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# Connes' Analogue of the Thom Isomorphism for the Kasparov Groups

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## Introduction - Notations

Let $(A, G, \alpha)$ be a $C^*$-dynamical system ($G=\mathbb{R}$ or $\mathbb{Z}$).

In [4], Pimsner and Voiculescu give six term exact sequences allowing to compute $K$ and Ext groups of the crossed product $A \rtimes_\alpha G$ in the case $G=\mathbb{Z}$.

In [1], A. Connes shows the existence (and the uniqueness) of an isomorphism of degree 1 between $K^*(A \rtimes_\alpha \mathbb{R})$ and $K^*(A)$. He deduces a new proof of the Pimsner-Voiculescu result about $K$-groups.

Both Ext and $K$ groups are particular cases of the Kasparov groups [3]. In this paper, we generalise the Thom isomorphism of [1] to these groups. Then, using the mapping torus construction, we generalise Pimsner and Voiculescu exact sequences to the Kasparov groups.

Let $(A, \mathbb{R}, \alpha)$ be a $C^*$-dynamical system. $\mathbb{R}$ acts on the crossed product by the dual action $\hat{\alpha}$, and $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$ is naturally isomorphic to $A \otimes \mathscr{K}$, where $\mathscr{K}$ is the algebra of compact operators.

For $C^*$-algebras $A$ and $B$, we note $KK^i(A,B)(i \in \mathbb{Z}/2\mathbb{Z})$ the Kasparov group (cf. [3], §5, Def. 2, p. 606). The product $x \otimes_D y$ is the intersection product of Kasparov (cf. [3], §5, Thm. 6, p. 607).

Recall that for $A$ nuclear separable and $B$ with a countable approximate unit, $\mathrm{Ext}(A,B) \approx KK^1(A,B)$ (cf. [3], §7, Thm. 1, p. 626). The Ext groups are Kasparov ones and coïncide with Brown-Douglas-Fillmore ones for $B=\mathbb{C}$. The isomorphism $\mathrm{Ext}(A,B) \approx KK^1(A,B)$ ($A$ nuclear) used here is described in [3] (Thm. 1, §7, p. 626). $\mathrm{Ext}(\mathbb{C}, B)$ is isomorphic to $K_1(B)$ in the following way: to each exact sequence $\varphi$ of the form

$$0 \to B \otimes \mathscr{K} \to \mathscr{E} \to \mathbb{C} \to 0$$

is associated a connecting map $\delta_\varphi \colon K_0(\mathbb{C}) \to K_1(B)$, and $\varphi \mapsto \delta_\varphi$ (positive generator of $K_0(\mathbb{C})$) is an isomorphism from $\mathrm{Ext}(\mathbb{C}, B)$ into $K_1(B)$. We note $c_1$ the positive generator of $KK(\mathbb{C}, \mathbb{C}) = K^0(pt) = \mathbb{Z}$.

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## The Thom Map

Let us first state the axioms for the Thom map we want to construct.

To each $C^*$-dynamical system $(A, \mathbb{R}, \alpha)$ with $A$ separable, we want to associate maps

$$\varphi_\alpha^i: KK^i(B, A) \to KK^{i+1}(B, A \rtimes_\alpha \mathbb{R}) \quad (B \text{ separable})$$

and

$$\Psi_\alpha^i: KK^i(A, B) \to KK^{i+1}(A \rtimes_\alpha \mathbb{R}, B)$$

($B$ with countable approximate unit) which should satisfy:

*Axiom 1.* If $\alpha_{\mathbb{C}}^0$ is the trivial action of $\mathbb{R}$ on $\mathbb{C}$, then $\varphi_{\alpha_{\mathbb{C}}^0}^0(c_1)$ is the positive generator $e_0$ of $K^1(\mathbb{R}) = \mathbb{Z}$ and $\Psi_{\alpha_{\mathbb{C}}^0}^0(c_1)$ is the positive generator $e_1$ of $\operatorname{Ext}(\mathbb{R}) = \mathbb{Z}$.

*Axiom 2* (naturality). If $\rho: (A, \alpha) \to (B, \beta)$ is an equivariant homomorphism, then

$$(\hat{\rho})_* \circ \varphi_\alpha^i = \varphi_\beta^i \circ \rho_*$$

and

$$(\hat{\rho})^* \circ \Psi_\alpha^i = \Psi_\beta^i \circ \rho^* \qquad (i \in \mathbb{Z}_{/2\mathbb{Z}})$$

where $\hat{\rho}: A \rtimes_\alpha \mathbb{R} \to B \rtimes_\beta \mathbb{R}$ is associated with $\rho$.

These two first axioms are exactly the same as those in [1]. The third axiom will be a little more restrictive:

*Axiom 3* (tensor product). For $x \in KK^i(A, B)$ and $y \in KK^j(D, E)$, we have

$$\Psi_{\alpha \otimes \mathrm{id}_D}^{i+j}(x \otimes_{\mathbb{C}} y) = \Psi_\alpha^i(x) \otimes_{\mathbb{C}} y.$$

For $x \in KK^i(B, A)$ and $y \in KK^j(D, E)$, we have

$$\varphi_{\mathrm{id}_E \otimes \alpha}^{i+j}(y \otimes_{\mathbb{C}} x) = y \otimes_{\mathbb{C}} \varphi_\alpha^i(x).$$

Here, $D$ is assumed to be separable and $E$ to have a countable approximate unit.

*Remarks.* 1) In fact, one needs to use Axiom 3 only with $x = c_1$.

2) Our axioms being stronger than those of [1], we see that for $B = \mathbb{C}$,

$$\varphi_\alpha^i: KK^i(\mathbb{C}, A) = K_i(A) \to K_{i+1}(A \rtimes_\alpha \mathbb{R})$$

is A. Connes' Thom map.

The main result of this paper is:

**Theorem 1.** *There exists a unique Thom map. Moreover, it is an isomorphism.*

Let us first give the existence part of this theorem. For a $C^*$-dynamical system $(A, \mathbb{R}, \alpha)$, the algebra $A$ is canonically imbedded in the multipliers algebra $M(A \rtimes_\alpha \mathbb{R})$. Let $t \to U_t$ be the canonical representation of $\mathbb{R}$ in $M(A \rtimes_\alpha \mathbb{R})$ with $U_t x U_t^* = \alpha_t(x)$ $(x \in A)$.

In what follows, $A$ is assumed to be separable.

Let $F_\alpha$ be the element defined in any representation of $A \rtimes_\alpha \mathbb{R}$ by

$$F_\alpha = \frac{1}{i\Pi} pV \int_{-1}^{+1} \frac{U_t}{t} dt.$$

It is easily seen to be an element of $M(A \rtimes_\alpha \mathbb{R})$ [2] (Prop. 9). We have:

**Proposition 1.** i) *The pair* (id, $F_\alpha$) *defines an element* $t_\alpha$ *of* $KK^1(A, A \rtimes_\alpha \mathbb{R})$.
ii) *If* $\rho: (A, \alpha) \to (B, \beta)$ *is an equivariant map, then* $\hat{\rho}_*(t_\alpha) = \rho^*(t_\beta)$.

*Proof.* i) We have $F_\alpha = F_\alpha^*$ and, by Fourier transform, $(F_\alpha^2 - 1)\, a \in A \rtimes_\alpha \mathbb{R}$ for $a \in A$. Moreover, if a is $C^\infty$ for the action, $[F_\alpha, a] \in A \rtimes_\alpha \mathbb{R}$.
ii) trivial. □

**Proposition 2.** $t_{\alpha_{\mathbb{C}}^0}$ *is the positive generator of* $K^1(\mathbb{R})$.

*Proof.* As an element of Ext$(\mathbb{C}, C_0(\mathbb{R}))$, $t_{\alpha_{\mathbb{C}}^0}$ is the following exact sequence:

$$0 \to C_0(\mathbb{R}) \to C_0(\mathbb{R} \cup \{+\infty\}) \xrightarrow{\sigma} \mathbb{C} \to 0,$$

where $\sigma$ is evaluation at $+\infty$. The corresponding element of $K^1(\mathbb{R})$ is the image of the positive generator of $K_0(\mathbb{C})$ by the index map associated with this exact sequence. Let then $f \in C_0(\mathbb{R} \cup \{+\infty\})$ with $f(+\infty) = 1$; then $\exp(2i\Pi f)$ determines the positive generator of $K^1(\mathbb{R})$. □

**Proposition 3.** $t_{\hat{\alpha}_{\mathbb{C}}^0}$ *is the positive generator of* Ext$(\mathbb{R})$.

*Proof.* As an element of $KK^1(C_0(\mathbb{R}), \mathbb{C})$, $t_{\hat{\alpha}_{\mathbb{C}}^0}$ is the pair $(\lambda, F)$ where $\lambda: C_0(\mathbb{R}) \to \mathscr{L}(L^2(\mathbb{R}))$ is the regular representation, and

$$F = \frac{1}{i\Pi} Pv \int_{-1}^{+1} \frac{V_t}{t} dt$$

where $(V_t \xi)(s) = e^{its} \xi(s)$ $(\xi \in L^2(\mathbb{R}))$.

Let $i$ be the representation of $L^\infty(\mathbb{R})$ by pointwise multiplication. Then $F = i(\varphi)$ with

$$\varphi(s) = \frac{1}{\Pi} \int_{-s}^{+s} \frac{\sin t}{t} dt$$

Put $\varepsilon(s) = \operatorname{sgn}(s)$ and $G = i(\varepsilon)$.

Then $G^2 = 1$ and

$$G(F - G) \in i(C_0(\mathbb{R})) \text{ because } \varphi(0) = 0.$$

It follows that $(F - G)\lambda(f)$ is compact for all $f \in C_0(\mathbb{R})$ and $(\lambda, G)$ determines also $t_{\hat{\alpha}_{\mathbb{C}}^0}$.

Put $P = \frac{G+1}{2}$. As an element of Ext$(\mathbb{R})$, $t_{\hat{\alpha}_{\mathbb{C}}^0}$ is then $P\lambda(.)P: C_0(\mathbb{R}) \to \mathscr{C}(L^2(\mathbb{R}_+))$ where $\mathscr{C}$ is the Calkin algebra. Put $S_t = P\lambda_t P$ $(t \in \mathbb{R}_+)$; one has

$$S_t(\xi)(s) = \begin{cases} \xi(s-t) & \text{if } s \geqq t \\ 0 & \text{if } s < t \end{cases} \qquad (\xi \in L^2(\mathbb{R}_+)).$$

Let

$$f \in C_0(\mathbb{R}), \quad f(t) = \frac{1}{1-it}.$$

Then

$$\lambda(f) = \int_0^{+\infty} e^{-t} S_t \, dt$$

and

$$P\lambda(f)P = \int_0^{+\infty} e^{-t} S_t \, dt.$$

It is clear that $1-2f = -\frac{t-i}{t+i}$ determines the negative generator of $K_1(\mathbb{R})$. Let us compute the index of $S = 1 - 2P\lambda(f)P$. Let $(e_n)$ be the orthonormal basis of $L^2(\mathbb{R}_+)$ obtained by Gram-Schmidt orthonormalization of $\{t^n e^{-t}\}_{n \geqq 0}$. One sees that $Se_n = e_{n+1}$, so that $\operatorname{Ind}(S) = -1$. □

With the notations of [3] (§4, Def. 4, p. 597), we have:

**Theorem 2.** *Put* $\sigma_\alpha = t_\alpha \otimes_{A \rtimes_\alpha \mathbb{R}} t_{\hat{\alpha}} \in KK(A, A)$.

*Then* $\sigma_\alpha$ *does not depend on* $\alpha$ *and is equal to* $\tau_A(c_1)$.

*Proof.* Let $\alpha^t$ ($t \in [0,1]$) be the action on $A$ defined by $(\alpha^t)_s = \alpha_{ts}$ ($s \in \mathbb{R}$). Note $\beta$ the action of $\mathbb{R}$ on $B = A \otimes C([0,1])$ defined by

$$\beta_t(x)(s) = \alpha_t^s(x(s)).$$

Evaluation at $t$, $g_t$: $B \to A$ is an equivariant homomorphism from $(B, \beta)$ to $(A, \alpha^t)$. We deduce

$$\hat{g}_{t*}(t_\beta) = g_t^*(t_{\alpha^t})$$

and

$$\hat{\hat{g}}_{t*}(t_{\hat{\beta}}) = \hat{g}_t^*(t_{\hat{\alpha}^t}).$$

The functional properties of the intersection-product (cf. [3], Thm. 4, §4, p. 597) imply

$$\hat{\hat{g}}_{t*}(t_\beta \otimes_{B \rtimes_\beta \mathbb{R}} t_{\hat{\beta}}) = g_t^*(t_{\alpha^t}) \otimes_{A \rtimes_{\alpha^t} \mathbb{R}} (t_{\hat{\alpha}^t}).$$

Let $f: A \to B$, $f(x) = x \otimes 1$. We have $g_t \circ f = \mathrm{id}_A$, hence $\sigma_{\alpha^t} = f^*(\hat{\hat{g}}_{t*}(\sigma_\beta))$. By homotopy invariance of the $KK$-functor ([3]), Thm. 3, §4, p. 597), $\sigma_{\alpha^t} = \sigma_{\alpha^0}$. But $\sigma_{\alpha^0} = \tau_A(\sigma_{\alpha^0_{\mathbb{C}}}) = \tau_A(c_1)$ (*Prop.* 2 *and* 3). □

*Remark.* In an unpublished proof of his Thom isomorphism, *A.* Connes had already used this homotopy algebra.

## Proof of the Existence and Isomorphism

By Theorem 2, we have

$$t_\alpha \otimes_{A \rtimes_\alpha \mathbb{R}} t_{\hat{\alpha}} = \tau_A(c_1)$$

and

$$t_{\hat{\alpha}} \otimes_A t_{\hat{\hat{\alpha}}} = \tau_{A \rtimes_\alpha \mathbb{R}}(c_1),$$

so that $t_{\hat{\alpha}} = t_\alpha$. Put

$$\Psi^i(x) = x \otimes_A t_\alpha \quad \text{and} \quad \varphi^i_\alpha(x) = x \otimes_A t_\alpha$$

By [3] (Thm. 6, §4), $\Psi^i_\alpha$ and $\varphi^i_\alpha$ are isomorphisms.

Axiom 1 is Proposition 2 and 3. Axiom 2 is Proposition 1, ii) and Axiom 3 is given by the associativity of the intersection-product. □

**Proof of the Uniqueness**

By Axioms 1 and 3, one has for the trivial action $\alpha^0$ on $A$:

$$\begin{aligned}\varphi^i_{\alpha^0}(x) &= \varphi^i_{\alpha^0}(x \otimes_{\mathbb{C}} c_1)\\ &= x \otimes_{\mathbb{C}} \varphi^0_{\alpha^0_{\mathbb{C}}}(c_1)\\ &= x \otimes_{\mathbb{C}} e_0 \qquad (x \in KK^i(D, A)).\end{aligned}$$

This determines $\varphi^i_{\alpha^0}$ for any trivial action.

Let $(B, \beta)$ and $g_t$ $(t \in [0, 1])$ as in Theorem 2.

By equivariance and Axiom 2, one has

$$\hat{g}_{t*} \circ \varphi^i_\beta = \varphi^i_{\alpha^t} \circ g_{t*}.$$

But $g_{t*}$ and $\hat{g}_{t*}$ are isomorphisms by the existence part, so that

$$\varphi^i_\alpha = \hat{g}_{1*} \hat{g}^{-1}_{0*} \varphi^i_{\alpha^0} \ (g_{0*} = g_{1*})$$

is uniquely determined. The same proof works for $\Psi^i_\alpha$. □

**Consequences**

Theorem 1 is a generalization of [1] (Thm. 2, §IV).

An other immediate consequence is the following:

**Corollary 1.** *Let $(A, \mathbb{R}, \alpha)$ be a $C^*$-dynamical system with $A$ nuclear and separable. Then,* $\operatorname{Ext}(A \rtimes_\alpha \mathbb{R})$ *is naturally isomorphic to* $\operatorname{Ext}_0(A) = \operatorname{Ext}(A \otimes C_0(\mathbb{R}))$

By induction, one has the following corollary

**Corollary 2** (cf. [1], corollary 7, §V, p. 27). *Let $(A, G, \alpha)$ be a $C^*$-dynamical system with $A$ separable and $G$ solvable, connected and simply connected Lie group of dimension $j$* (mod 2).

*Then, we have natural isomorphisms*

$$KK^i(A, B) \approx KK^{i+j}(A \rtimes_\alpha G, B)$$

*and*

$$KK^i(D, A) \approx KK^{i+j}(D, A \rtimes_\alpha G)$$

(Usual assumptions on $B$ and $D$).

*Remark.* Let us consider the exact sequence of the pseudo differentiel operators associated with an action of $\mathbb{R}^n$ on $A$ (see [2], Prop. 9):

$$0 \to A \rtimes_\alpha \mathbb{R}^n \to \mathscr{E} \xrightarrow{\sigma} A \otimes C(S^{n-1}) \to 0.$$

To compute the index of an elliptic pseudo differentiel operator $P \in \mathscr{E}$, one may use the intersection product.

Let $\Psi^{n-1} \in KK^{n-1}(S^{n-1}, \mathbb{C})$, be the Bott element, and $t_\alpha \in KK^n(A, A \rtimes_\alpha \mathbb{R}^n)$ be the Thom element. Then,

$$\operatorname{Ind}(P) = [\sigma(P)] \otimes_{\mathbb{C}} \Psi^{n-1} \otimes_A t_\alpha. \text{ (See [2], Thm. 10.)}$$

To get the analogue of the Pismer and Voiculescu six term exact sequences, we need the following:

**Lemma.** *Let $B$ be a $C^*$-algebra and $\theta$ be an automorphism of $B$. Let $e_0 \in KK^1(\mathbb{C}, \mathbb{R})$ be the element associated with the exact sequence:*

$$0 \to C_0(]0,1[) \to C_0(]0,1]) \to \mathbb{C} \to 0.$$

*Then* $(\mathrm{id}^* - \theta^*)\,\tau_B(e_0)$ *is associated with the mapping Torus exact sequence.*

$$0 \to C_0(]0,1[) \otimes B \xrightarrow{i} A \xrightarrow{\sigma} B \to 0$$

*where $A = \{x \in C(\mathbb{R}) \otimes B \mid x(s+1) = \theta(x(s))\}$ and $\sigma$ is evaluation at 0.*

*Proof.* $\tau_B(e_0)$ is associated with the exact sequence

$$0 \to C_0(]0,1[) \otimes B \to C_0(]0,1]) \otimes B \xrightarrow{\Pi_1} B \to 0.$$

For an automorphism $\gamma$ of $B$, put $\mu(\gamma) \in KK^1(B, B \otimes C_0(\mathbb{R}))$ given by

$$0 \to C_0(]0,1[) \otimes B \to C_0([0,1[) \otimes B \xrightarrow{\gamma \circ \Pi_0} B \to 0$$

where $\Pi_0$ and $\Pi_1$ are evaluations at 0 and 1.

Then $\tau_B(e_0) + \mu(\gamma)$ is given by the exact sequence

$$0 \to C_0(]0,1[) \otimes B \otimes M_2 \to \mathscr{E} \xrightarrow{\Pi} B \to 0$$

where

$$\mathscr{E} = \left\{ \begin{bmatrix} x_1, x_2 \\ x_3, x_4 \end{bmatrix} \middle| \begin{matrix} x_1 \in C_0(]0,1]) \otimes B,\ x_4 \in C_0([0,1[) \otimes B, \\ x_2, x_3 \in C_0(]0,1[) \otimes B,\ \Pi_1(x_1) = \gamma \circ \Pi_0(x_4) \end{matrix} \right\}$$

and

$$\Pi\left( \begin{bmatrix} x_1, x_2 \\ x_3, x_4 \end{bmatrix} \right) = \Pi_1(x_1) = \gamma \circ \Pi_0(x_4).$$

Let

$$R = \begin{bmatrix} \sin \frac{\Pi}{2} t & -\cos \frac{\Pi}{2} t \\ \cos \frac{\Pi}{2} t & \sin \frac{\Pi}{2} t \end{bmatrix} \in M(C_0(]0,1[) \otimes B \otimes M_2).$$

$\tau_B(e_0)+\mu(\gamma)=R(\tau_B(e_0)+\mu(\gamma))R^{-1}$, which is the mapping torus exact sequence for $\gamma$. This shows that $\mu(\mathrm{id})=-\tau_B(e_0)$ and that $(\mathrm{id}^*-\theta^*)\tau_B(e_0)$ is associated with the mapping torus exact sequence. □

**Theorem 3** (cf. [4], Thm. 2.4 and 3.5). *Let $(B,\theta,\mathbb{Z})$ be a $C^*$-dynamical system with $B$ nuclear and separable. Let $D$ be nuclear and separable and $E$ with countable approximate unit.*

*Then, we have the following exact sequences:*

a)
$$\begin{array}{ccccc} & KK(D,B) & \xrightarrow{\mathrm{id}_*-\theta_*} & KK(D,B) & \\ \nearrow & & & & \searrow \\ KK^1(D,B\rtimes_\theta\mathbb{Z}) & & & & KK(D,B\rtimes_\theta\mathbb{Z}) \\ \nwarrow & & & & \swarrow \\ & KK^1(D,B) & \xleftarrow{\mathrm{id}_*-\theta_*} & KK^1(D,B) & \end{array}$$

b)
$$\begin{array}{ccccc} & KK(B,E) & \xleftarrow{\mathrm{id}^*-\theta^*} & KK(B,E) & \\ \swarrow & & & & \nwarrow \\ KK^1(B\rtimes_\theta\mathbb{Z},E) & & & & KK(B\rtimes_\theta\mathbb{Z},E) \\ \searrow & & & & \nearrow \\ & KK^1(B,E) & \xrightarrow{\mathrm{id}^*-\theta^*} & KK^1(B,E) & \end{array}$$

*Proof.* We have $KK^i(D,B\rtimes_\theta\mathbb{Z})=KK^i(D,A\rtimes_\alpha\mathbb{R})$ by strong Morita equivalence, so that

$$KK^i(D,B\rtimes_\theta\mathbb{Z})=KK^{i+1}(D,A).$$

To compute $KK^i(D,A)$, we only have to compute the connecting maps of the exact sequence

$$\begin{array}{ccccc} & KK(D,B\otimes C_0(\mathbb{R})) & \longrightarrow & KK(D,A) & \\ \delta\nearrow & & & & \searrow \\ KK^1(D,B) & & & & KK(D,B) \\ \nwarrow & & & & \swarrow\delta \\ & KK^1(D,A) & \longleftarrow & KK^1(D,B\otimes C_0(\mathbb{R})) & \end{array}$$

which are by [3] (§7, Thm. 2, p. 629) intersection products by $(\mathrm{id}^*-\theta^*)\tau_B(e_0)$. Analogue proof for *b*. (Use [3], §7, Thm. 3, p. 631.) □

*Remarks.* 1) One may use Theorem 3 to compute $\mathrm{Ext}(\mathcal{O}_A,\mathcal{O}_B)$, where $\mathcal{O}_A$ is Cuntz' algebra (cf. [5], Thm. 3.11, p. 19).

2) By using Theorem 1 and 3, one may also extend the universal coefficient formula and the Künneth formula [6] allowing crossed products by $\mathbb{R}$ or $\mathbb{Z}$.

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