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Cancellation Problem of Complete Varieties

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We consider the following problem: Let M, V and W be complete varieties such that $M \times V \cong M \times W$. Then $V \cong W$?

We will see that the obstruction is caused by the Picard schemes of them.

Note. In the following, *variety* means an irreducible reduced proper k-scheme where k is a fixed field of any characteristic.

Definition 1. Let $P(X)$ denote the Picard scheme of a variety X parametrizing all the invertible sheaves on X algebraically equivalent to zero. For any variety S, we denote by $q_S(X)$ the dimension of the closed group subscheme of $P(X)$ generated by the images of all the morphisms $\varphi: S \to P(X)$ such that $\varphi(S) \ni 0$, the point on $P(X)$ corresponding to $\mathcal{O}_X$.

Obviously we have the following

Proposition 2. a) $q_S(X \times Y) = q_S(X) + q_S(Y)$. b) $q_S(X) \leqq q_T(X)$ *if there is a surjective morphism* $T \to S$.

Proposition 3. *Let X and Y are varieties. Then the following conditions are equivalent to each other:* a) $q_X(Y) = 0$. a′) $q_Y(X) = 0$. b) *For any invertible sheaf $\mathcal{L}$ on $X \times Y$, there are $\mathcal{A} \in \mathrm{Pic}(X)$ and $\mathcal{B} \in \mathrm{Pic}(Y)$ such that $\mathcal{L} = p_1^* \mathcal{A} \otimes p_2^* \mathcal{B}$, where $p_1: X \times Y \to X$ and $p_2: X \times Y \to Y$ are projections.*

Proof is easy.

Definition 4. X and Y are said to be *Picard independent* if the conditions in Proposition 3 are satisfied.

Proposition 5. *Let X, Y be Picard independent varieties and let $f: X \times Y \to M$ be a morphism onto a projective variety M such that $f_* \mathcal{O}_{X \times Y} = \mathcal{O}_M$. Then there exist varieties S and T, an isomorphism $i: M \to S \times T$, morphisms $g: X \to S$ and $h: Y \to T$ such that $i \circ f = g \times h: X \times Y \to S \times T$.*

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Proof. Let H be a hyperplane section on $M \subset \mathbb{P}^N$. Then $f^*|H| = |f^* H|$ since $f_* \mathcal{O}_{X \times Y} = \mathcal{O}_M$. So $X \times Y \to M \subset \mathbb{P}^N$ may be identified with the rational mapping ρ associated with the linear system $|f^* H|$. By assumption we have $\mathcal{A} \in \mathrm{Pic}(X)$ and $\mathcal{B} \in \mathrm{Pic}(Y)$ such that $f^* H = p_1^* \mathcal{A} \otimes p_2^* \mathcal{B}$. Restricting to fibers of p_1 and p_2, we infer that $\mathcal{A}$ and $\mathcal{B}$ are generated by global sections. Let α: $X \to \mathbb{P}^a$ and β: $Y \to \mathbb{P}^b$ be the rational mappings (morphisms in this case) associated with $\mathcal{A}$ and $\mathcal{B}$. We have natural isomorphisms $H^0(M, H) \cong H^0(X \times Y, f^* H) \cong H^0(X \times Y, p_1^* \mathcal{A} \otimes p_2^* \mathcal{B}) \cong H^0(X, \mathcal{A}) \otimes H^0(Y, \mathcal{B})$. This induces an imbedding σ: $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^N$ such that $\sigma(\alpha(x), \beta(y)) = \rho(x, y)$ for any $x \in X$, $y \in Y$. Letting $S = \mathrm{Im}(\alpha)$ and $T = \mathrm{Im}(\beta)$, we easily see that $M = \mathrm{Im}(\rho) = S \times T$. Now our assertion is obvious.

Theorem 6. *Let M, V and W be varieties such that $M \times V \cong M \times W$. Suppose that M and V are Picard independent and that M is projective. Then $V \cong W$.*

Proof. We use the induction on $m = \dim M$. Let f: $M \times V \to M \times W$ be the given isomorphism and let π_1: $M \times V \to M$, π_2: $M \times V \to V$, p_1: $M \times W \to M$ and p_2: $M \times W \to W$ be projections. Applying Proposition 5 to $p_1 \circ f$: $M \times V \to M$, we obtain varieties S and T together with morphisms g: $M \to S$, h: $V \to T$ and i: $M \to S \times T$ such that $i \circ p_1 \circ f = g \times h$ and i is an isomorphism. Consider first the case in which $\dim S = m$. Then $\dim T = 0$ and T is a point. So $p_1 \circ f = g \circ \pi_1$: $M \times V \to M (= S)$. For $0 \in M$, we have $W \cong p_1^{-1}(0) \cong f^{-1}(p_1^{-1}(0)) = (p_1 \circ f)^{-1}(0) = \pi_1^{-1}(g^{-1}(0))$, which is a union of fibers of π_1, each of which is isomorphic to V. Hence $V \cong W$.

Second consider the case in which $0 < \dim S < m$. We have $q_T(S) \leqq q_T(M) \leqq q_V(M) = 0$ and $q_V(T) \leqq q_V(M) = 0$. Hence $q_T(S \times V) = q_T(S) + q_T(V) = 0$. Therefore we can apply the induction hypothesis to $T \times (S \times V) \cong M \times V \cong M \times W \cong T \times (S \times W)$ to obtain $S \times V \cong S \times W$. Again by the induction hypothesis we infer $V \cong W$, since $q_V(S) \leqq q_V(M) = 0$.

Thus we have reduced the problem to the case in which $\dim S = 0$. In other words, S is a point and $p_1 \circ f = h \circ \pi_2$ for some h: $V \to M$. Note that $q_M(W) = q_M(M \times W) - q_M(M) = q_M(M \times V) - q_M(M) = q_M(V) = 0$. Namely M and W are Picard independent. So, by the same argument as above, we reduce the problem to the case in which $\pi_1 \circ f^{-1} = j \circ p_2$ for some morphism j: $W \to M$. Now, for a simple point 0 on M, let $U = (h \circ \pi_2)^{-1}(0) \cap \pi_1^{-1}(0)$. Then $W \cong p_1^{-1}(0) \cong (p_1 \circ f)^{-1}(0) = (h \circ \pi_2)^{-1}(0) \cong M \times U$. Note that $U \cong f(U) = p_1^{-1}(0) \cap (j \circ p_2)^{-1}(0)$. So, similarly as above, we infer $V \cong M \times U$. Thus we prove $V \cong W$. q.e.d.

Corollary 7. *Let M, V and W be compact complex manifolds such that $M \times V \cong M \times W$. Suppose that M is projective and that* $\mathrm{Alb}(M) = 0$ *or* $\mathrm{Alb}(V) = 0$. *Then $V \cong W$.*

Indeed, the preceding arguments work also in the analytic category.

Remark 8. Cancellation is not always true in the category of abelian varieties (cf. [7]). But we have the following

Proposition 9. *Let M, V and W be abelian varieties such that $M \times V \sim M \times W$, where $\sim$ denotes the isogeny equivalence. Then $V \sim W$.*

This follows easily from the observation below.

a) Any abelian variety is isogeneous to a product of simple abelian varieties.

b) $q_A(B)$ depends only on the isogeny classes of the abelian varieties A, B.

c) If A and B are simple abelian varieties, $q_A(B)>0$ if and only if $A \sim B$, and $q_A(A) = \dim A$.

Problem 10. Write $X \sim Y$ in general if there exists a variety Z equipped with finite unramified morphisms $Z \to X$ and $Z \to Y$. Then, does $M \times V \sim M \times W$ imply $V \sim W$ for general varieties M, V and W?

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