed, it suffices to consider the case when $c=1$: the left-hand side inequality is given by the exact sequence (2.3), while the right-hand side one follows from the specific form of the resolution constructed in (2.5).

For any local ring $(R, \mathfrak{m}, k)$ we denote by $\hat{R}$ its m-adic completion. If k is infinite, we set $\tilde{R}=\hat{R}$; if k is finite, we set $\tilde{R}=(R[X]_{\mathfrak{m}R[X]})\hat{}$, where X is an indeterminate over $\hat{R}$. Thus, $\tilde{R}$ is a faithfully flat extension of R, which is complete and has infinite residue field. We write $\tilde{M}$ for the $\tilde{R}$-module $M \otimes_R \tilde{R}$ and note that depth and Krull dimension do not change when passing from R or M to $\tilde{R}$ or $\tilde{M}$ respectively, and that $b_n^R(M)=b_n^{\tilde{R}}(\tilde{M})$ for all n. In particular, one has $\mathrm{pd}_R M = \mathrm{pd}_{\tilde{R}} \tilde{M}$, and $\mathrm{cx}_R M = \mathrm{cx}_{\tilde{R}} \tilde{M}$.

(3.3) **Definition.** For an R-module M the *virtual projective dimension* $\mathrm{vpd}_R M$ is (the non-negative integer or ∞)

$$\mathrm{vpd}_R M = \min\{\mathrm{pd}_{Q'} \tilde{M} \mid Q' \text{ is a deformation of } \tilde{R}\}.$$

(3.4) **Lemma.** *For a finitely generated R-module M, one has:*

(1) $\mathrm{vpd}_R M=0$ *if and only if M is R-free.*

(2) *If* $\mathrm{pd}_R M<\infty$, *then* $\mathrm{vpd}_R M = \mathrm{pd}_R M$.

(3) *If* $\mathrm{vpd}_R(M)=\mathrm{pd}_{Q'}(\tilde{M})$, *then* $\mathrm{edim}\, Q' = \mathrm{edim}\, R$; *in particular*: $\mathrm{vpd}_R(M) = \min\{\mathrm{pd}_{Q'} \tilde{M} \mid Q'$ *is an embedded deformation of* $\tilde{R}$).

(4) *The following are equivalent*: (i) *R is a local complete intersection*; (ii) $\mathrm{vpd}_R(M)<\infty$ *for all R-modules M*; (iii) $\mathrm{vpd}_R(k)<\infty$.

Proof. (1) If $\mathrm{pd}_Q \tilde{M}=0$ for a deformation Q of R, then $\tilde{M}$ is Q-free, in particular $\mathrm{Ann}_Q \tilde{M}=0$, hence $Q=\tilde{R}$, hence M is R-free.

(2) If $Q/(x_1, \ldots, x_c)=R$, then $\mathrm{pd}_R M<\infty$ implies $\mathrm{pd}_Q M = \mathrm{pd}_R M + c \geqq \mathrm{pd}_R M$.

(3) Let ρ be the deformation $Q' \to \tilde{R}$. If $\mathrm{Ker}\, \rho \not\subset \mathfrak{n}^2$, there is a non-zero divisor $z \notin \mathfrak{n}^2$, $z \in \mathrm{Ker}\, \rho$, hence by (2.8.1) $\mathrm{pd}_{Q'/(z)} \tilde{M} = \mathrm{pd}_{Q'} \tilde{M} - 1$, contradicting the assumption on Q'.

(4) Recall R is called a *complete intersection* if for some (hence for any) Cohen presentation of $\hat{R}$ as a homomorphic image of a complete regular local ring Q, $\mathrm{Ker}(Q \to \hat{R})$ is generated by a Q-regular sequence. Thus, (4) is just a restatement of the Auslander-Buchsbaum [AB] and Serre [Se] characterization of regular local rings. □

Complexity and virtual projective dimension are linked by:

(3.5) **Theorem.** *When $M \neq 0$ is a finitely generated R-module of finite virtual projective dimension, there is equality*

$$\operatorname{vpd}_R M = \operatorname{depth} R - \operatorname{depth} M + \operatorname{cx}_R M.$$

When $M \neq 0$ is finitely generated and $\operatorname{pd}_R M < \infty$, one has the Auslander-Buchsbaum equality, cf. [AB, Theorem 3.7]

$$\operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth} M.$$

In view of (3.2.1) and (3.4.2), the Theorem represents an extension of this fundamental formula. The proof of the Theorem uses the Auslander-Buchsbaum equality, and yields in fact the more precise result stated below. Recall that edim R stands for the embedding dimension, $\dim_k \mathfrak{m}/\mathfrak{m}^2$, of R.

(3.6) **Theorem.** *Let M be a finitely generated module of finite virtual projective dimension. If $Q \to \tilde{R}$ is any deformation, such that $\operatorname{pd}_Q \tilde{M} < \infty$, then it can be factored as $Q \to P \to \tilde{R}$ in such a way, that Q is a deformation of P, and P is a deformation of $\tilde{R}$ for which* $\operatorname{pd}_R M = \operatorname{vpd}_P M$, depth P = depth $R + \operatorname{cx}_R M$, *and* edim P = edim R.

We can now show that vpd_R, like pd_R, may not take arbitrary finite values.

(3.7) **Corollary.** *When* $\operatorname{vpd}_R M < \infty$, *one has the inequality*:

$$\operatorname{vpd}_R M \leqq \operatorname{edim} R$$

Assume in addition R is not a complete intersection. The stronger inequality

$$\operatorname{vpd}_R M \leqq \operatorname{edim} R - 2,$$

then holds, and it can further be improved to

$$\operatorname{vpd}_R M \leqq \operatorname{edim} R - 3$$

when R is Gorenstein.

It will be proved in (6.6) that over a complete intersection any possible value of $\operatorname{vpd}_R M$ occurs. The proofs of (3.5), (3.6), and (3.7) are given at the end of this section.

(3.8) *Notation.* We consider the following situation: $(Q, \mathfrak{n}, k)$ is a noetherian local ring; $x = x_1, \ldots, x_c$ is a Q-regular sequence; $R = Q/(\mathbf{x})$; $\mathfrak{m} = \mathfrak{n}/(\mathbf{x})$; M is a finitely-generated R-module.

Let $L_{\mathbf{x}}$ denote the c-dimensional k-vector space $(\mathbf{x})/\mathfrak{n}(\mathbf{x})$. For an element y contained in the ideal $(\mathbf{x})$ we write $\bar{y}$ to denote its image in $L_{\mathbf{x}}$. If $\mathbf{y} = y_1, \ldots, y_m$ is a sequence of elements of $(\mathbf{x})$, the vector subspace of $L_{\mathbf{x}}$ spanned by $\bar{y}_1, \ldots, \bar{y}_m$ is denoted $L_{\mathbf{y}}$.

The polynomials from $\mathscr{R}_{\mathbf{x}} = k[\chi_1, \ldots, \chi_c]$ are viewed as functions on $L_{\mathbf{x}}$ by means of the condition $\chi_i(\bar{x}_j) = \delta_{ij}$ (Kronecker delta). If $\mathbf{x}'$ is another Q-regular sequence with $(\mathbf{x}') = (\mathbf{x})$, then (1.3.4) with $\phi' = \operatorname{id}_Q$ and $\phi = \operatorname{id}_R$ shows this condition is independent of the choice of the generating system $x_1, \ldots, x_c$ of $(\mathbf{x})$ (note the "variables" $\chi_1, \ldots, \chi_c$ are determined by this choice). Thus, $\mathscr{R}_{\mathbf{x}}$ can be described in invariant terms as being the symmetric algebra on the k-linear dual of $L_{\mathbf{x}}$.

Since $\operatorname{Ext}_R(M, k)$ is a graded module, its annihilator is a homogeneous ideal of $R_{\mathbf{x}}$. Fixing an algebraic closure K of $\bar{k}$, we denote by $V(Q, \mathbf{x}, M)$ the cone in $K \otimes_k L_{\mathbf{x}}$ defined by this ideal.

(3.9) **Theorem.** *With the notation of* (3.8), *let* $\mathbf{y} = y_1 \ldots y_m$ *be a set of elements in* $(\mathbf{x})$, *and let* $\eta_1, \ldots, \eta_d$ *be a basis for the linear space* $\{\chi \in \mathscr{R}_{\mathbf{x}}^2 \mid \chi(L_{\mathbf{y}}) = 0\}$.

The following are equivalent:

(i) $\mathbf{y}$ *is a Q-regular sequence, and* $\operatorname{pd}_{Q/(\mathbf{y})} M < \infty$;

(ii) $\mathbf{y}$ *can be extended to a minimal system of generators of* $(\mathbf{x})$, *and* $\operatorname{pd}_{Q/(\mathbf{y})} M < \infty$;

(iii) $\dim_k L_{\mathbf{y}} = m$, $(K \otimes L_{\mathbf{y}}) \cap V(Q, \mathbf{x}, M) = \{0\}$, *and* $\operatorname{pd}_Q M < \infty$;

(iv) $d = c - m$, *and the* $k[\eta_1, \ldots, \eta_d]$*-module* $\operatorname{Ext}_R(M, k)$ *is finitely generated.*

Two special cases deserve special mention:

(3.10) **Corollary.** *When* $R = Q/(\mathbf{x})$, $\operatorname{pd}_Q M < \infty$ *is equivalent to the finite generation of* $\operatorname{Ext}_R(M, k)$ *over* $\mathscr{R}_{\mathbf{x}}$.

Indeed, this is the equivalence of (i) and (iv) for $\mathbf{y}$ the empty set, i.e. $(\mathbf{y}) = 0$. Thus, the Theorem contains both a generalization of, and a converse to Gulliksen's Theorem (2.1). □

(3.11) **Corollary.** *Assume* $\operatorname{pd}_Q M < \infty$, *and let* z *be a non-zero divisor in* $(\mathbf{x})$. *Then* $\bar{z} \in V(Q, \mathbf{x}, M)$ *if and only if* $\operatorname{pd}_{Q/(z)} M = \infty$.

This is the equivalence of (ii) and (iii) for $\mathbf{y}$ consisting of a single element. □

Proof of Theorem (3.9). (i)$\Leftrightarrow$(ii) Assume (i) holds, but there exists a $z \in (\mathbf{y}) \cap \mathfrak{n}(\mathbf{x})$, such that $z \notin \mathfrak{n}(\mathbf{y})$. Note that $\operatorname{pd}_{Q/(z)} Q/(\mathbf{y}) < \infty$ and $\operatorname{pd}_{Q/(\mathbf{y})} M < \infty$ imply $\operatorname{pd}_{Q/(z)} M < \infty$. This, however, is impossible in view of (2.8.2), hence (ii) follows from (i) as claimed. The converse is clear, since any minimal system of generators of a complete intersection ideal forms a regular sequence.

Next we make some adjustments in order to facilitate the exposition. Observe, that the first requirement in each of conditions (ii), (iii), and (iv) is that $\mathbf{y}$ form part of a minimal system of generators for the ideal $(\mathbf{x})$. By the remarks made in (3.8), we can then assume $y_i = x_i$ for $1 \leqq i \leqq m$, and $\eta_j = \chi_{j+m}$ for $1 \leqq j \leqq c - m$. Thus, in order to finish the proof, we have to show the following conditions are equivalent:

(ii)′ $\operatorname{pd}_{Q/(x_1, \ldots, x_m)} M < \infty$

(iii)′ $K(\bar{x}_1, \ldots, \bar{x}_m) \cap V(Q, \mathbf{x}, M) = \{0\}$ and $\operatorname{pd}_Q M < \infty$.

(iv)′ The $k[\chi_{m+1}, \ldots, \chi_c]$-module $\operatorname{Ext}_R(M, k)$ is finitely generated.

That (ii)′ implies (iv)′ is seen directly from Gulliksen's Theorem (2.1). Next we set the stage for proving the converse, which is the core of the whole argument.

With $Q' = R' = Q/(x_1, \ldots, x_m)$, consider the commutative diagram of ring homomorphisms:

$$\begin{array}{ccccc} Q & \xrightarrow{\text{id}} & Q & \xrightarrow{\phi'} & Q' \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi\phi'} & & \downarrow{\scriptstyle \phi} \\ R' & \xrightarrow[\phi]{} & R & \xrightarrow[\text{id}]{} & R \end{array}$$

where ϕ and ϕ' are the canonical projections. Applying (1.5) to it one obtains the commutative diagram

$$\begin{array}{ccccc} \bar{\mathscr{R}} \otimes \operatorname{Ext}^*_{R'}(M, k) & \xleftarrow{\Phi \otimes \operatorname{Ext}_\phi(M,k)} & \mathscr{R} \otimes \operatorname{Ext}^*_R(M, k) & \xleftarrow{\Phi' \otimes \text{id}} & \mathscr{R}' \otimes \operatorname{Ext}^*_R(M, k) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Ext}^*_{R'}(M, k) & \xleftarrow[\operatorname{Ext}_\phi(M,k)]{} & \operatorname{Ext}^*_R(M, k) & \xleftarrow[\text{id}]{} & \operatorname{Ext}^*_R(M, k) \end{array}$$

where the vertical maps are given by the respective module structures, and the following notation has been used:

$\mathscr{R} = k[\chi_1, \ldots, \chi_c]$ with χ_i induced by $t_i(Q, \mathbf{x}, \mathbb{F})$ for some R-free resolution $\mathbb{F}$ of M;

$\mathscr{R}' = k[\chi'_{m+1}, \ldots, \chi'_c]$ with χ'_j induced by $t_j(R', \mathbf{x}', \mathbb{F})$, where $\mathbf{x}' = x'_{m+1}, \ldots, x'_c$, with x'_j denoting the image of x_j in R';

$\bar{\mathscr{R}} = k[\bar{\chi}_1, \ldots, \bar{\chi}_m]$ with $\bar{\chi}_h$ induced by $t_h(Q, \mathbf{y}, M)$ for some R'-free resolution $\mathbb{F}'$ of M;

Φ': $\mathscr{R}' \to \mathscr{R}$ is the k-algebra homomorphism defined by $\Phi'(\chi'_j) = \chi_j$ for $m+1 \leqq j \leqq c$;

Φ: $\mathscr{R} \to \bar{\mathscr{R}}$ is the k-algebra homomorphism defined by $\Phi(\chi_i) = 0$ for $m+1 \leqq i \leqq c$, $\Phi(\chi_i) = \bar{\chi}_i$ for $1 \leqq i \leqq m$.

We are now ready to prove that (iv)′ implies (ii)′. Let $\mathbb{F}$ be as above, and choose an injective resolution $\mathbb{I}$ of k over R'. The standard filtration $F^p = \{f \in \mathbb{H}\mathrm{om}_R(\mathbb{F}, \mathbb{H}\mathrm{om}_{R'}(R, \mathbb{I})) \mid f(F_i) = 0 \text{ for } i < p\}$ yields the change-of-rings spectral sequence with second term:

$${}_2E^{p,q} = \operatorname{Ext}^p_R(M, \operatorname{Ext}^q_{R'}(R, k)) \Rightarrow \operatorname{Ext}^{p+q}_{R'}(M, k).$$

Equipping $\mathbb{F}_*$ with the operators $t_i = t_i(Q, \mathbf{x}, \mathbb{F})$ we see they satisfy $t_i(F^p) \subset F^{p+2}$, hence they induce an action of $R[\chi_1, \ldots, \chi_c]$ on the spectral sequence, with $\chi_i({}_rE^{p,q}) \subset {}_rE^{p+2,q}$ for $r \geqq 2$. Note that for this action the term ${}_2E$ is simply $\operatorname{Ext}_R(M, k) \otimes_k \operatorname{Ext}_{R'}(R, k)$ viewed as an $\mathscr{R}$-module via the left-hand factor. By our assumption, $\operatorname{Ext}_R(M, k)$ is finitely generated as a $\Phi'(\mathscr{R}')$-module. Since $\operatorname{Ext}_{R'}(R, k)$ is the exterior algebra over the d-dimensional vector space $\operatorname{Ext}^1_{R'}(R, k)$, it follows that ${}_2E$ is a finitely generated $\Phi'(\mathscr{R}')$-module as well. But now, because of the $\mathscr{R}$-linearity of the differentials ${}_r d$ of the spectral sequence, this property is seen to be inherited by the consecutive subquotients ${}_3E$, ${}_4E$, ..., ${}_{c-m+1}E$.

It is time to note that since ${}_2E^{p,q}=0$ when $q<0$ or $q>c-m$, one has ${}_{c-m+1}E={}_{\infty}E$. Thus, the limit term $\mathrm{Ext}_{R'}(M, k)$ has a finite filtration by finitely generated $\Phi'(\mathscr{R}')$-modules. It follows that $\mathrm{Ext}_{R'}(M, k)$ is itself finitely generated as a module over the subring $\Phi'(\mathscr{R}')$ of $\mathscr{R}$. However, the left-hand square of the commutative diagram shows the variables $\chi_j=\Phi'(\chi'_j)$ annihilate $\mathrm{Ext}_{R'}(M, k)$ when $m+1 \leqq j \leqq c$, hence for this module finite generation is equivalent to finite dimension as a k-vector space, which in turn obtains if and only if $\mathrm{pd}_{R'} M<\infty$.

Next we prove that (iv)′⇔(iii)′. We have just shown (iv)′ implies $\mathrm{pd}_{Q'} M<\infty$, and since $\mathrm{pd}_Q Q'$ is finite, $\mathrm{pd}_Q M<\infty$ follows. Furthermore, the fact that $\mathrm{Ext}_R(M, k)$ is finitely generated as an $\mathscr{R}$-module, follows both from (iv)′ (obvious) or from (iii)′ (by (2.1)). Thus, in order to show the equivalence of (iii)′ and (iv)′, we assume $\mathrm{Ext}_R(M, k)$ is finitely generated over $\mathscr{R}$, and have to establish that $K(\bar{x}_1, \ldots, \bar{x}_m) \cap V(Q, \mathbf{x}, M)=\{0\}$ if and only if $\mathrm{Ext}_R(M, k)$ is finitely generated over $k[\chi_{m+1}, \ldots, \chi_c]$. However, since the points in the intersection correspond bijectively to the maximal ideals in the support of $K \otimes [\mathrm{Ext}_R(M, k)/(\chi_{m+1}, \ldots, \chi_c)\,\mathrm{Ext}_R(M, k)]$, this is just Nakayama's lemma. □

In terms of growth invariants, the theorem has the following interpretation:

(3.12) **Corollary.** *Let $M \neq 0$ be a finitely-generated R-module with* $\mathrm{vpd}_R M<\infty$, *say* $\mathrm{pd}_Q \tilde{M}<\infty$ *for* $\tilde{R}=Q/(\mathbf{x})$. *The following integers are then (defined and) equal:*

(i) *the complexity* $\mathrm{cx}_R(M)$;

(ii) *the order of the pole of* $P_M^R(t)$ *at* $t=1$;

(iii) *the Krull dimension of the* $\mathscr{R}_{\mathbf{x}}$*-module* $\mathrm{Ext}_R(M, k)$.

(iv) *the dimension of the algebraic set* $V(Q, \mathbf{x}, \tilde{M})$.

When the field k is infinite, they are moreover, equal to:

(v) $\min\{\mathrm{codim}\, H \mid H$ *is a k-rational subspace of* $K \otimes L_{\mathbf{x}}$, *such that* $H \cap V(Q, \mathbf{x}, \tilde{M})=\{0\}\}$;

(vi) $c-\max\{m \mid$ *there exist elements* $y_1, \ldots, y_m$, *which form part of a minimal system of generators of* $(\mathbf{x})$ *and satisfy* $\mathrm{pd}_{Q/(y_1, \ldots, y_m)} M<\infty\}$;

(vii) $\min\{d \mid Q'$ *is a codimension d deformation of* $\tilde{R}$ *such that* $\mathrm{pd}_{Q'} \tilde{M}<\infty\}$.

Proof. By the remarks preceding (3.3), the first three conditions do not change when R and M are replaced by $\tilde{R}$ and $\tilde{M}$; we make this substitution.

Let d_j denote the integer defined by the j-th condition: the finiteness of d_1 (resp. d_2) follows from (3.2.3) (resp. (2.6)). Since, according to Gulliksen's Theorem (2.1), $\mathrm{Ext}_{\tilde{R}}(\tilde{M}, k)$ is a finitely generated $\mathscr{R}_{\mathbf{x}}$-module, the equality of d_1 through d_5 follows by elementary commutative algebra and algebraic geometry. Remarking that every subspace H as in (v) has the form $L_{\mathbf{y}} \otimes K$ for some sequence $\mathbf{y} \subset (\mathbf{x})$, whose elements extend to a minimal system of generators of $(\mathbf{x})$, we conclude directly from the theorem that $d_5=d_6$. Finally, the definitions yield the inequality $d_7 \leqq d_6$, while $d_1 \leqq d_7$ follows from (3.2.3); with the equality $d_1=d_6$ already in hand, this completes the proof. □

Proof of Theorems (3.5) *and* (3.6). In the notation of (3.6), assume $\tilde{R} = Q/(x_1, \ldots, x_c)$, choose $y_1, \ldots, y_m$ to satisfy (3.12. vi), and set $P = Q/(y_1, \ldots, y_m)$. We then have:

$$\begin{aligned} \operatorname{vpd}_R M &\leqq \operatorname{pd}_{Q/(\mathbf{y})} M \quad \text{(by (3.3))} \\ &= \operatorname{depth} Q/(\mathbf{y}) - \operatorname{depth} M \\ &= \operatorname{depth} R + (c-m) - \operatorname{depth} M \\ &= \operatorname{depth} R + \operatorname{cx}_R M - \operatorname{depth} M \quad \text{(by (3.12))} \end{aligned}$$

On the other hand, choosing a deformation Q' with $\operatorname{pd}_{Q'} M$ minimal, we can write:

$$\begin{aligned} \operatorname{vpd}_R M &= \operatorname{pd}_{Q'} M \quad \text{(by (3.3))} \\ &= \operatorname{depth} Q' - \operatorname{depth} M \\ &= \operatorname{depth} R + (\operatorname{depth} Q' - \operatorname{depth} R) - \operatorname{depth} M \\ &\geqq \operatorname{depth} R + \operatorname{cx}_R M - \operatorname{depth} M \quad \text{(by (3.12))} \end{aligned}$$

It follows all the quantities considered in both chains are equal to each other. In particular, (3.5) holds. Furthermore, we also have by the choice of P that

$$\operatorname{cx}_R M = c - m = (\operatorname{depth} Q - \operatorname{depth} R) - (\operatorname{depth} Q - \operatorname{depth} P) = \operatorname{depth} P - \operatorname{depth} R,$$

and we see from the first chain of equalities that $\operatorname{vpd}_R M = \operatorname{pd}_P M$. By (3.4.3) this implies edim $P =$ edim R, which was the last thing left to prove. □

Proof of Corollary (3.7). Choose P, by (3.6), to satisfy $\operatorname{cx}_R M = \operatorname{depth} P - \operatorname{depth} R$, and edim $P =$ edim R. Applying (3.5) one obtains:

$$\begin{aligned} \operatorname{vpd}_R M &= \operatorname{depth} P - \operatorname{depth} M \\ &= \operatorname{edim} P - \operatorname{depth} M - (\operatorname{edim} P - \operatorname{depth} P) \\ &\leqq \operatorname{edim} R - (\operatorname{edim} P - \operatorname{depth} P). \end{aligned}$$

The first inequality in Cor. 3.7 follows because edim $P \geqq$ depth P for any local ring P. The remaining two are immediate consequences of the fact, that if edim $P -$ depth $P \leqq 1$ (resp.: P is Gorenstein and edim $P -$ depth $P \leqq 2$), then P, hence also R, is a complete intersection. □

4. Periodicity

Before turning to Eisenbud's conjecture on modules with bounded Betti numbers, cf. (4.6), we show that the condition $\operatorname{vpd}_R M < \infty$ allows for a very precise determination of the asymptotic behaviour of the Betti sequence of M. In the formulation of the next result, we use the notation $O(u(n))$ to indicate a function $v(n)$ such that $|v(n)| < \gamma |u(n)|$ for some $\gamma \in \mathbb{R}$ and all $n \gg 0$.

(4.1) **Theorem.** *Let M be a finitely generated R-module of finite virtual projective dimension. The following then holds:*

– $\mathrm{cx}_R M = 0$ *if and only if the sequence* $\{b_n^R(M)\}$ *is eventually zero;*
– $\mathrm{cx}_R M = 1$ *if and only if the sequence* $\{b_n^R(M)\}$ *is eventually constant and non-zero;*
– $\mathrm{cx}_R M = d \geqq 2$ *if and only if there exists a positive integer* A *such that*

$$b_n^R(M) = \frac{A}{2^{d-1}(d-1)!}\, n^{d-1} + O(n^{d-2}).$$

Proof. By (2.6) the Poincaré series $P_M^R(t) = \sum_{n \geqq 0} b_n(M)\, t^n$ can be written in irreducible form $f(t)/(1-t)^u(1+t)^e$ with $f(t)$ a polynomial with integer coefficients. Furthermore, by (3.12) u equals d, the complexity of M. Decomposing this rational fraction into prime fractions, we have:

(4.1.1)
$$P_M^R(t) = \frac{a_d}{(1-t)^d} + \dots + \frac{a_1}{(1-t)} + \frac{b_e}{(1+t)^e} + \dots + \frac{b_1}{(1+t)} + p(t)$$

for a polynomial $p(t)$ and some rational numbers $a_1, \dots, a_d, b_1, \dots, b_e$ such that $a_d = a \neq 0$, $b_e = b \neq 0$. The case $d = e = 0$ occurs precisely when $\mathrm{pd}_R M$ is finite, so in the sequel we suppose one of d or e is positive. Denoting by $n_0 - 1$ the degree of $p(t)$, and writing out the binomial expansions, one obtains polynomials

$$g(X) = \frac{a}{(d-1)!} X^{d-1} + \textit{lower order terms},$$
$$h(X) = \frac{b}{(e-1)!} X^{e-1} + \textit{lower order terms},$$

such that for all $n \geqq n_0$ one has:

(4.1.2)
$$b_n(M) = \begin{cases} g(n) + h(n) & \text{when } n \text{ is even;} \\ g(n) - h(n) & \text{when } n \text{ is odd.} \end{cases}$$

If $d < e$, one of the series of the even or odd Betti numbers is eventually given by a polynomial with negative leading term, hence takes on negative values for $n \gg 0$, which is absurd.

Assume $d = e \neq 0$. For the same reason as just stated, one sees that $a + b \geqq 0$ and $a - b \geqq 0$, hence $a \geqq 0$; together with our assumption that $a \neq 0$ this means $a > 0$. Choose now an integer j such that $a < (2j+1)|b|$, and let n be an even integer such that $n - (j+1) \geqq n_0$. Localizing at a minimal prime, $\mathfrak{p}$, of R, and using the exactness of the sequence

$$0 \leftarrow G' \leftarrow (F_{n-j})_{\mathfrak{p}} \leftarrow \dots \leftarrow (F_n)_{\mathfrak{p}} \leftarrow \dots (F_{n+j})_{\mathfrak{p}} \leftarrow G'' \leftarrow 0$$

derived from a minimal free resolution $\mathbb{F} = \{F_n\}$ of M, one can write down the equality:

$$\sum_{q=-j}^{j} (-1)^q \,\mathrm{length}_{R_{\mathfrak{p}}} (F_{n-q})_{\mathfrak{p}} = (-1)^j (\mathrm{length}\, G' + \mathrm{length}\, G'').$$

Dividing both sides by length $R_\mathfrak{p}$, and setting $\chi_{i,n} = \sum_{q=-i}^{i} (-1)^q b_{n-q}(M)$, one obtains for even j the inequalities:

(4.1.3) $$\chi_{j,n} \geqq 0; \quad \text{and}$$

(4.1.4) $$\chi_{j+1,n} \leqq 0.$$

However, (4.1.2) shows $\chi_{i,n}$ is given by a polynomial in n, whose leading term is

$$\begin{aligned} &(a+(2j+1)b)\,n^d && \text{for } i=j; \text{ and} \\ &(-a+(2j+3)b)\,n^d && \text{for } i=j+1. \end{aligned}$$

Now if $b<0$ this implies $\chi_{j,n}$ is negative for all j when $n \gg 0$, and this contradicts (4.1.3). If $b>0$, it implies $\chi_{j+1,n}$ is positive for all $j+1$ when $n \gg 0$, and this is ruled out by (4.1.4). Consequently, b is equal to zero.

Thus, we have shown $d>e$. However d (resp. e) is the order of the pole of $P_M^R(t)$ at $t=1$ (resp. $t=-1$), so that when $\operatorname{pd}_R M = \infty$ one can write the Poincaré series in the form

(4.1.5) $$P_M^R(t) = \frac{f(t)}{(1-t)^d(1+t)^e} \quad \text{with } d>e, \quad f(t) \in \mathbb{Z}[t],\ f(1)>0.$$

It remains to see the inequality for $f(1)$. Multiplying both sides by $(1-t)^d$, invoking (4.1.1), and passing to the limit as $t \to 1$, one obtains the expression $0<a=f(1)/2^e$.

The theorem follows by setting $A=f(1)\cdot 2^{d-e-1}$. □

(4.2) *Remark.* Returning to the formulas (4.1.2) one sees – taking (4.1.5) into account – that each one of the polynomials, expressing (for big enough n) the even or the odd Betti numbers of M, has the same leading term. Thus, we obtain yet another interpretation for the complexity of M: it is equal to the Krull dimension of either $\operatorname{Ext}_R^{\text{even}}(M,k)$ or $\operatorname{Ext}_R^{\text{odd}}(M,k)$, when viewed in the natural way as a module over $\mathscr{R}_\mathbf{x}$.

In [Av_1, p. 34] the question was raised whether the inequality $b_n^R(M) \leqq b_{n+1}^R(M)$ holds for all $n \gg 0$ for any finitely generated module M over a local ring R. The answer is known (and is positive) in only two cases, both due to Lescot: when $\mathfrak{m}^3=0$, or when R is a Golod ring. For modules virtually of finite projective dimension, the question can now be reduced to a very concrete problem:

(4.3) **Proposition.** *Let M be a module of finite virtual projective dimension, and let its Poincaré series be written in irreducible form $f(t)/(1-t)^d(1+t)^e$ with $f(t) \in \mathbb{Z}[t]$. When $d \leqq 1$ the Betti numbers of M are eventually constant. When $d>e+1$, the Betti numbers of M are eventually increasing. When $d-1=e$, they are eventually increasing iff $f(1) > \pm f(-1)$.*

Proof. We use the notation introduced in the proof of (4.1). The case $d \leqq 1$ being clear from (4.1), we assume in the sequel that $d \geqq 2$. Using (4.1.1), the difference $b_{n+1}^R(M) - b_n^R(M)$ can be expressed in the form:

$$a\binom{n+d-1}{d-2} + (-1)^{n+1} b\left(\binom{n+e}{e-1} + \binom{n+e-1}{e-1}\right) + r(n) + (-1)^{n+1} s(n),$$

where $r(n)$ (resp. $s(n)$) is a polynomial of degree $<d-2$ (resp. $<e-2$). By (4.1.5) we know that $d \geqq e-1$ and that $a=f(1)/2^e$ is positive, hence for $n \gg 0$ this difference is positive either when $d>e-1$, or when $d=e-1$ and $a>2|b|$. It remains to note that multiplying both sides of (4.1.5) by $(1+t)^e$, passing to the limit as $t \to -1$, and looking at (4.1.1), one obtains the equality $b=f(-1)/2^d$, so that the inequality involving a and b can be rewritten as $f(1)>|f(-1)|$. □

A weaker question than the one addressed in (4.3) is due to Ramras: is it true that either $\lim_{n\to\infty} b_n^R(M)=\infty$, or else the Betti numbers are eventually constant? Theorem (4.1) answers it affirmatively for modules virtually of finite projective dimension.

To close this section, we inspect these modules for periodic behaviour.

Recall that a complex of R-modules $\mathbb{F}_*$ is called *periodic of period* q, if there exist an integer r and an endomorphism of complexes $t\colon \mathbb{F} \to \mathbb{F}$ of degree $-q$, such that $F_i=0$ for $i<r$, and t maps F_{i+q} isomorphically onto F_i for $i \geqq r$. A resolution $\mathbb{F}$ is said to become periodic after r steps, if the truncated complex $\mathbb{F}_*/\mathbb{F}_{<r}$ is periodic. A module is said to be *eventually periodic* (*after* r *steps*) if its minimal free resolution becomes periodic (after r steps). When one can take $r=0$, the module is called periodic.

An interesting class of exact periodic complexes of period 2 has been constructed by Eisenbud [Ei, Sect. 5] on the basis of his notion of *matrix factorization*, which refers to an ordered pair of maps of free P-modules $\phi'\colon F' \to G'$ and $\psi'\colon G' \to F'$ such that $\phi'\psi' = x'\mathrm{id}_{G'}$ and $\psi'\phi' = x'\mathrm{id}_{F'}$ for some element $x' \in P$. Denoting reduction modulo (x') by forgetting the "prime" symbol, consider the periodic complex

$$\mathbb{F}(\phi', \psi')\colon \ldots \to F \xrightarrow{\phi} G \xrightarrow{\psi} F \xrightarrow{\phi} G \to 0 \to 0 \to \ldots$$

in which the first 0 appears in degree -1. It its easily verified [Ei, Proposition 5.1], that when x' is a non zero divisor on P, this is a resolution of the $P/(x')$-module $\operatorname{Coker} \phi = \operatorname{Coker} \phi'$.

(4.4) **Theorem.** *Let M be a finitely generated R-module of finite virtual projective dimension.*

(1) *The following conditions are equivalent:*

(i) *M has bounded Betti numbers;*

(ii) $b_n^R(M)=b_{n+1}^R(M)$ *for all* $n \geqq \operatorname{depth} R - \operatorname{depth} M + 1$;

(iii) *M is eventually periodic of period* 2 *after at most* $\operatorname{depth} R - \operatorname{depth} M + 1$ *steps.*

(2) *If* $R=P/(x')$ *for some non zero divisor x', and* $\operatorname{pd}_P M<\infty$, *there is a matrix factorization* (ϕ', ψ') *of* x', *such that* $\mathbb{F}(\phi', \psi')$ *is the minimal resolution of the* ($\operatorname{depth} R - \operatorname{depth} M + 1$)*'st syzygy N of M.*

When k is infinite, R is complete, $\operatorname{pd}_R M=\infty$, *and condition* (1.*i*) *holds, such a P always exists.*

(4.5) **Corollary.** *When* $\operatorname{vpd}_R M < \infty$ *and* $M \neq 0$, *the following conditions are equivalent:*

(i) *M is periodic*

(ii) $\operatorname{vpd}_R M = 1$, *and M has no non-trivial free direct summand;*

(iii) *$\tilde{M}$ has a minimal $\tilde{R}$-free resolution of the form* $\mathbb{F}(\phi', \psi')$.

(4.6) *Remarks.* A well known conjecture of Eisenbud [Ei, p. 37] asserts, that if M has bounded Betti numbers, it is eventually periodic of period 2 (cf. Note added in proof). In view of the theorem, if $\operatorname{vpd} M$ is finite, then the even stronger statement holds, that the minimal resolution of such a module is eventually given by a matrix factorization.

As noted in (3.4.4), this will be the case when R is a complete intersection; indeed, then (4.4) and (4.5) specialize to some key results of [Ei], namely Theorems (4.1), (5.2), (6.1), and Proposition (5.3). Thus, they may be viewed as "relative" versions of some assertions of [Ei], in which a hypothesis on the ring – i.e., one which is being imposed on *all* R-modules – is replaced by a hypothesis on the individual module under consideration. It should be noted that several proofs in [Ei] proceed by reduction to dimension zero and subsequent use of the self-injectivity of artinian complete intersections. This technique being no longer available in the present context, it is Theorem (3.9) which – in a rather discreet way – plays a pivotal role in our arguments.

Proof of Theorem (4.4). Consider first the case of $\operatorname{pd}_R M < \infty$. By the Auslander-Buchsbaum formula then $b_n^R(M) = \operatorname{rank} F_n = 0$ for $n \geqq \operatorname{depth} R - \operatorname{depth} M + 1$, so that conditions (i), (ii), and (iii) of (1) all hold, in particular they are equivalent. The conclusion of (2) holds for the same reason. Thus, for the rest of the proof we shall assume $\operatorname{pd}_R M = \infty$.

It has been noted before (3.3) that conditions (1.i) and (1.ii) are invariant under the passage from R and M to $\tilde{R}$ and $\tilde{N}$. Denoting by N' the (depth R $-$depth $M+3$)'rd syzygy of M, condition (iii) means that N and N' are isomorphic as R-modules. However, by faithful flatness, this is equivalent to the isomorphism $\tilde{N} \cong \tilde{N}'$ as $\tilde{R}$-modules. Since $\tilde{N}$ is the (depth R $-$depth $M+1$)'st syzygy of $\tilde{M}$, and $\tilde{N}'$ is its (depth $R-$depth $M+3$)'rd syzygy, one sees that condition (iii) also is invariant under the switch from R to $\tilde{R}$. Thus – exactly as in [Ei] – we have reduced the proof of (1) to the case when R is complete with infinite residue field. We shall assume this to be the case.

If (iii) holds, the Betti numbers with indices $\geqq \operatorname{depth} R - \operatorname{depth} M + 1$ take on at most two values, both of which are non-zero, since $\operatorname{pd}_R M = \infty$. Thus $\operatorname{cx}_R M = 1$, and we see from (4.1) that these two values coincide, which is precisely the claim of (ii). Since the implication (ii)$\Rightarrow$(i) is trivial, and (iii) represents a weakening of the statement in (2), both parts of the Theorem will be proved once we establish (2).

Since (1.i) implies that M has complexity 1, (3.5) and (3.6) provide a presentation $R = P/(x')$ with $\operatorname{pd}_P M = \operatorname{depth} R - \operatorname{depth} M + 1$. Working our way up the resolution from M to N, we find $\operatorname{pd}_P N = 1$. To show the minimal resolution of N has the desired form, we use an argument due to Eisenbud [Ei, p. 53].

Let $0 \to F' \xrightarrow{\phi'} G' \to N \to 0$ be P-free resolution of N. Since $x'N=0$, multiplication of this resolution by x' is homotopic to zero, hence there exists a map $\psi' : G' \to F'$ such that $\phi'\psi' = x' \operatorname{id}_{G'}$ and $\psi'\phi' = x' \operatorname{id}_{F'}$. Let $\mathfrak{m}'$ denote the maximal ideal of P, and assume $\mathbb{F}(\phi', \psi')$ is not minimal, i.e. $\operatorname{Im}\psi' \not\subset \mathfrak{m}F'$. This implies $\operatorname{Im}\psi'$ contains a non-trivial R'-free summand, hence $\operatorname{Im}\psi$ contains a non-trivial R-free summand. However, the exactness of the complex $\mathbb{F}(\phi', \psi')$ shows $\operatorname{Im}\psi = \operatorname{Coker}\phi \cong N$, which is not possible due to the following:

(4.7) **Lemma.** *If* $(\mathbb{F}, \partial)$ *is a minimal free resolution of a finitely generated module* M *over an arbitrary local ring* R, *then the modules* $\operatorname{Im}\partial_i$ *contain no non-trivial free direct summand when* $i > q = \max(\operatorname{depth} R - \operatorname{depth} M, 0)$.

Proof. The statement is a slightly more precise version of [Ei, Lemma (0.1.ii)], and the proof given there can easily be adapted to fit the present formulation. For diversity (and completeness) we offer an alternate argument.

Let $\mathbf{a}$ be a maximal R-regular sequence, chosen in such a way that its initial segment of min(depth R, depth M) elements forms an M-regular sequence. If $\mathbb{E}$ is the Koszul complex on $\mathbf{a}$, consider the homomorphisms of complexes

$$\bar{\mathbb{F}} = \bar{R} \otimes_R \mathbb{F} \leftarrow \mathbb{E} \otimes_R \mathbb{F} \to \mathbb{E} \otimes_R M$$

defined by the augmentation maps $\bar{R} = R/(\mathbf{a}) \leftarrow \mathbb{E}$ and $\mathbb{F} \to M$. Both homomorphisms induce isomorphisms in homology. By the depth-sensitivity of the Koszul complex, $H_i(\mathbb{E} \otimes_R M) = 0$ when $i > q$. Since depth $\bar{R} = 0$, for each i there exists a non-zero element $r_i \in \bar{R}$ which annihilates $\operatorname{Im}(\bar{R} \otimes \partial_i)$, so that $\operatorname{Im}(\bar{R} \otimes \partial_i)$ cannot contain a non-trivial $\bar{R}$-free direct summand. But then $\operatorname{Im}\partial_i$ contains no non-trivial R-free direct summand, as claimed. □

Proof of Corollary (4.5) (i)⇒(ii). By assumption, M is isomorphic to its own syzygy which can be made to sit as far back in the resolution as one wishes it to. On the one hand, by the preceding lemma, this implies M has no free summands $\neq 0$. On the other hand, it forces the equality depth M = depth R, and since nontrivial periodic modules obviously have complexity one, $\operatorname{vpd}_R M = 1$ follows from (3.5).

(ii)⇒(iii). Let $\tilde{R} = P/(\mathbf{x})$ be given by (3.6), with $\operatorname{pd}_P \tilde{M} = \operatorname{vpd}_R M = 1$, and with $(\mathbf{x})$ generated by a regular sequence of length $\operatorname{cx}_R M = d$. Since $\tilde{M}$ is a torsion P-module of projective dimension 1, its annihilator in P has grade 1; since $(\mathbf{x})\tilde{M} = 0$, it also has grade $\geq d$, hence $1 = d = \operatorname{cx}_R M$. To see that the resolution of M has the desired form, one can now apply the first part of the argument used to prove (4.4.2).

The remaining implication (iii)⇒(i) being trivial, the proof of the corollary is complete. □

5. Plexity and virtual injective dimension

There is no reason not to subject to growth of the Bass numbers (cf. (2.7)) to the same kind of analysis we have used on the Betti numbers. Since the results can sometimes be reduced to the ones for free resolutions, and can usually be proved in a similar way, this short section mostly focuses on the specific features of the minimal injective resolutions.

(5.1) **Definition.** The *plexity* of M, denoted $\mathrm{px}_R M$ is the smallest integer d such that $\mu_r^n(M) \leqq \gamma n^{d-1}$ for some real $\gamma > 0$, and for all sufficiently large n; if no such d exists, we set $\mathrm{px}_R M = \infty$.

The *virtual injective dimension*, $\mathrm{vid}_R M$, is defined to be equal to $\min\{\mathrm{id}_Q \tilde{M} \mid Q$ is a deformation of $\tilde{R}\}$.

As before, $\mathrm{px}_R M = 0$ if and only if M is injective. If Q is a codimension c deformation of R, then $\mathrm{px}_Q M \leqq \mathrm{px}_R M \leqq \mathrm{px}_Q M + c$. When M is finitely generated and $\mathrm{id}_R M < \infty$, then $\mathrm{vid}_R M = \mathrm{id}_R M$. In this case, a well known result of Bass [Ba] states that $\mathrm{id}_R M = \mathrm{depth}\, R$. Now the analogue of Theorem (3.5), which extends Bass's equality to infinite injective dimension, is

(5.2) **Theorem.** *Assume M is a finitely generated R-module of finite virtual injective dimension. There is then equality*

$$\mathrm{vid}_R M = \mathrm{depth}\, R + \mathrm{px}_R M. \quad \square$$

It should be noted, that in view of the "Bass conjecture", proved by Peskine and Szpiro, Hochster, and P. Roberts, if $\mathrm{vid}_R M < \infty$ for some $M \neq 0$, then R is Cohen-Macaulay.

Let $W(Q, \mathbf{x}, M)$ denote the algebraic variety defined by the annihilator of $\mathrm{Ext}_R^*(k, M)$ in $\mathscr{R}_\mathbf{x}$. The precise analogue of Theorem (3.9) obtains – with a similar proof – by exchanging projective with injective dimension. In particular (cf. Corollary (3.11)), a non zero divisor $z \in (\mathbf{x})$ has the property that $\bar{z} \in W(Q, \mathbf{x}, M)$ if and only if $\mathrm{id}_{Q/(z)} M = \infty$. Since over a Gorenstein ring the conditions $\mathrm{pd}\, M < \infty$ and $\mathrm{id}\, M < \infty$ are equivalent, one has:

(5.3) **Proposition.** *Let R be a Gorenstein local ring, and let M be a finitely generated module with $\mathrm{pd}_Q(M) < \infty$. Then the varieties $V(Q, \mathbf{x}, M)$ and $W(Q, \mathbf{x}, M)$ coincide.* $\square$

It seems relevant to ask here the question whether the finiteness of *both* $\mathrm{vpd}_R M$ and $\mathrm{vid}_R M$ for the *same* module M imply that R is Gorenstein: this is indeed known to be the case when the finiteness condition is placed on the actual – rather than the virtual – dimensions, cf. [Fo].

6. The cohomological variety of a module

In the preceding sections, the main tool has been the module structure of $\mathrm{Ext}_R(M, k)$ over the graded polynomial algebra $\mathscr{R}_\mathbf{x} = k[\chi_1, \ldots, \chi_c]$. The construction in (1.5) of the χ_i's from Eisenbud's operators and their duals has the advantage of easy definition and useful interpretation at the cochain level. However, it has the major drawback of *presupposing* that R has been given to us along with its deformation Q. This section shows how to remedy such a situation. We only discuss projective invariants, since statements and arguments dualize without problem to yield results on injective resolutions.

(6.1) *The homotopy Lie algebra.* Recall that $\mathrm{Ext}_R^*(k, k)$ is in a natural way the universal enveloping algebra of a graded k-Lie algebra $\pi^*(R)$, called the homotopy Lie algebra of R (for details cf. [Av_1]). Furthermore, every homomorphism $R' \to R$ of local rings, which induces the identity on their (common) residue field k, defines a map $\pi^*(R) \to \pi^*(R')$ of graded Lie algebras. When $\rho: Q \to R$ is a deformation, $\pi^*(\rho)$ is an isomorphism in degrees $\geqq 3$, and defines the five-term exact sequence

$$(6.1.1)\quad 0 \to \pi^1(R) \xrightarrow{\pi^1(\rho)} \pi^1(Q) \to \mathrm{Hom}_k(L_{\mathbf{x}}, k) \xrightarrow{\sigma(\rho)} \pi^2(R) \xrightarrow{\pi^2(\rho)} \pi^2(Q) \to 0$$

where $L_{\mathbf{x}} = (\mathbf{x})/\mathfrak{n}(\mathbf{x})$ as in (3.8); furthermore, Ker $\pi^2(\rho)$ is a subspace in the center $\zeta^*(R)$ of $\pi^*(R)$, cf. [Ja] or [Av_1]. Thus, there are two natural actions of $\mathscr{R}_{\mathbf{x}} = k[\chi_1, \ldots, \chi_c]$ on $\mathrm{Ext}_R(M, k)$ (or $\mathrm{Ext}_R(k, M)$): the one defined in Sect. 1, and the one coming from the k-algebra homomorphism $\Sigma(\rho): \mathscr{R}_{\mathbf{x}} \to \mathrm{Ext}_R(k, k)$, $\Sigma(\rho)(\chi_i) = \sigma(\rho)(\chi_i)$, with subsequent application of the Yoneda pairing. It can be shown both actions yield the same product.

Summing up, we obtain the following result, whose last statement is due to the Poincaré-Birkhoff-Witt theorem:

(6.1.2) **Lemma.** *The action on* $\mathrm{Ext}_R(M, k)$ *or* $\mathrm{Ext}_R(k, M)$ *of the k-algebra* $\mathscr{R}_{\mathbf{x}}$ *obtained from a deformation* $Q \to Q/(\mathbf{x})$ *of R, factors canonically through an action of the subalgebra* $\mathscr{R}$ *of* $\mathrm{Ext}_R(k, k)$, *generated by the subspace* $\zeta^2(R)$ *of central elements of degree 2 of* $\pi^*(R)$. *Furthermore,* $\mathscr{R}$ *is a polynomial algebra.* □

(6.1.3) *Remark.* The groups $\pi^*(R)$ which are of interest to us in this paper are π^1 and π^2. One always has $\pi^1(R) = \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. When $R = Q/I$ for a regular local ring $(Q, \mathfrak{n})$ and $I \subset \mathfrak{n}^2$, then $\pi^2(R) = \mathrm{Hom}_k(I/\mathfrak{n}I, k)$.

In particular, when R is a complete intersection, then $\pi^2(R) = \mathrm{Hom}_k(L_{\mathbf{x}}, k)$ in the notation of (3.8). Furthermore, in this case $\pi^i(R) = 0$ for $i \geqq 3$, and this is known to characterize complete intersections.

(6.2) *The cohomological variety of a module.* In the sequel we consider finitely-generated R-modules M only. Let K be a fixed algebraic closure of $\tilde{k}$. Writing $\zeta_2(R)$ for $\mathrm{Hom}_k(\zeta^2(R), k)$, and viewing the elements of $\mathscr{R}$ as functions on the affine space $K \otimes_k \zeta_2(R)$, we set

$$V_R^*(M) = \{z \in K \otimes \zeta_2(R) \mid f(z) = 0 \text{ for all } f \in \mathrm{Ann}_{\mathscr{R}} \mathrm{Ext}_R(M, k)\}$$

and call this the *cohomological variety* of the R-module M.

In the notation of (3.3), $\mathrm{Ext}_{\tilde{R}}(\tilde{M}, \tilde{k}) \simeq \tilde{k} \otimes_k \mathrm{Ext}_R(M, k)$ by an isomorphism compatible with the identification $\zeta^2(\tilde{R}) = \tilde{k} \otimes_k \zeta^2(R)$. Thus, $V_{\tilde{R}}^*(\tilde{M}) = V_R^*(M)$ canonically, hence in studying the variety one often can assume R is complete with residue field infinite (or – for that matter – equal to K).

Some illustrations are immediately at hand:

(6.2.0) $V_R^*(0)=\emptyset$.

(6.2.1) If $\mathrm{pd}_R M<\infty$, and $M\neq 0$ then $V_R^*(M)=\{0\}$.

(6.2.2) $V_R^*(k)=K\otimes_k \zeta_2(R)$, since $\mathrm{Ext}_R(k,k)$ is a free $\mathscr{R}$-module by Poincaré-Birkhoff-Witt.

In order to relate $V_R^*(M)$ to the variety $V(Q, \mathbf{x}, M)$ associated to a deformation $\rho: Q\to R$ with kernel $(\mathbf{x})$, consider the natural linear map $\sigma(\rho)\check{}$

$$K\otimes_k \zeta_2(R)=\mathrm{Hom}_k(\zeta^2(R),K)\xrightarrow{\mathrm{Hom}(\sigma(\rho),K)}\mathrm{Hom}_k(\mathrm{Hom}_k(L_{\mathbf{x}},k),K)=K\otimes_k L_{\mathbf{x}}$$

defined by the homomorphism $\sigma(\rho)$ from (6.1.1).

(6.2.3) **Proposition.** *A deformation $Q\to Q/(\mathbf{x})=\tilde{R}$ induces a morphism of varieties $\sigma(\rho)\check{}: V_R^*(M)\to V(Q,\mathbf{x},M)$, which is finite if $\mathrm{pd}_Q\tilde{M}<\infty$. When Q is of minimal codimension with this property, then $\sigma(\rho)\check{}$ is a Noether normalization of $V_R^*(M)$.*

In particular, when $\mathrm{vpd}_R M<\infty$, one has $\dim V_R^*(M)=\mathrm{cx}_R M$.

Proof. The k-algebra homomorphism $\Sigma(\rho)$ defines the morphism $\sigma(\rho)\check{}$. Assuming $\mathrm{pd}_Q M<\infty$, we know from (2.1) that $\mathrm{Ext}_R(M,k)$ is finitely generated over $\mathscr{R}_{\mathbf{x}}$, hence also over $\mathscr{R}$. It follows that $\mathscr{R}/\mathrm{Ann}_{\mathscr{R}}\mathrm{Ext}_R(M,k)$ is finitely generated over $\mathscr{R}_{\mathbf{x}}/\mathrm{Ann}_{\mathscr{R}_{\mathbf{x}}}\mathrm{Ext}_R(M,k)$, so that $\sigma(\rho)\check{}$ is surjective. Let Q be of minimal codimension $\mathrm{cx}_R M$, with $\mathrm{pd}_Q\tilde{M}<\infty$. By the preceding and (3.12), we see $V(Q,\mathbf{x},M)=L_{\mathbf{x}}$, hence $\sigma(\rho)\check{}$ is a Noether normalization. Finally, since minimal deformations exist by (3.6) for any M with $\mathrm{vpd}_R M<\infty$, we obtain the last claim. □

The next statement contains the only non-trivial general result I am aware of, concerning the form cohomological varieties of modules can take. The construction is adapted from one due to J. Carlson for representations [Ca_3, Lemma (2.3)]. The proof is – necessarily – completely different, since none of the standard tools of group cohomology is available to us.

(6.3) **Theorem.** *Let f be an arbitrary homogeneous element of the algebra $\mathscr{R}$, defined in* (6.1.2). *Then there exists an R-module $M(f)$ such that $V_R^*(M(f))$ is the hypersurface in $K\otimes\zeta_2(R)$, defined by the equation $f=0$.*

Proof. Choose a minimal R-free resolution $(\mathbb{F},\partial)$ of k. Then $\mathrm{Ext}_R(k,k)=\mathbb{H}\mathrm{om}_R(\mathbb{F},k)=\mathbb{H}\mathrm{om}_k(\mathbb{F}/\mathfrak{m}\mathbb{F},k)$: under these identifications, the polynomial $f\in\mathscr{R}^{2a}\subset\mathrm{Ext}_R^{2a}(k,k)$ defines a k-linear homomorphism $\alpha: F_{2a}/\mathfrak{m}F_{2a}\to k$. Writing N for $\mathrm{Im}\,\partial_{2a}=\mathrm{Ker}\,\partial_{2a-1}$, and noting that because of the minimality of $\mathbb{F}$, $k\otimes\partial_{2a}: k\otimes F_{2a}\to k\otimes N$ is an isomorphism, we set ϕ to be the composition of R-linear homomorphisms:

$$N\longrightarrow N/\mathfrak{m}N\xrightarrow{(k\otimes\partial_{2a})^{-1}}F_{2a}/\mathfrak{m}F_{2a}\xrightarrow{\alpha}k.$$

Now $M(f)$ is defined from the exact sequence:

(6.3.1) $$0\to M(f)\to N\xrightarrow{\phi}k\to 0.$$

Consider the long exact sequence

$$0 \to N \to F_{2a-1} \to F_{2a-2} \to \dots \to F_0 \to k \to 0$$

Breaking it down into short exact sequences and applying $\mathrm{Ext}_R^*(\ ,k)$ repeatedly, one ends up with a degree zero isomorphism of $\mathrm{Ext}_R^*(k,k)$-modules $\mathrm{Ext}_R^*(N,k) \cong s^{-2a}\mathrm{Ext}_R^{\geqq 2a}(k,k)$, where $s^b E$ is the graded module with $(s^b E)_n = E_{n-b}$. This explains the bottom row of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}_R^*(k,k) & \xrightarrow{f} & s^{-2a}\mathrm{Ext}_R^*(k,k) & \longrightarrow & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{Ext}_R^*(\phi,k)} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}_R^*(N,k) & \longrightarrow & s^{-2a}\mathrm{Ext}_R^*(k,k) & \longrightarrow & s^{-2a}\mathrm{Ext}_R^{<2a}(k,k) & \longrightarrow & 0 \end{array} \qquad (6.3.2)$$

whose top row is multiplication by $f \in \mathrm{Ext}_R^{2a}(k,k)$, followed by the projection onto $C = \mathrm{Coker} f$. The exactness of the bottom row has already been noted; that of the top row reflects the fact that f is a non-zero divisor in $\mathrm{Ext}_R(k,k)$ (by Poincaré-Birkhoff-Witt). The diagram is commutative: for the left-hand square this is due to the naturality of Yoneda pairings, and for the other one it holds by construction. Finally, all maps are degree zero homomorphisms of $\mathrm{Ext}_R(k,k)$-modules, because of the centrality of f.

As a first input from (6.3.2), note the injectivity of $\mathrm{Ext}_R^*(\phi,k)$; applying this to the exact triangle of $\mathrm{Ext}_R^*(k,k)$-modules

$$\begin{array}{ccc} \mathrm{Ext}_R(k,k) & \xrightarrow{\mathrm{Ext}_R(\phi,k)} & \mathrm{Ext}_R(N,k) \\ & \nwarrow \qquad \swarrow & \\ & \mathrm{Ext}_R(M(f),k) & \end{array}$$

derived from the exact sequence (6.3.1), one sees that Coker $\mathrm{Ext}_R^*(\phi,k)$ is naturally isomorphic with $\mathrm{Ext}_R^*(M(f),k)$. Returning to (6.3.2), we now obtain from the snake lemma the exact sequence of $\mathrm{Ext}_R(k,k)$-modules:

$$0 \to \mathrm{Ext}_R^*(M(f),k) \to s^{-2a}\left(\frac{\mathrm{Ext}_R^*(k,k)}{f \cdot \mathrm{Ext}_R^*(k,k)}\right) \to s^{-2a}\mathrm{Ext}^{<2a}(k,k) \to 0.$$

Since $\mathrm{Ext}_R^*(k,k)$ is free over $\mathscr{R}$ (Poincaré-Birkhoff-Witt again), the module in the brackets is free over $\mathscr{R}/(f)$, on a homogeneous basis of the vector space $k \otimes_{\mathscr{R}} \mathrm{Ext}_R(k,k)$. It follows that

$$\begin{aligned} &\mathrm{Ann}_{\mathscr{R}}\mathrm{Ext}_R^*(M(f),k) \supset (f) \\ &= \mathrm{Ann}_{\mathscr{R}}(\mathrm{Ext}_R^*(k,k)/f\,\mathrm{Ext}_R^*(k,k)) \supset (\mathrm{Ann}_{\mathscr{R}}\mathrm{Ext}_R^*(M(f),k))^{2a}, \end{aligned}$$

hence $V_R^*(M(f))$ is the hypersurface $f=0$. □

For further reference, we save the argument between (6.3.1) and (6.3.2) in the form of the following statement:

(6.4) **Lemma.** *Assume there exists a long exact sequence*

$$0 \to M' \to F_{n-1} \to \dots \to F_0 \to M'' \to 0$$

with F_i free R-modules for $1 \leqq i \leqq n$. There is then equality $V_R^*(M') = V_R^*(M'')$. □

Even for complete intersections, we do not know whether every k-rational variety in $K \otimes \pi_2(R)$ has the form $V_R^*(M)$ (cf., however, (7.6) below). That the simplest ones are of this form, is shown by the following argument, suggested to me by D. Eisenbud.

(6.5) **Proposition.** *Assume R is a complete intersection. If $H \subset K \otimes_k \pi_2(R)$ is a k-rational linear subspace, then there is an R-module M of finite length, such that $V_R^*(M) = H$.*

Proof. Since the $\hat{R}$-modules of finite length have finite length as modules over R as well, we may assume R is complete, and take it in the form $R = Q/(x_1, \dots, x_c)$, with $(Q, \mathfrak{n}, k)$ regular local, and $x_i \in \mathfrak{n}^2$. Let $d = \dim_k H \geqq 0$. Changing coordinates if necessary, we may assume H is spanned by $\bar{x}_1, \dots, \bar{x}_d$ (cf. (3.8) for notation). Denote by Q' the complete intersection $Q/(x_1, \dots, x_d)$, and let M' be the n'th syzygy in the minimal resolution $\mathbb{F}'$ of k over Q', with n chosen large enough. Then $x_{d+1}, \dots, x_c$ is a regular sequence both on M' and on Q', so that $\mathbb{F}_{\geqq n}/(x_{d+1}, \dots, x_c)\mathbb{F}_{\geqq n}$ is a minimal R-free resolution of $M = M'/(x_{d+1}, \dots, x_n) M'$. In particular, $\operatorname{Ext}_R^*(M, k)$ and $\operatorname{Ext}_{Q'}^*(M', k)$ are related by an isomorphism compatible with the homomorphism of k-algebras $\operatorname{Ext}_R(k, k) \to \operatorname{Ext}_{Q'}(k, k)$, and with the module structures over these algebras, i.e. with the map

$$\Phi = \mathscr{R}: k[\chi_1, \dots, \chi_c] \to k[\chi_1, \dots, \chi_d] = \mathscr{R}', \qquad \Phi(\chi_i) = 0 \qquad \text{for } i > d.$$

Since $\operatorname{Ann}_{\mathscr{R}'} \operatorname{Ext}_{Q'}(M', k) = 0$ by (6.4) and (6.2.2), $\operatorname{Ann}_{\mathscr{R}} \operatorname{Ext}_R(M, k) = (\chi_{d+1}, \dots, \chi_c)$, i.e. $V_R^*(M) = H$, as claimed. □

(6.6) **Corollary.** *If R is a complete intersection, then all possible values of* $\operatorname{cx}_R M$, *namely*

$$0 \leqq \operatorname{cx}_R M \leqq \operatorname{edim} R - \operatorname{depth} R,$$

and all possible values of $\operatorname{vpd}_R M$, *namely*

$$0 \leqq \operatorname{vpd}_R M \leqq \operatorname{edim} R,$$

are obtained for suitable finitely generated R-modules M.

Proof. In the proposition we have constructed modules of finite length, with complexity between 0 and edim R − depth R. By Theorem (3.5), they have virtual projective dimensions between depth R and edim R. Taking modules with finite projective dimensions varying between 0 and depth R, we get the same range of $\operatorname{vpd}_R M$ by (3.4.2). Finally, (3.2.3) and (3.7) assert that we have now obtained all possible values for the complexity and for the virtual projective dimension. □

We finish this section by using (6.3) in order to answer a question of Eisenbud. Namely, in [Ei, Theorem 3.1] he proves that for Q regular, $R=Q/(\mathbf{x})$ for some regular sequence $\mathbf{x}$, and *infinite* residue field k, there exists a linear combination of the form $t=t_1(Q,\mathbf{x},M)+\sum_{i\geq 2} a_i t_i(Q,\mathbf{x},M)$ with $a_i\in Q$, such that $t\colon F_{i+2}\to F_i$ is surjective for sufficiently large i ($\mathbb{F}$ denotes the R-minimal resolution of M). Furthermore, he raises the problem [Ei, p. 44] of whether the restriction to the case of infinite k is necessary. That it is, is shown by the next

(6.7) **Proposition.** *Let $(Q,\mathfrak{n},k)$ be a regular local ring, such that k contains q elements. Let $I\subset\mathfrak{n}^2$ be an ideal generated by a Q-regular sequence of length c.*

When $c\geq 2$, there exists an R-module M, such that for any regular sequence $\mathbf{x}$ generating I, no Q-linear combination, t, of the operators $\{t_j(Q,\mathbf{x},M)\mid j=1,\dots,c\}$, is eventually surjective.

Proof. Let $\chi_1,\dots,\chi_c$ be a k-basis of $\mathscr{R}^2$, and set

$$f=\det\begin{pmatrix}\chi_1 & \chi_2 & \cdots & \chi_c\\ \chi_1^q & \chi_2^q & \cdots & \chi_c^q\\ \vdots & \vdots & & \vdots\\ \chi_1^{q^{c-1}} & \chi_2^{q^{c-1}} & \cdots & \chi_c^{q^{c-1}}\end{pmatrix},\quad M=M(f).$$

It is a known and easily verified fact that f is equal to the product of all forms $\chi=\kappa_1\chi_1+\dots+\kappa_{j-1}\chi_{j-1}+\chi_j$, where $1\leq j\leq c$, and $\kappa_i\in k$. By (6.3) $V_R^*(M)$ is the set of points in K^c, for which $f=0$, i.e. the union of all hyperplanes in K^c, which are defined over k. Thus, for each $\chi\in\mathscr{R}^2$, the multiplication $\operatorname{Ext}_R^i(M,k)\xrightarrow{\chi}\operatorname{Ext}^{i+2}(M,k)$ has a non-trivial kernel for infinitely many values of i.

Let now t be an arbitrary Q-linear combination of the $t_j(Q,\mathbf{x},M)$. Then $\mathbb{H}\mathrm{om}(t,k)\colon \mathbb{H}\mathrm{om}(\mathbb{F},k)\to\mathbb{H}\mathrm{om}(\mathbb{F},k)$ induces, according to (6.2.3), maps $\operatorname{Ext}_R^i(M,k)\to\operatorname{Ext}_R^{i+2}(M,k)$ given by multiplication with some $\chi\in\mathscr{R}^2$. We already know it is not injective for infinitely many i's, hence (by Nakayama) $t\colon F_{i+2}\to F_i$ is not surjective for infinitely many i's. □

7. Applications to modular representations

In this section k denotes an algebraically closed field of characteristic $p>0$, G is a finite group and M stands for a finite-dimensional representation of G over k. The case in which the preceding results are directly applicable is easily described: $k[G]$ is a commutative local ring if and only if G is an abelian p-group A.

The seeming isolation of these specific groups is misleading: due to results of Quillen, Alperin-Evans, and Avrunin-Scott, many problems for general G are reduced to studying precisely this situation. What is important to us is that $k[A]$ is a complete intersection of a very special sort. Recall that in [AE] the complexity $c_G M$ is defined to be the smallest integer d, such that there exist a $k[G]$-projective resolution $\mathbb{P}$ of M and a $\gamma > 0$, with $\dim_k P_n \leqq \gamma n^{d-1}$ for all $n \geqq 1$. For $G = A$ as above, and $R = k[A]$, this clearly agrees with $\mathrm{cx}_R M$ as defined in (3.1).

(7.1) **Theorem.** *Let $M \neq 0$ be a finite-dimensional representation of the abelian p-group A. Then $c_A(M)$ is equal to the least integer d', such that the group of normed units of $k[A]$ contains a direct product $B \times C$ of subgroups with the following properties:*

(1) *B is minimally generated by d' elements;*

(2) *M is a free $k[C]$-module.*

(3) *$k[A] = k[B \times C]$.*

Proof. For B and C as above, the canonical isomorphisms $\mathrm{Tor}_n^{k[A]}(M, k) \cong \mathrm{Tor}_n^{k[B]}(M \otimes_{k[C]} k, k)$, $n \geqq 0$, show that $c_A(M) = c_B(M \otimes_{k[C]} k) \leqq d'$. Thus, in order to prove the theorem, we have to show there exist B and C as above, with B minimally generated by $d = c_A(M)$ elements.

Decompose A into a direct product $\langle a_1 \rangle \times \ldots \times \langle a_c \rangle$ with a_i of order p^{e_i} and $0 < e_1 \leqq e_2 \leqq \ldots \leqq e_c$. The map $X_j \to a_j - 1$ $(1 \leqq j \leqq c)$ extends to a surjective ring homomorphism $Q = k[\![X_1, \ldots, X_c]\!] \to k[A]$ with kernel generated by $x_1 = X_1^{p^{e_1}}, \ldots, x_c = X_c^{p^{e_c}}$. By (3.12), there exist $m = c - d$ linearly independent over k elements

$$y'_i = \sum_{j=1}^{c} a'_{ij} x_j, \qquad a'_{ij} \in k,$$

such that $\mathrm{pd}_{Q'} M < \infty$, for $Q' = Q/(y'_1, \ldots, y'_m)$.

Next we modify the basis of the k-vector space V spanned by the y'_i's as follows. Let $j_1, \ldots, j_q$ be the integers for which $e_{j_r} < e_{j_r+1}$; thus, setting $f_r = e_{j_r}$ for $1 \leqq r \leqq c$, the monomials which generate $(\mathbf{x})$ are written as:

$$X_1^{f_1}, \ldots, X_{j_1}^{f_1}, X_{j_1+1}^{f_2}, \ldots, X_{j_2}^{f_2}, \ldots, X_{j_q}^{f_q}, \ldots, X_c^{f_q}$$

Denote by i_1 the k-rank of the matrix $(a'_{ij})_{\substack{1 \leqq i \leqq m \\ 1 \leqq j \leqq j_1}}$. Pick i_1 linearly independent rows of this matrix, and call $y_1, \ldots, y_{i_1}$ the y'_i's which have produced them. Subtract suitable linear combinations of $y_1, \ldots, y_{i_1}$ from the rest of the y'_i's, in order to obtain a k-basis $y_1, \ldots, y_{i_1}, y''_{i_1+1}, \ldots, y''_c$ of V, in which y''_i are k-linear combinations of $X_{j_1+1}^{f_2}, \ldots, X_c^{f_q}$, for $i+1 \leqq i \leqq c$. Repeating the procedure on the submatrix $(a_{ij})_{\substack{i_1 < i \leqq m \\ j_1 < j \leqq c}}$ and iterating, we end up with a basis of V of the form

$$y_i = \sum_{j=1}^{c} b'_{ij} x_j, \qquad 1 \leqq i \leqq m,$$

where the $m \times c$ matrix (b'_{ij}) is block upper-triangular, with diagonal block B_{ss} having size $(i_s - i_{s-1}) \times (j_s - j_{s-1})$, and rank $B_{ss} = i_s - i_{s-1} \leqq j_s - j_{s-1}$ for $1 \leqq s \leqq q$ (we set $i_0 = j_0 = 0$).

Since k is algebraically closed, there exist $b_{ij} \in k$ such that $b'_{ij}=(b_{ij})^{p^{f_j}}$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq c$. Now, for i such that $i_{s-1} < i \leqq i_s$, set

$$Y_i = \sum_{j=1}^{c} b_{ij} X_j^{p^{(f_j - f_s)}}$$

this is well defined for $1 \leqq i \leqq m$, since $b_{ij}=0$ when $j \leqq j_{s-1}$, and $f_j \geqq f_s$ when $j > j_{s-1}$.

The polynomials $Y_1, \ldots, Y_m \in k[\![X_1, \ldots, X_c]\!]$ have no constant term, and their linear parts are linearly independent over k. Consequently, one can find $c-m$ among the X_i's, call them $Y_{m+1}, \ldots, Y_c$, such that $k[\![Y_1, \ldots, Y_c]\!] = k[\![X_1, \ldots, X_c]\!]$. For $1 \leqq i \leqq m$, set $g_i = f_s$, when $i_{s-1} < i \leqq i_s$; for $m+1 \leqq i \leqq c$, set $g_i = e_t$, when $Y_i = X_t$. Finally, set $R' = k[Y_1, \ldots, Y_m]/(Y_1^{p^{g_1}}, \ldots, Y_m^{p^{g_m}})$. In this notation we have

$$Q' = Q/(y_1, \ldots, y_m) = R'[\![Y_{m+1}, \ldots, Y_c]\!].$$

The fact that M has finite projective dimension over Q' implies it has the same property over R'. Being finitely generated over the local artinian ring R', it is necessarily free over it. Next note that the projection $Q \to R$ factors through

$$R'[\![Y_{m+1}, \ldots, Y_c]\!]/(Y_{m+1}^{p^{g_{m+1}}}, \ldots, Y_c^{p^{g_c}}) = R' \otimes_k R'',$$

where $R'' = k[Y_{m+1}, \ldots, Y_c]/(Y_{m+1}^{p^{g_{m+1}}}, \ldots, Y_c^{p^{g_c}})$. Since $\dim_k(R' \otimes R'') = \prod_{i=1}^{c} p^{g_i} = \prod_{i=1}^{c} p^{e_i} = \dim_k R$, we have $R' \otimes_k R'' \cong R$ as k-algebras. Denote now by C (resp. B) the image in R of the multiplicative group H' (resp. H'') generated in R' (resp. R'') by the classes of $1 + Y_i$ for $1 \leqq i \leqq m$ (resp. $m+1 \leqq i \leqq c$). Since $R' = k[H']$ and $R'' = k[H'']$, the theorem is proved. □

For an abelian p-group A, and for a non-negative integer d, write $\omega^d(A) = e_1 + \ldots + e_{c-d}$ if $A \cong C_{p^{e_1}} \times \ldots \times C_{p^{e_c}}$, where C_h is a cyclic group of order h, and $0 < e_1 \leqq e_2 \leqq \ldots \leqq e_c$. If G is an arbitrary finite group, set $\omega_p^d(G) = \max\{\omega^d(A)\}$ when A ranges over all abelian p-subgroups of G. Finally, recall that the p-rank of G is the maximum n, for which G contains a subgroup E isomorphic to $(C_p)^n$. With this notation, we have:

(7.2) **Corollary.** *Let M be a representation of G with $c_G(M) = d$. Then $p^{\omega_p^d(G)}$ divides $\dim_k M$, and $\omega_p^d(G) \geqq (p\text{-rank } G) - d \geqq 0$.*

Proof. Let A be a subgroup with $\omega_p^d(G) = e_1 + \ldots + e_{c-d}$. Since $d' = c_A(M) \leqq c_G(M)$ (straightforward from the definitions), then (7.1.3) provides a direct product decomposition $A \cong B \times C$, with C minimally generated by $c - d'$ elements by (7.1.1). The minimal order of such a C being $p^{e_1 + \ldots + e_{c-d'}} \geqq p^{e_1 + \ldots + e_{c-d}}$, (7.1.2) shows that $p^{\omega_p^d(G)}$ divides $\dim_k M$. The first inequality of the corollary is obvious.

For the second one choose – by the Quillen-Alperin-Evans dimension theorem [AE] – an elementary abelian subgroup E of rank $n=p$-rank G, and such that $c_E(M)=c_G(M)=d$, and finish by invoking e.g. (3.2.3). □

(7.3) *Remarks.* (1) In the special case when $c_G(M)=1$, Theorem (7.1) gives the central result – Theorem 5.1 – of Carlson [Ca_1]; the proof given there is different, and involves the analysis of a number of special cases. With the same assumption on M, Corollary (7.2) coincides with [Ca_1, Corollary 5.6] and with [Ei, Theorem 9.1].

(2) Assume $A=E$ is an elementary abelian group of rank c and M is arbitrary. In this case one recovers all the main results of Kroll's paper [Kr] as follows: Theorems 1.2 and 1.3 from (7.1), Theorem 1.5 from (7.2), and Proposition 1.7 from the equality of (i) and (iii) in (3.12). The last Proposition has been sharpened in (4.2).

(3) The result of (7.2) is "best possible", since $M=k[A]/(a_{c-d+1}-1, \ldots, a_c - 1)$ has dimension $\omega^d(A)$ and complexity d.

(7.4) *The cohomological varieties associated to a representation.* Recall that $H^*(G, M)$, the cohomology of the group G with coefficients in M, is a graded module over the graded ring $H^*(G, k)$ in such a way, that the isomorphisms $H^*(G, -)\cong \mathrm{Ext}^*_{k[G]}(k, -)$ transform these structures into Yoneda products. In compliance with the notation of [Qu], [AS], etc., we write $H^*(G)$ to denote the commutative ring $H^*(G, k)$ when $p=2$, and $H^{\mathrm{even}}(G, k)$ when p is odd.

The *cohomological variety* $V_G(M)$ of a p-group G is defined to be the maximal ideal spectrum of $H^*(G)/\mathrm{Ann}_{H^*(E)} H^*(G, M)$ with the Zariski topology: cf. [Qu], [Ca_2].

Next we fix an elementary abelian p-group E, minimally generated by $a_1, \ldots, a_c$ and denote by J the k-subspace of $k[E]$ spanned by $a_1-1, \ldots, a_c-1$.

The *rank variety* $V_E^r(M)$ of a representation $M \neq 0$ of E consists of 0 and of these $f \in J$, for which M is *not* a free $k[f]$-module [Ca_2]; one also sets $V_E^r(0)=\emptyset$.

It is not obvious from the description that $V_E^r(M)$ is an algebraic variety: this is proved by Carlson [Ca_2, Theorem (4.3)]. That these two varieties coincide was conjectured by him on the basis of evidence obtained in [Ca_2], and was proved by Avrunin and Scott [AS, Theorem (1.1)]. We shall now show this fact is contained in the results of Sect. 3.

To fix notation, we identify $k[E]$ with $k[\![X_1, \ldots, X_c]\!]/(X_1^p, \ldots, X_n^p)$, by sending X_i to a_i-1. The ring $H^*(E, k)=\mathrm{Ext}_R(k, k)$ has a well known structure, described in terms of the c-dimensional vector spaces $T=\mathrm{Hom}_k(J, k)$, and $U=\mathrm{Hom}_k(L_x, k)$ (cf. (3.8)), situated in degrees 1 and 2 respectively. Namely:

– when p is odd, $H^*(E, k)=\Lambda^* T \otimes_k S^* U$ (Λ^* = exterior algebra, S^* = symmetric algebra);

– when $p=2$, $H^*(E, k)=S^* T$, and the natural injection of U in $H^*(E, k)$, cf. (6.1.3), identifies it with $\{t^2 \mid t \in T\}$.

In either case, the polynomial algebra $\mathcal{R}$ of (6.1.2) can be described as the subalgebra $k[U]$ of $H^*(E)$. Since both algebras act on $\mathrm{Ext}^*_R(M, k)$ via the embedding $H^*(E) \subset H^*(E, k)=\mathrm{Ext}_R(k, k)$, the inclusion $\mathcal{R} \to H^*(E)$ defines a homomorphism of rings α: $\mathcal{R}/\mathrm{Ann}\,\mathrm{Ext}_R(M, k) \to H^*(E)/\mathrm{Ann}\,\mathrm{Ext}_R(M, k)$.

On the other hand, consider the map $\beta: J \to L_{\mathbf{x}}$, given by $f \to \bar{g}^p$, where g is an element of $\mathfrak{n}=(X_1, \ldots, X_c)$ mapping to f, and the bar refers to the image in $(\mathbf{x})/\mathfrak{n}(\mathbf{x})$, $\mathbf{x}=X_1^p, \ldots, X_c^p$.

(7.5) **Theorem.** *The homomorphism α and the map β induce isomorphisms*

$$V_E(M) \xrightarrow{\alpha^*} V_R(M) \xleftarrow{\beta_*} V_E^r(M)$$

of algebraic varieties.

Proof. To see that α^* is an isomorphism it suffices to show that α induces an isomorphism of rings, $\mathscr{R}/\sqrt{\operatorname{Ann}\operatorname{Ext}_R(M,k)} \to H^*(E)/\sqrt{\operatorname{Ann}\operatorname{Ext}_R(M,k)}$. When p is odd, this is obvious from the fact that $H^*(E)=\mathscr{R}\oplus$ *(nilpotent elements)*. When $p=2$, an inverse homomorphism is induced by the Frobenius endomorphism.

By Corollary (3.11), $g^p \notin V_R(M)$ if and only if the projective dimension of M over $R'=k[\![X_1, \ldots, X_c]\!]/(g^p)$ is finite. We shall show this condition is equivalent to the freeness of M over $k[f]$, i.e. to $f \notin V_R^r(M)$: this will finish the proof, since β is bijective because of the algebraic closedness of k.

Let $\mathbb{F}'$ be a minimal free resolution of M over R'. Since $R' \cong k[f] \otimes_k k[\![X_1', \ldots, X_{c-1}']\!]$, this is also a free resolution of M over $k[f]$. Thus $\operatorname{pd}_{R'} M < \infty$ implies $\operatorname{pd}_{k[f]} M < \infty$, which is only possible if M is free over $k[f]$. Conversely, if M is $k[f]$-free, then the exact sequence $\ldots \to F_1' \to F_0' \to M \to 0$ splits over $k[f]$, hence $\mathbb{F}' \otimes_{k[f]} k$ is a $k[\![X_1', \ldots, X_{c-1}']\!]$-free resolution of M/fM, which is necessarily minimal. By the regularity of $k[\![X_1', \ldots, X_{c-1}']\!]$ this implies $F_n'/fF_n'=0$ for $n \geqq c$. Thus one obtains for $n \geqq c$ the equalities $F_n'=fF_n' = f^2 F_n' = \ldots = f^p F_n' = 0$, that is, $\operatorname{pd}_{R'} M \leqq c-1$. □

(7.6) *Remarks.* Using the description of $V_E^r(M)$, it is easily seen that $V_E^r(M \otimes_k N) = V_E^r(M) \cap V_E^r(N)$ [Ca_2, Theorem 3.5]. Combining this with [Ca_3, Lemma 2.3] (cf. also (6.3) above), Carlson observes that *every* subvariety of $V_E^r(k)=k^c$ occurs in the form $V_E^r(M)$ for some finite – dimensional representation M. Finally, standard reduction techniques are used to show that for any finite G, every variety defined by a homogeneous ideal of $H^*(G)$ is of the form $V_G(M)$ for some finite-dimensional representation M.

Thus, in view of (7.5), we can conclude that over rings of the form $R = k[\![X_1, \ldots, X_c]\!]/(X_1^{ep_1}, \ldots, X_c^{ep_c})$, any subvariety of k^c is of the form $V_R(M)$ for some finitely generated R-module M.

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Note added in proof

Examples are now known of rings which have periodic modules of arbitrary periods $q \geq 1$, and also of rings which have non-periodic modules with constant Betti numbers, c.f.: Gasharov, V.N., Peeva, I.N.: Boundedness versus periodicity over commutative local rings. Trans. Am. Math. Soc. (to appear)